A Guided Tour Through Buoyancy Driven Flows and Mixing

Achim Wirth

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Die Natur verbirgt ihr Geheimnis durch die Erhabenheit ihres Wesens, aber nicht durch List

(Nature hides her secret because of her essential loftiness, but not by means of ruse)

Albert Einstein

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Preface

The Webster Dictionary gives these two definitions for buoyancy: (a) the tendency of a body to float or to rise when submerged in a fluid and (b) the power of a fluid to exert an upward force on a body placed in it. So it has to do with a body (of fluid), a surrounding fluid and the force of gravity, as it is the direction of gravity which defines the downward direction.

The major source of energy for fluid motion on the earth surface is the thermal heating by the sun, leading to temperature differences in the atmosphere and the ocean. These temperature difference and differences of other fluid properties (as e.g. salinity, humidity, particles, ...) lead to differences in density which generate fluid motion when subject to the gravitational force. In the interior of our planet the motion of the magma is also generated by density differences. the same is true for the dynamics of the sun. Although the resulting fluid motion is influenced by a variety of processes as the interaction with the boundaries, the rotation of the earth and others, the primary source of the fluid motion are density differences. The same is true for other planets and the stars in the universe.

At smaller scale, the buoyancy force lets rivers flow downslope, rise (descend) hot (cold) air along mountain slopes, generate waves within the atmosphere, the ocean and at their interface.

Buoyancy driven flows are also key in many engineering applications and the heating and circulation of air in houses. This guided tour is inspired by the book “Buoyancy Effects in Fluids” by J.S. Turner.

In this guided tour I try to present the subject: “as simple as possible, but not simpler”

Chapter 1

Introduction and Preliminaries

1.1 Buoyancy of an object in a homogeneous fluid

Some authors (Gill) define the *buoyancy* of an object or fluid parcel as the negative weight per unit volume

$$B = -gm/V = -g\rho \tag{1.1}$$

(the negative sign is used as an object is less buoyant when it is more dense), where $g \approx 9.81\mathrm{Nkg}^{-1}$, is gravity m is mass and is V the volume of an object. The unit of this buoyancy is $[B] = \mathrm{Nm}^{-3}$, force per volume. When an object is submerged in a fluid the *(relative) buoyancy* in the fluid is then

$$b = -g\frac{\rho_{\text{object}} - \rho_{\text{fluid}}}{\rho_{\text{fluid}}}, \tag{1.2}$$

which is also called buoyancy by some authors (Vallis) its units are those of acceleration $[b] = \mathrm{ms}^{-2}$. Both definitions reflect that buoyancy has to do with density AND gravity. The negative of the relative buoyancy $g' = -b$ is called the reduced gravity.

Objects that are less dense than the fluid, float (b is positive) on the surface of the fluid, objects that are denser, drown (b is negative). The force responsible for the floating of objects is called the buoyancy force and it is due to the increase of pressure with depth in a fluid. In a hydrostatic (motion less) fluid the (hydrostatic) pressure increase with depth as

$$\partial_z P = -g\rho_{\text{fluid}}, \tag{1.3}$$

where P is the pressure, g the acceleration of gravity, ρ the density and z is the vertical coordinate (positive upward). Pressure is due to the weight of the fluid above a unit surface, it is a scalar quantity and is measured in Nm^{-2} (Newton per square metre). Equation (1.3) is also very well approximated in moving fluids as long as the acceleration of the fluid is smaller than the gravitational acceleration g. Calculating the (hydrostatic) pressure as:

$$P(z) = P(z_0) - \int_{z_0}^{z} g\rho_{\text{fluid}} dz, \tag{1.4}$$

and thus neglecting pressure variations due to accelerations in the fluid motion, is called the *hydrostatic approximation.* The force exerted by the pressure P on a surface area dS is $F = PdS$ and it is directed normal to the surface dS ($\mathbf{F} = P\mathbf{n}dS$ with $\mathbf{n} \perp dS$ and $|\mathbf{n}| = 1$. The *buoyancy force* exerted by a fluid on a body is equal to the weight of the fluid displaced by the body

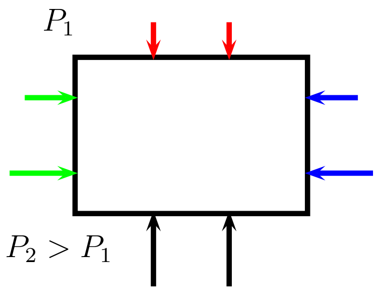


Figure 1.1: Buoyancy is a result of the increasing pressure with depth. Pressure forces from the left and right cancel (green and blue) but those from top and bottom (red and black) do not.

$F_B = g\rho_{\text{fluid}}V = gm_{\text{fluid}}$. This can easily be verified for the case of a cylindrical body in a fluid as seen in fig. 1.1.

Exercise 1: Do the calculation of the buoyancy force for a cylindrical body standing upright, as shown in fig. 1.1.

For arbitrarily shaped objects it is a consequence of *Gauss theorem* :

$$\begin{aligned} F_B &= -\int_S P(z)dS = -\int_V \nabla P(z)dV = -\int_V \partial_z P(z)dV = \int_V g\rho_{\text{fluid}}dV \\ &= g\rho_{\text{fluid}}V = gm_{\text{fluid}}, \end{aligned} \tag{1.5}$$

For a floating (un accelerated body) the buoyancy force balances the weight and we have *Archimedes principle* : Any floating object displaces its own weight of fluid.

Other authors define the buoyancy force as the force exerted by a fluid on a body minus the weight of the body, $F_b = g(\rho_{\text{fluid}} - \rho_{\text{object}})V = g(m_{\text{fluid}} - m_{\text{object}})$, the definition I will adopt in this course. So positive buoyancy means, the object is rising in the fluid, negative, the object is diving and zero buoyancy means, the object is floating.

If the density of the body is larger than the fluid density, the bodies weight, which outside the fluid is given by $F_g = g\rho_{\text{object}}V$, is in the fluid "reduced" by the buoyancy force. The felt weight of the object is

$$F_b = g(\rho_{\text{fluid}} - \rho_{\text{object}})V = g\frac{\rho_{\text{fluid}} - \rho_{\text{object}}}{\rho_{\text{object}}}\rho_{\text{object}}V = -g'\rho_{\text{object}}V = b\rho_{\text{object}}V, \tag{1.6}$$

with:

$$g' = g\frac{\rho_{\text{object}} - \rho_{\text{fluid}}}{\rho_{\text{object}}} = -b. \tag{1.7}$$

So that the dynamics of a body in the fluid is as if the gravity were reduced from g to g'. This concept is only true when the object is homogeneous, completely submerged and not moving. This property is used when training astronauts for space missions under water.

Exercise 2: Explain the preceding sentence ("This concept is only true when the object is homogeneous, completely submerged and not moving.") Find counter examples of the statement for bodies that are not homogeneous and/or a moving).

Exercise 3: A diver of mass 82kg has an average density of 998kgm^{-3}. He dives in a water of constant density 1023kgm^{-3}. What is his volume and weight ? What is the buoyancy force acting on him when he is at the surface, at 10m, 100m depth? The same diver puts on a weight-belt increasing his average density of 1100kgm^{-3}. What is the buoyancy force acting on him when he is at 10m, 100m depth, at the surface? How much time does it take him to "fall" too a depth of 100m.

Exercise 4: A floating iceberg which is formed of sea water, is melting. How does this change the sea-level of the worlds ocean.

Exercise 5: A floating iceberg which is formed of fresh water, is melting. How does this change the sea-level of the worlds ocean.

Exercise 6: A floating iceberg which is formed of sea water and contains a substantial amount of solid rock, is melting. How does this change the sea-level of the worlds ocean.

Exercise 7: (***) The hull of a boat has a parabolic cross-section, it is 50m large and 50m high. Consider the stability of the boat as a function of its density (the boat is supposed to be homogeneous).

Exercise 8: Show that in a homogeneous and incompressible fluid the buoyancy of an (incompressible) object does not depend on depth nor on the orientation of the object in the fluid (Gauss theorem).

1.2 Buoyancy of an object in heterogeneous fluid

We have seen that the buoyancy force acting on an object is given by: $F_b = bV\rho$ when the fluid has varying density the force is also varying in space. The simplest case of a fluid with variable density is to consider the case where the density variation is a linear function of depth: $\rho(z) = \rho_0 + (\partial_z\rho)(z - z_0)$, where $\partial_z\rho$ (< 0) is called the vertical density stratification. If an object of density ρ_{object} is released in a fluid with a density stratification it will settle at an equilibrium depth at which the fluid density equals its density $z_{\text{eq}} = z_0 + (\rho_0 - \rho_{\text{object}})/(\partial_z\rho)$, that is, where its buoyancy vanishes. It will approach this equilibrium by performing oscillations. If friction is neglected the (angular) frequency of these oscillation is given by:

$$N = \sqrt{\partial_z b}, \tag{1.8}$$

it is called the *buoyancy frequency* or *Brunt-Väisälä frequency* . The period of these oscillations is $T = 2\pi/N$ (there is a 2π-factor as it is an angular-frequency). Indeed the buoyancy force per volume is $F_b = g(\partial_z\rho)z$, where z is the vertical distance from the neutral density level (note that $\partial_z\rho < 0$). Newtons second law reads: $\rho\partial_{tt}z = g(\partial_z\rho)z$ which has solutions of the form: $z = A\sin(Nt) + B\cos(Nt)$, where N is given by eq. (1.8), the constants A and B depend on the initial conditions (position and velocity).

In the troposphere $N \approx 10^{-2}\text{s}^{-1}$ leading to oscillations with a period of $\approx$ 10min.

Exercise 9: In the ocean the thermal expansion coefficient is $\alpha \approx 2\,10^{-4}\text{K}^{-1}$. Calculate N for an ocean with a stratification of: (a) 10^{-4}Km^{-1} (deep ocean), (b) 10^{-2}Km^{-1}, (c) 10^{-1}Km^{-1} (thermocline).

Exercise 10: A diver of mass 82kg has an average density of 1020kgm^{-3}. He dives in a water of density 1018kgm^{-3} at the surface and a density gradient of $\partial_z\rho = -2.\,10^{-2}\text{kgm}^{-4}$. What is the buoyancy force acting on him when he is floating at the surface, at 10m, 100m and 200m depth? Give the variation of his depth as a function of time when the diver is initially motion less at the surface. Viscous forces will act on the diver while moving vertically in the water. We suppose that the friction can be described by a linear Rayleigh friction $F_r = rm\partial_t z$. Give the variation of his depth as a function of time when the diver is initially motion less at the surface for the friction values of $r = 1$, $5.\,10^{-2}$ and 10^{-2}s^{-1}.

Exercise 11: In the ocean water-masses are often separated by rather sharp density interfaces. In this exercise we suppose the density interface to be of vanishing thickness. Suppose that in the upper 50m the temperature of a water mass is 20^{o}C and below it is 10^{o}C, the thermal expansion coefficient of sea water is $\alpha = 2.\,10^{-4}\mathrm{K}^{-1}$. Discuss the motion of an object of density ρ that is released at a depth z_0.

1.3 Equation of State and Potential Temperature and Density

Temperature is measured in degrees Celsius (^{o}C) and temperature differences in Kelvin (K), oceanographers are however slow in adapting to the SI unit Kelvin to measure temperature differences.

The temperature of the world ocean typically ranges from -2^oC (-1.87^oC freezing point for $S = 35$ at surface) (freezing temperature of sea water) to 32^oC. About 75% of the world ocean volume has a temperature below 4^{o}C. Before the opening of the Drake Passage 30 million years ago due to continental drift, the mean temperature of the world ocean was much higher. The temperature difference in the equatorial ocean between surface and bottom waters was about 7K compared to the present value of 26K. The temperature in the Mediterranean Sea is above 12^oC even at the bottom and in the Red Sea it is above 20^{o}C.

If one takes a mass of water at the surface and descends it adiabatically (without exchanging heat with the environment) its *in situ* (Latin for: in position; the temperature you actually measure if you put a thermometer in the position) temperature will increase due to the increase of pressure. Indeed if you take a horizontal tube that is 5km long and filled with water of salinity $S = 35$psu and temperature $T = 0^o$C and put the tube to the vertical then the temperature in the tube will monotonically increase with depth reaching $T = 0.40^o$C at the bottom. To get rid of this temperature increase in measurements oceanographers often use *potential temperature* θ (measured in ^{o}C) that is the temperature of a the water mass when it is lifted adiabatically to the sea surface. It is always preferable to use potential temperature, rather than *in situ* temperature, as it is a conservative tracer. Differences between temperature and potential temperature are small in the ocean < 1.5K, but can be important in the deep ocean where temperature differences are small.

In the previous sections we looked at objects of constant density in a fluid of varying density. Things become more involved if the submerged object is itself subject to density changes due to changes in pressure, temperature or other influences. This is the case for a fluid parcel submerged in a fluid. One difficulty is that the density of a fluid parcel is not a quantity that is conserved with the flow. If a parcel of fluid is displaced vertically in the fluid column its density changes due to pressure changes. Another difficulty is that density is difficult to measure directly for air in the atmosphere or sea water in the ocean. The first step is to obtain the density of a fluid as a function of a certain number of its properties. Such a relation is called an equation of state:

$$\rho = \rho(S, T, P) \tag{1.9}$$

where the density of sea water, chosen here as an example, is a function S,T,P, that is salinity, temperature and pressure, respectively. It can be shown that the density of sea water and air is determined by three independent physical properties. For the case of the atmosphere the density is a function of temperature, humidity (water content) and pressure. For an avalanche the density is mostly determined by temperature and water (snow) content. The density is often a non-linear and complicated function of the other fluid properties which is mostly determined empirically in laboratory experiments and looked up in tables or calculated by functions which are a best fit to laboratory measurements.

If one takes a mass of water at the ocean surface or a mass of air at high altitude and descends it adiabatically (without exchanging heat with the environment) its *in situ* (Latin for: in position; the density you actually measure if you put a thermometer in the position) density will increase due to the increase of pressure. To get rid of this density increase in measurements scientists often use *potential density* ρ_θ that is the density of a fluid parcel when it is moved adiabatically to a reference level (the sea level). It is always preferable to use potential density, rather than *in situ* density, as it is a conservative tracer. Differences between density and potential density are unimportant in the laboratory (why?), but can be important in the atmosphere and the deep ocean.

The same concept applies to temperature of sea water or air in the atmosphere. When a mass of fluid is moved adiabatically downward the increase in pressure will increase the temperature ($\approx$ 10K/km in the atmosphere). The *potential temperature* θ is thus introduced. It is the temperature of the fluid when moved adiabatically to a reference pressure (level). A consequence is that the potential density is given as a function of potential temperature. An adiabatic atmosphere is an atmosphere where the potential temperature θ is constant, an isothermal atmosphere is when ($T = const$).

Exercise 12: A diver who weighs 82kg has an average density of 1015kgm^{-3}. He dives in a water of density 1018kgm^{-3} at the surface and a density gradient of $\partial_z \rho = -2.\,10^{-2}$kgm^{-4}. At the surface he has 6litres of air (perfect gas)in his lounges. What is his density, weight and compressibility ? What is the buoyancy force acting on him at 0m, 10m, 100m depth? What happens if he dives to a depth of 10m and exhales half of the air in his lounges?

Exercise 13: Explain the physics of a "Cartesian diver".

1.4 Static Stability

A mass of fluid subject to gravitational acceleration is in equilibrium if the gravitational force is balanced by the pressure force this is called the hydrostatic equilibrium as expressed by eq. (1.4). This shows that for a static fluid the density has to be constant in every horizontal plane, as horizontal pressure differences create horizontal pressure gradients which can not be balanced by a vertical gravitational force (in rotating flows the Coriolis force can balance horizontal pressure gradients leading to a "geostrophic equilibrium"). If lighter fluid superposes heavier fluid this equilibrium is stable as restoring forces counter act to departures of this equilibrium. In the inverse case forces destroy the unstable equilibrium and a stable equilibrium will be established after some time, after a redistribution of fluid masses due to (turbulent) fluid motion occurred. This, often violent, vertical exchange of water masses is called *convection.* (Warning: In older literature and in the engineering community the term "convection" is often used for what is today called "advection")

Exercise 14: Is an isothermal atmosphere stable?

Exercise 15: Is an atmosphere of constant density stable?

Convection is a process of paramount importance in the ocean and atmosphere as it exchanges fluid masses in the vertical.

Static stability and static stability parameters are of great importance to scientist to understand convective weather patterns in the atmosphere and circulation patterns in the ocean.

If the atmosphere is well mixed it has a constant potential temperature and a constant potential density. In this case the atmosphere is called *neutrally stable* as no forces (except friction) act to oppose vertical movement and no energy is necessary to exchange or move around fluid parcels. Considering the stability by adiabatic movement of fluid parcels is called the parcel method. The parcel is a hypothetical expandable box that does not allow any transfer of heat into or out of the box. The stability of the parcel is dependent upon the parcel's motion after a forced displacement from its original location. A parcel that is forced back to its original position is considered stable while one that is forced away from its original position is unstable. One that is displaced with no force acting on it is considered neutral. An important quantity to investigate the stability is the laps rate $\Gamma = -\partial_z T$, which gives the decrease of temperature with height. If the potential temperature θ is constant the atmosphere is adiabatic ($\Gamma_{\text{adiabatic}} \approx 10\text{K/km}$) and the atmosphere is neutrally stable, if $\Gamma < \Gamma_{\text{adiabatic}}$ the atmosphere is stable (the temperature decreases less rapidly than for a mixed atmosphere) and unstable if $\Gamma > \Gamma_{\text{adiabatic}}$.

Exercise 16: Further complication arise because $\Gamma_{\text{adiabatic}}$ is a function of potential temperature and humidity. What is the meaning of absolute stability, absolute instability, conditional instability and potentially unstable.

Exercise 17: What is the layer method of determining stability

1.5 Buoyancy in a fluid of varying density

Gravity is constantly seeking to put fluid in a stably stratified equilibrium. This process is counteracted by other phenomena constantly trying to perturb this equilibrium. The competition between this tendencies creates fluid motion.

Buoyancy is the product of density and gravity, the latter shows only minor variations within the thin layer of fluid motion at the surface of our planet and can thus be treated as a constant for applications in environmental fluid dynamics. Variations in buoyancy are thus equivalent to density variations.

There are conceptually two ways of changing the buoyancy of a fluid parcel, the first is to change its properties as for example pressure, temperature, salinity (for water) and humidity (for air) by displacement, fluxes of heat and fluxes of fresh water (rain) evaporation. Another way is to mix to masses of fluid of different density. Mixing is not possible for an object in a fluid and we will see that mixing plays an important part in the dynamics of buoyancy driven flows in the environment.

1.6 Boussinesq approximation

Density differences in fluids are usually small and their influence on the inertia of a fluid can often be neglected. In the ocean potential density variations are mostly smaller than 3/1000 and in the atmosphere variations of potential density are typically small in the troposphere. The small density differences are however important when considering the buoyancy of fluid volumes. At every horizontal level the average buoyancy force is counter acted by a pressure force due to the presence of horizontal boundaries and only the deviations of buoyancy from the horizontal mean (the relative buoyancy) is dynamically important. The *Boussinesq approximation* consists in neglecting density differences in the equations except if they are multiplied by g which is usually much bigger than the vertical accelerations within the fluid. The Boussinesq approximation is made in the major part of this lecture.

The buoyancy is a convenient quantity when writing down the Bousinesq equations where the acceleration term per fluid volume is usually divided by a reference density. When the buoyancy force per fluid volume is usually divided by a reference density it is buoyancy, so that the equations read:

$$\frac{d}{dt}w = ... + b, \tag{1.10}$$

so the main motivation to introduce buoyancy is to make equations look easier.

When the fluid is homogeneous, of constant density, the gravity force is often omitted in the equations and calculations. This is possible as there is a the hydrostatic pressure gradient that balances the gravitational force $\partial_z P_{hydrostat} = -g\rho_0$. The real preassure can than be written as $P = P_{hydrostat} + P'$. Replacing P by P' and omitting gravity in the equations leads to the same solutions. If there are density differences in the fluid this is no-longer possible.

1.7 Heat Transport

Heat (thermal energy) can be transported by three different mechanisms.

Radiation: Thermal radiation is generated by the thermal motion of charged particles in matter. Every object is radiating heat at a rate which is proportional to the fourth power of its temperature $P = \sigma \cdot A \cdot T^4$ where $\sigma = 5.670373(21)\, 10^{-8}\mathrm{W\ m^{-2}\ K^{-4}}$ is the StefanBoltzmann constant, A the surface area of the object and T its temperature measured in Kelvin.

Molecular transport: is the transport of heat due to molecular motion it is given by its thermal diffusivity κ, which depends on the substance and its physical parameters as for example temperature, pressure, composition $\kappa_{\mathrm{air}} \approx 2.\,10^{-5}\mathrm{m^2s^{-1}}$, $\kappa_{\mathrm{air}} \approx 1.\,10^{-7}\mathrm{m^2s^{-1}}$. It is equivalent to thermal conduction in solids.

Convection: is the advective transport of heat by fluid motion.

Exercise 18: A spherical air parcel of radius of 1m has a temperature anomaly depending on the radius r of $T = 1K(1 - r/(1m)) + 20^oC$ in a surrounding fluid of 20^oC and is moving with a speed of $1\mathrm{ms^{-1}}$. Compare the loss of heat due to radiation, diffusion and the convection due to the motion of the parcel.

Chapter 2

Hydraulics

In this chapter we will suppose that the the fluid is incompressible, that is, density does not depend on pressure. If the density of a fluid is constant the conservation of mass leads to the conservation of volume, that is, the flow is divergence free. We furthermore suppose that the flow is time independent.

2.1 Bernoulli Equation

In this section we will consider a stationary (=time-independent) flow in tubes of varying section in a gravitational field. We neglect friction in the flow due to viscosity. The conservation of volume states that the transport of volume at a time interval Δt through every cross section is equal along the flow ($\Delta t\, u_1 A_1 = \Delta t\, u_2 A_2$). This leads to: $uA = const$ along the tube. This means that in parts of the tube with a smaller cross-section the fluid has to move faster. Lets look at a fluid volume between cross-section 1 and 2. If a fluid particle has moved Δl_1 at location 1 a particle at location 2 has moved Δl_2. The change of kinetic energy of the fluid volume is: $\Delta E_{kin} = (\rho/2)(\Delta l_2 A_2 u_2^2 - \Delta l_1 A_1 u_1^2)$. The force that accelerates the fluid is due to the pressure difference at P_2 and P_1. The difference of work done at location 2 and 1 by the pressure is: $\Delta W = P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2$. Conservation of energy gives: $P_1 - P_2 = (\rho/2)(u_2^2 - u_1^2)$ or $P_1 + (\rho/2)u_1^2 = P_2 + (\rho/2)u_2^2$. The same reasoning can be made for the work done by pressure against gravity and we obtain: $P_1 + g\rho h_1 + (\rho/2)u_1^2 = P_2 + g\rho h_2 + (\rho/2)u_2^2$ or more generally:

$$P + \rho g h + (\rho/2)u^2 = const. \tag{2.1}$$

along any stream-line, which is the Bernoulli equation.

If we consider a flow with a free surface than the Bernoulli equation can be applied to the free surface, which is a streamline with no pressure, and we get

$$g(h + b) + u^2/2 = const. \tag{2.2}$$

We now consider the flow over a topographic obstacle as shown in fig. **??**. Far from the obstacle we have the velocity u_0 and the layer thickness h_0. Bernoulli equation teaches us that: $g(h(x) + b(x)) + u(x)^2/2 = gh_0 + u_0^2/2 = const.$ for all x. This leads to:

$$\frac{g(h+b)}{u_0^2} + \frac{u^2}{2u_0^2} = \frac{gh_0}{u_0^2} + \frac{1}{2} \tag{2.3}$$

$$\frac{gh_0}{u_0^2}(h' + b') + \frac{1}{2h'^2} = \frac{gh_0}{u_0^2} + \frac{1}{2} \tag{2.4}$$

$$h' + \frac{F_0^2}{2h'^2} = 1 - b' + \frac{F_0^2}{2} \tag{2.5}$$

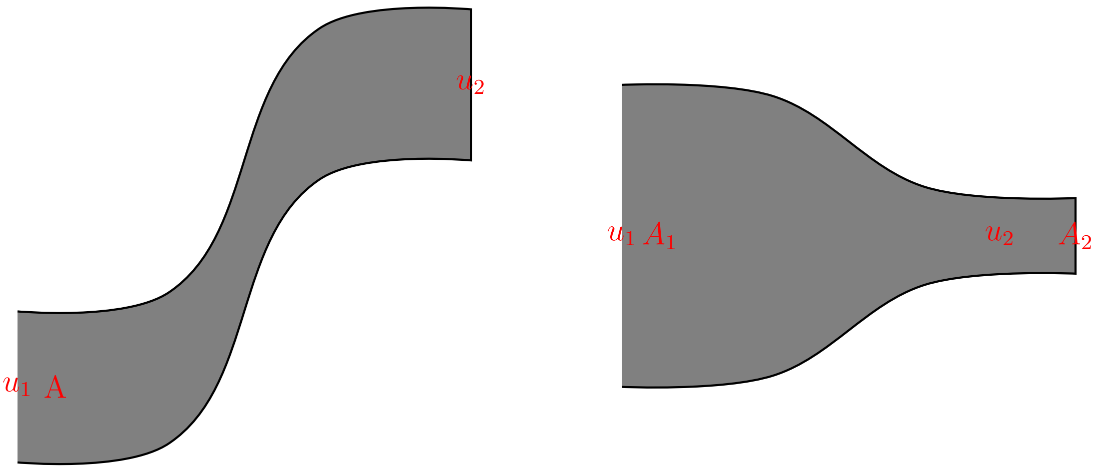


Figure 2.1: Flow through a pipe with varying height in a gravitational field (left). Flow through a pipe with varying diameter.

Where $F_0 = \sqrt{u_0^2/(gh_0)}$ is the Froude number and $h' = h/h_0$ and $b' = b/h_0$.

Together with the equation of conservation of mass we obtain a cubic equation for h. On of the solution has a negative h and is thus not physical. If b increases above a critical value b_c no solution remains and the flow is blocked by the topography.

Exercise 19: Calculate b_c.

Exercise 20: When initially there is no vertical gradient of the horizontal velocity this property is conserved (why?).

If the fluid evolves under a motion less deep-layer of fluid of lower density ρ_0 the pressure term $P = -g\rho_0(h+b)$has to be added to the Bernoulli equation (2.2) and we obtain:

$$g'(h+b) + u^2/2 = const. \tag{2.6}$$

with the reduced gravity $g' = g(\rho - \rho_0)/\rho$.

2.2 Derivation of the Shallow Water Equations from First Principels *ab initio*

To derive the shallow water equtions we assume that the horizontal velocity does not depend on the vertical coordinate and that the pressure can be obtained using the hydorstatic equation (1.4). In figure 2.3 we see the fluid volume A between the coordinates x_1 and x_2 its mass is $m = \rho \int_{x_1}^{x_2} h(x)dx$. Its time evolution is:

$$\partial_t m = \rho(h_1 u_1 - h_2 u_2). \tag{2.7}$$

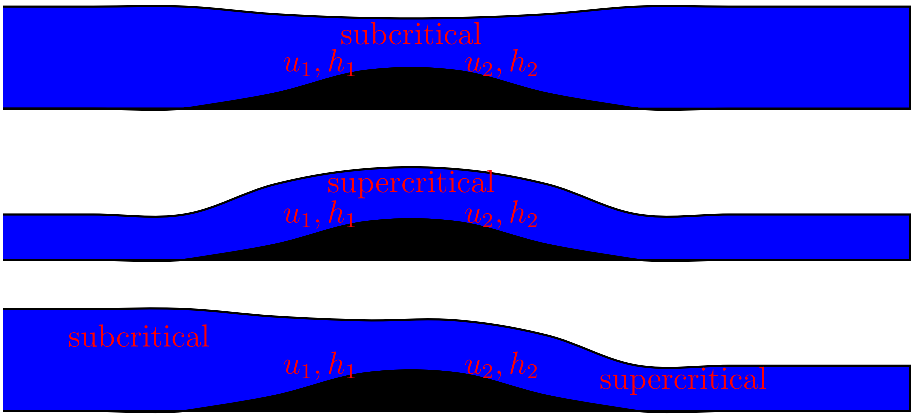


Figure 2.2: Bernoulli equation

If $\Delta x = x_2 - x_1$ is very small we see that to first order $m = \rho h \Delta x$, where h is a value of the height within the interval $[x_1, x_2]$. Introducing this in eq. (2.7) and supposing that the variables are sufficiently smooth, gives:

$$\partial_t h + \partial_x (uh) = 0. \tag{2.8}$$

The inertia of the volume A is $M = \rho \int_{x_1}^{x_2} u(x)h(x)dx$. Its time evolution is:

$$\partial_t M = \rho \left(h_1 u_1^2 - h_2 u_2^2 + \frac{g}{2}(h_1^2 - h_2^2) \right). \tag{2.9}$$

The last term is the pressure force acting at x_1 and x_2, it is calculated using hydrostaticity. The above equation is Newtons second law applied to the are A. If Δx is very small we see that to first order $M = \rho h u \Delta x$. In troducing this in eq. (2.9) gives:

$$\partial_t uh + \partial_x (u^2 h) + gh\partial_x h = 0. \tag{2.10}$$

Exercise 21: Show that the system given by eqs. (2.8) and (2.10) is equivalent to eq. (2.8) and $\partial_t u + u\partial_x u + g\partial_x h = 0$.

2.3 From the Navier-Stokes to the Shallow Water Equations

The dynamics of and incompressible fluid is described by the Navier-Stokes equations:

$$\begin{aligned}
\partial_t u + \quad & u\partial_x u + v\partial_y u + w\partial_z u + \tfrac{1}{\rho_0}\partial_x P &= \nu \nabla^2 u \qquad (2.11)\\
\partial_t v + \quad & u\partial_x v + v\partial_y v + w\partial_z v + \tfrac{1}{\rho_0}\partial_y P &= \nu \nabla^2 v \qquad (2.12)\\
\partial_t w + \quad & u\partial_x w + v\partial_y w + w\partial_z w + \tfrac{1}{\rho_0}\partial_z P &= -g\frac{\rho}{\rho_0} + \nu \nabla^2 w \qquad (2.13)\\
& \partial_x u + \partial_y v + \partial_z w &= 0 \qquad (2.14)\\
+ \quad & \text{boundary conditions} &
\end{aligned}$$

where u is the zonal, v the meridional and w the vertical (positive upward even in oceanography) velocity component, P the pressure, ρ density, ρ_0 the average density, ν viscosity of sea water, g gravity, and $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ is the Laplace operator.

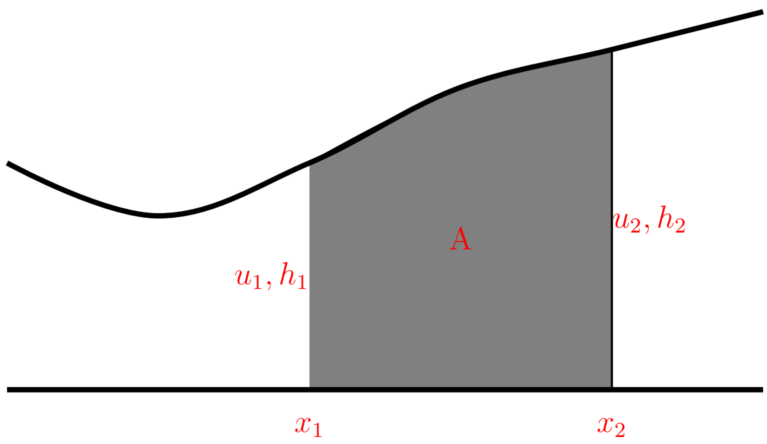


Figure 2.3: Shallow water layer

The equation of a scalar transported by a fluid is:

$$\begin{aligned} \partial_t T + \quad & u\partial_x T + v\partial_y T + w\partial_z T && = \kappa_T \nabla^2 T && (2.15) \\ & +\text{boundary conditions} && && (2.16) \\ \partial_t S + \quad & u\partial_x S + v\partial_y S + w\partial_z S && = \kappa_S \nabla^2 S && (2.17) \\ & +\text{boundary conditions} && , \end{aligned}$$

where T is temperature, S is salinity and κ_T, κ_S are the diffusivities of temperature and salinity. The state equation:

$$\rho = \rho(S, T, P) \tag{2.18}$$

allows to obtain the density from salinity, temperature and pressure.

The above equations describe the motion of the ocean to a very high degree of accuracy, but they are much too complicated to work with, even today's and tomorrows numerical ocean models are and will be based on more or less simplified versions of the above equations.

These equations are too complicated because:

- Large range of scales; from millimetre to thousands of kilometres
- Nonlinear interactions of scales
- How is pressure P determined, how does it act?
- Complicated boundary conditions; coastline, surface fluxes ...
- Complicated equation of state (UNESCO 1981)

A large part of physical oceanography is in effect dedicated to finding simplifications of the above equations. In this endeavour it is important to find a balance between simplicity and accuracy.

How can we simplify these equations? Two important observations:

- The ocean is very very flat: typical depth (H=4km) typical horizontal scale (L=10 000 km)
- Sea water has only small density differences $\Delta\rho/\rho \approx 3 \cdot 10^{-3}$

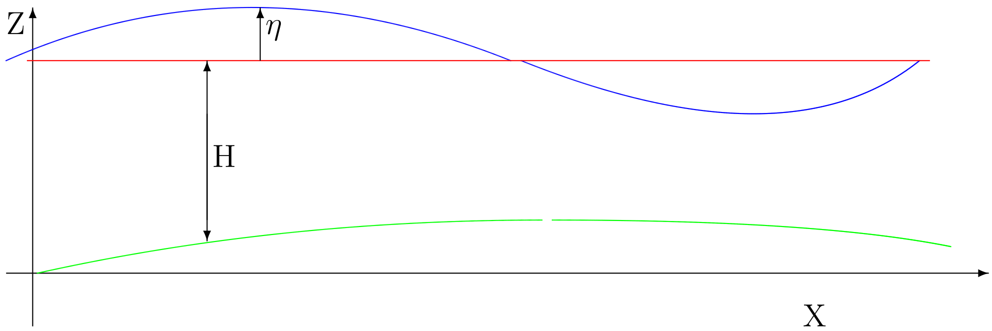


Figure 2.4: Shallow water configuration

Using this we will try to model the ocean as a *shallow homogeneous* layer of fluid, and see how our results compare to observations.

Using the shallowness, equation (2.14) suggests that w/H is of the same order as u_h/L, where $u_h = \sqrt{u^2+v^2}$ is the horizontal speed, leading to $w \approx (Hu_h)/L$ and thus $w \ll u_h$. So that equation (2.13) reduces to $\partial_z P = -g\rho$ which is called the *hydrostatic approximation* as the vertical pressure gradient is now independent of the velocity in the fluid.

Using the homogeneity $\Delta\rho = 0$ further suggest that:

$$\partial_{xz} P = \partial_{yz} P = 0. \tag{2.19}$$

If we derive equations (2.11) and (2.12) with respect to the vertical direction we can see that if $\partial_z u = \partial_z v = 0$ at some time this property will be conserved such that u and v do not vary with depth (We have neglected bottom friction). Putting all this together we obtain the following equations:

$$\begin{aligned}
\partial_t u + \quad & u\partial_x u + v\partial_y u + \tfrac{1}{\rho}\partial_x P && = \nu\nabla^2 u && (2.20)\\
\partial_t v + \quad & u\partial_x v + v\partial_y v + \tfrac{1}{\rho}\partial_y P && = \nu\nabla^2 v && (2.21)\\
& \partial_x u + \partial_y v + \partial_z w && = 0 && (2.22)\\
\text{with} \quad & \partial_z u = \partial_z v = \partial_{zz} w && = 0 && (2.23)\\
+ \quad & \text{boundary conditions} &&
\end{aligned}$$

What are those boundary conditions? Well on the ocean floor, which is supposed to vary only very slowly with the horizontal directions, the vertical velocity vanishes $w = 0$ and it varies linearly in the fluid interior (see eq. 2.23). The ocean has what we call a free surface with a height variation denoted by η. The movement of a fluid particle on the surface is governed by:

$$\frac{d_H}{dt}\eta = w(\eta) \tag{2.24}$$

where $\frac{d_H}{dt} = \partial_t + u\partial_x + v\partial_y$ is the horizontal Lagrangian derivation. We obtain:

$$\partial_t\eta + \quad u\partial_x\eta + v\partial_y\eta - (H+\eta)\partial_z w = 0 \tag{2.25}$$

or

$$\partial_t\eta + \quad u\partial_x(H+\eta) + v\partial_y(H+\eta) + (H+\eta)(\partial_x u + \partial_y v) = 0. \tag{2.26}$$

Using the hydrostatic approximation, the pressure at a depth d from the unperturbed free surface is given by: $P = g\rho(\eta + d)$, and the horizontal pressure gradient is related to the horizontal gradient of the free surface by:

$$\partial_x P = g\rho\partial_x\eta \text{ and } \partial_y P = g\rho\partial_y\eta \tag{2.27}$$

Some algebra now leads us to the shallow water equations (sweq):

$$\begin{aligned}
\partial_t u + & \quad u\partial_x u + v\partial_y u + g\partial_x\eta = \nu\nabla^2 u && (2.28)\\
\partial_t v + & \quad u\partial_x v + v\partial_y v + g\partial_y\eta = \nu\nabla^2 v && (2.29)\\
\partial_t\eta + & \quad \partial_x\left[(H+\eta)u\right] + \partial_y\left[(H+\eta)v\right] = \quad 0 && (2.30)\\
& +\text{boundary conditions} \quad .
\end{aligned}$$

All variables appearing in equations (2.28), (2.29) and (2.30) are independent of z!

2.4 The Linearised One Dimensional Shallow Water Equation

We will now push the simplifications even further, actually to its non-trivial limit, by considering the linearised one dimensional shallow water equations. If we suppose the dynamics to be independent of y and if we further suppose $v = 0$ and that H is constant, the shallow water equations can be written as:

$$\begin{aligned}
\partial_t u + & \quad u\partial_x u + g\partial_x\eta = \quad \nu\nabla^2 u && (2.31)\\
\partial_t\eta + & \quad \partial_x\left[(H+\eta)u\right] = \quad 0 && (2.32)\\
+ & \quad \text{boundary conditions.}
\end{aligned}$$

if we further suppose that $u^2 \ll g\eta$ that the viscosity $\nu \ll g\eta L/u$ and $\eta \ll H$ then:

$$\begin{aligned}
\partial_t u + g\partial_x\eta = 0 && (2.33)\\
\partial_t\eta + H\partial_x u = 0 && (2.34)\\
+\text{boundary conditions,}
\end{aligned}$$

which we combine to:

$$\begin{aligned}
\partial_{tt}\eta = gH\partial_{xx}\eta && (2.35)\\
+\text{boundary conditions.}
\end{aligned}$$

This is a one dimensional linear non-dispersive wave equation. The general solution is given by:

$$\begin{aligned}
\eta(x,t) &= \eta_0^-(ct - x) + \eta_0^+(ct + x) && (2.36)\\
u(x,t) &= \frac{c}{H}(\eta_0^-(ct - x) - \eta_0^+(ct + x)), && (2.37)
\end{aligned}$$

where η_0^- and η_0^+ are arbitrary functions of space only. The speed of the waves is given by $c = \sqrt{gH}$ and perturbations travel with speed in the positive or negative x direction. Note that c is the speed of the wave not of the fluid!

Rem.: If we choose $\eta_0^-(\tilde{x}) = \eta_0^+(-\tilde{x})$ then initially the perturbation has zero fluid speed, and is such only a perturbation of the sea surface! What happens next?

An application of such equation are Tsunamis if we take: $g = 10\text{m/s}^2$, $H = 4\text{km}$ and $\eta_0 = 1\text{m}$, we have a wave speed of $c = 200\text{m/s} = 720\text{km/h}$ and a fluid speed $u_0 = 0.05\text{m/s}$. What happens when H decreases? Why do wave crests arrive parallel to the beach? Why do waves break?

You see this simplest form of a fluid dynamic equation can be understood completely. It helps us to understand a variety of natural phenomena.

Exercise 22: Does the linearised one dimensional shallow water equation conserve energy?

Exercise 23: Is it justified to neglect the nonlinear term in eq. (2.31) for the case of a Tsunami?

2.5 Reduced Gravity

Suppose that the layer of fluid (fluid 1) is lying on a denser layer of fluid (fluid 2) that is infinitely deep. $H_2 \to \infty \Rightarrow c_2 \to \infty$, that is perturbations travel with infinite speed. This implies that the lower fluid is always in equilibrium $\partial_x P = \partial_y P = 0$. The lower fluid layer is passive, does not act on the upper fluid but adapts to its dynamics, so that $\eta_1 = \frac{\rho_1 - \rho_2}{\rho_1}\eta_2$. If we set $\eta = \eta_1 - \eta_2$ then $\eta = \frac{\rho_2}{\rho_2 - \rho_1}\eta_1$ and the dynamics is described by the same sweqs. 2.28, 2.29 and 2.30 with gravity g replaced by the *reduced gravity* $g' = \frac{\rho_2 - \rho_1}{\rho_2} g$ ("sw on the moon").

Example: $g' = 3 \cdot 10^{-3} g$, $H = 300\text{m}$, $\eta_0 = .3\text{m}$ we get a wave speed $c = \sqrt{g'H} = 3\text{m/s}$ and a fluid speed of $u = 1\text{m/s}$.

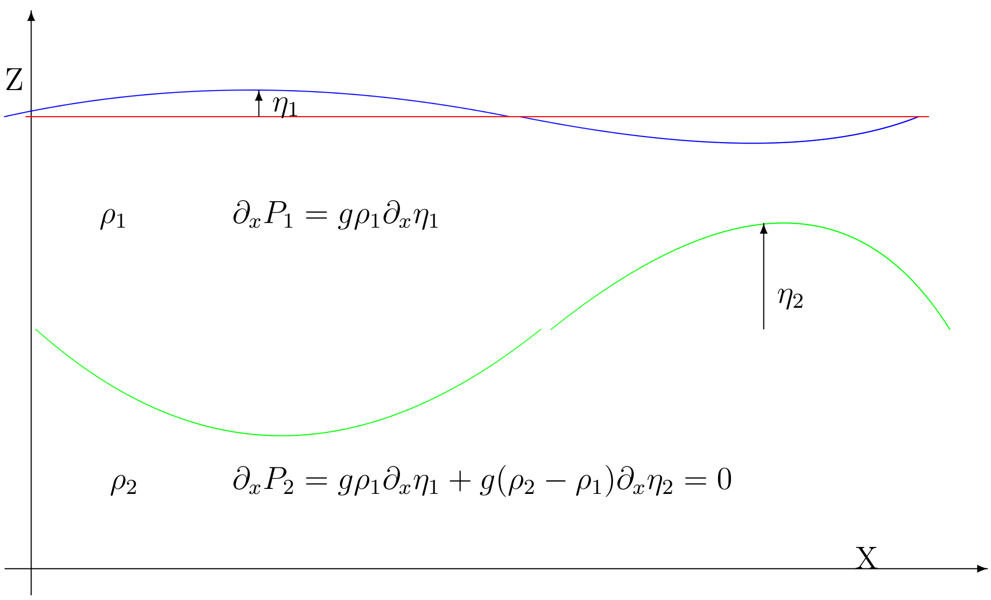


Figure 2.5: Reduced gravity shallow water configuration

Comment 1: When replacing $g\eta$ by $g'\eta$ it seems, that we are changing the momentum equations, but in fact the thickness equation is changed, as we are in the same time replacing the deviation of the free surface η (which is also the deviation of the layer thickness in not-reduced-gravity case) by the deviation of the layer thickness η , which is $(\rho_2 - \rho_1)/\rho_2$ times the surface elevation in the reduced gravity case. This means also that every property which is derived only from the momentum equations not using the thickness equation is independent of the reduced gravity.

Comment 2: Fig. 2.5 demonstrates, that the layer thickness can be measured in two ways, by the deviation at the surface (η_1) or by density structure in the deep ocean (η_2). For ocean dynamics the surface deviation for important dynamical features, measuring hundreds of kilometres in the horizontal, is usually less than 1 m whereas variations of (η_2) are usually several hundreds of meters. Historically the measurement of the density structure of the ocean to obtain η_2 are the major source of information about large scale ocean dynamics. Nowadays satellites measure the surface elevation of the ocean (altimetry) at a spatial and temporal density unknown before and are today our major source of information.

2.6 Bernoulli Revisited

Lets take a second look at the 1D Bernoulli equation which can now be derived from the stationary and in-viscid shallow water equations:

$$u\partial_x u + g\partial_x \eta = 0 \rightsquigarrow (\partial_x u^2)/2 + g\partial_x \eta = 0 \rightsquigarrow (u^2)/2 + g(h+b) = B = const. \tag{2.38}$$

$$\partial_x [hu] = 0 \rightsquigarrow hu = Q = const, \tag{2.39}$$

where $h = H + \eta - b$, thickness of the fluid layer. We combine these two equations to:

$$\frac{Q^2}{2h^2} + g(h+b) = B \rightsquigarrow P(h) = gh^3 + (gb - B)h^2 + \frac{Q^2}{2} = 0 \tag{2.40}$$

this cubic polynome has one or three real solutions. The local extrema of the cubic are at

$$3gh^2 + 2(gb - B)h = 0 \rightsquigarrow h_1 = 0 \text{ and } h_2 = \frac{2(B - gb)}{3g} \tag{2.41}$$

As $P(h_1) > 0$, one solution for $P(h) = 0$ is negative, which is not physical. At h_2 we get $Fr = 1$:

$$h_2 = \frac{2(B - gb)}{3g} = \frac{u_2^2 + 2gh_2}{3g} \rightsquigarrow 3gh_2 = u_2^2 + 2gh_2 \rightsquigarrow Fr^2 = \frac{u^2}{gh_2} = 1 \tag{2.42}$$

There is no real positive solution if $gh_2^3 + (gb - B)h_2^2 + Q^2/2 > 0$, one solution for equality and two solutions (a sub-critical and a super-critical one), when the inequality is inverted. If for $b = 0$ we have two solutions and $B = B_0$ then the maximum value for b to have at least one solution is when:

$$B_0 - gb = \frac{3gh_2}{2} \rightsquigarrow b = \frac{2B_0 - 3gh_2}{2g}. \tag{2.43}$$

For higher topographies the flow is blocked.

2.6.1 Hydraulic Jump

When a super critical flow is slowed down by an obstacle or a change in the slope, it can suddenly become sub-critical. This phenomena is called a hydraulic jump. They can be observed in rivers,

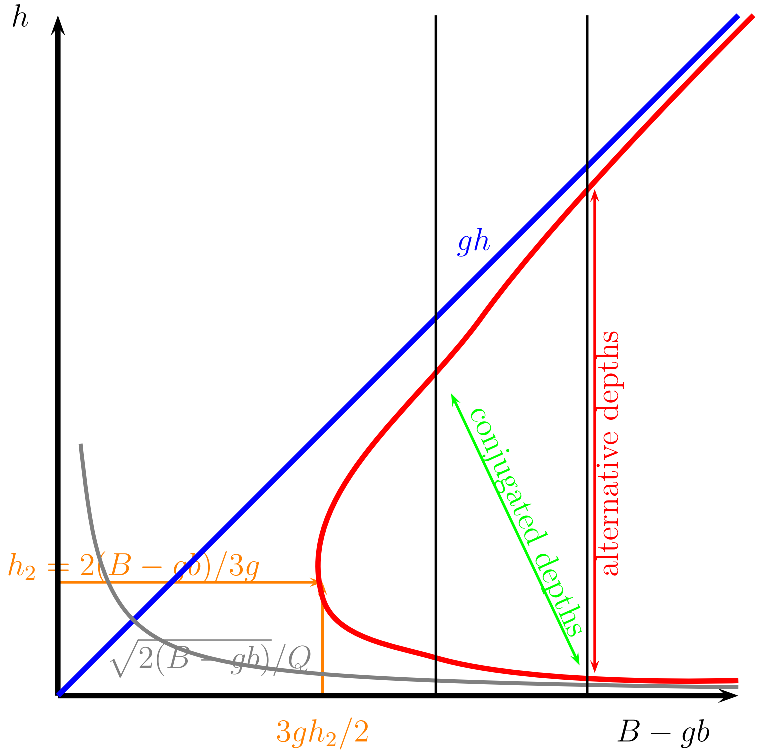


Figure 2.6: The two positive solutions of the thickness h as a function of $B - gb$ are given. The asymptotics for long and small h are also shown. The green arrow indicates the hydraulic jump.

especially after a flow has passed from sub to super-critical over a dam and then recovers to sub-critical afterwards. It has many applications in this context (reducing energy, mixing, sediment transport, oxygenizing, ...). In the atmosphere and ocean hydraulic jumps also occur at many locations. They are important in fluid dynamics as they lead to very violent mixing of the fluid, because the energy lost to the mean flow is put into small scale turbulent motion. The hydraulic jump is an example of a very localised mixing process.

This phenomena can NOT be described by the shallow water equations as the horizontal velocity is no longer independent of the vertical direction. Energy is not conserved but destroyed by the jump. The quantities conserved are the mass and the momentum. Please note that in the shallow water context the conservation of momentum and mass leads to a conservation of energy, if no forces (by the topography) are applied. Before the jump we have (u_1, h_1) and after the jump (u_2, h_2). The Bernoulli function is no longer conserved. The jump provides a dissipative process. Transport

$$u_1 h_1 = u_2 h_2 \tag{2.44}$$

is conserved (the jump does not make water disappear) and the momentum budget

$$h_1 u_1^2 + g h_1^2/2 = h_2 u_2^2 + g h_2^2/2 \tag{2.45}$$

(momentum per density is uhl and the momentum per density passing at the point 1 per time is $h_1 u_1^2$, the depth-averaged pressure per density at point 1 is $gh_1/2$ and force difference acting at point 1 and point 2 is $gh_1^2/2 - gh_2^2/2$. Note that in a hydraulic jump the (turbulent) surface does not excert an acceleration force on the water colomn as the flow is not hydrostatic. The above leads to (Bélanger equation):

$$\frac{u_1}{u_2} = \frac{h_2}{h_1} = \left(\frac{Fr_1}{Fr_2}\right)^{2/3} = \frac{\sqrt{1 + 8Fr_1^2} - 1}{2}. \tag{2.46}$$

First equality is an immediate consequence of conservation of transport as is the second.

Exercise 24: Show the third equality

For the third equality

$$\begin{aligned} h_1 u_1^2 + g h_1^2/2 = h_2 u_2^2 + g h_2^2/2 \rightsquigarrow 2u_1^2\left(h_1 - \frac{h_1^2}{h_2}\right) + g(h_1^2 - h_2^2) = 0 \\ \rightsquigarrow 2Fr_1^2\left(1 - \frac{h_1}{h_2}\right) + \left(1 - \frac{h_2^2}{h_1^2}\right) = 0 \rightsquigarrow 2Fr_1^2 = \frac{h_2}{h_1}\frac{(1 - \frac{h_1}{h_2})(1 + \frac{h_2}{h_1})}{(1 - \frac{h_1}{h_2})} \\ \rightsquigarrow \frac{h_2}{h_1}\left(\frac{h_2}{h_1} + 1\right) - 2Fr_1 = 0 \end{aligned} \tag{2.47}$$

The change in B is:

$$\Delta B = g\frac{(h_2 - h_1)^3}{4h_2 h_1}. \tag{2.48}$$

Exercise 25: Calculate ΔB

Hydraulic jumps are used in engineering applications to dissipate energy or to entrain air into the water. A sub-critical flow can not suddenly become super-critical because this would require a sudden increase of energy.

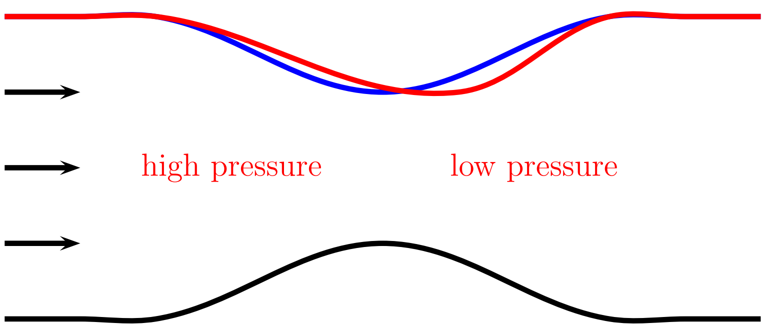


Figure 2.7: Pressure drag exerted by a sea-mount. The blue line shows the symmetric free-surface (=pressure distribution) at a symmetric sea-mount (black line). If friction is included (red line) the symmetry is lost. The pressure on the up-stream side of the seamount is larger than on the down-stream side.

2.6.2 Effects of Friction

When friction is present the Bernoulli property disappears and so do several characteristics that rely on it. The most conspicuous is the local dependence of u, h on b. The flow over a symmetric bump is no longer symmetric, this leads to a generation of a *pressure drag*, which slows down the flow even further. Indeed, as can be seen in fig. 2.7 the dip in the surface is displaced to the right which leads to a higher pressure at the left of the sea-mount. The horizontal component of the force exerted on the sea-mount per horizontal unit area is $F/A = P\partial_x b$. Having higher pressure up-stream thus leads to a force to the right. The sea-mount exerts the same force to the left and this is how the pressure drag is created.

2.6.3 Flow Down a Slope

We will consider here flows down an incline subject to turbulent drag law $(F_D/A = -\rho C_D u^2)$ For a time-independent flow down a slope of small angle θ ($\theta \approx \tan\theta$) the momentum equation is:

$$\partial_x(u^2 h) = gh(\theta - \partial_x h) - C_D u^2, \tag{2.49}$$

as $uh = Q$ =const. we have $\partial_x(u^2 h) = u\partial_x(uh) + uh\partial_x u = uh\partial_x u$. We also have, $\partial_x(u^2 h) = \partial_x(Q^2/h) = -(Q^2/h^2)\partial_x h = -u^2\partial_x h$ and thus obtain:

$$-u^2\partial_x h = gh(\theta - \partial_x h) - C_D u^2 \quad \rightsquigarrow \quad (1 - F^2)\partial_x h = \theta - C_D F^2 \tag{2.50}$$

We can see that if $F = 1$ then $\theta = C_D$, which is called the critical angle. It does not depend on density or gravity! More generally, if the flow properties do not change with x then $\partial_x h = 0$ and F =const., we get $F^2 = \theta/C_D$: This indicates, that when θ decreases a super-citical flow can become sub-critical, performing an hydraulic jump.

2.7 Bernoulli in 2D

We start with the shallow water equations (2.28), (2.29) and (2.30) adding topography b. Neglecting time dependence and viscosity we get:

$$u\partial_x u + v\partial_y u + g\partial_x \eta = \quad 0 \tag{2.51}$$
$$u\partial_x v + u\partial_x v + g\partial_y \eta = \quad 0 \tag{2.52}$$
$$\partial_x \left[(H+\eta-b)u\right] + \partial_y \left[(H+\eta-b)v\right] = \quad 0, \tag{2.53}$$

note that $(H+\eta) = h+b$. If we define the Bernoulli function as:

$$B = g(H+\eta) + \frac{u^2+v^2}{2} \tag{2.54}$$

we obtain using eqs. (2.51), (2.52) and (2.53) that

$$u\partial_x B + v\partial_y B = \quad 0 \tag{2.55}$$

which states that the Bernoulli function is conserved along stream lines in a 2D stationary non-viscous shallow water flow.

Exercise 26: Show that the Bernoulli function is conserved along stream lines in a 2D stationary non-viscous shallow water flow.

An interesting feature is that the solutions of (u, η) depend locally on b, that is if you know b at a location in 2D space and the Bernoulli constant and the transport along a stream line you can calculate u and η (well you have the choice between the sub and the super critical solution) and you do not have to care what happens at other locations. This nice property is lost when friction is considered.

Chapter 3

Vector Fields

3.1 Two Dimensional Flow

We have seen in the previous sections, that the dynamics of a shallow fluid layer can be described by the two components of the velocity vector $(u(x, y, t), v(x, y, t))$ and the surface elevation $\eta(x, y, t)$ (these are called the *dynamic variables*). The vertical velocity $w(x, y, t)$, is in this case, determined by these 3 variables (it is called a *diagnostic variable*). . The vertical velocities in a shallow fluid layer are usually smaller than their horizontal counterparts and we have to a good approximation a two dimensional flow field.

The dynamics of a fluid is described by scalar (density, pressure) and vector quantities (velocity). These quantities can be used to construct other scalar quantities (tensors, of order zero) vectors (tensors, of order one) and higher order tensors. Higher order tensors can than be contracted to form lower order tensors. An example is the velocity tensor (first order) which can be used to calculate the speed (its length), it is a tensor of order one. The most useful scalar quantities are those which do not change when measured in different coordinate systems, which might be translated by a given distance or rotated by a fixed angle. Such quantities are called *well defined* . For example it is more reasonable to consider the length of the velocity vector (the speed, well defined) rather than the first component of the velocity vector (ill defined), which changes when the coordinate system is rotated. The length of the velocity vector is used to calculate the kinetic energy. The most prominent second order tensor is the strain tensor, considering the linear deformation of a fluid volume. In two dimension it is:

$$(\nabla \mathbf{u}^t)^t = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \tag{3.1}$$

if a coordinate system $\mathbf{e}'$ is rotated by an angle α with respect to the original coordinate system $\mathbf{e}$ the components in the new system are given by:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{A}\mathbf{u} \tag{3.2}$$

and $\mathbf{u} = \mathbf{A}^t\mathbf{u}'$. Note that $\mathbf{A}^t = \mathbf{A}^{-1}$.

For a scalar field (as for example temperature) we can express in the two coordinate systems: $f'(x', v') = f(x(x', v'), y(x', v'))$ we have:

$$\partial_{x'} f' = \frac{\partial f'}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial f}{\partial y} \tag{3.3}$$

its gradient transforms as:

$$(\partial_{x'} f',\, \partial_{y'} f') = (\partial_x f,\, \partial_y f) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = (\nabla f)\mathbf{A^t} \tag{3.4}$$

and we can write by just adding a second line with a function g (say salinity)

$$\begin{pmatrix} \partial_{x'} f' & \partial_{y'} f' \\ \partial_{x'} g' & \partial_{y'} g' \end{pmatrix} = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = (\nabla \begin{pmatrix} f \\ g \end{pmatrix})\mathbf{A^t} \tag{3.5}$$

Now u, v are not scalar functions as are g, f but components of a vector so we have to rotate them from the prime system. And we obtain for the transformation of the strain tensor

$$\begin{pmatrix} \partial_{x'} u' & \partial_{y'} u' \\ \partial_{x'} v' & \partial_{y'} v' \end{pmatrix} = \mathbf{A} \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \mathbf{A^t} =$$
$$\begin{pmatrix} c^2\partial_x u + s^2\partial_y v + (\partial_y u + \partial_x v)sc & c^2\partial_y u - s^2\partial_x v + (\partial_y v - \partial_x u)sc \\ c^2\partial_x v - s^2\partial_y u + (\partial_y v - \partial_x u)sc & c^2\partial_y v + s^2\partial_x u - (\partial_y u + \partial_x v)sc \end{pmatrix}, \tag{3.6}$$

where $s = \sin\alpha$ and $c = \cos\alpha$. The well defined scalar quantities that depend linearly on the strain tensor are the trace of the tensor $d = \partial_x u + \partial_y v$ and the skew-symmetric part of the tensor, which gives the vorticity $\zeta = \partial_x v - \partial_y u$. this can be easily verified looking at eq. (3.6). A well defined quadratic quantity is the determinant of the tensor $D = \partial_x u \partial_y v - \partial_y u \partial_x v$. Other well defined quadratic quantities are the square of vorticity which is called enstrophy, the square of all components of the strain matrices $H = d^2 + \zeta^2 + 2D$ or the square of the strain rate $s^2 = d^2 + \zeta^2 - 2D$, the Okubo-Weiss parameter $OW = \zeta^2 - s^2$. they are all dependent on the trace, vorticity and determinant. You can construct your own well-defined quantity and become famous!

Exercise 27: Show that the vorticity is well defined.

Exercise 28: Show that every well defined quantity which is a linear combination of the components of the strain tensor, is a linear combination of the divergence and the vorticity.

If a variable does not depend on time it is called stationary. The trajectory of a small particle transported by a fluid is always tangent to the velocity vector and its speed is given by the magnitude of the velocity vector. In a stationary flow its path is called a stream line. If the flow has a vanishing divergence it can be described by a stream function Ψ with $v = \partial_x \Psi$ and $u = -\partial_y \Psi$. If the flow has a vanishing vorticity it can be described by a potential Θ with $u = \partial_x \Theta$ and $v = \partial_y \Theta$. Any vector field in 2D can be written as $u = \partial_x \Theta - \partial_y \Psi, v = \partial_y \Theta + \partial_x \Psi$, this is called the Helmholtz decomposition.

Exercise 29: Show that every flow that is described by a stream function has zero divergence.

Exercise 30: Show that every flow that is described by a potential has zero vorticity.

Exercise 31: Express the vorticity in terms of the stream function.

Exercise 32: Express the divergence in terms of the potential.

Exercise 33: Which velocity fields have zero divergence and zero vorticity?

Exercise 34: draw the velocity vectors and streamlines and calculate vorticity and divergence. Draw the a stream function where ever possible:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}; \begin{pmatrix} y \\ 0 \end{pmatrix}; \begin{pmatrix} -y \\ x \end{pmatrix}; \begin{pmatrix} -x \\ y \end{pmatrix}; \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix}; \begin{pmatrix} \cos y \\ \sin x \end{pmatrix}. \tag{3.7}$$

3.2 Three Dimensional Flow

Conceptually there is not much different when going from two to three dimensions. divergence is: $d = \nabla \cdot \mathbf{u} = \partial_x u + \partial_y v + \partial_z w$. vorticity is no longer a scalar but becomes a vector:

$$\zeta = \nabla \times \mathbf{u} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} \partial_y w - \partial_z v \\ \partial_z u - \partial_x w \\ \partial_x v - \partial_y u \end{pmatrix}. \tag{3.8}$$

Chapter 4

Buoyancy Driven Flows

4.1 Molecular Transport

The molecular thermal diffusivity of sea water is $\kappa \approx 10^{-7}\mathrm{m}^2\mathrm{s}^{-1}$. The diffusion equation in the vertical is given by,

$$\partial_t T = \partial_z(\kappa \partial_z T). \tag{4.1}$$

We further suppose that there is a periodic heat flux of magnitude Q at the surface (boundary condition), that is:

$$\partial_z T|_{z=0} = \frac{Q}{c_p \rho \kappa} \cos(2\pi t/\tau + \pi/4), \tag{4.2}$$

$$\lim_{z \to -\infty} \partial_z T = 0. \tag{4.3}$$

The linear equation (4.1) with the boundary conditions (4.3) has the solution:

$$T(z,t) = T_A e^{z/L} \cos(2\pi t/\tau + z/L), \tag{4.4}$$

with:

$$T_A = \frac{Q}{c_p \rho}\sqrt{\frac{\tau}{2\pi\kappa}} \text{ and } L = \sqrt{\frac{\tau\kappa}{\pi}}. \tag{4.5}$$

Where $Q \approx 200\mathrm{Wm}^{-2}$, $c_p = 4000\mathrm{JK}^{-1}\mathrm{kg}^{-1}$, $\rho = 1000\mathrm{kgm}^{-3}$ if we take τ to be one day we get: $T_A \approx 20\mathrm{K}$ and $L = 5.2\mathrm{cm}$, this means that the surface temperature in the ocean varies by 40K in one day and the heat only penetrates a few centimetres. If τ is one year, considering the seasonal cycle, $T_A \approx 400\mathrm{K}$ and $L \approx 1\mathrm{m}$. This means that the surface temperature in the ocean varies by 800K in one year and the heat only penetrates about one meter. This does not at all correspond to observation!

Exercise 35: Derive eqs. (4.4) and (4.5).

When using the molecular viscosity, ν we can also calculate the thickness of the Ekman layer $\delta = \sqrt{2\nu/f}$ which is found to be a few centimetres. The observed thickness of the Ekman layer in the ocean is however over 100 times larger.

This shows that molecular diffusion can not explain the vertical heat transport, and molecular viscosity can not explain the vertical transport of momentum! But what else can?

4.2 Turbulent Transport

In the early 20th century fluid dynamicists as L. Prandtl suggested that small scale turbulent motion mixes scalars and momentum very much like the molecular motion does, only that the turbulent mixing coefficients are many orders of magnitude larger than their molecular counterparts. This is actually something that can easily be verified by gently poring a little milk into a mug of coffee. Without stirring the coffee will be cold before the milk has spread evenly in the mug, with a little stirring the coffee and milk are mixed in less than a second.

In this section we like to have a quantitative look at the concept of eddy diffusivity in a very simplified frame work that nevertheless contains all the important pieces. The starting point of our investigation is the transport equation of a scalar (2.15). We start by considering the two dimensional motion in the $x-z$-plane. Motion and dependence in the y direction are neglected only to simplify the algebra, and do not lead to important changes. We further suppose that the large scale scalar field depends only on the z-direction and that the large-scale flow in the z direction vanishes. The x and z component of the motion is given by u and w, respectively.

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} U(z) + u' \\ w' \end{pmatrix} \quad \text{and} \quad s = S(z) + s' \tag{4.6}$$

with $U(z) = \langle u \rangle_x$, $W(z) = \langle w \rangle_x$ and $S(z) = \langle s \rangle_x$.

The $\langle . \rangle_x$ operator denotes the average over a horizontal slice:

$$A(y) = \langle a(x,z) \rangle_x = \frac{1}{L} \int_L a(x,z) dx \tag{4.7}$$

In the sequel we will use the following rules:

$$\langle \lambda(z) a \rangle_x = \lambda(z) \langle a \rangle_x \tag{4.8}$$

$$\langle \partial_z a \rangle_x = \partial_z \langle a \rangle_x \tag{4.9}$$

and

$$\langle \partial_x a(x,z) \rangle_x = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \partial_x a(x,z) dx = \frac{a(x_2,z) - a(x_1,z)}{x_2 - x_1}, \tag{4.10}$$

which vanishes if $a(x,z)$ is bounded and we take the limit of the averaging interval $L = (x2 - x1) \to \infty$. But of course:

$$\langle ab \rangle_x \neq \langle a \rangle_x \langle b \rangle_x. \tag{4.11}$$

If we suppose $u' = w' = 0$, that is no turbulence, the transport equation (see eq. (2.15) becomes:

$$\partial_t S = \kappa \partial_{zz} S. \tag{4.12}$$

If we allow for small scale turbulent motion we get:

$$\partial_t(S + s') + \partial_x \left((U + u')(S + s') \right) + \partial_z \left(w'(S + s') \right) = \kappa \partial_{zz}(S + s'). \tag{4.13}$$

Where we have used the identity

$$u \partial_x s + w \partial_z s = \partial_x (us) + \partial_z (ws), \tag{4.14}$$

which is a direct consequence of the incompressibility ($\partial_x u + \partial_z w = 0$). Applying the horizontal averaging operator to eq. (4.13) we get:

$$\partial_t S + \partial_x \langle (U + u')(S + s') \rangle_x + \partial_z \langle w'(S + s') \rangle_x = \kappa(\partial_{xx} + \partial_{zz}) \langle (S + s') \rangle_x \tag{4.15}$$

which simplifies to:

$$\partial_t S + \partial_z \langle w's' \rangle_x = \nu \partial_{zz} S \tag{4.16}$$

If we now compare eqs. (4.12) and (4.16) we see that the small scale turbulent motion adds one term to the large scale equations. The value of this term depends on the small scale turbulence and the large scale flow and is usually unknown. There are now different ways to parametrise this term, that is, express it by means of the large scale flow. The problem of finding a parametrisation is called *closure problem.*

None of the parametrisations employed today is rigorously derived from the underlying Navier-Stokes equations, they all involve some “hand-waving.” We will here only discuss the simplest closure, the so called *K-closure* (also called *Boussinesq assumption*).

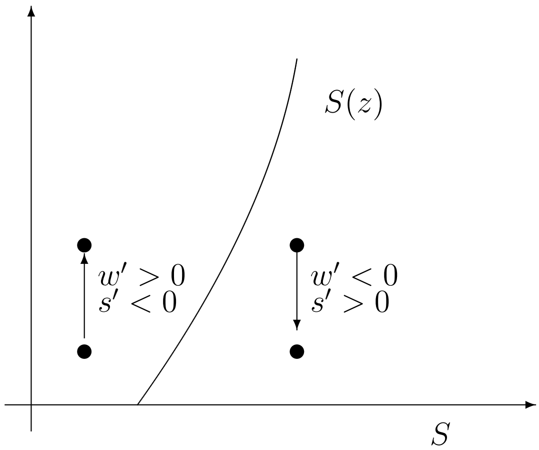


Figure 4.1: K-closure

The K-closure assumes the turbulent flux term to be proportional to the large-scale gradient of S:

$$\langle w's' \rangle_x = -\kappa'_{eddy} \partial_z S \tag{4.17}$$

where $-\kappa'_{eddy}$ is the proportionality coefficient. Looking at fig. 4.1 this choice seems reasonable: firstly the coefficient should be negative as upward moving fluid transport a fluid parcel that originates from an area with a lower average S value to an area with a higher average S , such that s' is likely to be negative. The reverse is true for downward transport. Such that $\langle w's' \rangle_x$ is likely to be negative. Secondly, a higher gradient is likely to increase $|s'|$ and thus also $-\langle w's' \rangle_x$.

Using the K-closure we obtain:

$$\partial_t S = (\kappa + \kappa'_{eddy}) \partial_{zz} S, \tag{4.18}$$

which is identical to eq. (4.12) except for the increased effective diffusivity $\kappa_{eddy} = \kappa + \kappa'_{eddy}$ called the eddy diffusivity.

Exercise 36: Perform the calculations without neglecting the motion and dependence in the y-direction.

4.3 Rayleigh-Bénard Convection

The conceptually simplest convective situation is when we have two horizontal plates which are separated by a distance d. The lower one is heated and the upper one cooled. For small temperature differences, cooling rates, the heat transport is done by molecular diffusion at a rate $-\kappa\partial_z T = \kappa(T_{hot} - T_{cold})/d$ (minus sign because the flux is down gradient). When the temperature difference increases above a certain threshold fluid motion sets in leading to a "turbulent" transport of heat which ads to the molecular transport. Averaged over a horizontal plane we obtain for the tempreature transport $H(z) = \langle wT - \kappa\partial_z T\rangle_{x,y}$. In a statistically stationary state we can also average over time and the heat transport H becomes independent of z. The turbulent transport is most efficient far from the plates, where the friction of the plates transmitted by viscosity of the fluid is less felt, and w has large values. Please note, that in an incompressible fluid the average vertical velocity is vanishing $\langle w\rangle_{x,y} = 0$ and the heat transport is well defined (independent on whether temperature is measured in Celsius or Kelvin). In the vicinity of the plates the heat transport is still governed by molecular motion and thus less efficient than in the turbulent interior. This leads to strong temperature gradients at the plates showing that the overall heat transport in the turbulent case is determined by the boundary layer dynamics (see fig. 4.2).

The external parameters of the system are: diffusivity κ, viscosity ν, distance between the plates d, temperature difference between the plates $T_{hot} - T_{cold}$, gravity g and the thermal expansion coefficient α of the fluid. The last three are combined to form a buoyancy difference $b = g\alpha(T_{hot} - T_{cold})$. Two non dimensional parameters can be formed, the Prandtl number $Pr = \nu/\kappa$ and the Rayleigh number:

$$Ra = \frac{bd^3}{\nu\kappa} = \frac{g\alpha(T_{hot} - T_{cold})d^3}{\nu\kappa}, \tag{4.19}$$

it is the ratio of two time scales squared $t^2_{molec}/t^2_{buoyancy}$. Where $t_{molec} = \sqrt{d^4/\nu\kappa}$ is the characteristic time scale of the stabilising action of molecular viscosity and diffusion and $t_{buoyancy} = \sqrt{d/b}$ is the characteristic time scale of the destabilising buoyancy gradient. For small Rayleigh numbers molecular exchange coefficients lead to stability for large Rayleigh numbers the buoyancy gradient wins. In problems where diffusion can be neglected or is unimportant the Grashof number $Gr = Ra/Pr$, is defined in which the diffusivity in the Rayleigh number is replaced by viscosity. The Rayleigh-Bénard convection can be understood as a competition of buoyancy versus dissipation.

In laboratory experiments the heat flux between the plates is given as a function of the temperature differences. This can be done in two ways, keeping the temperature difference constant and measuring the heat flux or imposing the heat flux and measuring the temperature differences.

The measured heat flux H divided by the molecular heat flux is called the Nusselt number,

$$Nu = \frac{Hd}{\kappa(T_{hot} - T_{cold})}, \tag{4.20}$$

it is an increasing function of the Rayleigh number. To determine the behaviour of the Nusselt number as a function of the Rayleigh number no fluid measurement is necessary, only the heat transport and the temperature difference at the plates has to be measured. This makes the Rayleigh-Bénard experiment so appealing as fluid measurements are difficult and have usually larger error bars than measurements of power input and temperature of a solid.

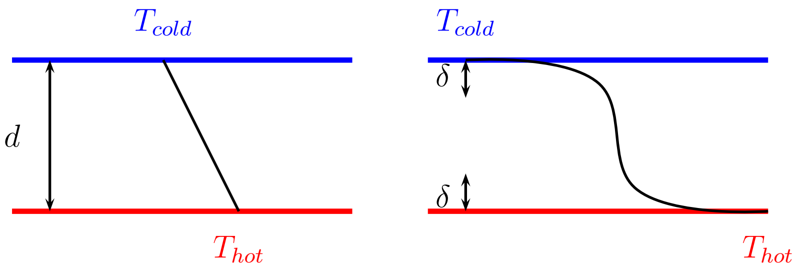


Figure 4.2: Temperature profile between to plates in Rayleigh-Bénard convection with only diffusive transport, no fluid motion (left) and with convective fluid motion and formation of boundary layers on the plates (right).

Dimensional Analysis

Above I stated that there are only two external independent dimensional parameters in the system, Ra and Pr. How can I be so sure, how can this statement be derived systematically?

There are four dimensional external parameters, b, d, ν, κ, if we use the symbols $[.]$ to denote "dimension of" then: $[b^{e_1}] = [b]^{e_1} = \mathrm{m}^{e_1}\mathrm{s}^{-2e_1}$, $[d^{e_2}] = \mathrm{m}^{e_2}$, $[\nu^{e_3}] = \mathrm{m}^{2e_3}\mathrm{s}^{-e_3}$ and $[\kappa^{e_4}] = \mathrm{m}^{2e_4}\mathrm{s}^{-e_4}$. All these units are formed by the fundamental units m and s, so that: $[b^{e_1}d^{e_2}\nu^{e_3}\kappa^{e_4}] = \mathrm{m}^{l_m}\mathrm{s}^{-l_s}$. If we write the exponents as a vector we can see that:

$$\begin{pmatrix} l_m \\ l_s \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ -2 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \quad \text{or } \vec{l} = \mathbf{B}\vec{e}, \tag{4.21}$$

and we have reduced the problem to linear algebra. Finding the non-dimensional parameters is equivalent to determining the kernel of $\mathbf{B}$. As the two lines of $\mathbf{B}$ are independent the dimension of the kernel is 2. It is easily verified, that the Ra and the Pr are in the kernel as the corresponding vectors of exponents are:

$$\begin{pmatrix} 1 \\ 3 \\ -1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \tag{4.22}$$

As these vectors are linearly independent they span the kernel of B and every other vector is a linear combination of these two vectors. In terms of non-dimensional parameters that means, that every non-dimensional parameter can be formed by a combination of Ra and Pr. These dimensional considerations are usually revered to as *Buckingham-Pi theorem* .

4.3.1 Instability

An Example in $1D$

For those who have never calculated the stability of a state, here an example from mechanics. Suppose a glider at position x of mass one in a gravitational field on a surface of the form $P(x) = x^3/3 - x$ (see fig. 4.3). The governing equation equation is,

$$\partial_{tt}x = -\partial_x P = 1 - x^2. \tag{4.23}$$

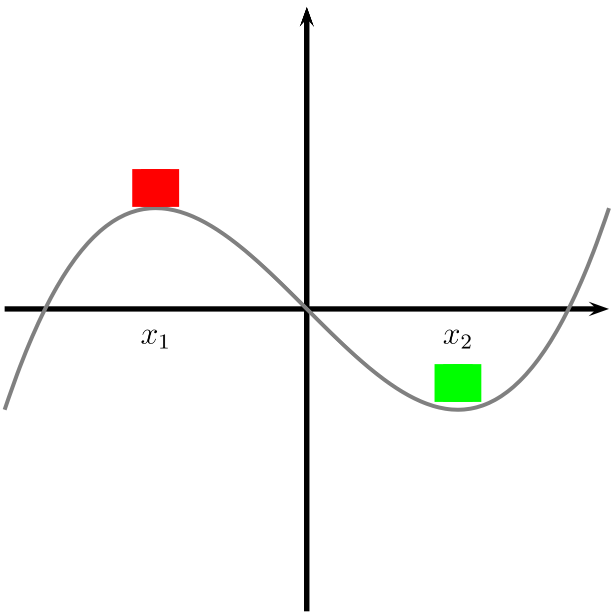


Figure 4.3: Two blocks on a surface at stationary points, the green block is stable, the red block is unstable.

The two stationary solutions are $x_1 = -1$ and $x_2 = 1$. We now suppose that the solutions are slightly perturbed (at $t = 0$) by $x(0) = x_i + \epsilon x'(0)$, with $\epsilon \ll 1$. Equation (4.23) now reads:

$$\epsilon \partial_{tt} x' = -(x_i + \epsilon x')^2 + 1. \tag{4.24}$$

At $O(1)$ the equation is trivial and at $O(\epsilon)$ it is:

$$\partial_{tt} x' = -2x_i x'. \tag{4.25}$$

which has the solution:

$$x'(t) = x'(0)\exp(\sqrt{-2x_i}t). \tag{4.26}$$

This shows that at x_1 the solution is unstable (grows exponentially in time) and for x_2 it performs oscillations with a constant amplitude. This is called "overstability", the restoring force is so strong that the block over-shoots from the stable situation, this is characteristic for a stable state in a non-dissipative system.

Exercise 37: Perform the above calculations including linear friction: $\partial_{tt} x = 1 - x^2 - D\partial_t x$, where the constant D is the inverse of a friction time.

Exercise 38: Discuss the stability of: $\partial_{tt} x = \lambda - x^2$ as a function of λ. Show the stable and unstable stationary solutions in a $\lambda - x$-diagram (bifurcation diagram).

Rayleigh-Bénard (∞D)

For simplicity we will consider the problem in two dimensions (x, z)-plane and suppose that $v = 0$ (the first is not a loss of generality, as you can always turn your coordinate system such that the normal modes do not depend on y, whereas the second is). Results of 3D calculations show, however, that the most unstable mode is purely 2D and such included in our calculations). The governing equations are:

$$\begin{aligned}
\partial_t u + u\partial_x u + w\partial_z u + \partial_x P &= \nu\Delta u && (4.27)\\
\partial_t w + u\partial_x w + w\partial_z w + \partial_z P &= \nu\Delta w + g\alpha T && (4.28)\\
\partial_t T + u\partial_x T + w\partial_z T &= \kappa\Delta T && (4.29)\\
\partial_x u + \partial_z w &= 0 && (4.30)
\end{aligned}$$

with $\Delta = \partial_{xx} + \partial_{zz}$. We now linearise the equations around the basic state. This means that we write every variable $a = a_b + \epsilon\tilde{a}$ and introduce them in the above equations. The variables with the subscript $_b$ represent the time independent basic state and $\epsilon\tilde{a}$ are the perturbations. When we now write the equations we see that at $O(1)$ we obtain the stationary part of the equations above, which are satisfied by the basic flow. At $O(\epsilon)$ we obtain the equations of the perturbations linearised around the basic flow. Higher order in ϵ are neglected. As the equations at $O(\epsilon)$ are linear, we can consider the temporal behaviour of the eigen modes, if at least one mode grows (exponentially) in time the flow is unstable, if all eigen modes decay the flow is stable. The instability of RB-convection is especially easy as the basic flow is vanishing. so at $O(\epsilon)$ we get:

$$\begin{aligned}
\partial_t\tilde{u} + \partial_x\tilde{P} &= \nu\Delta\tilde{u} && (4.31)\\
\partial_t\tilde{w} + \partial_z\tilde{P} &= \nu\Delta\tilde{w} + g\alpha\tilde{T} && (4.32)\\
\partial_t\tilde{T} + \tilde{w}\partial_z T_b &= \kappa\Delta\tilde{T} && (4.33)\\
\partial_x\tilde{u} + \partial_z\tilde{w} &= 0. && (4.34)
\end{aligned}$$

Using a stream-function $\tilde{u} = -\partial_z\psi$, $\tilde{w} = \partial_x\psi$ eliminates pressure, $\Gamma = -\partial_z T_b = (T_{hot} - T_{cold})/d$ (note that $\Gamma > 0$):

$$\partial_t\Delta\psi = \nu\Delta^2\psi + g\partial_x\alpha\tilde{T}. \qquad (4.35)$$

$$\partial_t\tilde{T} - \Gamma\partial_x\psi = \kappa\Delta\tilde{T} \qquad (4.36)$$

If we use the dimensionless variables: $(x', z') = (x, z)/d$, $t' = t\nu/d^2$ (viscous time), $\psi' = \psi/\nu$ and $T' = \tilde{T}/(T_{hot} - T_{cold})$, we obtain:

$$\begin{aligned}
\partial_{t'}\Delta'\psi' &= \Delta'^2\psi' + \tfrac{Ra}{Pr}\partial_{x'}T' \rightsquigarrow (\partial_{t'} - \Delta')\Delta'\psi' &&= \frac{Ra}{Pr}\partial_{x'}T' && (4.37)\\
Pr\partial_{t'}T' &= \Delta'T' + Pr\partial_{x'}\psi' \rightsquigarrow (Pr\partial_{t'} - \Delta')T' &&= Pr\partial_{x'}\psi' && (4.38)
\end{aligned}$$

As the differential equations are linear with constant coefficients and as there are no boundary conditions in x we look for the stability of solutions of the form.

$$\begin{aligned}
\psi' &= \Psi(z,k)\exp(ikx + \sigma t) && (4.39)\\
T' &= \Theta(z,k)\exp(ikx + \sigma t), && (4.40)
\end{aligned}$$

where $i = \sqrt{-1}$ and obtain:

$$\begin{aligned}
-(\sigma + k^2 - \partial_z^2)(k^2 - \partial_z^2)\Psi &= ik\frac{Ra}{Pr}\Theta && (4.41)\\
(\sigma Pr + k^2 - \partial_z^2)\Theta &= ikPr\Psi && (4.42)
\end{aligned}$$

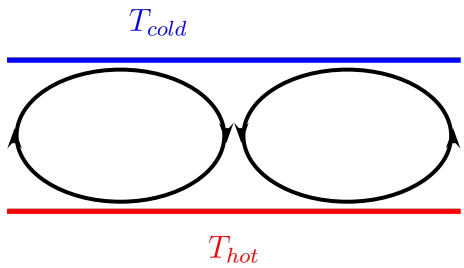


Figure 4.4: Circulation pattern of the most unstable mode in Rayleigh-Bénard convection.

which can be combined to:

$$\left[(\sigma + k^2 - \partial_z^2)(k^2 - \partial_z^2)(\sigma\, Pr + k^2 - \partial_z^2) - k^2 Ra\right] \Psi(z,k) = 0 \tag{4.43}$$

Please note that the Prandtl number Pr appears only in front of the stability parameter σ and as instability occurs when σ crosses zero, the stability of the flow does not depend on the Prandtl number.

Boundary conditions: are $\Theta = 0$, no temperature anomalie at the interface. At a free-slip boundary are $w = 0$ and $\partial_{zz} w = 0$ ($\partial_{zz} w = -\partial_{xz} u$ eq. (4.34) and $\partial_z u = 0$ along free slip boundary). This translates to $\Psi = 0$ and $\partial_{zz}\Psi = 0$. We can furthermore use eqs. (4.41) and (4.42) to show that all even derivations of Ψ with respect to z have to vanish at the boundaries.

Exercise 39: Show that all even derivatives of Ψ vanish ($\partial_{zz}^{2n}\Psi = 0$)

At a no-slip boundary we have $\Psi = 0$ and $\partial_z \Psi = 0$.

Free-slip boundaries are easier to consider as one can set $\Psi(k,z) = A(k,j)\sin(\pi j z)$ and look at the stability for all pairs (k,j).

Looking for marginally stable solutions ($\sigma = 0$) in eq. (4.43) (It can be shown that if the real-part of σ vanishes, so does the imaginary part, this is called exchange of stability.) we obtain:

$$(k^2 + \pi^2 j^2)^3 - k^2 Ra = 0, \quad \text{and} \tag{4.44}$$

$$Ra = \frac{(\pi^2 j^2 + k^2)^3}{k^2} \tag{4.45}$$

We can see that $Ra(j=1) < Ra(j=2) < Ra(j=3)$... if all other parameters are kept fix. Meaning that the most unstable mode occures for $j = 1$.

Exercise 40: Think of a physical argument explaining why the most unstable mode occurs at $j = 1$.

To obtain the lowest value for which the r.h.s. of the Rayleigh number for which instability occurs we calculate the critical wavelength k_c for which $\partial_k Ra = 0$ put the value ($k_c = \pi/\sqrt{2}$) into eq. (4.45) and obtain the critical Rayleigh number $Ra_c = 27\pi^4/4 \approx 657.5$.

Exercise 41: Derive k_c and Ra_c from eq. (4.45)

Please note, that the horizontal scale of the most unstable mode is $\sqrt{2^3}d$ only slightly larger than twice the the plate distance, so the rolls formed by this instability are only slightly flattened ellipses. The most unstable mode is (in dimensional variables):

$$\psi = \psi_0 \sin\left(\frac{\pi x}{\sqrt{2}d}\right) \sin\left(\frac{\pi z}{d}\right) \tag{4.46}$$

$$T = \theta_0 \cos\left(\frac{\pi x}{\sqrt{2}d}\right) \sin\left(\frac{\pi z}{d}\right). \tag{4.47}$$

For no-slip boundary conditions the problem is mathematically more involved, the critical Rayleigh number in this case is $Ra = 1707.7$ and the critical wavelength $k_c = \pi/(1.008d)$, so that the rolls have a horizontal scale that equals almost perfectly the plate distance.

For one free-slip and one no-slip boundary condition the critical Rayleigh number is $Ra = 1100.7$ and the critical wavelength $k_c = 2.682$

Exercise 42: What makes the no-slip case more complicated?

Exercise 43: My kitchen stove has a heating power of 1500W and the plate has a diameter of 22cm. In the pan on the stove there is a layer of water which is 2cm thick. We suppose that the transport is performed only by molecular diffusion. (a) What is the temperature difference between the top and the bottom of the fluid layer? (b) What is the Ra?, What do you think about the initial hypothesis, that the transport is performed only by molecular diffusion.

4.3.2 Coherent Structures and Patterns

We have seen that the first instability in Rayleigh-Bénard convection leads to roll structures. Such structures are often observed in the atmosphere and the ocean (rolls in the atmosphere and ocean are not always created by convection). When the Rayleigh number is further increased (typically a few percent above to a few times the critical value) these roll structures start to move and evolve. For the evolution and the stability of the roll structures the Prandtl number is very important, although it was unimportant in the instability leading to the rolls. The Prandtl number is around 1/100 for liquid metals, around one for most gases, ranges from 2 to 12 for water and is several 1000s for silicone oils. At even higher Rayleigh number the motion becomes chaotic in time but still shows a coherent (roll or hexagonal cell) picture in space. At even higher Rayleigh number the coherent rolls break up and their appearance becomes more and more intermittent. The spatial structure of the flow looses its coherence becomes chaotic in time and space, a state called *turbulence* . Convective plumes start to appear which are the dominant structures in turbulent Rayleigh Benard convection.

4.3.3 Chaos and the Lorenz Model

Based on the most unstable modes for the stream function eq. (4.46) and the temperature perturbation eq. (4.47) E. Lorenz designed a simple mathematical model to study the chaotic evolution of the coherent structures. The starting point are eqs. (4.27 - 4.30) using the stream function and neglecting the non-linear terms in the momentum equations but not in the temperature equation one gets:

$$\partial_t \Delta\psi = \nu\Delta^2\psi + g\alpha\partial_x\tilde{T} \tag{4.48}$$
$$\partial_t\tilde{T} = \partial_x\psi(\gamma + \partial_z\tilde{T}) + \nu\Delta^2\tilde{T} \tag{4.49}$$

We then consider two modes, the fist is the most unstable mode: $m_1 = \sin\left(\frac{\pi x}{\sqrt{2}d}\right)\sin\left(\frac{\pi z}{d}\right)$ and the second mode is the first mode in the vertical direction: $\sin\left(\frac{2\pi z}{d}\right)$. The dynamics of the stream function projected on m_1 and the dynamics of temperature annomaly projected on m_1 and m_2 is considered. This model relies on only three time dependent real scalar variables giving the amplitude of the stream function $X(t)$, the amplitude of the temperature difference between ascending and descending motion $Y(t)$ and the distortion of the vertical temperature profile from linearity $Z(t)$:

$$\psi = \lambda_1 X(t) m_1 \tag{4.50}$$
$$\tilde{T} = \lambda_2 Y(t)(\partial_x m_1) + \lambda_3 Z(t) m_2. \tag{4.51}$$

We furthermore use the trigonometric relations:

$$(\partial_x m_1)(\partial_{xz} m_1) = \frac{\pi^3}{8d^3} m_2 + oh \tag{4.52}$$

$$(\partial_x m_1)(\partial_z m_2) = -\frac{\pi}{d}(\partial_x m_1) + oh \tag{4.53}$$

where the last term stands for other harmonics. When put into equations (4.48) and (4.49) and choosing λ_i suitably they become:

$$\dot{X} = -\sigma X + \sigma Y \tag{4.54}$$

$$\dot{Y} = -XZ + rX - Y \tag{4.55}$$

$$\dot{Z} = XY - bZ. \tag{4.56}$$

Where $\sigma = Pr$, $r = Ra/Ra_c$ the normalised Rayleigh number, $b > 0$ some geometric factor and the dot presents differentiation with respect to a normalised time. This *Lorenz model* is the most famous and well studied and used model for chaotic dynamics.

Exercise 44: Determine λ_1, λ_2 and λ_3

The model has some elementary properties:

(i) All solutions are bounded and large values of X, Y, Z are damped towards zero.

(ii) The phase space volume $\partial_X \dot{X} + \partial_Y \dot{Y} + \partial_Z \dot{Z} = -\sigma - 1 - b < 0$ contracts at a constant rate

(iii) for $0 < r < 1$ the origin is the only fixed point, it is attracting. This corresponds to steady-state heat conduction in RB convection. for $r > 1$ the origin looses stability and two new fixed points arise they are attracting up to $r = r_2 = 470/19$. This corresponds to steady convection heat conduction in RB convection. For $r > r_2$ all three points are unstable and there is no single point attractor. The system is chaotic, that is, two infinitesimal close points in phase space separate exponentially when the two systems are integrated forward in time for a very long time.

Exercise 45: Show that the three points, (0,0,0) and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ are stationary solutions. Discuss their stability.

4.3.4 Turbulence, Soft and Hard (Scaling Theory)

For sub-critical Rayleigh numbers $Ra < Ra_c$, there is no fluid motion and $Nu = 1$. After that the fluid motion increases with the Rayleigh number and so does the Nusselt number. Boundary layers develop on both plates (see fig.4.4) and their maximal thickness can be estimated by imposing that the Rayleigh number based on the boundary layer thickness δ (rather than plate distance d), is sub-critical $Ra_\delta < Ra_c$. In this case the heat transport becomes independent of d as all dependence lies in the boundary layers. This is equivalent to state that $Nu \propto d$ (see eq. (4.20)) and thus $Nu \propto Ra^{1/3}$ (see eq. (4.19)) where the factor of proportionality depends on the Pr number. The thickness of the boundary layer δ (see fig. 4.2) is such that the Rayleigh number based on the boundary layer thickness is critical $Ra^\delta = g\alpha(T_{hot} - T_{cold})\delta^3/(\nu\kappa) \approx 1100$ where we chose a value close the no-slip-free-slip case. Note that $Ra/Ra^\delta = d^3/\delta^3$. If the heat flux is dominated by the molecular flux though the boundary layer we have $Nu \propto d/\delta = (Ra/Ra_c)^{1/3}$. Such type of turbulent scaling (so with an exponent smaller than 1/3) is observed for Rayleigh numbers larger 2×10^5.

It is further supposed that for very high Ra numbers the dynamics is so violent, that it destroys the boundary layers and the scaling of the heat transport becomes independent of ν and κ and thus $Nu \propto Ra^{1/2} Pr^{1/2}$. This scaling is often called, the ultimate state of thermal convection. Such scaling has never been observed in Rayleigh Bénard laboratory experiments, even at very high Rayleigh number as obtained in liquid helium. At $Ra \approx 10^{11}$ a transition to steeper scalings than $Nu \propto Ra^{1/3}$ is observed in some experiments. For such Rayleigh number the Nusselt number is $Nu \approx 100$. This scaling is important in geophysical applications for two reasons: (a) it describes the behaviour in the interior of the flow, away from boundaries; (b) at the ocean surface when there are strong winds there is no thin thermal boundary layer as the the breaking waves stir the flow mechanically and constantly thicken the boundary layer. In this case its boundary-layer thickness is not determined by the heat flux but by the wave breaking.

The ultimate state of thermal convection applies in the centre region of the domain away from the boundaries, the scaling of the heat transport is $\propto Ra^{1/2}$. In the boundary layer the scaling is $\propto Ra^{1/3}$, which is a slower increase with the Rayleigh number. With increasing Rayleigh number the heat transport is, therefore, increasingly dominated by the behaviour in the boundary layer.

Around a Rayleigh number $Ra \approx 4 \times 10^7$, another transition from soft to hard turbulence occurs abruptly. The Nusselt number at this transition is around 30. Soft turbulence is characterised by a Gaussian statistics of the temperature distribution while hard turbulence shows temperature statistics with an exponential tail. This means that in hard turbulence extreme events are more likely as compared to Gaussian dynamics. A good fit (for soft and hard turbulence) to some experimental data is given by $Nu = 0.2Ra^{0.28}$ for $Pr \approx 1$. This behaviour is however not reproduced in other experiments. Many aspects of Rayleigh-Bénard convection are not well understood today.

There is also a dependence on the Prandtl number (not discussed here), as it governs the relative thicknesses of the viscous and the thermal boundary layer.

4.4 Horizontal Convection

In Rayleigh-Bénard convection the heating and cooling act at a different geo-potential height. The height difference in height d, in a gravitational field g is the governing parameter in dynamics of the system, which can be seen by the fact that its third power appears in the Rayleigh number. In the environment the heating and cooling is often acting at locations which are separated in the horizontal but not in the vertical direction. Examples are heating and cooling of the atmosphere at the surface of the earth. At large scales the hotter surface of the earth near the equator and the colder surface at high latitudes leads to the atmospheric circulation. At smaller scale, the sea-breeze is generated by the different surface temperature over land and sea. The same is true for the ocean which is heated and cooled at its surface, at the same geo-potential height but at a large variety of horizontal scales. The situation in the ocean is similar to the atmosphere turned upside-down. Such kind of situation with heating and cooling at the same geo-potential height is referred to as *horizontal convection*.

4.4.1 Governing Equations

When we restrict ourselves to the two dimensional case the governing equations are:

$$\begin{aligned}
\partial_t u + u\partial_x u + w\partial_z u + \partial_x P &= \nu\Delta u && (4.57)\\
\partial_t w + u\partial_x w + w\partial_z w + \partial_z P &= \nu\Delta w + g\alpha T && (4.58)\\
\partial_t T + u\partial_x T + w\partial_z T &= \kappa\Delta T && (4.59)\\
\partial_x u + \partial_z w &= 0 && (4.60)
\end{aligned}$$

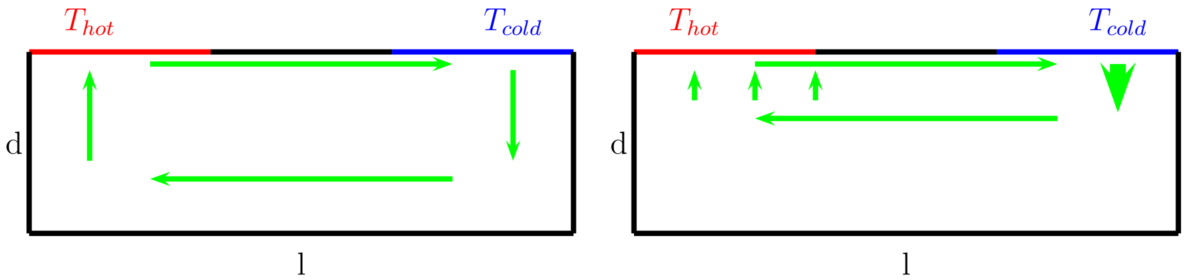


Figure 4.5: Horizontal convection of the "ocean type" (cooling-heating at the upper surface) with "intuitive" circulation pattern (green). The left panel shows low Ra case with a symmetric and deep circulation. At high Ra (right panel) the circulation flattens and becomes more asymmetric with a localised and strong downward convection and extended and slow up-welling.

At the boundary we can either prescribe the temperature $T_B = T$ or the temperature flux $H = \partial_{\mathbf{n}} T$ (where $\mathbf{n}$ is the vector normal to the boundary). We suppose that $H = 0$ except at the upper boundary (see fig. 4.5). Please note that the heat flux through the boundary is $H = \kappa \partial_n T$. Often the temperature at the boundary is relaxed to some value T^* (which varies along the boundary) and the boundary condition becomes: $H = \lambda(T^* - T)$ where λ is the inverse of a relaxation time. Please note that for a stationary state the integral of the heat flux around the domain has to vanish. I again choose to write the equations in terms of temperature rather than buoyancy $b = g\alpha T$ which is strictly equivalent when the thermal expansion coefficient α is constant.

The governing dimensional parameters are $g\alpha(T_{hot} - T_{cold}), \nu, \kappa, l$ and h. The three non-dimensional parameters are now the Rayleigh number:

$$Ra = \frac{g\alpha(T_{hot} - T_{cold})l^3}{\nu\kappa}, \tag{4.61}$$

where the important length scale is the distance of the separation of the cold and hot source, that is the distance in the horizontal. The Prandtl number $Pr = \nu/\kappa$ and the aspect ratio of the domain l/d.

Exercise 46: In the Rayleigh-Bénard case there was no dependence on the aspect ratio of the domain. What is different here?

When the horizontal convection experiment is performed there are several important observations: The "critical" Rayleigh number is vanishing. The tiniest temperature gradient leads to a horizontal pressure gradient, which leads to motion. So a horizontal temperature gradient means acceleration, means motion. There are thermal boundary layers at the surface, where heating and cooling is applied. The hottest and coldest temperatures are found in this boundary layer. The coldest water is densest and sinks. At depth the box fills up with dense fluid formed at the surface. On its way down diffusion and mixing leads to a certain warming of the water. Diffusion of heat downward under the heated surface also heats the deep layers. There is a competition between accelerating downward flow due to buoyancy differences and mixing and viscous and dissipative processes opposing the downward acceleration. It is observed that the downwelling (convective) region are smaller in size than the up-welling regions and that there is a single overturning cell, with cold waters descending in a small region under the cooled surface area and rising water on a larger region under the heated surface area. The difference in size of the up and downwelling increases with the Rayleigh number.

4.4.2 Energetics (Sandström theorem)

I first define the horizontal average by the $\langle . \rangle_x$ and the vertical by $\langle . \rangle_z$ the potential energy (per average density and per unit volume) is given by:

$$E_{\text{pot}} = -g\alpha \langle z \langle T \rangle_x \rangle_z \tag{4.62}$$

and the kinetic energy (per average density and per unit volume):

$$E_{\text{kin}} = \frac{1}{2} \langle \langle u^2 + w^2 \rangle_x \rangle_z \tag{4.63}$$

For their time evolution we have to consider:

$$\frac{d}{dt}(zT) = z\frac{d}{dt}T + wT = z\kappa\Delta T + wT \tag{4.64}$$

$$\frac{d}{dt}(u^2 + w^2) = -\partial_x(uP) - \partial_z(wP) - g\alpha wT - \nu(u\Delta u + w\Delta w) \tag{4.65}$$

When averaging these equations over the whole domain we obtain:

$$\partial_t E_{\text{pot}} = -g\alpha \langle wT \rangle_{xz} + \frac{g\alpha\kappa}{d} \langle T(z=0) - T(z=-d) \rangle_x \tag{4.66}$$

$$\partial_t E_{\text{kin}} = g\alpha \langle wT \rangle_{xz} - \epsilon \tag{4.67}$$

with $\epsilon = -\nu \langle u\partial_{xx}u + u\partial_{zz}u + w\partial_{xx}w + w\partial_{zz}w \rangle_{xz} = \nu \langle (\partial_x u)^2 + (\partial_z u)^2 + (\partial_x w)^2 + (\partial_z w)^2 \rangle_{xz} > 0$,

Exercise 47: Show that $-\langle u\partial_{xx}u + u\partial_{zz}u + w\partial_{xx}w + w\partial_{zz}w \rangle_{xz} = \langle (\partial_x u)^2 + (\partial_z u)^2 + (\partial_x w)^2 + (\partial_z w)^2 \rangle_{xz}$ (When we have free slip boundary conditions in a rectangular box).

Please not that choosing $z = 0$ at the upper boundary the forcing of the potential energy vanishes and the fluxes at the upper boundary do not appear. The last term in first equation above results from: $\langle z\Delta T \rangle_{xz} = \langle \langle z\partial_{zz} \langle T \rangle_x \rangle_z = -\langle \langle \partial_z \langle T \rangle_x \rangle_z$ as $0 = \langle \partial_z (z\partial_z T) \rangle_z = \langle z\partial_{zz} T \rangle_z + \langle \partial_z T \rangle_z$.

We see that when high temperature (less dense) fluid is moved upward, potential energy is transfered to kinetic energy, with a conservation of the sum. When considering the total energy (pot +kin) we see that the only source of energy is due to diffusion and the average temperature difference between the top and the bottom of the box and the sink of energy is the dissipation of kinetic energy by viscosity.

Exercise 48: If a water-body is 1km deep and if the surface is on average 10K hotter than the bottom, what is the mechanical work done by diffusion ($\kappa = 10^{-7}\text{m}^2\text{s}^{-1}$)?

If there is no dissipation of heat a steady circulation can only be maintained against friction if the heat source is located at a lower height than the cooling. This statement is called the *Sandström theorem.*

From the above result we can derive the so called *non-turbulence theorem.* In a turbulent flow we suppose that the kinetic-energy flux to smaller scales (ϵ) stays constant in the limit of the viscosity going to zero. When Prandtl number is fixed than horizontal convection can not be turbulent as the energy injection goes to zero with the viscosity and so does the kinetic-energy flux to smaller scales.

We next discuss the connection between the surface heat flux and the surface temperature. We start by multiplying eq. (4.59) by temperature T and using (4.60) we obtain:

$$\frac{1}{2}\left(\partial_t T^2 + \partial_x(uT^2) + \partial_z(wT^2)\right) = T\kappa\Delta T. \tag{4.68}$$

When we average over the whole box and if we suppose that the averages are in a statistically stationary state (=time averages over long enough time intervals do not depend on time) the left side vanishes and we get:

$$
\begin{aligned}
0 &= \langle T\Delta T\rangle_{xzt} = \langle \partial_x(T\partial_x T) - (\partial_x T)^2\rangle_{xzt} + \langle \partial_z(T\partial_z T) - (\partial_z T)^2\rangle_{xzt} && (4.69)\\
&= -\langle(\partial_x T)^2 + (\partial_z T)^2\rangle_{xzt} + \langle T\partial_z T|_{z=0}\rangle_{xt}/d. && (4.70)
\end{aligned}
$$

Where we used that the temperature gradient normal to the boundary vanishes everywhere except at the surface. This tells us that $\langle T\partial_z T|_{z=0}\rangle_{xzt} > 0$ which says that temperature and temperature flux has to be positively correlated, heat enters through hot surface areas. This comes at no surprise, for buoyancy driven flows we need buoyancy differences, which can only be created at the surface. Buoyancy differences can only be created by making more buoyant fluid more buoyant and less buoyant fluid less buoyant.

The extensive use of averaging over the whole domain has several reasons: (i) Integrating (averaging) of quantities is necessary as the energy is only conserved if integrated over a closed system. (ii) Non integrated quantities are often difficult to determine and their knowledge is often of secondary importance. (iii) Integrating systems often requires knowledge of variables (or their derivatives) only at the boundaries, which are often well known.

4.4.3 Scaling of the Circulation Depth

We have not considered the vertical extension of the flow so-far. The aspect ratio of the box and thus the vertical extension of the flow were introduced in the beginning of the section as one of the non-dimensional parameters, but we have not used it so-far. The picture of horizontal convection is given in fig. 4.5. with increasing buoyancy forcing (Ra number) the circulation cell becomes: (i) more asymetric (fast and localized down-ward movement and slow upward movement over a large area) and (ii) the circulation cell becomes flatter, so that the determining vertical scale is not the depth of the box d but a scale h which is a result of the dynamics rather than an external parameter. Our starting point is again eqs. (4.57) – (4.60). The vertical acceleration is $g\alpha(T_{hot} - T_{cold})$ and it happens on a horizontal length scale proportional to h (strong assymetry of descending and ascending motion. The horizontal acceleration is thus $g\alpha(T_{hot} - T_{cold})h/l$. The major retarding acceleration is due to vertical shear of the horizontal velocity $\nu u/h^2$. Equation (4.59) shows that the upwelling velocity $w_{up} \sim \kappa/h$ (where $\sim$ means "scales as") and $w = (l/h)w_{up}$ and also $u = (l/h)w_{up}$ due to (4.60). So that the acceleration scales as $g\alpha(T_{hot} - T_{cold})h/l$ which is balanced by $\kappa\nu l/h^4$.

$$
g\alpha(T_{hot} - T_{cold})h/l \sim \kappa\nu l/h^4 \rightsquigarrow Ra \sim l^5/h^5 \tag{4.71}
$$

As every parameter is fixed except the vertical extension of the motion h we get $h/l \sim Ra^{-1/5}$ That is the thickness of the circulation cell is a (slowly) decreasing function of the Rayleigh number, but Ra is usually huge in nature. This shows that for large Rayleigh number flow the vertical extension of the domain (d) is no longer an important parameter of the circulation as the dynamics creates its own vertical scale ($h < d$).

4.4.4 Horizontal Convection in the Ocean and Atmosphere

In the ocean and atmosphere the Rayleigh numbers are huge, so we expect the overturning circulation in the ocean to be very shallow.

Exercise 49: Estimate the Rayleigh number for the ocean circulation in the North Atlantic.

Exercise 50: Estimate the Rayleigh number for the atmospheric circulation at global scale.

Exercise 51: Estimate the Rayleigh number for a sea breeze.

The overturning circulation of the ocean and atmosphere diverts from the simple picture of horizontal convection. The major difference in the atmosphere is, that it is mostly heated at its lower boundary but cooled also above it. Clouds radiate energy into space and cool the atmosphere. So, on average, heating occurs at a lower geopotential level than cooling and the convection has some vertical-convection-component (Rayleigh-Bénard convection) and there is a clear input of mechanical energy from heating and cooling.

For the ocean the convection is completely horizontal, but the mechanical forcing of the dynamics due to wind shear at the surface is important as it injects mechanical energy at large and small scales. At large scale it entrains a large scale circulation and at small scale it increases the mixing, leading to a stronger and deeper overturning circulation as predicted by purely horizontal convection.

4.4.5 Boussinesq Approximation (revisited)

In our calculations we used the *Boussinesq approximation* in which the conservation of mass is replaced by the conservation of volume (zero divergence). In this context heating means reducing the mass and cooling increasing it. So when the integrated heating is vanishing, mass is transported (by smurfs) from the heating to the cooling area, this is of course non-physical. If the mass is subtracted and added at the same geopotential height, no net work is done on the system. In the real world the density is reduced in the heated area by expansion of the fluid parcel, meaning that the expansion does work on the system given by $W_h = \int p_h dV_h$ and the opposite happens in the cooling area $W_c = \int p_c dV_c$, where now the volume decreases $dV_c < 0$. We see now that if cooling equals (minus) heating, if the thermal expansion coefficient is constant and if it happens at the same geopotential height ($p_h = p_c$) then no net work is done on the system. If heating and cooling is done at different levels the input of work in the *Boussinesq approximation* is: $g\Delta h\Delta m$ and without it is $\Delta W = (p_c - p_h)\Delta V = g\Delta h\rho_0\Delta V = g\Delta h\Delta m$ where ρ_0 is some average density. This shows that energetically it is justified to leading order (meaning that using a average density ρ_0 is a good approximation) to use the Boussinesq approximation. The water masses produced by the heating with or without the Boussinesq approximation are also identical to leading order. So in both cases the the effect of heating and the work done on the system is the same (when the thermal expansion coefficient is constant and if variations of density are small) and it is (to my opinion) justified to use the Boussinesq approximation.

4.5 Convection into a Stable Environment

In many applications a mass of fluid subject to heat fluxes is initially in a stable situation (see fig 4.6). If such kind of fluid is heated from above or cooled from below the stratification becomes even more stable and there is no convective motion if the surface heat fluxes is homogeneous. If the flux is inhomogeneous the cooled mass of fluid spreads along the boundary in form of a gravity current (discussed latter in this "guided tour").

If a stable environment is cooled from above (or heated from below) the stable stratification is destroyed. Indeed a convective front is established which progresses downward (upward) into the fluid leaving an almost perfectly mixed water-mass behind (see fig. 4.6). The turbulent motion above the front creates a small scale turbulent motion that leads to an increased eddy diffusivity which completely homogenises the water masses above the front. The velocity of

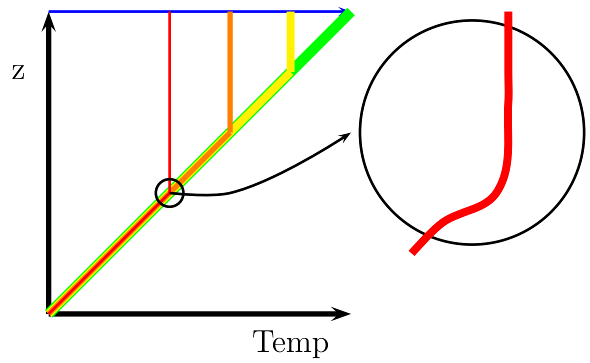


Figure 4.6: When cooling an ocean with a positive temperature gradient (stable) at the ocean surface, a well mixed layer appears above the stratified region. Initial temperature profile in green followed in time by yellow, orange and red. A mixed layer develops above the stratified deeper ocean. The depth of the mixed layer increases in time.

advancement of the front can (to first order) be easily obtained using only the conservation of heat and the fact that the water above the stratified zone is well mixed.

Exercise 52: Suppose that the ocean near the surface has a temperature stratification of $\partial_z T = 10^{-2}\mathrm{Km}^{-1}$ is subject to a heat flux of $500\mathrm{Wm}^{-2}$. Calculate the descent of the mixed-layer depth. (A typical value of the heat capacity of sea water is $c_p = 4000.\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$)

A closer look reveals, however, that the descending water overshoots and entrains water from the stratified region into the mixed layer leading to a temperature jump at the interface between the mixed layer and the stratified region (see right of fig. 4.6).

4.6 Convection from point sources

So far we have considered convection from a spatial extended, large scale surfaces, we will now consider the opposite case of spatial localised point sources. Applications are numerous in the environment. We can roughly distinguish two cases (see fig. 4.7): a sudden release of buoyancy ("point" source also in time) and release of heat that is continuous in time. In the former case a *thermal* is created and in the latter we have a *plume*. The case of a starting plume, the sudden start of a continuous buoyancy release is somehow between this two cases. In all these cases the flow will be axis symmetric when averaged over a sufficient time scale (or many realizations of the experiment). In the sequel we will use dimensional arguments in the determination of the flow field. In all the cases considered the flow structure will expand when released from the point source. Strictly speaking, the flow at the point source is singular as the buoyancy anomaly is infinite at that point. At some distance above a small source the flow behaves as if it originated from such a virtual point source. So the notion of a point source is a useful theoretical concept. I will always suppose here that buoyancy is positive, the case of negative buoyancy is obtained by inverting the z-direction.

We further suppose that the height z above the point source the Reynolds number $Re = (ur)/\nu$ is large so that the flow is turbulent. Where u and r are the velocity and the horizontal scale, respectively, at the height z and ν is viscosity.

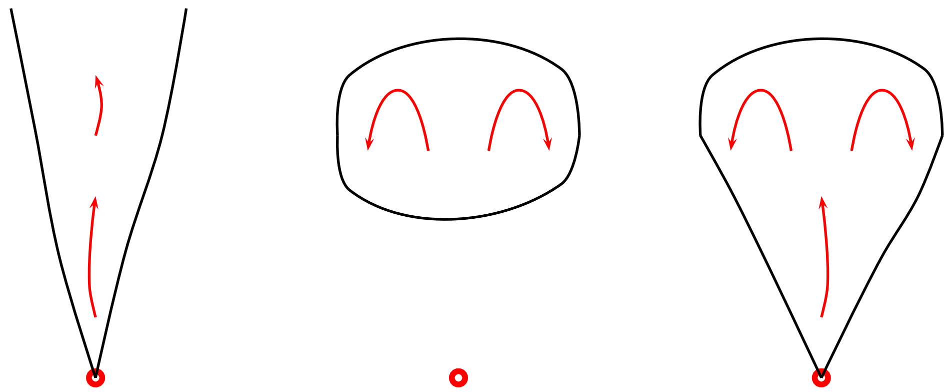


Figure 4.7: Sketch of three situations of convection from a point source (red dots): plume (left), thermal (middle) and starting plume (right).

4.6.1 Plume

The only governing physical parameter is the buoyancy production per time F_0 of the point source measured in $\mathrm{m}^4\mathrm{s}^{-3}$ (volume $\times$ buoyancy per time). Please note that the time averaged buoyancy flux through every cross section z is equal to F_0. There is no non-dimensional parameter.

So a typical velocity scale at a distance at z is given by:

$$w \propto F_0^{1/3} z^{-1/3} \tag{4.72}$$

$$g' \propto F_0^{2/3} z^{-5/3}. \tag{4.73}$$

Using the fact that the system is axial symmetric we get:

$$w = F_0^{1/3} z^{-1/3} f_1(r) \tag{4.74}$$

$$b = F_0^{2/3} z^{-5/3} f_2(r) \tag{4.75}$$

$$r = \sqrt{\frac{x^2+y^2}{z^2}} \tag{4.76}$$

where r is the distance from the axis divided by the height. In experiments it is found that:

$$f_1(r) \approx 4.7\exp(-96r^2) \tag{4.77}$$

$$f_2(r) \approx 11\exp(-71r^2) \tag{4.78}$$

$$\tag{4.79}$$

Note that the conservation of buoyancy is assured by $\int_A f_1(r) f_2(r) dxdy = 1$ and that buoyancy has a slightly wider spread than momentum. I emphasise once more, that the above description is for values averaged over very long time or many realizations of the experiment (ergodicity).

If in a system the averages over many realizations (repeating the same experiment many times) equals time averages in one realization of the experiment, the experiment is called *ergodic*. Most systems are supposed to be ergodic but it is mathematically proofed only for very few systems. A supposed ergodicity is used in most laboratory and numerical experiments.

If we consider the upward mass transport W we obtain:

$$W(z) = F_W F_0^{1/3} z^{5/3} \tag{4.80}$$

which is an increasing function of height. This leads to a vertical divergence of $\partial_z W \propto z^{2/3}$. This mass is entrained from the surrounding fluid. A visual inspection of plumes (cigarette smoke) shows the there are large eddies that capture surrounding fluid which is then quickly mixed and homogenised by smaller turbulence within the plume. As the circumference is proportional to z, we find that the *entrainment velocity*, that is the velocity with which the outside fluid passes into the plume is proportional to the plume velocity $u_E = \alpha w$, where α is the *entrainment constant*. It is this entrainment constant that determines the plume dynamics, its value must be obtained from experiment.

If we suppose that the plume has a top-hat structure ($(w, g') = 0$) outside the plume and $= (w(z), g'(z)))$ inside. If we further suppose that the angle of the plume boundary to the vertical is γ we have:

$$W(z) = w(z)\pi R(z)^2 = w(z)\pi\gamma^2 z^2 = F_0^{1/3}\pi\tilde{f}_1\gamma^2 z^{5/3} \quad \text{and} \tag{4.81}$$

$$\frac{dW}{dz} = \frac{5}{3}F_0^{1/3}\gamma^2\pi\tilde{f}_1 z^{2/3} \tag{4.82}$$

considering the entrainment we have:

$$\frac{dW}{dz} = u_E(z)2\pi R(z) = u_E(z)2\pi\gamma z = \alpha w(z)2\pi\gamma z = 2\pi\gamma\alpha F_0^{1/3}\tilde{f}_1 z^{2/3} \tag{4.83}$$

Equating eq. (4.82) and eq. (4.83) leads to:

$$\gamma = \frac{6}{5}\alpha \tag{4.84}$$

Applied to our Gaussian case with $\gamma \approx 1/9$ we get $\alpha \approx 0.1$ which is very close to observed values. In the Gaussian case it is however ambiguous to define a plume boundary and a plume velocity.

4.6.2 Thermal

If there is an initial buoyancy anomaly of a fluid volume $B_0 = g'V_0$ ($[B_0]$ =m^{4}s^{-2}, dimensional analysis gives:

$$w \propto B_0^{1/2} z^{-1} \tag{4.85}$$

$$g' \propto B_0^1 z^{-3}. \tag{4.86}$$

When the dependence of time is considered we get:

$$w \propto t^{-1/2} \tag{4.87}$$

$$r \propto t^{1/2}. \tag{4.88}$$

The shape of a thermal is a slightly flattened sphere with a vortex ring in its interior. Its surrounding flow is that of a flow around a sphere with a superimposed constant inflow and no boundary layer is present as it would be sucked into the thermal.

4.6.3 Forced Plume

In the previous subsection we considered the dynamics that develops above a source of buoyancy (plume). We now combine what happens when the source also emits momentum. A source of momentum without buoyancy leads to a jet. The case in between this two extremes is a *forced plume*. There are now two parameters, buoyancy production F_0 and momentum production $\rho_0 M_0$. With units: $[F_0] = \mathrm{m}^4\mathrm{s}^{-3}$, $[M_0] = \mathrm{m}^4\mathrm{s}^{-2}$. Note that a length scale is given by $z_0 = M_0^{3/4} F_0^{-1/2}$.

The vertical velocity in a vertical jet scales as

$$w_{jet} = M_0^{1/2} z^{-1} \tag{4.89}$$

and the spread is given by 2α.

Exercise 53: Show that for a "top-hat" jet the velocity scales as eq. (4.89), and the spread is 2α.

We can discuss two cases F_0 and M_0 have the same sign or opposite signs. In the former case the forced plume starts as a jet and becomes a plume after a distance that scales with z_0, it starts with a spread of 2α which reduces to $(6/5)\alpha$. If $M_0 > 0$ and $F_0 < 0$ (fountain case) a heavy upward jet with its upward momentum which is continuously reduced by buoyancy until the jet reverses at a height of about $1.3 z_0$ (found from experiments).

Exercise 54: Why is the entrainment flux into the plume and not out of the plume (detrainment) ?

4.6.4 Buoyant Vortex Ring (Forced Thermal)

When a fluid volume is pushed into a motionless surrounding fluid, a vortex ring is created.

4.6.5 Starting Plume

When a buoyancy source is suddenly turned on we have a starting plume. Its form stays the same but its size is increasing in time. At some distance behind the top of the starting plume the dynamics is that of a plume. At the front is behaves as a thermal which is constantly fed by buoyancy and momentum from behind, so the plume cap dilutes slower.

In the flows considered so far, convective flows, buoyancy had a destabilising effect, was at the origin of the (turbulent) fluid motion. We will now consider situations where the buoyancy has a stabilising effect, opposing instability.

4.6.6 Plume in a stable stratified environment

When a plume rises in a stratified environment the buoyancy flux is no longer constant, as the surrounding density is a function of height.

There are two dimensional parameters the buoyancy production F_0 and square of the Brunt Väsäilä frequency $G = N^2$. So we can define a length scale $z_0 = F_0^{1/4} G^{3/8}$. Experimental results show that the plume at values of $z < 2z_0$ the effect of stratification is small. At $z \approx 3z_0$ the plume starts to extend horizontally and the maximal height of the plume is given by $z \approx 5z_0$.

4.7 Richardson Number

In the flows considered so far, convective flows, buoyancy had a destabilising effect, was at the origin of the (turbulent) fluid motion. We will now consider situations where the buoyancy has a stabilising effect, opposing instability.

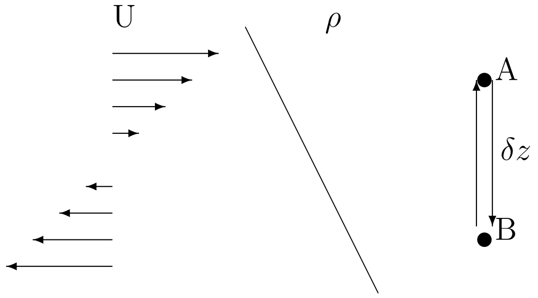


Figure 4.8: Exchanging volumes A and B in a sheared stably stratified flow.

When considering the vertical mixing in the ocean we usually have a large scale horizontal flow that has a vertical shear $\partial_z U$ which has a tendency to destabilise the flow and generate turbulence. On the other hand the flow usually has a stable stratification that suppresses instability and also turbulence. This means that there are two competing phenomena and it is key for vertical mixing to determine under which circumstances one of the processes dominates. To this end we look at a stably stratified sheared flow, and consider the energy budget when to equal volumes A and B, as shown in fig. 4.8, separated by a distance δz are exchanged. The potential energy ΔE_{pot}, necessary to exchange the heavier and lower volume B with the lighter and higher volume A is supposed to be provided by the kinetic energy ΔE_{kin}, in the shear. For this to be possible it is clear, that $\Delta E_{total} = \Delta E_{kin} + \Delta E_{pot} > 0$, which are given by,

$$\Delta E_{kin} = 2\frac{\rho V}{2}((\delta z/2)\partial_z U)^2 \tag{4.90}$$

$$\Delta E_{pot} = -gV(\delta z)^2 \partial_z \rho. \tag{4.91}$$

Indeed, $E_{pot} = gh\Delta m$, and for our case $h = \delta z$ and $m = \delta z V \partial_z \rho$, is the mass difference between volume B and A. $\Delta E_{total} > 0$ if the Richardson number ,

$$Ri = \frac{g\,\partial_z \rho}{\rho(\partial_z U)^2} < \frac{1}{4}, \tag{4.92}$$

or if we write $\delta U = \delta z\ \partial_z U$ and $\delta\rho = \delta z\ \partial_z \rho$ we obtain,

$$Ri = \frac{g\,\delta\rho\ \delta z}{\rho(\delta U)^2} < \frac{1}{4}, \tag{4.93}$$

Which means that using the kinetic energy of the volumes A and B it is possible to interchange the volumes A and B when $Ri < 1/4$. Although that this calculation is very simple, only comparing kinetic to potential energy, and does not tell us how the volumes A and B should be exchanged, it is found in laboratory experiments that sheared stratified flow does indeed become unstable around a critical Richardson number of one quarter.

The above, and more involved, calculations together with laboratory experiments and oceanic observations have led to a variety of parametrisations of the vertical mixing based on the Richardson number.

One of the simplest, and widely used, parametrisations for vertical mixing based on the Richardson number was proposed by Philander and Pacanowski (1981):

$$\nu_{eddy} = \frac{\nu_0}{(1+\alpha Ri)^n} + \nu_b, \tag{4.94}$$

where typical values of the parameters, used in today's numerical models of the ocean dynamics, are $\nu_0 = 10^{-2}\mathrm{m}^2\mathrm{s}^{-1}$, $\nu_b = 10^{-4}\mathrm{m}^2\mathrm{s}^{-1}$, $\alpha = 5$ and $n = 2$.

Exercise 55: Explain the meaning of "Slippery Sea".

4.8 Gravity Currents

Examples are katabatic winds, dense ocean currents on the ocean floor after passage through a strait. The buoyancy transport per unit cross-stream-distance is $A = g'hU$ with units $\mathrm{m}^3\mathrm{s}^{-3}$ and the angle α (no units) are the only external parameters. So dimensional analysis says:

$$U \propto A^{1/3}x^0 = A^{1/3} \tag{4.95}$$
$$g' \propto A^{2/3}x^{-1} \tag{4.96}$$
$$h \propto x^1 \tag{4.97}$$
$$\tag{4.98}$$

where the constant of proportionality depends on the angle α. Note also that $g'h \propto x^0$ is constant. In a stationary entraining gravity current which entrainment constant E, the continuity and momentum equations (neglecting friction) are:

$$\frac{dUh}{dx} = EU \tag{4.99}$$
$$\frac{dU^2h}{dx} = g'h\sin\alpha, \tag{4.100}$$

When we used eq. (4.95) we see that $dU/dx = 0$. So that the entrainment is given by:

$$E = \frac{g'hsin\alpha}{U^2} = Ri\tan\alpha \tag{4.101}$$

and entrainment is a function of the Richardson number

$$Ri = \frac{g'h\cos\alpha}{U^2} \tag{4.102}$$

Chapter 5

Mixing

In previous chapters we have seen that it seems impossible to describe the motion of a turbulent fluid in all its detail. Furthermore, such description, if found, would not be to useful as we are often only interested in what happens on average or at large scales. To entangle that large scale, or average, behaviour we have extensively used averaging and dimensional analysis in the previous chapters. We will now introduce the mathematical concept of stochastic process to investigate the turbulent motion further. If we flip a coin we can calculate its trajectory and also the outcome ("head" or "tail") when we know all the parameters involved (initial location, speed, rotation, friction, ...) to a sufficient accuracy. This is rarely the case we always have some degree of uncertainty in all the variables and so the outcome of the experiment is random. We however know that if we toss the coin, say a hundred times, with a high probability the outcome will be that heads appear between 40 and 60 times. The same concept appears to turbulent flow, which is always observed with some uncertainty and so it can be mathematically described as a random experiment. When a random variable is evolving in time it is called a random process.

5.1 Random Walk

A drunken person coming out of a bar is often taken as the analogy of a *random walk*. If one supposes that at every time interval Δt the drunkard does a step forward or backward, with equal probability, the average (averaged over many drunkards) displacement of the drunkard will be vanishing. The average square-displacement however will be an increasing function of time.

The position of the drunkard $X(n\Delta t)$ at the discrete time $n\Delta t$ ($n = 0, 1, 2, 3, ...$) can be described as a (discrete) random process. When we have many drunkards (called realizations) we can give them numbers $\omega \in \Omega$. The position of drunkard ω, at time $n\Delta t$, is $X_\omega(n\Delta t)$. The average of a function f of X over all the elements is given by

$$\langle f(X)\rangle_\Omega = \frac{1}{|\Omega|}\sum_\Omega f(X_\omega) = \frac{1}{|\Omega|}\int_\Omega f(X_\omega)d\omega \tag{5.1}$$

if we assign a probability $P(\omega)$ to every drunkard we get in the general case:

$$\langle f(X)\rangle_\Omega = \int_\Omega f(X_\omega)dP(\omega) \tag{5.2}$$

where $P(\omega)$ is called a probability measure that has $|\Omega| = \int_\Omega dP(\omega) = 1$.

So coming back to our drunkards we can say: $\langle X\rangle_\Omega = 0$, meaning that the centre of gravity of all drunkards does not move. Indeed,

$$X_\omega((n+1)\Delta t) = X_\omega(n\Delta t) + \zeta_\omega(t)\Delta x \tag{5.3}$$

with $\zeta_\omega(t) = \pm 1$ with probability 1/2, so that $\langle \zeta_\omega(t)\rangle_\Omega = 0$. so that,

$$\begin{aligned}\langle X_\omega((n+1)\Delta t)\rangle_\Omega &= \langle X_\omega(n\Delta t)\rangle_\Omega + \langle \zeta_\omega(t)\Delta x\rangle_\Omega = \langle X_\omega(n\Delta t)\rangle_\Omega = \langle X_\omega((n-1)\Delta t)\rangle_\Omega \quad (5.4)\\ &= \ldots = \langle X_\omega(0)\rangle_\Omega. \quad (5.5)\end{aligned}$$

What about the square distance to the bar: $\langle X^2\rangle_\Omega = 0$? We have, using eq. (5.3):

$$X^2_\omega((n+1)\Delta t) = X_\omega(n\Delta t)^2 + 2X_\omega(n\Delta t)\zeta_\omega(t)\Delta x + (\zeta_\omega(t)\Delta x)^2 \tag{5.6}$$

and

$$\begin{aligned}\langle X^2_\omega((n+1)\Delta t)\rangle_\Omega &= \langle X_\omega(n\Delta t)^2\rangle_\Omega + 2\langle X_\omega(n\Delta t)\zeta_\omega(t)\Delta x\rangle_\Omega + \langle(\zeta_\omega(t)\Delta x)^2\rangle_\Omega \quad (5.7)\\ &= \langle X_\omega(n\Delta t)^2\rangle_\Omega + (\Delta x)^2 = (n+1)(\Delta x)^2 + \langle X^2_\omega(0)\rangle_\Omega. \quad (5.8)\end{aligned}$$

Where the second term in the first line vanishes as $\langle X_\omega(n\Delta t)\zeta_\omega(t)\rangle_\Omega = \langle X_\omega(n\Delta t)\rangle_\Omega\langle\zeta_\omega(t)\rangle_\Omega$, the random variables are statistically independent and $\langle\zeta_\omega(t)\rangle_\Omega = 0$.

So the average square distance of the drunkard from the bar grows linearly at a rate $(\Delta x)^2/\Delta t$. If we take the limits $\Delta x \to 0$ and $\Delta t \to 0$ with keeping $c = (\Delta x)^2/\Delta t$ fixed, our discrete random walk converges towards a Wiener process.

If many steps are taken the distribution of a random walk approaches a Gaussian.

Exercise 56: Calculate $\langle X^m_\omega((n{+}1)\Delta t)\rangle_\Omega$ for all positive integers m, supposing that $\langle X^m_\omega(0)\rangle_\Omega = 0$.

Random walks have different properties of which, I want to mention the two most important. A stochastic process is called a Markov process if for the prediction of a system from a present state $X(t_0)$ to the future $X(t_1)$ the knowledge of $X(t)$ at times $t < t_0$ leads to no improvement. A stochastic process is called a martingale, if the average of a stochastic process starting at t_0 from $X_0 = X(t_0)$ is X_0. A random walk is a Markov process and a martingale.

So far we have considered only the 1D-random walk. The generalisation to 2 or 3 dimensions is straightforward, by taking a different random variable for each dimension $\mathbf{X}(t) = (X(t), Y(t), Z(t))$.

5.2 From the Random-Walk to Diffusion

We can modify the random walk introduced above by allowing each particle to stay at its location with probability $(1-D)$ and to move a distance Δx to the left with a probability $D/2$ and to move a distance Δx to the right with a probability $D/2$, during the time interval Δt. We have discretised space ($x = i\Delta x$) and time ($t = n\Delta t$). If we denote by $T_n(i)$ the density of particles at location i at time n we can write:

$$\begin{aligned}T_{n+1}(i) &= (D/2)T_n(i+1) + (1-D)T_n(i) + (D/2)T_n(i-1) \quad (5.9)\\ &= T_n(i) + (D/2)\left[T_n(i+1) - 2T_n(i) + T_n(i-1)\right] \rightsquigarrow \quad (5.10)\\ T_{n+1}(i) &- T_n(i) = (D/2)\left[T_n(i+1) - 2T_n(i) + T_n(i-1)\right] \quad (5.11)\end{aligned}$$

which is the discretized form of the diffusion equation:

$$\partial_t T(x,t) = \kappa\partial_{xx}T(x,t) \quad \text{with} \quad \kappa = \frac{D(\Delta x)^2}{2\Delta t}. \tag{5.12}$$

Equation (5.12) is the continuous macroscopic version of the discretised microscopic random walk.

Exercise 57: What is the macroscopic equation (equivalent of eq. (5.12)) if the probability of a particle to move left is A to move right B and to stay $1 - A - B$?

It is straightforward to generalise the above derivations to two and three dimensions.

Exercise 58: Derive eq. (5.12) for the two dimensional case.

5.3 Solutions of the Diffusion Equation

An explicit example: If the tracer is initially in a single point, that is, represented by a delta function, then its density at all times is given by:

$$T(x,t) = \frac{H_0}{\sqrt{4\pi\kappa t}} \exp(-x^2/(4\kappa t)) \tag{5.13}$$

and its mean-square displacement (variance) is:

$$\langle X^2 \rangle = \frac{H_0}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{+\infty} x^2 \exp(-x^2/(4\kappa t) dx = 2\kappa t H_0. \tag{5.14}$$

In three dimensions the same example becomes:

$$T(r,t) = \frac{T_0}{(\sqrt{4\pi\kappa t})^3} \exp(-r^2/(4\kappa t)) \tag{5.15}$$

and its mean-square displacement is:

$$\langle X^2 \rangle = \frac{H_0 4\pi}{(\sqrt{4\pi\kappa t})^3} \int_{R^3} \mathbf{r}^2 \exp(-\mathbf{r}^2/(4\kappa t) d\mathbf{r} = 6\kappa t H_0, \tag{5.16}$$

which shows, again, that the mean-square displacement grows linearly in time.

Exercise 59: Show that eqs. (5.13) and (5.16) are the solutions of the heat equation in one and three dimensions, respectively.

Exercise 60: Show eqs. (5.14) and (5.16).

Exercise 61: Calculate the equivalent of equations (5.16) and (5.16) in two dimensions.

Exercise 62: In a cylindrical container filled with water which is standing upright you keep the temperature T_b constant at the bottom and the top $T_t > T_b$. You wait a long time. How does the temperature change along the vertical. What happens when the container is a cone (cross-section changing linearly with the vertical). What is the differnce of a case without gravity and strong gravity which forces the iso-density lines to be horizontal? (We suppose that the diffusion constant is independent of temperature).

5.4 Solutions of the Advection-Diffusion Equation

5.4.1 2D

Let $\mathbf{u} = (u, 0, 0)$ and C the concentration of some passive tracer, which is injected at a point source at a constant rate S. At large enough velocity u the stream-wise diffusion is negligible. The dominant term in the time-independent tracer equations are:

$$u\partial_x C = \kappa \partial_{yy} C \tag{5.17}$$

We can introduce $x = ut$ and we reduced the problem to a one dimensional heat equation (in the y-direction) the point source broadens in time as:

$$\tilde{C}(t, y) = \frac{S}{u\sqrt{4\pi\kappa t}} \exp\left(-\frac{y^2}{4\kappa t}\right) \tag{5.18}$$

Re-replacing $t = x/u$ the solution in 2D-space is:

$$C(x, y) = \frac{S}{\sqrt{4\pi\kappa x/u}} \exp\left(-\frac{y^2 u}{4\kappa x}\right) \tag{5.19}$$

Exercise 63: At what rate does the average density decrease along the centre line $(x, 0, 0)$.

5.4.2 3D

Exercise 64: Do the calculations for the 3D case.

Exercise 65: At what rate does the average density decrease along the centre line $(x, 0, 0)$.

From what we will learn in the next two sections, the diffusion coefficient depends on the size of the structure (relative diffusion) so that the plume spreads faster and the average concentration decreases faster than predicted by the above calculations.

Exercise 66: Why is there no entrainment?

5.5 Turbulent Diffusion (one particle)

In the previous section the diffusion process was derived from a random walk in which there was no correlation between two consecutive steps, the correlation time was vanishing ($\langle \zeta_\omega(t_1)\zeta_\omega(t_2)\rangle_\Omega = 0$ if $t_1 \neq t_2$). Turbulent motion is different as there is motion on different scales, in time and space. A particle is captured by small and larger eddies that move it around. In 3D turbulence larger eddies have larger correlation times and the exact displacement law of one particle depends of how strong are the eddies of a certain size.

The velocity of a particle is:

$$\partial_t X(t) = V(t) \tag{5.20}$$

where $V(t)$ is the (Lagrangian) velocity of the particle at time t. The displacement of a particle is then:

$$X(t) = X(0) + \int_0^t V(t')dt', \tag{5.21}$$

and the square displacement

$$X^2(t) = \int_0^t \int_0^t V(t_1')V(t_2')dt_1'dt_2'. \tag{5.22}$$

When an ensemble average is taken we have:

$$\langle X^2(t)\rangle = \int_0^t \int_0^t \langle V(t_1')V(t_2')\rangle dt_1'dt_2'. \tag{5.23}$$

If the system is stationary we have the correlation $v^2R(t_2 - t_1) = \langle V(t_1')V(t_2')\rangle$, where $v^2 = \langle V(t)^2\rangle = \langle V(0)^2\rangle$ is a constant and $R(t_2 - t_1)$ is the velocity auto-correlation function with $R(\tau) = R(-\tau)$, $\tau = t_2 - t_1$. We obtain by change of variables (and using $R(\tau) = R(-\tau)$):

$$\langle X^2(t)\rangle = v^2 \int_0^t \int_0^t R(t_2' - t_1')dt_1'dt_2' = v^2 \int_0^t \int_{-t'}^{t-t'} R(\tau)d\tau dt'. \tag{5.24}$$

In a turbulent flow $R(\tau)$ is a decreasing function of τ as the the particle "forgets" its velocity from long time ago. The way $R(\tau)$ decreases determines the diffusion process. A correlation time

$$\tau_{\text{corr}} = \int_0^\infty R(\tau)d\tau = \frac{1}{2}\int_{-\infty}^\infty R(\tau)d\tau \tag{5.25}$$

can be calculated.

Exercise 67: Show the second equality of eq. (5.24).

For the behaviour of $\langle X^2(t)\rangle$ two limiting cases can be considered: $t \ll \tau_{\text{corr}}$, in this case $R(\tau) \approx 1$ leading to:

$$\langle X^2(t)\rangle = v^2t^2. \tag{5.26}$$

such (advective) transport where the distance is proportional with time is also called ballistic.

For the case $t \gg \tau_{\text{corr}}$ we can see that $\int_0^t R(t_2 - t_1)dt_1' \approx (1/2)\int_{-\infty}^\infty R(t_2 - t_1)dt_1'$ leading to:

$$\langle X^2(t)\rangle = 2v^2\tau_{\text{corr}}t \tag{5.27}$$

for the second case we can define the turbulent diffusion coefficient to be:

$$\kappa_{\text{turb}} = v^2\tau_{\text{corr}}. \tag{5.28}$$

This shows that for times larger than the correlation time the transport is diffusive with:

$$\langle X^2(t)\rangle = 2\kappa_{\text{turb}}t. \tag{5.29}$$

(in 3D). If we introduce a correlation length scale $l_{\text{corr}} = (v^2)^{1/2}\tau_{\text{corr}}$ we get $\kappa_{\text{turb}} = v_{\text{rms}}l_{\text{corr}}$ ($v_{\text{rms}} = (v^2)^{1/2}$ root-mean-square). So for long times (much larger than τ_{corr}), and large distances, the behaviour of a fluid particle resembles that of a random-walk leading to a diffusive process.

The case between the two extremes is relevant for turbulent transport, where the large eddies transport ballistically over times shorter than their characteristic time scale and small eddies lead to diffusion over times longer than their characteristic time scale.

5.6 Turbulent Diffusion (two particles)

All the above does not, however, tell us how a cloud of fluid spreads as we have only considered the behaviour in time, but not the effect of separation in space. The calculations in the previous section apply to random processes with no spatial correlation, as the random walk of drunkards (the displacement between two drunkards is un-correlated whether they are close of or not) or the case where the displacement is constant in space. In the former case the average square-distance between two realisations (drunkards) grows proportional to $\langle Y^2(t)\rangle = \langle (X_1(t) - X_2(t))^2\rangle \propto t$, in the latter it is constant.

The analysis performed in the one-particle case is not possible as the separation depends now on time and the spacial separation of the two particles. Indeed:

$$Y(t) = X_1(0) - X_2(0) + \int_0^t V_1(t') - V_2(t')dt', \tag{5.30}$$

$$\langle Y^2(t)\rangle = (X_1(0) - X_2(0))^2 + \int_0^t \int_0^t (V_1(t') - V_2(t'))(V_1(t'') - V_2(t''))dt'dt''. \tag{5.31}$$

the expression in the integral is now a function of difference in time and distance in space which prohibits a simple analysis.

We can however distinguish again three cases: (i) the separation between the particles is larger than all the turbulent length scales; (ii) the separation between the particles is within the turbulent length scales and (iii) the separation between the particles is smaller than all the turbulent length scales;

In the first case we have diffusive separation of the particles and the mean-square-separation is $\propto t$. In case (iii) there is, to first order a linear variation of the relative velocity of the two particles and the mean-square-separation is $\propto t^2$. Case (ii) is the most involved.

In the energy cascade (3D and large scales in 2D) we have: In a turbulent flow the larger eddies lead to advection the smaller eddies have a diffusive effect and the eddies of size comparable to the cloud of particles has some effect in between these two regimes. Richardson found in 1926 that the evolution of such a cloud can be modelled with a size dependent eddy-diffusivity coefficient. This phenomena is usually revered to as *relative diffusion*. He determined that the eddy-diffusivity coefficient $\kappa_{\text{turb}} \propto l^{4/3}$, where l is the size of the cloud. The constant of proportionality depends on the strength of the turbulence measured by the energy dissipation rate. Indeed, using dimensional analysis we have:

$$\frac{d\langle Y^2\rangle}{dt} = A\epsilon^{1/3}\langle Y^2\rangle^{2/3}, \tag{5.32}$$

where ϵ ($[\epsilon]$ =m^{2}s^{-3}, $[Y]$ =m) is the energy dissipation rate (per unit mass) and A a non-dimension constant. Integrating the above equation in time gives: $\langle Y^2\rangle \propto t^3$. In a diffusive process we also have $d(\langle Y^2\rangle)/dt \propto \kappa$ and so $\kappa_{\text{turb}} \propto \langle Y^2\rangle^{2/3}$.

In the enstrophy cascade (2D) we have:

$$\frac{d\langle Y^2\rangle}{dt} = B\eta^{1/3}\langle Y^2\rangle, \tag{5.33}$$

where η ($[\eta]$ =s^{-3}) is the enstrophy cascade rate and $\langle Y^2\rangle \propto exp(B\eta^{1/3}t)$. and so the separation is exponential and $\kappa_{\text{turb}} \propto Y^2$.

5.7 Absence of Extrema and Homogenization

An advected and diffused scalar is governed by the equation:

$$\frac{d\theta}{dt} = \nabla \cdot (\kappa \nabla \theta), \tag{5.34}$$

where κ is the diffusivity. If the flow is incompressible and stationary, we get:

$$\nabla \cdot (\mathbf{u}\theta) = \nabla \cdot (\kappa \nabla \theta). \tag{5.35}$$

Suppose that there is a extrema (say a maximum) of $\theta = \theta_{max}$ in the flow. Then there exists a isoline of $\theta = \theta_{max} - \epsilon$, such that $\nabla\theta \cdot \mathbf{n} < 0$, with $\mathbf{n}$ the outward normal vector to the isoline of $\theta = \theta_{max} - \epsilon$. The left hand side of equation (5.35) integrated over the domaine within the isoline of $\theta = \theta_{max} - \epsilon$, gives:

$$\int_A \nabla \cdot (\mathbf{u}\theta) dA = \oint (\mathbf{u}\theta) \cdot \mathbf{n} dl = (\theta_{max} - \epsilon) \oint (\mathbf{u}) \cdot \mathbf{n} dl = (\theta_{max} - \epsilon) \int_A \nabla \cdot \mathbf{u} dA = 0. \tag{5.36}$$

Whereas the right hand side gives:

$$\int_A \nabla \cdot (\kappa \nabla \theta) dA = \oint \kappa \nabla \theta \cdot \mathbf{n} dl < 0. \tag{5.37}$$

So eq. (5.35) can not be true and we conclude, that a stationary diffusive flow can not have an extrema in the interior, that is, all extremas are on the boundary.

The above is true in any dimension. In two dimensions we can go even further. If Ψ is the stream function of a divergence free stationary flow we can write the equation governing the dynamics of a stationary scalar as:

$$J(\Psi, \theta) = \nabla \cdot (\kappa \nabla \theta) \tag{5.38}$$

If the Peclet number $Pe = lu/\kappa \gg 1$ then $J(\Psi, \theta) \approx 0$ and $\theta \approx G(\Psi)$ wher G is some function. This means that isolines of θ are stream lines. Combining with the absence of extrema (due to the effect of (small) diffusion) this means that θ is constant within a closed stream line meaning that it is *homogenized* within every closed stream-line.

5.8 Turbulent Mixing in Stratified Fluids

So far we have considered the mixing of a passive tracer, one that does not act on the velocity dynamics. We will now look at the case of an active trace, that acts on the velocity due to its buoyancy.

Mixing of a substance involves a non-reversible process that is usually provided by molecular diffusion. Mixing usually acts in three phases: (i) initially the two phases are clearly separated with a small interface between them, not much happens. (ii) The velocity field distorts the substance and leads to an increase in the length and the magnitude of gradients (stirring-phase). (iii) Molecular motion can then act along this gradient in an efficient way, gradients suddenly disappear and the mixture becomes homogeneous. When there is no such molecular diffusion taking place the process is reversible and mostly referred to as stirring rather than mixing. A nice example is when a mixture of oil and water is stirred. If the motion ceases oil and water re-separate, no mixing has occured.

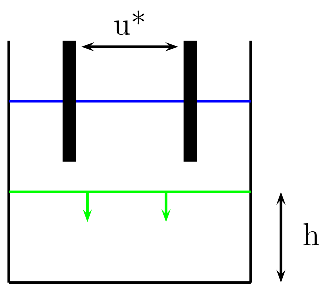


Figure 5.1: Mixing-box experiment: a box is filled with a fluid with two densities separated by an interface (green line). The upper fluid is stirred to generate turbulence. The resulting entrainment flux moves the interface downward as turbulent fluid "eats" itself into the non-turbulent fluid. The entrainment flux also increases the density in the upper layer, which stays homogeneous due to the turbulent motion. If the lower layer is stirred the interface moves upward.

5.8.1 Mixing in Shear Free Turbulence

This situation is often explored using the "mixing-box experiment" (see fig. 5.1).

The entrainment mechanism was studied using vortex rings which were directed towards the interface. The vortex rings hit the interface. A Richardson number can be formed using the buoyancy jump across the interface Δb the scale of the vortex ring l and its velocity u:

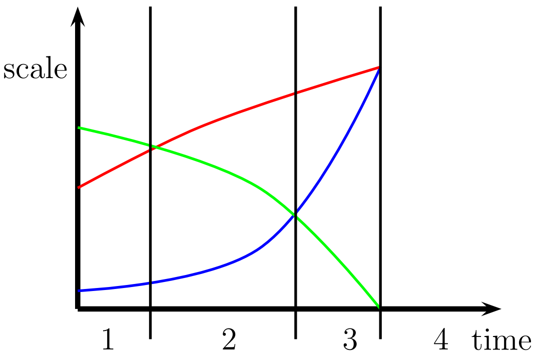
$$Ri = \frac{\Delta b l}{u^2}. \tag{5.39}$$

For small values of $Ri < 1$ we have splashing. The ring hits the interface, water from the other side of the interface splashes and is rapidly mixed into the turbulent fluid. $Ri > 15$ collision with "rigid" interface, the ring is squeezed but waves are created on the interface. For large Richardson number $Ri > 30$ entrainment due to interfacial wave breaking (Kelvin-Helmholtz instability) in highly strained regions of the internal waves can happen. This heigh Richardson number entrainment does not contradict the critical Richardson number criteria derived in the context of shear flows as, in the mixing-box experiment, there is a constant (external) source of energy, measured by its power P.

In the mixing-box experiment a wide variety of scaling laws for the entrainment constant,

$$e = \frac{u_E}{u} \propto Ri^\alpha, \tag{5.40}$$

often contradicting each other, have been observed. The exponents, put forward, range within $\alpha \in [-1.75, -1.]$. In other experiments a scaling depending on the Richardson number Ri and



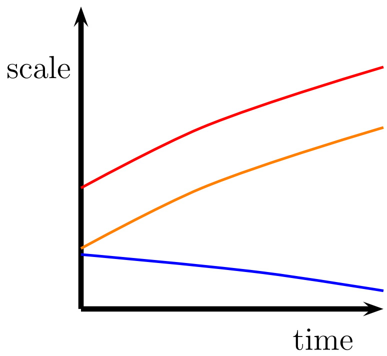


Figure 5.2: Evolution of turbulent length scales: $10 \times$ Kolmogorov scale, largest turbulent length scale, Ozmidov scale and Corsin scale. Left picture: shows the evolution of length scales in stratified turbulence phases 1-4 are explained in the text. Right picture: shows the evolution of length scales in shear turbulence.

the Peclet number $Pe = ul/\kappa$ was found:

$$e = \frac{u_E}{u} \propto Ri^{-3/2}(K_1 + K_2(Ri/Pe)^{1/2}), \tag{5.41}$$

where K_1 and K_2 are non-dimensional constants.

In the mixing-box experiment power P_{input} is provided by mechanical stirring. Part of this power (P_{pot}) is used up by irreversibly increasing the potential energy E_{diff} through mixing and part is irreversibly dissipated by viscosity P_{visc}. The ratio:

$$R_f = \frac{P_{\text{pot}}}{P_{\text{input}}}, \tag{5.42}$$

is called the *mixing efficiency* or the *Richardson flux number*. Its knowledge is of paramount importance in the parameterisation of mixing. It is a key quantity when the energy cycle in the ocean and atmosphere is considered. Its value is often fixed to 0.2 but evidence for such a value is scant. For unstably stratified cases (convection) it can reach up to 0.4. Observations, laboratory experiments and numerical experiments suggest that R_f depends on Re, Ri, Pr and on how the turbulence is generated. In the mixing-box experiments R_f is only constant when $e \propto Ri^{-1}$ as: The power (all per unit surface area) injected is equal to the energy flux $P_{input} = C_D d\rho u_*^3$, where u_* is the speed of the generating device (see Fig.5.1) and d is the area of its cross-section. The potential energy is $E_{pot} = ag(\Delta\rho)h^2/2$, where h is the height of the interface (see Fig.5.1). Note that $(\Delta\rho)h = const.$ and we obtain $\dot{E}_{pot}P_{pot} = agh\Delta\rho u_E/2 = -ah\rho\Delta b u_E/2$, where a is the surface area of the container and so:

$$R_f = \frac{P_{pot}}{P} \propto \frac{h\Delta b u_E}{u^3} \propto Ri\frac{u_E}{u}. \tag{5.43}$$

which says that R_f is independent of Ri iff $e \propto Ri^{-1}$.

5.8.2 Mixing in Stratified Shear Flow

If in a stratified fluid a shear flow is present instability occurs when the Richardson number falls below $1/4$. The primary large scale instability is the 2 dimensional Kelvin-Helmholtz instability, depending on the Reynolds number secondary (3D) instabilities occur. When the Reynolds number $Re > 10^4$, the *mixing transition* is passed, a 3D turbulent inertial range develops and the mixing completely homogenises the fluid captured by the Kelvin-Helmholtz instability. Below this Reynolds number some of the fluid "un-mixes", heavy droplets move back down to reunite with the dense fluid and light droplets move back up to reunite with the light fluid.

In the following we will suppose, that the mixing transition has passed and we have a state of stratified turbulent motion. The stationary turbulence is characterised by the energy transfer (at scale l) and the dissipation rate:

$$\epsilon = \frac{4}{5}\frac{\langle u_l^3\rangle}{l} = \nu\langle\sum_{i,j}(\partial_i u_j + \partial_j u_i)^2\rangle \tag{5.44}$$

($\langle u_l^3\rangle$ is the third-order longitudinal structure function). The stratification is characterised by the Brunt-Väisälä frequency N. As their units are $[\epsilon]$ =m^{2}s^{-3} and $[N]$ =s^{-1} we can form the Ozmidov scale $L_{Oz} = \sqrt{\epsilon/N^3}$. Turbulent eddies larger than L_{Oz} are strongly influenced by the stratification while smaller eddies are less. If the stirring is very strong so that L_{Oz} is much larger than the largest scale in the flow the stratification has a minor influence. In decaying turbulence the *Ozmidov scale* will decrease, so that buoyancy will eventually become important, deforming the largest scales first and the smaller scales later. When turbulence is impulsively generated in a layer of thickness h (by instability) the resulting decaying turbulence can thus be partitioned in four phases (see fig. 5.2):(1) the largest scales h grows and turbulence is uneffected by stratification, (2) largest scales h grow and are effected by stratification when they are larger than the Ozmidov scale and (3) turbulence is suppressed by buoyancy on all scales larger than about ten times the Kolmogorov scale $\eta = (\nu^3/\epsilon)^{1/4}$. This stage is also called "fossil turbulence". Phase (4) is when all the motion is damped by viscosity as it is on scales smaller than 10 times η. The passage from phase (1) to (2) is called "onset of buoyancy control" and the transition from (2) to (3) the "buoyancy-inertial-viscous (BIV) transition" as buoyancy and viscous damping are equally important at the intersection of L_{Oz} and 10η.

When the turbulence is dominated by large scale shear S the important length scale is the *Corsin scale* $L_{Co} = \sqrt{\epsilon/S^3}$. Shear, however does the opposite to buoyancy, it enhances turbulence, makes ϵ grow. So L_{Co} increases in time and the range of scales between L_{Co} and the Kolmogorov scale (the inertial range) grows.

Note that the bulk Richardson number is given by the ratio:

$$Ri = \frac{N^2}{S^2} = \left(\frac{L_{Co}}{L_{Oz}}\right)^{4/3} \tag{5.45}$$

and

$$R_b = \left(\frac{L_{Oz}}{\eta}\right)^{4/3} = \frac{\epsilon}{\nu N^2} \tag{5.46}$$

is called the buoyancy Reynolds number. If it is large, there is space between the scales where turbulence is attenuated by buoyancy (at large scale) and viscosity (at small scale), so that turbulent motion is possible. In experiment it is found that for values of $R_b > 20$ turbulent flow is possible. In analogy a shear turbulent Reynolds number

$$R_S = \left(\frac{L_{CO}}{\eta}\right)^{4/3} = \frac{\epsilon}{\nu S^2} \tag{5.47}$$

can be defined.

Another important length scale is the parcel-displacement scale also called Thorpe displacements $L_{Th}(z)$. It is computed by adiabatically reordering a discretised vertical profile so as to render it stable. More precisely, if a heavier fluid superposes a lighter fluid they are exchanged until we obtain a stable stratification. The distance a parcel at height z has to be displaced in the vertical to obtain a stable stratification, is the Thorpe displacement $L_{Th}(z)$. This reordering can be done for every (x, y) (one dimensional reordering) or over the whole domain (3D reordering leading to $L^{3D}_{Th}(z)$).

Exercise 68: Discuss the difference of $L_{Th}(z)$ and $L^{3D}_{Th}(z)$ for waves, that is, why are these two scales used to distinguish between wave motion and overturning motion

Comment: It is often impossible to exactly determine $L^{3D}_{Th}(z)$ when the equation of state is non-linear and depends on three or more variables (P,T and salinity in the ocean or P,T and humidity in the atmosphere). There can also be several stable configurations.

5.8.3 Combination

In nature we often have a combination of turbulence created away from the interface and migrating towards it and turbulence created by the shear at the interface. One example is the mixed layer in the ocean: wave breaking at the surface creates turbulence as does the shear at the interface to the deep ocean. The atmospheric mixed layer is another example, turbulence generated at the rough boundary and shear at its interface to the upper troposphere combine to entrain fluid.

5.8.4 Double Diffusion and Salt-fingering (an Exercise)

Three water masses are given: M_1 with $T_1 = 0.0C$, $S_1 = 32.5$; M_2 with $T_2 = 14^0C$, $S_2 = 34$; M_3 with $T_3 = 7^0C$, $S_3 = 37$.

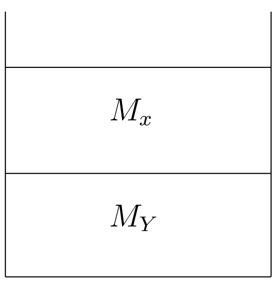
1: Draw these water masses in the diagram: (All drawings in this exam have to be done with extreme care):

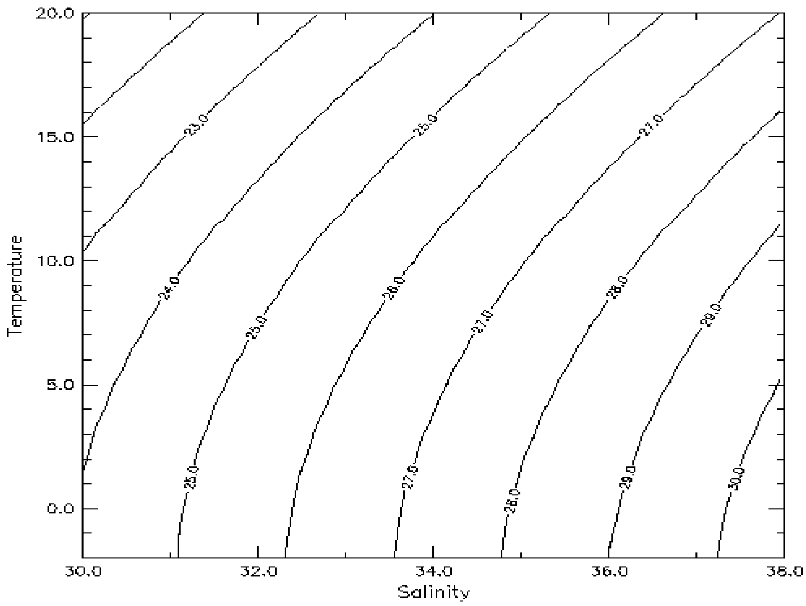
Give their densities:

$\rho_1 =$ .... kg/m^3 ; $\rho_2 =$ .... kg/m^3 ;

$\rho_3 =$ .... kg/m^3.

2: An equal volume of two of the water masses are carefully placed in a recipient (of a depth of one meter) one above the other, so that there is initially no fluid motion:





Potential density 0 dbar

When ignoring diffusion:

Placing M_1 above M_2 is : □ stable □ unstable

Placing M_1 above M_3 is : □ stable □ unstable

Placing M_2 above M_1 is : □ stable □ unstable

Placing M_3 above M_2 is : □ stable □ unstable

For the rest of the exam we suppose that *initially* there is no fluid movement and thus no mixing by turbulent motion (but there is of course molecular motion). Viscosity is neglected in the following.

3: (Molecular) diffusion of salt and heat at the interface between the two water masses leads to an exchange of heat and salinity. We now suppose that the diffusivity of temperature equals the diffusivity of salt $\kappa_T = \kappa_S$. We place M_2 above M_1. Draw the subset L_1 in the T-S-diagram on which will be found all measurements during the molecular exchange of heat and salt. If we wait for a very long time the two fluids will perfectly mix and form one water mass M_4. Mark M_4 in the diagram.

4: However, the diffusivity of temperature is about 100 times larger than the diffusivity of salt $\kappa_S \ll \kappa_T$! This is called "double diffusion" and leads to several important (and strange) phenomena in ocean dynamics. In the following we will suppose that $\kappa_S = 0$ ($\kappa_T > 0$ of course). We place M_2 above M_1. Draw (carefully) the subset L_2 in the T-S-diagram on which will be found all measurements during the molecular exchange of heat (we suppose that there is NO fluid motion). WARNING: This is different of what you saw in my course, where we supposed the (turbulent) diffusivities to be equal.

In the following questions we allow for possible fluid motion due to instability.

5: What happens during the process described in question 4 when we include the possibility of fluid motion due to instability? The process is called "salt fingering", why?

6: We now place M_1 above M_3 and still suppose $\kappa_S = 0$ ($\kappa_T > 0$). What happens (stability!)? why is this process called "double diffusive convection"?

We now place M_2 above M_3 and still suppose $\kappa_S = 0$ ($\kappa_T > 0$). What happens (stability!)?

Chapter 6

Solution of Exercises

Exercise ??:

Brunt-Väisälä frequency: The restoring buoyancy force per mass of the particle (of density ρ displaced by a distance z from the equilibrium in a fluid of stratification $\partial_z b$ is is $F_b/m = (\partial_z b)z$. Newtons second law leads to $\partial_{tt} z = -(\partial_z b)z$ which has solutions of the form: $z(t) = A\cos(\omega t) + B\sin(\omega t)$ with $\omega = \sqrt{\partial_z b}$.

Exercise 11:

The energy of a fluid of density ρ between the two points a and b in a channel of width L is composed of kinetic energy:

$$E_{kin} = \rho L \int_a^b \frac{H}{2} u^2 dx, \tag{6.1}$$

and potential energy:

$$E_{pot} = \rho L \int_a^b \frac{g}{2} \eta^2 dx. \tag{6.2}$$

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