Mathematics for Dynamicists - Lecture Notes

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Mathematics for Dynamicists

Lecture Notes

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Dedicated to the memory of my parents Michael and Hildegard Michelitsch

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Preface

These notes are the essence of the lecture series "Mathematics of Dynamicists" started in November 2004 and held regularly at the University of Sheffield as a MSc course module.

The module "Mathematics for Dynamicists" addresses to any students of Engineering and Natural Sciences who are interested in mathematical-analytical solution methods to tackle problems in physics and engineering dynamics.

The lecture notes consist of two parts. Part I is a warmup of the basic mathematics which is prerequisite for the understanding of part II. Some students will be familiar with the basics reviewed in part I, however the experience showed that it is useful to repeat these basics before starting with dynamical problems of engineering which are described by Recurrences and Ordinary Differential Equations in part II. To ensure the appropriate maths level needed for part II, the reader is highly recommended to answer the questions to part I before going to part II.

Part II is the essential part and gives a brief introduction to the theory and solution methods for linear recurrences, Ordinary Differential Equations (ODEs) and Initial Value Problems (IVPs) which are also referred to as Cauchy Problems. The main goal is less to give proofs but more on applications of the methods on dynamical problems relevant to physical and engineering sciences.

Instead of strict proofs we rather confine to short derivations with emphasis on applications of the analytical techniques. The reader is referred to the standard literature for any further proofs which are beyond the scope of these lecture notes.

The selection of materials for this short-term module (2 weeks time) can only be an incomplete and sometimes arbitrary selection of materials. To improve the syllabus, suggestions from anybody are highly appreciated. The enclosed lecture notes are continuously updated and extended. Any communicated discrepancies and errors will be corrected steadily[2].

I am especially grateful to my wife Dr. Jicun Wang for proof-reading this manuscript, to Mr. Shahram Derogar (Dept. Civil & Struct. Eng., Univ. of Sheffield) and to Dr. Andrzej F. Nowakowski (Dept. of Mech. Eng., Univ. Sheffield) for their helpful critical comments and valuable inputs which significantly improved the contents of this lecture.

Sheffield, November 2006 Thomas Michael Michelitsch

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1 Part I

In this section we give a brief repetition of the basic maths which is prerequisite to understand the dynamical problems and their solution methods introduced in part II.

1.1 Binomial Theorem

In this section we derive the expansion of $(a+b)^n$ in terms $a^k b^{n-k}$ where a, b are arbitrary *complex numbers* and n is an integer $(n = 0, 1, 2, .., \infty)$. For $n = 1$, we have

$$(a+b)^0 = 1 \tag{1}$$

$$(a+b)^1 = a+b \tag{2}$$

for n=2

$$(a+b)^2 = (a+b)(a+b) = (0+1)a^2 + (1+1)ab + (1+0)b^2 = a^2 + 2ab + b^2 \tag{3}$$

for $n = 3$

$$(a+b)^3 = (a+b)(a+b)^2 = (0+1)a^3+(1+2)a^2b+(2+1)ab^2+(1+0)b^3 = a^3+3a^2b+3ab^2+b^3 \tag{4}$$

and for general $n = 0, 1, 2, ...,$ we have

$$(a+b)^n = \sum_{k=0}^{n} C_{n,k} a^k b^{n-k} \tag{5}$$

We here observe that

$$C(n,k) = C(n-1,k) + C(n-1,k-1) \tag{6}$$

and with $C(n,n) = C(n,0) = 1$. Relation (6) can be visualized in terms of *Pascal's triangle*. Hence we have for $k = 1$ and arbitrary $n \geq 1$

$$C(n,1) = C(1,1) + C(1,0) + C(2,0) + .. + C(n-1,0) = \underbrace{1+1+..+1}_{n\,times} = n \tag{7}$$

for $k = 2$ and arbitrary $n \geq 2$

$$C(n,2) = C(2,2)+C(2,1)+C(3,1)+..+C(n-1,1) = 1+2+..+n-1 = \frac{n(n-1)}{2} \tag{8}$$

for $k = 3$ and arbitrary $n \geq 3$

$$C(n,3) = C(3,3) + C(3,2) + C(4,2) + .. + C(n-1,2) = \\ \frac{2\times 1}{2} + \frac{3\times 2}{2} + .. + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)(n-2)}{2\times 3} \tag{9}$$

... for arbitrary $k \leq n$.

$$C(n,k) = C(k,k)+C(k,k-1)+C(k+1,k-1)+..+C(n-1,k-1) = \frac{n(n-1)...(n-k+1)}{1\times 2..\times k} \tag{10}$$

where $C(n,0) = C(n,n) = 1$ and moreover $C(n,k) = C(n,n-k)$. When we introduce the factorial function, then we have "m-factorial" being defined as

$$m! = 1 \times 2 \times .. \times (m-1) \times m \tag{11}$$

and further it is convenient to define $0! = 1$. Taking into account that

$$\underbrace{n(n-1)..(n-k+1)}_{k\,factors} = \frac{n!}{(n-k)!} \tag{12}$$

Then we have that (where we introduce the notation $a_1 + a_2 + .. + a_n = \sum_{k=0}^{n} a_k$)

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^k b^{n-k} \tag{13}$$

where one defines $0! = 1$. The binomial coefficient $\dfrac{n!}{(n-k)!\,k!}$ often is also denoted in the form of

$$\frac{n!}{(n-k)!k!} = \left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n \\ n-k \end{array} \right) \tag{14}$$

and referred to as n *over* k, where $\left(\begin{array}{c} n \\ 0 \end{array} \right) = \left(\begin{array}{c} n \\ n \end{array} \right) = \dfrac{n!}{n!\,0!} = \dfrac{n!}{n!} = 1$ (as $0! = 1$).

1.2 Derivatives, Integrals and Taylor Series

When we have a function $f(t)$ which we assume to be infinitely often continuously[3] differentiable, it is interesting how this function changes when t is increased by a bit, say h. That is we consider

$$f(t+h) = f(t) + \int_t^{t+h} f'(\tau_1) \mathrm{d}\tau_1 \tag{15}$$

where we have introduced the derivative

$$f'(\tau) = \frac{df(\tau)}{d\tau} = \lim_{\delta \to 0} \frac{f(\tau+\delta) - f(\tau)}{\delta} \tag{16}$$

Then we can again put

$$f'(\tau_1) = f'(t) + \int_t^{\tau_1} f''(\tau_2) \mathrm{d}\tau_2 \tag{17}$$

and a similar relation we have between the $n-1^{th}$ derivative $f^{(n-1)}$ and the n^{th} derivative $f^{(n)}$, namely

$$f^{(n-1)}(\tau) = f^{(n-1)}(t) + \int_t^{\tau} f^{(n)}(\tau') \mathrm{d}\tau' \tag{18}$$

Now we put expression for $f'(\tau_1)$ (17) into the integral of (15) and obtain

$$f(t+h) = f(t) + f'(t) \int_t^{t+h} \mathrm{d}\tau_1 + \int_t^{t+h} \mathrm{d}\tau_1 \int_t^{\tau_1} f''(\tau_2) \mathrm{d}\tau_2 \tag{19}$$

[3] A function $f(t)$ is referred to as continuous in a interval $x \in [a,b]$ if there is for any $\epsilon > 0$ a δ, so that $f(x+\delta) - f(x) < \epsilon$.

or

$$f(t+h) = f(t) + f'(t)h + f''(t)\int_t^{t+h} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2 + R_2 \tag{20}$$

$$\begin{aligned} f(t+h) &= f(t) + f'(t)h + f''(t)\int_t^{t+h} \mathrm{d}\tau_1(\tau_1 - t) + R_2 \\ &= f(t) + f'(t)h + f''(t)\frac{h^2}{2} + R_2 \end{aligned} \tag{21}$$

where the remainder term R_2 is given by

$$R_2 = \int_t^{t+h} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2 \int_t^{\tau_2} \mathrm{d}\tau_3 f'''(\tau_3) \tag{22}$$

To evaluate R_3 we make use of (18) for $n = 3$, i.e. we replace the variable function $f'''(\tau_3)$ by the with respect to the τ_3-integration constant $f'''(t)$ plus remaining integral R_3. Then we can write for (22)

$$\begin{aligned} R_2 &= f'''(t)\int_t^{t+h} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2 \int_t^{\tau_2} \mathrm{d}\tau_3 + R_3 \\ &= f'''(t)\int_t^{t+h} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2(\tau_2 - t) + R_3 \\ &= f'''(t)\int_t^{t+h} \mathrm{d}\tau_1 \frac{(\tau_1 - t)^2}{2} + R_3 \\ &= f'''(t)\frac{h^3}{3!} + R_3 \end{aligned} \tag{23}$$

Repeating this process successively, we obtain

$$f(t+h) = \underbrace{f(t) + hf'(t) + \frac{h^2}{2}f''(t) + .. + f^{(n)}(t)\frac{h^n}{n!}}_{S_n} + R_n \tag{24}$$

where the remainder term is given by

$$R_n = \underbrace{\int_t^{t+h} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2 .. \int_t^{\tau_n} \mathrm{d}\tau_{n+1} f^{(n+1)}(\tau_{n+1})}_{n+1 \ integrations} \tag{25}$$

which can be also rewritten as

$$R_n = \int_t^{t+h} \frac{(t+h-\tau)^n}{n!} f^{(n+1)}(\tau)\mathrm{d}\tau \tag{26}$$

If $\lim_{n\to\infty} R_n = 0$, the series S_n converges towards $f(t+h)$. (24) is also known as **Taylor Theorem**. If the remainder term R_n does not tends to zero, the series expansion can still be convergent, however not towards the value $f(t+h)$. Moreover, if we have a convergent power series

$$g(t) = \sum_{n=0}^{\infty} a_n t^n \tag{27}$$

which we assume to be convergent, then we find by differentiating (27) at $t = 0$ the relation between the coefficients a_n and the derivatives at $t = 0$, namely $a_n = \frac{1}{n!}\frac{d^n}{dt^n} f(t)|_{t=0}$.

Thus a power series of the form (27) always represents its own Taylor (Mac-Laurin) series. This remains also true for *polynomials*, i.e. for a series braking with a certain maximum power t^m.

To judge whether or not the remainder term R_n tends to zero for increasing n, we need not to evaluate R_n explicitly, instead it is sufficient to estimate its value to judge whether or not it tends to zero when n exceeds all limits. To this end we make use of the **Mean Value Theorem** of integral calculus which says for any in the interval $[a, b]$ continuous function $g(x)$ that

$$\begin{aligned} g(\tilde{x}) &= \frac{1}{(b-a)} \int_a^b g(x)\mathrm{d}x \\ \int_a^b g(x)\mathrm{d}x &= (b-a)g(\tilde{x}) \end{aligned} \tag{28}$$

where $\tilde{x} \in [a, b]$. Applying this relation to the last integral of (25) or equivalently to (26) we obtain

$$R_n = f^{(n+1)}(t + \Theta_n h)\frac{h^{n+1}}{(n+1)!} \tag{29}$$

where $0 \leq \Theta_n \leq 1$ $(t + \Theta_n h \in [t, t + h])$. Hence we have for $\lim_{n\to\infty} R_n \to 0$

$$f(t+h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t)}{k!} h^k \tag{30}$$

An expansion the of the form (30) is referred to as a **Taylor Series**. For the h range in which the remainder $R_n \to 0$ as $n \to \infty$ tends to zero, $S_{n\to\infty}$ in (24) yields exactly $f(t + h)$. Then S_n is an approximation for $f(t + h)$ up to the remainder R_n. Equation (30) is referred to as the *Taylor series* or *Taylor expansion* of function f around t with respect to h. If we expand around $t = 0$ then the Taylor series is also referred to as *Mac-Laurin series*. For any function it has to be checked whether or not the remainder tends to zero for increasing n. There is always only one *unique* power series of a function when expanding it with respect to a variable t. Many functions can be only defined and computed by Taylor series. Examples include Exponential-function, Logarithm, Hyperbolic and circular (trigonometric) functions. Taylor series have the useful property that one can generate derivatives and integrals of the function expanded by differentiating the powers of t and integrating powers, respectively. Note that the concept of Taylor series can also be extended to complex valued functions.

Power function: By using (30), we obtain the Mac-Laurin series of $(1 + x)^\alpha$ according to

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + .. + \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}x^n + R_n \tag{31}$$

where α is a *real* (not necessarily integer) number.

The remainder term has the form

$$R_n = \frac{\alpha(\alpha-1)..(\alpha-n+1)(\alpha-n)}{(n+1)!}(1+\Theta_n x)^{\alpha-n}x^{n+1} \tag{32}$$

where $0 \leq \Theta_n \leq 1$. The remainder obviously tends to zero for $\frac{x^n}{(1+\Theta_n x)^n} \leq x^n \to 0$,i.e. when $|x| < 1$. Hence the series

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + .. + \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}x^n + .. = \sum_{n=0}^{\infty} \left(\begin{array}{c} \alpha \\ n \end{array} \right) x^n \tag{33}$$

Expansion (33) converges for $-1 \leq x \leq 1$ towards $(1+x)^\alpha$. We observe from (33) that if $\alpha = m$ is an integer, the series (33) coincides with the binomial expansion for $(1+x)^m$, breaking with the power x^m. In this case it is a sum of a finite number of terms, holding for any x, not only $|x| < 1$.

1.3 Elementary functions

In this section we derive some basic functions which are inevitable for our further applications. Generally, Taylor series which are used to compute values of functions.
Exponential function: The exponential function describes a growth or decay process with its change rate being proportional to the value to the value itself. There are a huge set of examples showing exponential time evolution, among them the time-evolution of an asset with a certain interest and the decay of an radioactive isotope. Let us assume we have to a time $t = 0$ the amount x_0 and the bank will give us an interest rate h (100h percent), then our assets will obey the following *recurrence relation*[4]

$$x_{n+1} = (1+h)x_n \tag{34}$$

where x_n denotes the amount of money after n years with the *initial condition* (amount of money at $t = 0$) x_0. (34) is the simplest example for a **linear recurrence** relation, i.e. x_{n+1} is a linear function of x_n. We shall linear recurrences in more details below. The solution of the recurrence (34) obviously is

$$x_{n+1} = (1+h)^n x_0 \tag{35}$$

which gives using the binomial theorem (13) to evaluate $(1+h)^n$

$$(1+h)^n = \sum_{k=0}^{n} \frac{n(n-1)..(n-k+1)}{k!} h^k \tag{36}$$

If we introduce a continuous time variable $t = nh$ and consider the limiting case $h \to 0$ with $t = nh = const$ thus $n = \frac{t}{h} \to \infty$, then (36) becomes

$$(1+h)^n = \sum_{k=0}^{n=\frac{t}{h}} \frac{(nh)(nh-h)..(nh-kh+h)}{k!} = \sum_{k=0}^{n} \frac{t(t-h)..(t-kh+h)}{k!} \tag{37}$$

in the limiting case $h \to 0$, the powers $\underbrace{t(t-h)..(t-kh+h)}_{k\ factors} \to t^k$ and $\frac{t}{h} \to \infty$ and hence we obtain

$$\lim_{h\to 0}(1+h)^{\frac{t}{h}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t \tag{38}$$

which is the definition of the *Exponential function* e^t. Often one denotes it also as $e^t = \exp(t)$, and just calls it the *Exponential of* t. The series

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \tag{39}$$

[4]Without loss of generality our consideration holds for any *complex* h.

which we derived from the binomial theorem is the Taylor series (Mac-Laurin series) of this function with respect to t around $t = 0$. We observe from (39) that its derivative with respect to t yields the same series, thus

$$\frac{d}{dt}\, e^t = e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \tag{40}$$

that is $y = e^t$ solves the *differential equation* $\frac{d}{dt}x = x$ and correspondingly we find the solution of $\frac{d}{dt}x = \lambda x$ is $x(t) = e^{\lambda t}x_0$ ($x_0 = const$).

When we return to our recurrence problem (35), we see that this limiting case $h \to 0$ when we put $x_n = x(t)$ and $x_{n+1} = x(t + h)$

$$\lim_{h\to 0} x(t + h) = (1 + h)x(t) \tag{41}$$

or equivalently

$$\lim_{h\to 0} \frac{x(t + h) - x(t)}{h} = x(t) \tag{42}$$

The left hand side of this relation is just the definition of the time derivative at time t, namely $\lim_{h\to 0} \frac{x(t+h)-x(t)}{h} = \frac{dx(t)}{dt}$, thus the recurrence (42) yields in the limiting case the differential equation

$$\frac{dx}{dt} = x \tag{43}$$

which has the solution $x(t) = e^t x_0$ where x_0 is a constant indicating $x_0 = x(t = 0)$. Without loss of generality, this consideration remains true also for any starting time t_0, i.e. $x(t) = e^{(t\, -\, t_0)}x(t_0)$. By using the Binomial Theorem (13), we can similarly show as in (36) that

$$e^t = \lim_{n\to\infty}(1 + \frac{t}{n})^n \tag{44}$$

also yields the exponential function. For $t = 1$, the limiting case of (44) is called $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + .. \approx 2.7183$ **Euler number**. It is clear that we can represent any number by $a = e^{\ln(a)}$ and therefore $a^x = e^{x\ln(a)}$. Inverting (44) by putting $x = e^t$ yields per definition the natural *Logarithm function* $t = \ln(x)$ and is obtained by the limiting case

$$\ln(x) = \lim_{n\to\infty} n[x^{\frac{1}{n}} - 1] \tag{45}$$

Let us now assume that $x = 1 + u$ with $|u| \leq 1$, in order to derive a series for $\ln(1 + u)$. Then we have[5]

$$\ln(1 + u) = \lim_{n\to\infty} n[(1 + u)^{\frac{1}{n}} - 1] \tag{46}$$

and because of

$$(1 + u)^{\frac{1}{n}} = 1 + \frac{1}{n}u + \frac{1}{2!}(\frac{1}{n} - 1)u^2 + ... = \sum_{k=0}^{\infty} \left(\begin{array}{c} \frac{1}{n} \\ k \end{array} \right) u^k \tag{47}$$

[5]The restriction $|u| \leq 1$ is necessary to guarantee the convergence of the series.

converging for $|u| \leq 1$. As all terms for $k \geq 1$ in (47) contain the factor $\frac{1}{n}$, we can write for (46)

$$\ln(1+u) =$$

$$= \lim_{n\to\infty}\{u + (\frac{1}{n}-1)\frac{u^2}{2!} + (\frac{1}{n}-1)(\frac{1}{n}-2)\frac{u^3}{3!} + .. + (\frac{1}{n}-1)\times..\times(\frac{1}{n}-k+1)\frac{u^k}{k!} + ..\} \tag{48}$$

thus $\frac{1}{k!}(\frac{1}{n}-1)\times..\times((\frac{1}{n}-k+1) \to \frac{1}{k!}(-1)^{k-1}(k-1)! = (-1)^{k-1}\frac{1}{k}$ in the limiting case, thus

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + .. + (-1)^{k-1}\frac{u^k}{k} + .. \tag{49}$$

which is the Mac-Laurin series of $\ln(1+u)$ with respect to u converging for $-1 < u \leq 1$. The convergence of this series will be discussed in more detail in section 1.5.

Hyperbolic functions: Related to the *Exponential function* are the *Hyperbolic functions*, namely $\cosh(t), \sinh(t)$ and $\tanh(t)$. They are defined as

$$\begin{aligned}\cosh(t) &= \frac{1}{2}(e^t + e^{-t}) \\ \sinh(t) &= \frac{1}{2}(e^t - e^{-t}) \\ \tanh(t) &= \frac{\sinh(t)}{\cosh(t)}\end{aligned} \tag{50}$$

or equivalently $e^t = \cosh(t) + \sinh(t)$, $e^{-t} = \cosh(t) - \sinh(t)$. By expanding the exponential into a Mac-Laurin series

$$e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + .. \tag{51}$$

and as a consequence of (50) rearranging the terms according to even and odd powers of t, we get

$$\begin{aligned}\cosh(t) &= \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!} \\ \sinh(t) &= \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\end{aligned} \tag{52}$$

where $\cosh(t)$ contains only *even* powers and $\sinh(t)$ only *odd* powers of t. We emphasize that the definitions of above functions remain also true when t is *complex*. In the next section we give a brief review of complex numbers.

1.4 Complex numbers: Cartesian and polar representation

Cartesian representation of a complex number: Let x and y denote the axis of a Cartesian coordinate system, then we introduce the *complex number* z as follows

$$z = x + iy \tag{53}$$

where $i = \sqrt{(-1)}$ denotes the *imaginary unit*[6] and is conceived to be directed in the positive y-direction with $i^2 = -1$. (53) is referred to as Cartesian representation of the complex number z and $x = Re\{z\}$ is called the *real part* of z and $y = Im\{z\}$ is referred to as the *imaginary part* of z. Both x and y are real numbers, iy is a imaginary number. A visualization of complex numbers is obtained when we represent them as "vectors" in the **complex plane**, where the 1 is the basis vector in positive x-direction and the imaginary unit i the basis vector in positive y-direction.

Complex numbers were first introduced by Gauss, wherefore the complex plane is referred to as **Gaussian plane**. Complex numbers are together with the **Theory of complex functions** one of the most crucial tools in mathematics, engineering science and physics. To calculate with them is just as with real numbers, but separating real and imaginary parts. Let $z_1 = x_1 + iy_2$, and $z_2 = x_2 + iy_2$ be complex numbers, then we can define the *addition* the sum of them (*addition of complex numbers*)

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + i(y_1 + y_2) \tag{54}$$

and their product (*multiplication of complex numbers*)

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + i^2 y_1 y_2 + i(x_1 y_2 + x_2 y_1) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \tag{55}$$

where we have used $i^2 = -1$. The geometrical interpretation of the multiplication of 2 complex numbers becomes clear when performing the multiplication in polar representations shown below. Let $z = x + iy$ be a complex number, then $z^* = x - iy$ is referred to as the **complex conjugate** of z. Sometimes in the literature the complex conjugate is also denoted as $\bar{z}$ instead of z^*.

Polar representation of complex numbers: Let us introduce the polar coordinates

$$x = \rho \cos\varphi, \;\; y = \rho \sin\varphi \tag{56}$$

where $\rho = \sqrt{x^2 + y^2}$ and $\dfrac{y}{x} = \tan(\varphi)$. Then one makes the convention that the angle φ is in the range $\pi < \varphi \leq \pi$. Other conventions are possible which are equivalent because of the 2π-periodicity of the trigonometric functions, for instance $0 \leq \varphi < 2\pi$. Expressing z in polar coordinates then writes

$$z = \rho(\cos\varphi + i \sin\varphi) \tag{57}$$

Now we introduce the exponential function for an imaginary argument $i\varphi$ in the form

$$e^{i\varphi} = \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!} \tag{58}$$

Because of $i^{2m} = (-1)^m$ and $i^{2m+1} = i(-1)^m$ we obtain for $e^{i\varphi}$ the representation

$$e^{i\varphi} = \underbrace{\sum_{m=0}^{\infty} (-)^m \frac{\varphi^{2m}}{(2m)!}}_{\cos(\varphi)} + i \underbrace{\sum_{m=0}^{\infty} (-)^m \frac{\varphi^{2m+1}}{(2m+1)!}}_{\sin(\varphi)} = \cos\varphi + i \sin\varphi \tag{59}$$

or equivalently

$$\begin{aligned} \cos(\varphi) &= \cosh(i\varphi) = \tfrac{1}{2}(e^{i\varphi} + e^{-i\varphi}) \\ \sin(\varphi) &= \tfrac{1}{i} \sinh(i\varphi) = \tfrac{1}{2i}(e^{i\varphi} - e^{-i\varphi}) \end{aligned} \tag{60}$$

[6] In many textbooks, the imaginary unit is denoted as j instead of i.

where the hyperbolic functions (60) just indicate just the even and odd parts of the exponential function, respectively.

In view of decomposition (59) into circular (trigonometric) functions, we also can write the complex number in the form (57), namely

$$z = \rho\, e^{i\varphi} \tag{61}$$

$\rho = |z| = \sqrt{zz^*}$ is referred to as the *modulus* of z, and $\varphi = arg(z)$ is referred to as the **argument** of the complex number z or sometimes its **phase**. The exponential $e^{i\varphi}$ is also referred to as **phase factor**. Note that complex conjugation of a phase factor means to change the sign of its phase, namely $(e^{i\varphi})^* = \frac{1}{e^{i\varphi}} = e^{-i\varphi}$. Per definition the argument of the complex number z is within the interval $-\pi < arg(z) \leq \pi]$. To meet this convention $arg(z)$ is expressed in terms of the arctan-function (where $-\frac{\pi}{2} < \arctan(..) < \frac{\pi}{2}$) in the form

$$\begin{aligned} arg(z) &= \arctan(\tfrac{y}{x}), \ \text{ if } Rez > 0 \\ arg(z) &= \arctan(\tfrac{y}{x}) + \pi sign(y), \ \text{ if } Rez < 0 \end{aligned} \tag{62}$$

where $z = x + iy$ and $sign(y) = \pm 1$ is the sign of y ($y = |y| sign(y)$).

By using representation(61), it is straight-forward to see the geometrical interpretation of the multiplication of two complex numbers $z_1 = \rho_1 \exp i\varphi_1$ and $z_2 = \rho_2 \exp i\varphi_2$

$$z_1 \cdot z_2 = \rho_1\rho_2 \exp(i\varphi_1) \exp(i\varphi_2) = \rho_1\rho_2 \exp[i(\varphi_1 + \varphi_2)] \tag{63}$$

that is the moduli are being multiplied, whereas the angles are added. Or equivalently: Multiplying a complex number with another complex number means to multiply with its modulus and rotate by its argument.

In the next section we give a brief review of the properties of *complex numbers*. All considerations of above series and functions remain true in the complex case.

1.5 Sequences, Series, Integrals and Power-Series

In this section we give a brief review of the **convergence criteria** which are a useful tool to judge whether or not an infinite series converges towards a finite limit. All statements on sequences and series made in this paragraph hold also when they are *complex*.
Sequences: Before we start to look at series, let us first give a criteria for the convergence of a *sequence* $a_0, a_1, .., a_n, ..$ which is due to Cauchy: the sequence $\{a_n\}$ is convergent $a_n \to a$ if there is for any given $\epsilon > 0$ a n_ϵ so that $|a_n - a| < \epsilon$. A simple example is the sequence $a_n = \frac{1}{n}$ which converges towards $a = 0$.
Series: Let us consider the convergence of series. To this end, we consider first the *Geometrical series* which can be obtained from (33) for $\alpha = -1$, namely

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + .. = \sum_{n=0}^{\infty} (-1)^n z^n \tag{64}$$

where z can be a complex number $z = x + iy$. As we know from (33), this series is convergent for $|z| = \rho < 1$. The circle $\rho_c = 1$, which separates convergent z from nonconvergent z, is generally referred to as the *circle of convergence*. For $z = -x$ and $|x| < 1$ we have that

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + .. = \sum_{n=0}^{\infty} x^n \tag{65}$$

Let us now consider an arbitrary infinite sum of the form

$$A_n = \sum_{j=0}^{n} a_j \tag{66}$$

where a_j can be complex numbers. A_n converges for $n \to \infty$ if there exists a *convergent* series

$$B_n = \sum_{j=0}^{n} |b_j| \tag{67}$$

with $|b_n| > |a_n| \forall n$, i.e. B_n is an *upper limit* for $A_n \leq \sum_{n=0}^{\infty} |a_n|$.
The series $A_{n\to\infty}$ **converges absolutely** if the a_n tends to zero *at least as fast as* the power ζ^n with $0 < \zeta < 1$. *It is a necessary condition that* $a_n \to 0$ *for convergence, but not a sufficient one.* An example for a divergent series of this kind is the *harmonic series* (see question **6**). *Note* $a_n \to 0$ *for* $n \to \infty$ *does not necessarily indicate that* $\sum_n a_n$ *is convergent!.*

From this consideration we obtain the convergence criteria for the series (66): The series $A_{n\to\infty}$ *converges absolutely* (absolute convergence means $\sum_n^{\infty} |a_n|$) if there is a ζ $(0 < \zeta < 1)$ where $\lim_{n\to\infty} |a_n| \leq const\, \zeta^n$ or equivalently if

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1 \tag{68}$$

or as an equivalent criteria

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \tag{69}$$

Criteria (68) is due to *Cauchy* and (69) is due to *D'Alembert*. Both criteria are equivalent to each other and indicate *absolute convergence*, i.e. convergence of $\sum_n^{\infty} |a_n|$.
Integrals: We give the following theorem on the convergence of functions and existence of integrals: If $f(t)$ is a monotonous function and if the series converges

$$\sum_{n}^{\infty} f(n) < \infty \tag{70}$$

then also the integral

$$\int^{\infty} f(t)\mathrm{d}t < \infty \tag{71}$$

is convergent and vice verse. The lower limit of integration is omitted as it changes the integral only by a constant and thus does not influence the convergence behavior and the same is true for the lowest index n_0 in series (70). We only need to prove either the sum (70) or the integral (71) being convergent in order to judge the convergence for both of them. The proof is as follows: Let $f(t)$ be a monotonous decreasing function for $0 \leq t \leq \infty$, then we have that

$$\sum_{n=1}^{\infty} f(n) \leq \int_0^{\infty} f(t)\mathrm{d}t \leq \sum_{n=0}^{\infty} f(n) \tag{72}$$

from which follows if the integral is convergent, it is also the series, and vice versa.

Note: If the sequence $a_n \to 0$ converges towards zero as $n \to \infty$, it is a necessary however not sufficient condition that $\sum_n^{\infty} a_n$ is convergent. An example is the harmonic series $\sum_n^{\infty} \frac{1}{n}$ which is *divergent* (see question **6**).

Power-Series: It is also useful to use above convergence criteria to derive the circle of convergence for a complex power series. Consider the series

$$A(z) = \sum_{j=0}^{\infty} a_j z^j \tag{73}$$

where $z = \rho e^{i\varphi}$. Applying the above criteria to (73), we have convergence if their terms tend to zero faster than a geometrical series. We have convergence for $|z|$ fulfilling

$$\lim_{n\to\infty} (|z|^n |a_n|)^{\frac{1}{n}} = |z||a_n|^{\frac{1}{n}} < 1 \tag{74}$$

or

$$|z| < \lim_{n\to\infty} \frac{1}{|a_n|^{\frac{1}{n}}} = \rho_c \tag{75}$$

or equivalently by using (69), we have convergence when $|z|$ fulfills

$$\lim_{n\to\infty} \frac{|a_{n+1}||z|^{n+1}}{|a_n||z|^n} = |z|\frac{|a_{n+1}|}{|a_n|} < 1 \tag{76}$$

or

$$\lim_{n\to\infty} |z| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \rho_c \tag{77}$$

Both criteria are equivalent and it depends on the problem to decide which criteria may be more convenient in order to determine the *radius of convergence* ρ_c.
In conclusion: A power series $\sum_{n=0}^{\infty} a_n z^n$ converges if z is within the **circle of convergence** $|z| < \rho_c$, which is given by either of the relations (75) and (77). In the case when (69) and (77) tend to infinity, the power series is convergent for all z. If these limiting cases yield zero, it means that the power series is nowhere convergent except for $z = 0$. Note: (75) and (77) determine the radius of convergence, i.e. it determines that within the circle of convergence $|z| < \rho_c$, we have convergence. However it does not provide information about the behavior of convergence *at the circle of convergence*, i.e. for $|z| = \rho_c$. To judge the convergence on the circle of convergence, additional considerations have to be made. An example is the geometrical series

$$\frac{1}{1-x} = 1 + x + x^2 + .. + x^n + .. \tag{78}$$

with convergence circle $\rho_c = 1$, i.e. (78) converges for $x < 1$, but obviously this is not true any more for $x = 1$.
To judge the convergence behavior on the circle of convergence, it is often useful to employ another convergence criteria, which is due to **Leibniz** saying that: *If a series is alternating, i.e.* $sign(a_{n+1}) = -sign(a_n)$ *and* $|a_n|$ *monotonously decreasing* $|a_{n+1}| < |a_n|$ *with* $\lim_{n\to\infty} a_n = 0$ *tending to zero, then the series* $\sum_n^{\infty} a_n = \sum_n (-1)^n |a_n|$ *is convergent.* For instance we can use this criteria to judge what happens on the radius of convergence for the Logarithm series (49) for $u = 1$ which is alternating and meeting the *Leibniz criteria*, thus

$$\ln(2) = \ln(1+u)|_{u=1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + .. + (-1)^{k-1}\frac{1}{k} + .. \tag{79}$$

which converges according to the Leibniz criteria towards $\ln(2)$.

1.6 Polynomials and Partial Fractions

In this section we give a brief outline of the properties of polynomials and partial fractions.

Polynomials: Let us consider a polynomial of degree $n = 01, 2, ...$, where n is an integer number

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + .. + a_1\lambda + a_0 \tag{80}$$

where the highest power n determines the degree of the polynomial. A fundamental property of polynomials is that they can be factorized in terms of their zeros λ_i ($P(\lambda_i) = 0$), namely

$$P_n(\lambda) = (\lambda - \lambda_1)^{m_1} \times .. \times (\lambda - \lambda_r)^{m_r} \tag{81}$$

where m_i is referred to as the multiplicity of the zero λ_i and where $n = m_1 + .. + m_r$. If $m_s = 1$ then there is only one factor $(\lambda - \lambda_s)$ in (81). Note that in general the zeros λ_i of a polynomials are *complex* numbers. Without loss of generality we have put the coefficient of λ^n as equal to one.

Partial Fractions: Let $Q_m(\lambda)$ and $P_n(\lambda)$ be polynomials of degrees m and n, respectively, where $m < n$, then exists a decomposition into partial fractions according to

$$\frac{Q_m(\lambda)}{P_n(\lambda)} = \frac{A_{m_1}}{(\lambda - \lambda_1)} + .. + \frac{A_1}{(\lambda - \lambda_{m_1})^{m_1}} + \frac{B_{m_2}}{(\lambda - \lambda_2)} + .. + \frac{B_1}{(\lambda - \lambda_2)^{m_2}} + .. \tag{82}$$

where λ_s ($s = 1, .., r$) denote the r zeros of the denominator $P_n(\lambda)$ and m_s indicate the multiplicity of zeros λ_s, i.e.

$$P_n(\lambda) = \Pi_i^r(\lambda - \lambda_i)^{m_i} \tag{83}$$

and A_l denote constant coefficients to be determined. It is sufficient to look at the case $m < n$ only, as otherwise we can utilize polynomial division to bring the result into the form of a polynomial of degree $m - n$ plus a fraction of the form (82). In order to determine the coefficients $A_1, .., A_{m_1}$ we multiply (82) with $(\lambda - \lambda_1)_1^m$. Then (82) assumes the form

$$(\lambda-\lambda_1)^{m_1}\frac{Q_m(\lambda)}{P_n(\lambda)} = A_{m_1}(\lambda-\lambda_1)^{m_1-1}+..+A_1+(\lambda-\lambda_1)^{m_1}[\frac{B_{m_2}}{(\lambda-\lambda_2)}+..+\frac{B_1}{(\lambda-\lambda_2)^{m_2}}+..] \tag{84}$$

Note that the left hand side of this relation is no more zeros λ_1 in its denominator (see (83), $\frac{P_n(\lambda)}{(\lambda-\lambda_1)^{m_1}} = \Pi_{i\neq 1}^r(\lambda - \lambda_i)^{m_i}$). Hence we obtain the coefficients $A_1, .., A_{m_1}$ containing the terms $(\lambda - \lambda_1)^{-p_1}$ ($p_1 = 1, .., m_1$) as

$$\begin{aligned}
A_1 &= (\lambda - \lambda_1)^{m_1}\frac{Q_m(\lambda)}{P_n(\lambda)}|_{\lambda=\lambda_1} \\
A_2 &= \frac{1}{1!}\frac{d}{d\lambda}\left\{(\lambda - \lambda_1)^{m_1}\frac{Q_m(\lambda)}{P_n(\lambda)}\right\}|_{\lambda=\lambda_1} \\
&.. \\
A_{m_1} &= \frac{1}{(m_1-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\left\{(\lambda - \lambda_1)^{m_1}\frac{Q_m(\lambda)}{P_n(\lambda)}\right\}|_{\lambda=\lambda_1}
\end{aligned} \tag{85}$$

and so forth for all remaining coefficients related to the terms $(\lambda-\lambda_s)^{-p_s}$ ($p_s = 1, .., m_s$). The case when the multiplicity of a zero λ_1 is equal to one ($m_1 = 1$) is also covered. In this case only the first equation $(85)_1$ occurs with $m_1 = 1$.

Example: The fraction $\frac{x^2+2x}{(x-3)^3(x+1)}$ is to be decomposed into partial fractions. According to (82) we can write

$$\frac{x^2+2x}{(x-3)^3(x+1)} = \frac{A_1}{(x+1)} + \frac{B_1}{(x-3)^3} + \frac{B_2}{(x-3)^2} + \frac{B_3}{(x-3)} \tag{86}$$

To handle with partial fractions instead of a representation of the form (82) is often much more convenient, for instance if we want to integrate such functions. We can find an antiderivative by integrating its partial fractions, namely

$$\int \frac{\mathrm{d}x}{(x^2+px+q)} = A_1 \int \frac{\mathrm{d}x}{(x-x_1)} + A_2 \int \frac{\mathrm{d}x}{(x-x_2)} = A_1 \ln(x-x_1) + A_2 \ln(x-x_2) + const \tag{87}$$

where x_i are the zeros of x^2+px+q, where we have non-multiple zeros $\frac{p^2}{4} - q \neq 0$ is assumed. In the case of $x_1 = x_2$ we have

$$\int \frac{\mathrm{d}x}{(x-x_1)^2} = -\frac{1}{(x-x_1)} + const \tag{88}$$

1.7 Matrix Calculus

In this section we recall briefly the essentials of matrix calculus with special emphasis on eigenvectors and normal forms of matrices.

We confine here to *quadratic matrices*, which are as a rule sufficient for our purposes.

Quadratic Matrices: A quadratic $n \times n$ matrix $\mathbf{A}$ is a scheme

$$\mathbf{A} = \begin{pmatrix} a_{11}, & .., & a_{1n} \\ .. & .. & \\ .. & .. & .. \\ a_{n1}, & .., & a_{nn} \end{pmatrix} \tag{89}$$

Matrices are essential in order to deal with *linear recurrences* which we shall consider in the next section. Let $\vec{X} = x_1\mathbf{e}_1 + .. + \mathbf{e}_n x_n \in R^n$ be a vector with n components x_i with the basis vectors $\mathbf{e}_i$ in the n-dimensional vector space $R^n = R \times .. \times R$ Normally the basis vectors $\mathbf{e}_i$ are chosen in *Cartesian* representation, namely

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{90}$$

where we have introduced the **Scalar Product** of two vectors $\vec{a}, \vec{b}$ according to

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + .. + a_n b_n = |\vec{a}||\vec{b}| \cos \alpha_{a,b} \tag{91}$$

where $\alpha_{a,b}$ is the angle between $\vec{a}$ and $\vec{b}$ and $|\vec{a}|$ denotes the modulus of a vector $\vec{a}$ being defined as $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + .. + a_n^2}$. The condition (90) indicates mutual ortho-normality (it means mutually orthogonal and normalized, i.e. $\mathbf{e}_i \cdot \mathbf{e}_i = 1$) of the basis vectors where δ_{ij} denotes Kronecker δ-symbol being defined as

$$\delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases} \tag{92}$$

i.e., $\delta_{11} = \delta_{22} = .. = \delta_{nn} = 1$, and $\delta_{12} = \delta_{13} = .. \delta_{n-1,n} = 0$. For a Cartesian basis $\{\mathbf{e}_1, .., \mathbf{e}_n\}$ we have

$$\boldsymbol{D}et(\mathbf{e}_1,..,\mathbf{e}_n) = 1 \tag{93}$$

which means that the coordinate system spanned by $\{\mathbf{e}_1,..,\mathbf{e}_n\}$ is a rectangular one. (93) represents the volume of the n dimensional unit cube spanned by the ortho-normal basis vectors. In (93) we have introduced the *Determinant* of an $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$ $(i,j = 1,..,n)$ which is defined by

$$\boldsymbol{D}et\mathbf{A} = \sum_{\sigma}(-1)^{\sigma} a_{1,i_1} \times .. \times a_{n,i_n} \tag{94}$$

where the summation is performed over all $n!$ permutations of the indices. $\sigma = \sigma$ denotes the number of transpositions (mutual exchanges of indices) necessary to transform the sequence $\{1,2,..,n\}$ into the permutated sequence $\{i_1,i_2,..,i_n\}$.

For example the determinant of a 2×2 matrix $\boldsymbol{A} = (a_{ij}\ (i,j = 1,2)$ is given by

$$\boldsymbol{D}et\mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \tag{95}$$

The value of a determinant is a measure for their linear dependence (or independence). $\boldsymbol{D}et\mathbf{A} = 0$ indicates that $\mathbf{A}$ consists of n *linear dependent* (columns and line) vectors. $\boldsymbol{D}et\mathbf{A} \neq 0$ indicates they are *linear independent*. What means *linear independence*? A useful definition is: The vectors $\vec{X}_1, \vec{X}_2, .., \vec{X}_n$ are *linear independent* if the system of equations

$$C_1\vec{X}_1 + C_2\vec{X}_2 + .. + C_n\vec{X}_n = 0 \tag{96}$$

have no other than the *trivial solution* $C_1 = C_2 = .. = C_n = 0$. Linear independence of a set of vectors $\vec{X}_i \in R^n$ also means that any given vector $\vec{b} \in R^n$ can be uniquely represented by a linear combination $A_1\vec{X}_1 + .. + A_n\vec{X}_n$, which means the set $\vec{X}_i$ constitutes a *basis* of the vector space R^n with which all vectors of the R^n can be represented. In this sense the constants A_i represent the $\vec{X}_i$-components of vector $\vec{b}$. If the vectors $\vec{X}_1, \vec{X}_2, .., \vec{X}_n$ are linear independent, i.e. $rank\{\mathbf{X}\} = n$, $(\mathbf{X} = \vec{X}_1, \vec{X}_2, .., \vec{X}_n)$, then we can use the set of $\vec{X}_i$ as a basis to span any vector $\vec{Y} = (y_i)$ $(i = 1,..,n)$, that is the equation system

$$C_1\vec{X}_1 + C_2\vec{X}_2 + .. + C_n\vec{X}_n = \vec{Y} \tag{97}$$

or in matrix notation $\mathbf{X} = (\vec{X}_1, .., \vec{X}_n)$

$$\mathbf{X} \cdot \vec{C} = \vec{Y} \tag{98}$$

has one **unique** solution-vector $\vec{C} = (C_i)$ because of $\boldsymbol{D}et\mathbf{X} \neq 0$ which is crucial to invert (98). Formally we can write the inversion as

$$\vec{C} = \mathbf{X}^{-1}\vec{Y} \tag{99}$$

where $\mathbf{X}^{-1}$ is the **inverse matrix** to $\mathbf{X}$ fulfilling the property $\mathbf{X} \cdot \mathbf{X}^{-1} = \mathbf{X}^{-1} \cdot \mathbf{X} = \mathbf{I}$. Further details about solution techniques (*Cramer's rule, Gaussian elimination*) which can be found in many textbooks. Applying **Cramer's rule**, the coefficients C_k of vector $\vec{C}$ can equivalently be written as

$$
\begin{aligned}
C_1 &= \frac{Det\left(\vec{Y},\vec{X}_2,..,\vec{X}_n\right)}{Det\left(\vec{X}_1,\vec{X}_2,..,\vec{X}_n\right)} \\
C_2 &= \frac{Det\left(\vec{X}_1,\vec{Y},..,\vec{X}_n\right)}{Det\left(\vec{X}_1,\vec{X}_2,..,\vec{X}_n\right)} \\
&.. \qquad\qquad\qquad\qquad .. \\
C_k &= \frac{Det\left(\vec{X}_1,\vec{X}_2,..,\vec{X}_{k-1},\vec{Y},\vec{X}_{k+1},..\vec{X}_n\right)}{Det\left(\vec{X}_1,\vec{X}_2,..,\vec{X}_n\right)} \\
&.. \qquad\qquad\qquad\qquad .. \\
C_n &= \frac{Det\left(\vec{X}_1,\vec{X}_2,..,\vec{X}_{n-1},\vec{Y}\right)}{Det\left(\vec{X}_1,\vec{X}_2,..,\vec{X}_n\right)}
\end{aligned}
\tag{100}
$$

where the numerators on the right hand sides of the C_k are generated by determinants of matrices which are obtained when replacing $\vec{X}_k$ by $\vec{Y}$ in $\mathbf{X} = \left(\vec{X}_1,\vec{X}_2,..,\vec{X}_n\right)$.

Rank of a Matrix: In the case of linear independence of the vectors $\vec{X}_1,\vec{X}_2,..,\vec{X}_n$ the rank of the matrix $\mathbf{X} = [\vec{X}_1,\vec{X}_2,..,\vec{X}_n]$ is n. Generally, the rank of a matrix is the number of linear independent column vectors (coinciding with the number of linear independent row vectors). For a $n\times n$ matrix $\mathbf{A}$ with some non-zero elements the rank is between 1 and n, i.e. $1 \leq rank(\mathbf{A}) \leq n$. If the rank of the $n\times n$ matrix $(\vec{X}_1,\vec{X}_2,..,\vec{X}_n)$ is r with $1 \leq r \leq n$, then there exists a $n-r$ parametric solution of (96). For instance for $r = n-1$, we have one-parametric solution $\vec{C} = (C_i) = C\vec{V}$ (with one parameter C to be arbitrarily chosen). Another observation is useful: **If the rank r of an $n\times n$-matrix $\mathbf{A}$ is smaller than n, i.e. $r < n$, then its determinant $\boldsymbol{D}et\mathbf{A} = 0$ is zero**.
Further matrix operations: A linear relation between two vectors $\vec{X} = (x_i)$ and $\vec{Y} = (y_i)$ means any component y_i is a linear combination of the components x_i, hence

$$
y_i = A_{i1}x_1 + .. + A_{in}x_n = \sum_{j=1}^{n} A_{ij}x_j,\ \ i = 1,..,n \tag{101}
$$

Symbolically we can write this linear relation in matrix form, namely

$$
\vec{Y} = \mathbf{A}\cdot\vec{X} \tag{102}
$$

indicating the "multiplication" of a vector with a matrix. When we now assume $\vec{x} = \mathbf{B}\cdot\vec{w}$, then we arrive at the multiplication of two matrices $\mathbf{A}$ and $\mathbf{B}$, defined by

$$
(\mathbf{A}\cdot\mathbf{B})_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \tag{103}
$$

Note in general $\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$, that is, matrix multiplication is in general **non-commutative**. The sequence of multiplication is crucial, unlike the case when we multiply scalar numbers. A little later we will discuss under which conditions the matrix multiplication is commutative (i.e. can be traced back on scalar multiplication).
Eigenvectors and eigenvalues: In general a vector $\mathbf{A}\cdot\vec{X}$ differs by modulus and direction from vector $\vec{X}$. That leads us to the question: *Are there vectors whose direction remains unchanged when multiplying them with a matrix* $\mathbf{A}$? Clearly such vectors depend

on matrix $\mathbf{A}$. If $\vec{v}$ represents such a vector, then any scalar multiple $const\,\vec{v}$ of it also represents such a vector. Hence we look for vectors $\vec{v}$ fulfilling

$$\mathbf{A} \cdot \vec{v} = \lambda \vec{v} \tag{104}$$

A vector fulfilling (104) is referred to as an *Eigenvector* of matrix $\mathbf{A}$ and λ an *Eigenvalue* of matrix $\mathbf{A}$. Bringing $\lambda\vec{v}$ to the left hand side, (104) can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I}) \cdot \vec{v} = 0 \tag{105}$$

where $\mathbf{I} = (\delta_{ij})$ is the unity matrix ($\mathbf{I}\vec{X} = \vec{X}, \forall \vec{X}$)[7]. We see that $\vec{v}$ is obviously the nontrivial solution vector of the *homogeneous system of equations (104)*. A nontrivial (i.e. nonzero) solution only exists if λ is chosen such that the rank of $(\mathbf{A} - \lambda\mathbf{I})$ is at most $n - 1$. That means the condition

$$\boldsymbol{D}et\,(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{106}$$

for λ needs to be fulfilled. From this condition (106), λ is to be determined. This condition also is referred to as **Characteristic Equation**. From the definition of the determinant follows that $P(\lambda) = (\mathbf{A} - \lambda\mathbf{I})$ is a polynomial of degree n. $P(\lambda)$ is referred to as the *Characteristic Polynomial of the matrix* $\mathbf{A}$ *and (106) is referred to as the characteristic equation.* Using the properties on polynomials of section 1.6 we know that $P(\lambda)$ has n zeros λ_i. There are now two different possibilities: (i) All zeros of (106) are different, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$; and (ii) we have multiple zeros λ_s. Let us first of all discuss case (i), i.e. the multiplicity of any zero λ_i is equal to 1. We then have n different equations

$$(\mathbf{A} - \lambda_i\mathbf{I}) \cdot \vec{v_i} = 0 \tag{107}$$

with n different non-zero eigenvectors $\vec{v_i}$ in general[8]. In the appendix it is shown that if all eigenvalues $\lambda_i \neq \lambda_j$ for $i \neq j$ then the n eigenvectors $\vec{v_i}$ are *linear independent* and hence constitute a *complete basis* of the n-dimensional vector space, i.e. any arbitrary vector $\vec{b} = (b_i) = \sum_i^n c_i\vec{v_i}$ can be represented as a unique linear combination of eigenvectors. Because of $\boldsymbol{D}et\mathbf{V} \neq 0$ ($\mathbf{V} = (\vec{v_1}, .., \vec{v_n})$), the matrix $\mathbf{A}$ has the representation

$$\mathbf{A} = \mathbf{V} \cdot \mathcal{D} \cdot \mathbf{V}^{-1} \tag{108}$$

where

$$\mathcal{D} = diag(\lambda_1, ..\lambda_n) = (\lambda_i\delta_{ij}) = \begin{pmatrix} \lambda_1 & 0.. & .. & 0 \\ 0 & \lambda_2 & 0.. & 0 \\ 0 & 0.. & .. & 0 \\ 0 & .. & \lambda_{n-1} & 0 \\ 0 & 0.. & 0 & \lambda_n \end{pmatrix}_{n\times n} \tag{109}$$

is the diagonal form (normal form) consisting of the eigenvalues $(\lambda_1, .., \lambda_n)$ in the same sequence as $\mathbf{V}$ the corresponding eigenvectors $(\vec{v_1}, .., \vec{v_n})$. (109) is referred to as the *diagonal form* or *diagonal representation* of matrix $\mathbf{A}$. The diagonal matrix $\mathcal{D}$ is the most simple and natural representation of the matrix $\mathbf{A}$. Again we emphasize that the representation (109) exists in general only if we have n *linear independent eigenvectors* $\vec{v_i}$, i.e. if we have enough, namely n eigenvectors that they constitute a complete basis of the R^n. This is always true if the eigenvalues λ_i are all different and it is sometimes

[7] $\forall$ abbreviating "for all".

[8] Any eigenvector $\vec{v_i}$ is only determined up to a constant, i.e. $\vec{v_i} \cdot const$ is also a eigenvector $\forall\, const$

true if we have multiple eigenvalues (see below). To understand (109) we consider how it acts on an arbitrary vector $\vec{X}$ which has the unique representation $\vec{X} = C_1\vec{v_1} + .. + C_n\vec{v_n}$ in terms of eigenvectors. Then we have that $\mathbf{A}\cdot\vec{X} = C_1\lambda_1\vec{v_1} + .. + C_n\lambda_n\vec{v_n} = \mathbf{V}\cdot\mathcal{D}\mathbf{V}^{-1}$.

A matrix with $det(A) = 0$ is called *singular* thus its inverse does not exist. As a consequence a system of equations $\mathbf{A}\cdot\vec{X} = \vec{b}$ do not have a solution if $\vec{b}$ is linear independent from the columns of $\mathbf{A}$.

Let us now consider the case of *multiple (degenerate) eigenvalues*. The characteristic polynomial assumes then the form[9]

$$\boldsymbol{D}et\,(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n \prod_{j=1}^{r} (\lambda - \lambda_j)^{m_j} \tag{110}$$

where the m_j are referred to as the *algebraic multiplicity of the eigenvalue* λ_j(with $m_1 + .. + m_r = n$). Let λ_s be an eigenvalue with algebraic multiplicity $m_s > 1$ and

$$r_s = rank\,(\mathbf{A} - \lambda_s\mathbf{I}) \tag{111}$$

Now we have the following possible cases:

Case (i) $r_s = n - m_s$: In this case we can find m_s linear independent eigenvectors $\vec{v_s}^i$ $i = 1, 2, .., m_s$ fulfilling

$$(\mathbf{A} - \lambda_s\mathbf{I})\cdot\vec{v_s} = 0 \tag{112}$$

In this case the situation is in a sense the same as if we would have m_s different eigenvalues: We can find a m_s dimensional eigenbasis. If $r_j = n - m_j$ is true for all eigenvalues λ_j then a full basis consisting of eigenvectors exist thus the diagonal form of matrix $\mathbf{A}$ exists.

Case (ii) $r_s = n - m_s + \delta_s > n - m_s$: In this case we can find only $n - r_s = m_s - \delta_s$ ($\delta_s = 0, 1, .., m_s - 1$) linear independent eigenvectors as solutions of (112) (The case $\delta_s = 0$ corresponds to the above case that $r_s = n - m_s$. That is in this case there are not enough linear independent eigenvectors which spanning the complete m_s-dimensional space. In this case there is no basis consisting of eigenvectors thus matrix $\mathbf{A}$ is non-diagonalizable. However we can find a basis which transforms matrix $\mathbf{A}$ to its most simple representation which is the so called *Jordan Normal Form*, with

$$\mathbf{A} = \mathbf{V}\mathcal{J}\mathbf{V}^{-1} \tag{113}$$

where

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & 0.. & .. & 0 \\ 0 & \mathcal{J}_2 & 0.. & 0 \\ 0 & 0.. & .. & 0 \\ 0 & .. & \mathcal{J}_{r-1} & 0 \\ 0 & 0.. & 0 & \mathcal{J}_r \end{pmatrix} \tag{114}$$

where to each eigenvalue λ_s with multiplicity m_s (114) contains $n - r_s = m_s - \delta_s$ **Jordan blocks** of the form

[9] where we introduce the notation $\prod_{j=1}^{r}\alpha_j = \alpha_1\alpha_2..\alpha_r$.

$$\mathcal{J}_s = \lambda_s \delta_{ij} + \delta_{i,j-1} = \begin{pmatrix} \lambda_s & 1.. & .. & 0 \\ 0 & \lambda_s & 1.. & 0 \\ 0 & 0.. & .. & 0 \\ 0 & .. & \lambda_s & 1 \\ 0 & 0.. & 0 & \lambda_s \end{pmatrix} \tag{115}$$

For more details we refer to the literature.

For the sake of completeness we briefly recall the most important properties of determinants. All relations hold for both cases when the vectors $\vec{X}_i$ are row vectors *as well as* column vectors since $\boldsymbol{D}et\mathbf{A} = \boldsymbol{D}et\mathbf{A}^{tr}$ (where $\mathbf{A}^{tr}$ denotes the *transposed matrix* of $\mathbf{A}$ generated by exchange of rows and columns, i.e. $(\mathbf{A}^{tr})_{ij} = (\mathbf{A})_{ji}$. Then we have

$$\boldsymbol{D}et\left(\vec{X}_1, \vec{X}_2, ..\vec{X}_n\right) = -\boldsymbol{D}et\left(\vec{X}_2, \vec{X}_1, ..\vec{X}_n\right) \tag{116}$$

i.e. to exchange of *any* two vectors will change the sign of the determinant.

$$\boldsymbol{D}et\left(C_1\vec{X}_1, C_2\vec{X}_2, .., C_n\vec{X}_n\right) = C_1 \cdot C_2 \cdot .. \cdot C_n \left(\vec{X}_1, \vec{X}_2, ..\vec{X}_n\right) \tag{117}$$

and

$$\boldsymbol{D}et\left(\vec{X}_1, C_1\vec{X}_1 + C_2\vec{X}_3, \vec{X}_3..\vec{X}_n\right) = 0 \tag{118}$$

i.e. a determinant with linear dependent column or row vectors is vanishing. Moreover we have the product rule

$$\boldsymbol{D}et(\mathbf{A} \cdot \mathbf{B}) = (\boldsymbol{D}et(\mathbf{A})) \cdot (\boldsymbol{D}et(\mathbf{B})) \tag{119}$$

i.e. the determinant of a matrix product is the product of determinants of the matrices. This case includes also (117) when multiplying $\mathbf{A}$ with a diagonal matrix. From (119) follows that

$$\boldsymbol{D}et(\mathbf{A} \cdot \mathbf{A}^{-1}) = \boldsymbol{D}et\,\mathbf{I} = \boldsymbol{D}et(\mathbf{A}) \cdot \boldsymbol{D}et(\mathbf{A}^{-1}) = 1 \tag{120}$$

or

$$\boldsymbol{D}et(\mathbf{A}^{-1}) = \frac{1}{\boldsymbol{D}et\mathbf{A}} \tag{121}$$

and similarly

$$\boldsymbol{D}et(\mathbf{A}^n) = (\boldsymbol{D}et(\mathbf{A}))^n \tag{122}$$

When we have a matrix which possesses a diagonal form and hence the representation $\mathbf{A} = \mathbf{V}\mathcal{D}\mathbf{V}^{-1}$, where $\mathcal{D}$ is the diagonal matrix consisting of eigenvalues of the form (108), then it follows that

$$\boldsymbol{D}et\,\mathbf{A} = \boldsymbol{D}et\,(\mathbf{V}\mathcal{D}\mathbf{V}^{-1}) = \boldsymbol{D}et\,\mathcal{D} = \prod_{i=1}^{n} \lambda_i \tag{123}$$

and similarly we have with that

$$\boldsymbol{D}et(\mathbf{A} - \lambda\mathbf{I}) = \boldsymbol{D}et\left(\mathbf{V}(\mathcal{D} - \lambda\mathbf{I})\mathbf{V}^{-1}\right) = \boldsymbol{D}et(\mathcal{D} - \lambda\mathbf{I}) = \prod_{i=1}^{n}(\lambda_i - \lambda) \tag{124}$$

which is indeed the characteristic polynomial $P_n(\lambda)$ (Eq. (106). Moreover from (122) and (124) we find that

$$\boldsymbol{D}et(\mathbf{A}^m) = (\boldsymbol{D}et(\mathbf{A}))^m = \prod_{i=1}^{n} \lambda_i^m \tag{125}$$

Functions of matrices: Let us assume $\mathbf{A}$ has a diagonal-form (i.e. the geometrical multiplicity of all eigenvalues is n) and thus can be represented as $\mathbf{A} = \mathbf{V}\mathcal{D}\mathbf{V}^{-1}$. Let us now consider the m^{th} power ($m = 0, \pm 1, \pm 2, ..$)

$$\mathbf{A}^m = \underbrace{\mathbf{A} \cdot .. \cdot \mathbf{A}}_{m\ times} \tag{126}$$

Because of $\mathbf{A} = \mathbf{V} \cdot \mathcal{D} \cdot \mathbf{V}^{-1}$, (126) assumes the form

$$\mathbf{A}^m = \mathbf{V} \cdot \mathcal{D} \cdot \underbrace{\mathbf{V}^{-1} \cdot \mathbf{V}}_{\mathbf{I}} \cdot \mathcal{D} \cdot \mathbf{V}^{-1} \cdot .. \cdot \mathbf{V} \cdot \mathcal{D} \cdot \mathbf{V}^{-1} = \mathbf{V} \cdot \mathcal{D}^m \mathbf{V}^{-1} \tag{127}$$

where

$$\mathcal{D}^m = diag(\lambda_1^m, .., \lambda_n^m) = (\lambda_i^m \delta_{ij}) = \begin{pmatrix} \lambda_1^m & 0.. & .. & 0 \\ 0 & \lambda_2^m & 0.. & 0 \\ 0 & 0.. & .. & 0 \\ 0 & .. & \lambda_{n-1}^m & 0 \\ 0 & 0.. & 0 & \lambda_n^m \end{pmatrix} \tag{128}$$

In view of these relations we can define *functions of $n \times n$ matrices to be also $n \times n$ matrices* by their Taylor series, namely a *scalar function* $f(\lambda)$ has a Mac-Laurin series according to

$$f(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m,\ |\lambda| < \rho_c \tag{129}$$

with $a_m = \frac{1}{m!}\frac{d^m}{d\lambda^m} f(\lambda = 0)$ and converging for $\lambda < \rho_c$. Then the corresponding **matrix function** can be defined by

$$f(\mathbf{A}) = \sum_{m=0}^{\infty} a_m \mathbf{A}^m \tag{130}$$

having the *same coefficients as the scalar function (129) and where the scalar variable λ is replaced by matrix* $\mathbf{A}$. By using (127) and (128) we can further write (130) in the form

$$f(\mathbf{A}) = \mathbf{V}(\sum_{m=0}^{\infty} a_m \mathcal{D}^m)\, \mathbf{V}^{-1},\ |\lambda_i| < \rho_c \tag{131}$$

We can evaluate the series $f(\mathcal{D}) = \sum_{m=0}^{\infty} a_m \mathcal{D}^m$ by taking into account that $\mathcal{D}$ is diagonal and obtain

$$f(\mathbf{D}) = \left(\sum_{m=0}^{\infty} a_m \lambda_i^m \delta_{ij}\right) = \begin{pmatrix} f(\lambda_1) & 0.. & .. & 0 \\ 0 & f(\lambda_2) & 0.. & 0 \\ 0 & 0.. & .. & 0 \\ 0 & .. & f(\lambda_{n-1}) & 0 \\ 0 & 0.. & 0 & f(\lambda_n) \end{pmatrix} \tag{132}$$

We observe that the matrix $f(\mathbf{A})$ has the eigenvalues $f(\lambda_i)$ (where λ_i are the eigenvalues of A) and the same eigenvectors as A. *Note: the power series in the diagonals converge only towards the $f(\lambda_i)$ if all eigenvalues λ_i are within the circle of convergence ρ_c of expansion (129), i.e. if the eigenvalue with the maximum modulus is within the circle of convergence $Max(|\lambda_i|) < \rho_c$.* A prominent example of a function of a matrix $\mathbf{B}$ is the exponential being defined as

$$\exp(\mathbf{B}) = \sum_{m}^{\infty} \frac{\mathbf{B}^m}{m!} \tag{133}$$

with $\rho_c = \infty$, i.e. converging $\forall \lambda_i$. Often one considers functions of $\mathbf{B} = \mathbf{A}t$, where t is a scalar (time) variable. A very useful property of the matrix $\exp(\mathbf{A}t)$ is that it solves in full analogy to scalar functions the linear differential equation system (where $\mathbf{A}$ is assumed to be a constant time independent $n \times n$ matrix)

$$\frac{d}{dt}\exp(\mathbf{A}t) = \mathbf{A} \cdot \exp(\mathbf{A}t) \tag{134}$$

which is easily verified by using the expansion of the exponential. thus the vector $\vec{X}(t) = \exp(\mathbf{A}t) \cdot \vec{X}_0$ solves the differential equations system

$$\frac{d}{dt}\vec{X} = \mathbf{A} \cdot \vec{X} \tag{135}$$

with the initial vector $\vec{X}(t = 0) = \vec{X}_0$. This will be important in part II when we consider systems of differential equations (Sec. (2.6)).

2 Part II

In this section we give an introduction into analytical methods to tackle dynamical problems. We consider recurrences which describe discrete dynamical processes and their continuous counterpart which are described by *Ordinary Differential Equations* (ODE's) and *Initial Value Problems* (IVPs) with special emphasis on *dynamical engineering applications* and physics. We recall the general theory briefly, but for the sake of analytical feasibility we mainly confine to dynamical systems which can be described by ODEs with constant (time independent) coefficients. Fortunately many dynamical problems relevant in engineering science and physics are covered by this type.

2.1 Recurrences

Many processes in dynamics can be described by either (i) *differential equations* or by (ii) *recurrences*. Differential equations describe processes which evolve continuously in time, whereas *recurrences* are an appropriate description when we are looking on the time evolution of a variable X only to certain discrete times t_n, i.e. $X(t_n) = X_n$. Examples are the development of a population which we measure for instance every year at the same day, or an asset on the bank which grows not continuously. A recurrence can be described as a discrete mapping of the form

$$X_{n+1} = f(X_n, C) \tag{136}$$

with a given initial value X_0 and a so called *control parameter* C (or even a set of control parameters). For the definition of recurrences the introduction of a control parameter is not necessary, however in many cases the dynamic behavior of a variable becomes more clear by considering the dependence of the dynamical behavior of such a control parameter. $f(X, C)$ can be an arbitrary generally also *nonlinear function of* X. An example for the influence of a control parameter is the growth of a population under the influence of a variable food supply which enters the dynamic behavior as a control parameter. *The dynamic behavior, i.e. the development of* X_n *depends on both, the initial value* X_0 *and on the control parameter* C*, i.e.* $X_n = X_n(X_0, C)$. Formally we write the solution of (136) in the form

$$X_n = \underbrace{f(..f(X_0, C))}_{n\,times} \tag{137}$$

i.e. the initial value X_0 is inserted into the function, and the result is again inserted into the function; this process is performed n times. One also calls such a process when the results is again entered into the function itself as *iteration or feedback loop*. In most nonlinear cases (apart from a few exceptions), there is no other possibility as to compute X_n by performing the (nonlinear) feedback loop n-times which corresponds to (137). As a rule, one cannot predict the *long-term* dynamic behavior of such a nonlinear recurrence, (i.e. the behavior after a high number n of loops) of nonlinear mappings except one performs all the n iterations explicitly. This is also reflected by the high sensitivity on the initial conditions X_0 of nonlinear recurrence mappings. Therefore nonlinear recurrences are also referred to as **deterministic chaotic mappings**. To get an overview on the nonlinear dynamic behavior one needs to perform the iterations on a computer. A prominent example of the most simple nonlinear recurrence is the **Verhulst-Equation**

$$X_{n+1} = \mu(1 - X_n)X_n \tag{138}$$

with control parameter μ. If the behavior for $n \to \infty$ is bounded, i.e. non-divergent, then the sequence X_n (for large n) is referred to as an **attractor**. In (138) one can find attractors $X_n(\mu)$ for $0 < \mu < 4$ for a certain set of initial conditions. The representation of this attractor vs μ has become famous under the name **Feigenbaum diagram**, and was discovered first by *Mitchell Feigenbaum* about three decades ago. (138) is conceived as the most simple example for a *deterministic chaotic system*. If we chose instead of real variables X_n *complex variables*, then nonlinear recurrences generate dynamics with basins which remain bounded (non-divergent) in both the complex z_0-plane of initial values (**Julia Set**) as well as in the complex c-plane of control parameters (**Mandelbrot Set**) with so called **fractal** boundaries or simply **Fractals**. The terminology **Fractal** and related, the field of **Fractal Geometry** was first introduced by *Benoit Mandelbrot* describing geometrical objects having non-integer dimensions, indicating their highly irregular shapes. Mandelbrot was motivated by the wish to create a theory and quantitative measure of *roughness* in order to describe rough surfaces and boundaries quantitatively. Fractal objects have the remarkable property of **self-similarity**, which means that zooming into a fractal pattern reveals again a "similar" looking fractal pattern. Assigning different colors depending how fast a recurrence of the form (136) converges or diverges can generate **fractal images** of stunning beauty[10]. Well-known is the so called **Mandelbrot Set** generated by the complex mapping $z_{n+1} = z_n^2 + c$ in the complex c-plane for a constant initial condition $z_0 = 0$ which was first discovered and described by Benoit Mandelbrot. An example for a fractal image which was generated by a complex recurrence of the form (136) is shown on the cover page of these notes. Unfortunately we cannot devote here more space to the highly challenging field of **Nonlinear Dynamics, Chaos Theory, Complex Systems and Fractal Geometry**. The interested reader is highly encouraged to further deal with these strongly related subjects. Any support for this goal is provided.

For the sake of analytical calculability we confine us now to **linear recurrences** which can be solved in *closed form*, i.e. without performing n iterations explicitly. *Linear recurrences*, are mappings (136) where $F(X, C)$ is a *linear function of* X having the form $F(X, A, B) = AX + B$, where A, B are constant (control) parameters.

Linear recurrences: The most simple example of a linear recurrence we already met in section 1.3 (eq. 34)). Similarly, in the general case, a linear recurrence has the form

[10]Stunning examples of *Fractals* including downloadable references for further reading, programs and related links can be found in the web at http://michelitsch-fractals.staff.shef.ac.uk and http://en.wikipedia.org/wiki/Burning_Ship_fractal

$$X_{n+1} = AX_n + B \tag{139}$$

with a given initial value X_0 and control parameters (constants) A, B. Performing the n iterations from X_0 to X_n leads to

$$\begin{aligned} X_n &= A^n X_0 + B(1 + A + A^2 + .. + A^{n-1}) \\ X_n &= A^n X_0 + B\frac{(A^n - 1)}{(A - 1}) \end{aligned} \tag{140}$$

especially for $B = 0$ (140) yields $X_n = A^n X_0$ according to the recurrence $X_{n+1} = AX_n$ with given X_0.
So far we considered in this section dynamical systems of one variable (one-degree of freedom). However, we can also define recurrences for systems characterized by n variables (multi-degree of freedom systems)which is represented by a vector $\vec{X}$. In these cases it is clear that linear recurrences are defined by matrix relations of the form.

$$\vec{X}_{n+1} = \mathbf{A} \cdot \vec{X}_n + \vec{b} \tag{141}$$

where $\mathbf{A}$ is now a $N \times N$ matrix. In complete analogy to the scalar case (136), the evaluation of (141) yields analogously to (140) the solution

$$\begin{aligned} \vec{X}_n &= \mathbf{A}^n \cdot X_0 + (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + .. + \mathbf{A}^{n-1}) \cdot \vec{b} \\ \vec{X}_n &= \mathbf{A}^n \cdot X_0 + \frac{(\mathbf{A}^n - 1)}{(\mathbf{A} - 1)} \cdot \vec{b} \end{aligned} \tag{142}$$

where it is important that the matrices (142) stand left of the vectors, in accordance to the recurrence (141). Note, for $\vec{b} = 0$ (142). The matrix $\frac{(\mathbf{A}^n - 1)}{(\mathbf{A} - 1)}$ can be further evaluated by using (131) if $\mathbf{A}$ has a diagonal form, i.e. the geometrical multiplicity of its eigenvectors is n.
In above cases we considered only *scalar recurrences* of the form (136). Let us now consider linear scalar recurrences of the form

$$X_{n+1} = aX_n + bX_{n-1} + c, \;\; n = 1, 2, .. \tag{143}$$

with control parameters a, b, c and initial *two values* X_0, X_1[11] . (143) can be traced back by introducing the vectors for $n = 1, 2, ..$

$$\vec{X}_n = \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}, \;\; \vec{c} = \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{144}$$

on case (141). Then we can write instead of (143) the matrix recurrence of the form $\vec{X}_n = \mathbf{A} \cdot \vec{X}_{n-1} + \vec{c}$, namely

$$\begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{n-1} \\ X_{n-2} \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix}, \;\; n = 2, 3, .. \tag{145}$$

which is solved by (142). As an explicit example we consider the *Fibonacci numbers* which are defined as a recurrence of the form (143), namely

[11] Two initial values have to be given since a recurrence of the form(143) gives no information how X_1 is generated from X_0.

$$X_{n+1} = X_n + X_{n-1},\ n = 1, 2, .. \tag{146}$$

with given initial values X_0, X_1. By using (145) we can write instead of (146) the matrix form

$$\begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{n-1} \\ X_{n-2} \end{pmatrix},\ n = 2, 3.. \tag{147}$$

which yields by using (142) with $\vec{b} = 0$

$$\begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \cdot \begin{pmatrix} X_1 \\ X_0 \end{pmatrix},\ n = 2, 3.. \tag{148}$$

which can be evaluated explicitly by using (127). We will do so in questions **21** and **22**.

2.2 Integral Transformations

This section is devoted to the most common integral transformations such as Fourier transformation and series, Laplace transformation, as they are needed in engineering sciences and physics all over. Especially in part II these transformations are an inevitable tool to tackle dynamical problems. To this end it is very useful to introduce the Dirac's δ-function.

2.2.1 Dirac's δ-Function

Let us consider a function $w(t, \epsilon)$ where $\epsilon > 0$ denotes a positive parameter with $w(t_0, \epsilon) = Max(\epsilon)$ which has the following properties: The function is normalized according to

$$\int_{-\infty}^{\infty} w(t, \epsilon) \mathrm{d}t = 1,\ \forall \epsilon \tag{149}$$

where $\lim_{\epsilon \to 0} w(t_0, \epsilon) = \infty$ and $w(t_0, \epsilon) = Max\{w(t, \epsilon)\}$. That means w is a function which becomes more and more narrow with decreasing ϵ and with increasing maximum $w(t_0, \epsilon)$. Because of the conservation of the area below the curve $w(t, \epsilon)$ this also means that as a necessary condition, $\lim_{\epsilon \to 0} w(t, \epsilon) = 0 \forall t \neq t_0$. There is an infinite set of functions with these properties. The easiest example is a *square-pulse function* defined as

$$w(t, \epsilon) = \begin{cases} \dfrac{1}{\epsilon}; & \text{for} - \epsilon \leq t \leq +\epsilon \\ \\ 0; & \text{elsewhere} \end{cases} \tag{150}$$

which obviously fulfills all demanded properties. The **Dirac's δ-function** then can be defined as the limiting case

$$\delta(t) = \lim_{\epsilon \to 0} w(t, \epsilon) = \begin{cases} \dfrac{1}{\epsilon} \to \infty, & for - \epsilon \leq t \leq +\epsilon \\ \\ 0, & elsewhere \end{cases} \tag{151}$$

By applying the *Mean Value Theorem (28)* we can find for all $a < t < b$ and for any continuous function $f(t)$

$$\int_{-\infty}^{\infty} w(t-t',\epsilon)f(t')\mathrm{d}t' = f(t+\epsilon\Theta)\underbrace{\int_{-\infty}^{\infty} w(t-t',\epsilon)\mathrm{d}t'}_{=1} = f(t+\epsilon\Theta) \tag{152}$$

with $-1 < \Theta < 1$ where we have used (149). When we exchange the integration and $lim_{\epsilon\to 0}$ we obtain

$$\int_{-\infty}^{\infty} \underbrace{lim_{\epsilon\to 0} w(t-t',\epsilon)}_{\delta(t-t')} f(t')\mathrm{d}t' = \int_{-\infty}^{\infty} \delta(t-t')f(t')\mathrm{d}t' = f(t) \tag{153}$$

An integral of the form (153) is referred to as a *Convolution integral*[12] Convolutions are the continuous analogue to matrix multiplication (in view of the analogy of a discrete index i and the continuous index t). In view of (153) we observe that convoluting a function with the δ-function yields the function. In this sense the δ-function plays the role of the unity operator $\hat{1}$ which maps the function (which can be conceived as vector in the function space) on itself $\hat{1}f = f$. Because $\delta(t-t') = 0$ for $t \neq t'$ we can write if $a < t < b$

$$\int_a^b \delta(t-t')f(t')\mathrm{d}t' = f(t) \tag{154}$$

The Dirac-δ-function $\delta(t)$ actually is not a function in its usual sense but is only defined "under the integral". If t would not be in the interval $[a,b]$, then (154) would be zero. Physically the δ-function $\delta(t)$ represents an idealized unit force pulse taking place at $t = 0$. A δ-function can also occur in the space and can then represent for instance the mass density of a point-unity-mass[13]. Useful is also the antiderivative of the δ-function, namely

$$\int_{-\infty}^{t} \delta(t') = H(t) \tag{155}$$

where $\Theta(\tau)$ denotes the **Heaviside unit-step function** being defined as

$$H(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{156}$$

where the value $\frac{1}{2}$ for $t = 0$ is a reasonable extension which is due to the fact that the δ-function is a limiting process of an *even* function in t, i.e. $\delta(t) = \delta(-t)$. The derivative of the Heaviside unit step function is again the δ function ,namely $\frac{d}{dt}H(t) = \delta(t)$, i.e. the $\delta(t)$-function describes the sudden increase of the Heaviside step function at its discontinuity at $t = 0$.

In order to introduce *Fourier Transformation* below, the δ-function will be a highly convenient tool. We will make use of the following integral[14]

$$\mathcal{I} = \lim_{\epsilon\to 0^+} \int_0^{\infty} e^{-\epsilon t}\cos\omega t\,\mathrm{d}t =: \frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}t \tag{157}$$

[12] A convolution is generally defined as $\int g(t-t')f(t')\mathrm{d}t'$ with integral *kernel* $g(\tau)$.
[13] The response $X(t)$ on such a δ-pulse excitation is referred to as the **Green's function** or *Propagator*.
[14] With $\lim_{\epsilon\to 0^+}(..)$ we mean ϵ tends to zero whilst being positive.

Note that the integral (157) is only well-defined for at least small $\epsilon \neq 0$. To evaluate it we can write it in the form (where $Re(..)$ means the real part)

$$\mathcal{I} = Re \lim_{\epsilon \to 0^+} \int_0^\infty e^{-(\epsilon - i\omega t)} \, \mathrm{d}t = \lim_{\epsilon \to 0^+} Re \left\{ \frac{1}{(\epsilon - i\omega)} \right\} = \frac{\epsilon}{\epsilon^2 + \omega^2} \tag{158}$$

We observe that the right hand side of (158), $\frac{\epsilon}{\epsilon^2+\omega^2}$ tends to zero as ϵ for $\omega \neq 0$ but to infinity at $\omega = 0$ as $\frac{1}{\epsilon}$. This leads us to the conjecture that this function could end up in a δ-function as $\epsilon \to 0^+$ and thus has a "similar" behavior (with respect to ω) as (151). To verify this, we consider the integral

$$\int_{-\infty}^{\infty} \frac{\epsilon}{\epsilon^2 + \omega^2} \mathrm{d}\omega = \int_{-\infty}^{\infty} \frac{1}{1 + (\frac{\omega}{\epsilon})^2} \frac{\mathrm{d}\omega}{\epsilon} = \arctan\left(\frac{\omega}{\epsilon}\right)|_{-\infty}^{\infty} = \pi \tag{159}$$

which is independent of ϵ. Hence the function $\frac{1}{\pi} \frac{\epsilon}{(\omega^2+\epsilon^2)}$ fulfills the normalization condition (149), thus we can conclude

$$\frac{1}{\pi} \mathcal{I}(\omega, \epsilon) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \frac{\epsilon}{(\omega^2 + \epsilon^2)} = \delta(\omega) \tag{160}$$

and we can conclude from this result by using (157) that

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, \mathrm{d}t \tag{161}$$

which is an important and highly useful representation of the δ-function, especially also for the introduction of Fourier transformation.

2.2.2 Fourier Transformation: Fourier Integrals and -Series

Fourier Integrals: Let us assume that $f(t)$ is a function where

$$\int_{-\infty}^{\infty} f(t) \mathrm{d}t < \infty \tag{162}$$

exists. To introduce the Fourier integral we first represent $f(t)$ by the identity (154)

$$f(t) = \int_{-\infty}^{\infty} \delta(t - t') f(t') \mathrm{d}t' \tag{163}$$

where we know from (161) that

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-t')} \, \mathrm{d}\omega \tag{164}$$

inserting this representation for $\delta(t - t')$ into (163) we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} e^{i\omega(t-t')} \, f(t') \mathrm{d}t' = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, \mathrm{d}\omega \underbrace{\int_{-\infty}^{\infty} e^{-i\omega t'} \, f(t') \mathrm{d}t'}_{\hat{f}(\omega)} \tag{165}$$

where $\hat{f}(\omega)$ is referred to as the **Fourier Transform** of the function $f(t)$. Note that $\hat{f}(\omega)$ is a different function of ω as $f(t)$ of t. The integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, \mathrm{d}\omega \tag{166}$$

is referred to as the **Fourier Integral** or the **Fourier Representation** of the function $f(t)$. Note that (166) exists only for functions $f(t)$ fulfilling condition (162). The Fourier transform is given in (165) by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t'} f(t') \mathrm{d}t' \tag{167}$$

where $\hat{f}(\omega = 0)$ just recovers the condition (162). $f(t)$ is referred to as the *time domain* representation and $\hat{f}(\omega)$ is referred to as the *frequency domain* representation. Both representations are equivalent. (167) is referred to as the Fourier transformation of $f(t)$ into the frequency domain and (166) is referred to as the back-transformation into the time domain. Note that the phase factor occurs complex conjugate to each other in (162) and (167). Hence knowing either $f(t)$ or $\hat{f}(\omega)$ contains both the full equivalent information, if $\hat{f}(\omega)$ is known then $f(t)$ can be recovered by using (166) and vice verse. This is especially useful and important for the solution of dynamical problems which we shall discuss below: In many cases one can determine directly the Fourier transform of a solution, i.e. the solution in the frequency domain. By using (166) then one can determine the desired time-domain solution. Hence Fourier transformation is one of the most crucial tools in the solution of *linear differential equations* which is to be discussed in part II.

Fourier Series: We observe in (166) that all frequencies ω contribute to $f(t)$. This is the case when the only condition for $f(t)$ is (162), i.e. when $f(t)$ is sufficient rapidly decaying. In other cases in dynamics we have instead so called **periodic boundary conditions** for $f(t)$, which means $f(t)$ is defined over an interval $0 < t < T$ and then periodically continued, namely

$$f(t) = f(t + T) = f(t + nT) \tag{168}$$

where $n = 0, \pm 1, \pm 2, ..\infty$. When we look at (166) we see that (320) must be reflected in the condition $e^{i\omega(t+nT)} = e^{i\omega t} e^{i\omega nT}$. Hence we have to claim that $e^{i\omega nT} = 1$ defines a *selection rule* for omega, namely only the discrete frequencies $\omega = \omega_m = \frac{2\pi}{T} m$ ($m = \pm 1, \pm 2, .., \infty$) which are integer multiples of the basic frequency $\omega_1 = \frac{2\pi}{T}$ are compatible with (168). Hence, instead of integrating over all frequencies ω as in (166), $f(t)$ has here the representation

$$f(t) = \sum_{m=-\infty}^{\infty} a_m e^{\frac{2\pi i m}{T} t} \tag{169}$$

This representation is referred to as a **Fourier Series** of $f(t)$. A Fourier series always exists in the case of **periodic boundary conditions** (168), i.e. when $f(t)$ is periodically continued with period T. The Fourier coefficients a_m are obtained by using the orthogonality relation

$$\int_0^T e^{\frac{2\pi i(m-n)}{T} t} \mathrm{d}t = T \delta_{mn} \tag{170}$$

where $m, n = \pm 1, \pm 2, ..$ and δ_{mn} is the Kronecker δ-symbol (92). Hence we obtain

$$\frac{1}{T} \int_0^T f(t) e^{-\frac{2\pi i m}{T} t} \mathrm{d}t = \sum_{m=-\infty}^{\infty} a_m \underbrace{\frac{1}{T} \int_0^T e^{\frac{2\pi i(m-n)}{T} t} \mathrm{d}t}_{\delta_{mn}} = a_n \tag{171}$$

that is the Fourier-coefficients are obtained by

$$a_n = \frac{1}{T}\int_0^T f(t)\, e^{-\frac{2\pi i m}{T}t}\mathrm{d}t \tag{172}$$

where again the phase factor in (172) is the complex conjugate of that one in the Fourier-series (169). Remark: In case of a discontinuity of f at $t = t_0$ (often for instance at the boundaries $t_0 = nT$), the Fourier series (169) converges towards the mean value $\frac{f(t_0-\epsilon)+f(t_0+\epsilon)}{2}$, $(\epsilon \to 0)$. This effect is referred to as **Dirichlet theorem**, which reflects the limiting case of the fact that (169) for a finite number of terms always represents a continuous function.

2.2.3 Laplace Transformation

It is often the case that Fourier transformation is not possible, because neither of the conditions (162) and (168) applies. Nevertheless, the Laplace transformation is a special kind of Fourier transformation of a slightly modified function. Laplace transformation is useful when we deal with *causal functions* $f(t)$ which are only nonzero for $t > 0$. The reason for this is that often a sudden $\delta(t)$-type force impact takes place, say at time $t = 0$ and we are interested in the response of the system taking place only *after* the impact, i.e. for $t > 0$. Let us assume that $f(t) = H(t)f(t)$ is such a causal function where $H(t)$ denotes the *Heaviside unit step function* defined in (154), the the *Laplace Transform* $\mathcal{L}(f)$ of $f(t)$ exists if there is a σ_c with $\sigma_c > 0$ such that

$$\int_0^\infty f(t)\, e^{-\sigma_c t}\mathrm{d}t < \infty \tag{173}$$

This condition means that $f(t)$ may not increase faster than exponentially, with exponent $\exp(\sigma_c t)$. *The Laplace transform $\mathcal{L}(f)$ of a function $f(t)$ exists if*

$$|f(t)| < Me^{\sigma_c t},\ t > 0 \tag{174}$$

As we shall see, *Laplace transformation is nothing else as the Fourier transformation (166), (167) of the the modified function $f(t)H(t)e^{-st}$ with sufficient large $Re(s) > \sigma_c$.* That means (173) is condition (162) for the function $f(t)H(t)e^{-st}$. An example of such a function which is Laplace transformable is $e^{\lambda t}$. Then $\sigma_c = Re(\lambda)$. A function which increase faster than exponentially, e.g. e^{t^2} is not Laplace transformable, as there does not exist a σ_c so that the condition (173) can be met.
Let us assume that $f(t) = H(t)f(t)$ is causal and meets (173), then the Laplace transform of $f(t)$ is defined as

$$\mathcal{L}(f) = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t \tag{175}$$

where s is a *complex* variable, where $Re(s) > \sigma_c$ and σ_s is referred to as the **abscissa of convergence**. The Laplace transform (175) of $f(t)$ is only defined in the complex half-plane fulfilling $Re(s) > \sigma_c$.
Before we discuss some examples of Laplace transforms, let us tackle the question: If only the Laplace transform $\tilde{f}(s)$, is known, how can we get back the the time-domain representation $f(t)$, where we only get back the causal part of $f(t)H(t)$ $(t > 0)$. To this end, let us take into account that $s = \sigma + i\omega$, where $\sigma = Re(s) > \sigma_c$ and $\omega = Im(s)$, then (175) can be re-written as a Fourier transform, namely

$$\tilde{f}(s=\sigma+i\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}\underbrace{(e^{-\sigma t}H(t)f(t))}_{Fourier\,transformed\,function}\mathrm{d}t \tag{176}$$

where the $H(t)$-function guarantees the right lower integration limit at $t=0$. *This relation shows that the Laplace transform of $f(t)H(t)$ is nothing else than the Fourier transform of $e^{-\sigma t}H(t)f(t)$ where $\sigma=Re(s)$.* Thus, if the Laplace transform $\tilde{f}(\sigma+i\omega)=\tilde{f}(s)$ is given as a function of s, we only need to invert the Fourier transformation (176) according to (166) to recover the time domain representation function $e^{-\sigma t}H(t)f(t)$ of which (176) is the Fourier transform. Thus

$$H(t)f(t)=\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(s=\sigma+i\omega)e^{i\omega t}\mathrm{d}\omega \tag{177}$$

where we have multiplied both sides with $e^{-\sigma t}$ to recover the causal function $f(t)H(t)$, i.e. $f(t)$ for $t>0$.
We will take great advantage of Laplace transforms in the solution of *Initial Value Problems* which will be discussed in part II. The main advantage of the use of Laplace transforms for the solution of Initial Value Problems is that the back-transformation, namely the evaluation of the integral (177)(inversion of Laplace transformation into the time domain) as a rule needs not to be performed explicitly. Instead, in most practical cases the knowledge of the Laplace transform allows to give $H(t)f(t)$ as they are tabulated or easy to be traced back as we shall wee in the examples to follow.
As a highly useful example let us consider the Laplace transform of $e^{\lambda t}H(t)$, i.e. of the causal branch of an exponential functions with arbitrary complex λ. The the Laplace transform $\mathcal{L}(e^{\lambda t})$ is given by using (175)[15]

$$\mathcal{L}(e^{\lambda t})=\int_0^{\infty}e^{-(s-\lambda)t}\mathrm{d}t=\frac{1}{s-\lambda} \tag{178}$$

where the Laplace transform is defined only when $Re(s)>Re(\lambda)=\sigma_c$. From this relation we can read off a couple of useful Laplace transforms, namely by putting $\lambda=0$ we find

$$\mathcal{L}(1)=\frac{1}{s} \tag{179}$$

We can Moreover generate from (178) $\mathcal{L}(t)$ by differentiating both sides of (178) with respect to λ at $\lambda=0$, namely

$$\mathcal{L}(t)=\frac{d}{d\lambda}\left(\frac{1}{s-\lambda}|_{\lambda=0}\right)=\frac{1}{(s-\lambda)^2}|_{\lambda=0}=\int_0^{\infty}te^{-(s-\lambda)t}\mathrm{d}t=\frac{1}{s^2} \tag{180}$$

and by successive differentiation with respect to λ at $\lambda=0$

$$\mathcal{L}(t^n)=\frac{d^n}{d\lambda^n}\left(\frac{1}{s-\lambda}\right)|_{\lambda=0}=\frac{n!}{(s-\lambda)^2}|_{\lambda=0}\int_0^{\infty}t^ne^{-(s-\lambda)t}\mathrm{d}t=\frac{n!}{s^n},\ n=0,1,2,.. \tag{181}$$

where in this relation for $n=0$ and $n=1$ (179) and (180) are recovered. The Laplace transforms (181) are defined only for $Re(s)>0=\sigma_c$ (because $\lambda=0$). In the following we derive some useful relations between Laplace transforms which will be *needed* for the solution of dynamic problems in part II.

[15]We skip the Heaviside $H(t)$-function and write instead of $\mathcal{L}(H(t)e^{\lambda t})$ only $\mathcal{L}(e^{\lambda t})$ bearing in mind that only the causal part of $e^{\lambda t}$, i.e. for $t>0$ is Laplace transformed.

First, we assume throughout that $f(t)$ fulfills (174) and assume in all cases $Re(s) > \sigma_c$. Very useful is the relation between the Laplace transforms of $f(t)$ and its time derivative $Df(t)$ where $D = \frac{d}{dt}$ denotes the time derivative operator. The Laplace transform $\mathcal{L}(Df(t))$ is given by (where we again skip throughout the Heaviside function)

$$\mathcal{L}(Df) = \tilde{f}(s) = \int_0^\infty e^{-st} Df(t) \mathrm{d}t \tag{182}$$

where we can apply partial integration by using the properties of the *differential operator* $e^{-st}Df(t) = D[e^{-st}Df(t)] - f(t)De^{-st} = D[e^{-st}Df(t)] + sf(t)e^{-st} = (D+s)(e^{-st}f(t))$. More compactly we can write

$$e^{-st}Df(t) = (D+s)(e^{-st}f(t)) \tag{183}$$

to arrive at

$$\mathcal{L}(Df) = \int_0^\infty (D+s)(e^{-st}f(t)) \mathrm{d}t \tag{184}$$

which we can evaluate taking into account that $\int_0^\infty s(e^{-st}f(t))\mathrm{d}t = s\mathcal{L}(f(t))$ and the integral operator D^{-1} compensates D taking into account the lower integration limit $t = 0$,

$$\int_0^\infty D(e^{-st}f(t))\mathrm{d}t = (e^{-st}f(t))|_0^\infty = 0 - f(t=0) = -f(0) \tag{185}$$

thus we obtain for (184)

$$\mathcal{L}(Df) = s\mathcal{L}(f) - f(0) \tag{186}$$

which can be found in many textbooks. Hence the time derivative operator D modifies the Laplace transform by a pre-factor s minus the *initial value* $f(t=0)$.
Let us now consider the Laplace transform of the n^{th} time derivative $D^n f(t) = \frac{d^n}{dt^n} f(t)$, where we emphasize here that we always mean the Laplace transform of the causal part $H(t)D^n f(t)$ ($\neq D^n[H(t)f(t)]$!). To this end we make use of the corresponding relation (183) for the n^{th} derivative

$$e^{-st}D^n f(t) = (D+s)^n(e^{-st}f(t)) \tag{187}$$

thus

$$\mathcal{L}(D^n f) = \int_0^\infty (D+s)^n(e^{-st}f(t))\mathrm{d}t \tag{188}$$

To make the evaluation of (188) easier, we apply a little trick namely we use the trivial identity

$$(D+s)^n = s^n + (D+s)^n - s^n = s^n + D\sum_{k=1}^{n}(D+s)^{n-k}s^{k-1} \tag{189}$$

where we have used $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + .. + ab^{n-2} + b^{n-1})$ by putting $a = D+s$ and $b = s$. Let us now evaluate

$$\mathcal{L}(D^n f(t)) = \int_0^\infty \underbrace{e^{-st}D^n f(t)}_{(D+s)^n(e^{-st}f(t))} \mathrm{d}t = s^n\mathcal{L}(f) + \int_0^\infty D\{\sum_{k=1}^{n}(D+s)^{n-k}s^{k-1}e^{-st}f(t)\}\mathrm{d}t \tag{190}$$

where we have used (189). Further evaluation yields, where one operator D under the integral (190) is compensated by the integration

$$\mathcal{L}(D^n f(t)) = s^n\mathcal{L}(f) + \sum_{k=1}^{n}(D+s)^{n-k}s^{k-1}e^{-st}f(t)|_0^\infty \tag{191}$$

We can again make use of (187) (read from right to the left hand side) and obtain

$$\mathcal{L}(D^n f(t)) = s^n \mathcal{L}(f) + \{e^{-st} \sum_{k=1}^{n} D^{n-k} s^{k-1} f(t)\}|_0^\infty = s^n \mathcal{L}(f) - \sum_{k=1}^{n} s^{k-1} D^{n-k} f(t=0)$$

$$\mathcal{L}(D^n f(t)) =$$

$$= s^n \mathcal{L}(f) - [D^{n-1} f(t=0) + s D^{n-2} f(t=0) + .. + s^{n-2} D f(t=0) + s^{n-1} f(t=0)] \tag{192}$$

Relation (192) covers all cases for $n = 01, 2, ...$
For $n = 2$ it yields

$$\mathcal{L}(D^2 f(t)) = s^2 - D f(t=0) - s f(t=0) \tag{193}$$

The Laplace transform (192) of $H(t) D^n f(t)$ is determined by the Laplace transform of $H(t) f(t)$ and by the *initial values* $f(t=0), Df(t=0), .., D^{n-1} f(t=0)$ up to the $(n-1)^{th}$ order. Because the Laplace transform of the n^{th}-order time derivative contains all initial values $f(t=0), Df(t=0), .., D^{n-1} f(t=0)$ makes it especially useful for the solution of *Initial Value Problems* to be discussed below.

2.3 Linear Ordinary Differential Equations and Initial Value Problems

Linear Ordinary Differential Equations: We begin with the description of a dynamical process characterized by *one degree of freedom*, i.e. one variable $X(t)$ which is the unknown of the problem and which is to be determined as *solution* and t denotes the continuous *time variable*. Moreover, as dynamical processes are related to the change of variable X in time, which is mathematically the time derivative $\frac{d}{dt} X(t) = DX(t)$ where $D = \frac{d}{dt}$ is the *time derivative operator* which is defined by acting on variable $X(t)$ by producing its time derivative. The physical meaning of $X(t)$ could be for instance a displacement, then $DX(t)$ would be a velocity and $D^2 X(t)$ an acceleration. We emphasize that we confine here on *linear differential equations*.
A general dynamic process then can be described by a *differential equation*, physically it can be conceived as a **equation of motion** which has the general form

$$\left[D^n + a_{n-1}(t) D^{n-1} + .. + a_k(t) D^k + .. + a_1(t) D + a_0\right] X(t) = b(t) \tag{194}$$

or more compact, one can write (194) more compactly

$$\hat{L} X(t) = b(t) \tag{195}$$

with the **linear operator**

$$\hat{L} = D^n + a_{n-1}(t) D^{n-1} + .. + a_k(t) D^k + .. + a_1(t) D + a_0 \tag{196}$$

(194)is called a inhomogeneous *linear* **Ordinary Differential Equation** (ODE). The operator $\hat{L}$ is just defined by its action on $X(t)$. The highest derivative $D^n X(t)$ determines the *order* of the differential equation. Without loss of generality we can always divide the differential equation by the coefficient of $D^n X(t)$, thus we can chose it as equal to one. The $a_k(t)$ in (196) are *given* by problem. They are in general *time dependent* functions and also the *disturbance function* $b(t)$ is *given* by the dynamical problem defined by (194). $D^k X(t) = \frac{d^k}{dt^k} X(t)$ denotes the k^{th} derivative of variable

$X(t)$ with respect to time t. Note that if the $a_k = a_k(t)$ are time dependent, then $a_k(t)D^kX(t) \neq D^kX(t)a_k(t)$, thus the sequence is essential. The right hand side of (194) $b(t)$ is referred to as *disturbance function* and has physically often the meaning of an external force (194) with a *non-vanishing disturbance function* on the right hand side is referred to as a *inhomogeneous* differential equation. $X(t)$ is also often referred to as the *response* of the the dynamical system (on a exciting force $b(t)$). In case we have no disturbance function, i.e. $b = 0$ then (194) is referred to as a *homogeneous differential equation.*

A physical example for a dynamical system described by a differential equation of the form (194), is the *damped harmonic oscillator* which we will discuss in depth later. The harmonic oscillator obeys an equation of motion of the form

$$\left[D^2 + \frac{\gamma}{m}D + \frac{d}{m}\right] X(t) = \frac{F(t)}{m} \tag{197}$$

where $X(t)$ is the elongation of the oscillator from the equilibrium position $X = 0$, and $-\gamma DX$ denotes the friction force (assumed to be proportional to the velocity DX where $\gamma > 0$ is the friction coefficient, and $-dX$ the spring force with spring constant d. The disturbance function $F(t)$ is an external driving force and m denotes the mass of the oscillator. (197)is second order differential equation.

In order to construct the solution for a linear differential equation of the form (195), it is useful to review the properties of the *linear operator* $\hat{L}$ defined by (196), namely it follows directly from the linearity of $\hat{L}$ that

$$\hat{L}(C_1X_1(t) + C_2X_2(t)) = C_1\hat{L}X_1(t) + C_2\hat{L}X_2(t) \tag{198}$$

where C_i are arbitrary constants (independent on t). From this relation it follows for the *homogeneous problem*

$$\hat{L}X(t) = 0 \tag{199}$$

If $X_1(t)$ and $X_2(t)$ are both solutions of (199), then also *any linear combination* $C_1X_1 + C_2X_2$ with arbitrary constants C_i.

In general we can find n *linear independent solutions* $X_1(t), X_2(t), ..X_n(t)$ *to a homogeneous differential equation of* n^{th} *order* all fulfilling $\hat{L}X_i = 0$. Hence the *complete space of solution functions* of the homogeneous problem (199) is covered by

$$X_h(t) = C_1X_1(t) + C_2X_2(t) + .. + C_nX_n(t) \tag{200}$$

where C_i are arbitrary constants. A solution (200) of the homogeneous problem (199) is referred to as **Complementary Function**. Each basis function $X_i(t)$ is also a complementary function since it solves the homogeneous problem. A *complete set of (linear independent) basis functions* $X_1(t), .., X_n(t)$ with which we can generate any possible complementary function is referred to as a *Fundamental System* of solutions for the homogeneous problem. There are n such basis functions $X_i(t)$ for a differential equation of n^{th} order. If we have a set of n functions solving (199), how can we judge whether or not the set is a fundamental system? How can we actually define *linear independence or dependence of functions*? So far we only talked about linear (in)dependence when we considered vectors spanning an n-dimensional vector space R^n.

To construct solutions for the inhomogeneous problem of (195) we have to notice that there is *no general method* available to find a solution of a inhomogeneous problem (195). Nevertheless in many cases, the properties of solutions are useful to consider in order to construct them. Let us assume we have found such a particular solution $X_p(t)$ which is

also referred to as **Particular Integral** solving $\hat{L}X_p(t) = 0$, then the *general solution* of the inhomogeneous problem (195) is given by

$$\begin{aligned} X(t) &= X_h(t) + X_p(t) \\ X(t) &= \underbrace{C_1X_1(t) + C_2X_2(t) + .. + C_nX_n(t)}_{Complementary\,Function} + \underbrace{X_p(t)}_{Particular\,Integral} \end{aligned} \tag{201}$$

i.e. the **general solution of the ODE (194) is the sum of the Complementary Function and any Particular Integral**. By varying the C_i we obtain the whole variety of all possible solutions of (201). Especially if we have found by some way two different particular integrals X_{p1} and X_{p2} then the function $X_{p1} - X_{p2}$ solves the homogeneous problem, namely we have $\hat{L}(X_{p1} - X_{p2}) = \hat{L}X_{p1} - \hat{L}(X_{p2} = b(t) - b(t) = 0$. That is, the difference of two particular integrals always is a complementary function $X_h = X_{p1} - X_{p2}$ and can be written in terms of a fundamental system in the form (200). Hence the general solution (201) defines us also the *complete n-parametric (containing n arbitrary parameters C_i) are a set of possible particular integrals.*

Superposition Principle: The superposition principle is a consequence of linearity of the operator $\hat{L}$ and say: If $Y_1(t)$ is a particular integral solving $\hat{L}Y_1 = b_1(t)$ and $Y_2(t)$ a particular integral solving $\hat{L}Y_2 = b_2(t)$, the $Y = Y_1 + Y_2$ is a particular integral solving $\hat{L}Y = b_1 + b_2$.

Let us now recall a criteria for the linear independence of a set of function, i.e. with this criteria we can judge whether or not a set of complementary functions (solutions of the homogeneous problem) constitute a fundamental system. To this end we imagine that a function X can be conceived as a "vector" having infinitely many components $X(t)$, i.e. we have the analogy of a vector $\vec{v} = (v_i)$, namely the discrete index i corresponds in a function to the continuous "index" t. To this end we can recall the linear independence of a set of vectors $\vec{v_1}, .., \vec{v_n} \in R^n$ which means that we can always find non-trivial solutions for coefficients C_i so that any arbitrary given vector $\vec{b} \in R^n$ can be uniquely represented by $C_1\vec{v_1} + .. + C_n\vec{v_n} = \vec{b}$. Similarly, we claim for a set of independent functions $X_i(t), .., X_n(t)$, that any given function function $Y(t)$ (which is in the function space) can be uniquely represented by the *identity*[16]

$$C_1X_1(t) + .. + C_nX_n(t) = Y(t) \tag{202}$$

That is, selecting n *arbitrary* different times t_i, we see that (202) produces a $n\times$ equation system, which determines the coefficients C_i uniquely if the set of basis functions $X_i(t)$ is linear independent. For a set of linear independent functions $X_i(t)$ the determinant (where all times t_i are different to each other $t_i \neq t_j$, $i \neq j$) must be non-vanishing, namely

$$\mathcal{W}(t_1, .., t_n) = \boldsymbol{D}et \begin{pmatrix} X_1(t_1), & .., & X_n(t_1) \\ .. & .. & \\ .. & .. & .. \\ X_1(t_n), & .., & X_n(t_n) \end{pmatrix} \neq 0, \ \forall t_i \neq t_j \tag{203}$$

Now we can consider (203) for different times $t_m = t_1 + mh$ $(m = 0, 1, 2, .., n-1)$ and for $h \to 0$ by adding linear combinations of $X(t_m)$ to different columns, to obtain a criteria for linear independence of a set of functions which is known as

[16] An "identity" means an equation holds identical for a whole range of time t.

$$\boldsymbol{W}(t) = \boldsymbol{D}et \begin{pmatrix} X_1(t), & .., & X_n(t) \\ DX_1(t), & .., & DX_n(t) \\ .. & .. & \\ .. & .. & .. \\ D^{n-1}X_1(t), & .., & D^{n-1}X_n(t) \end{pmatrix} \neq 0 \tag{204}$$

where $D = \frac{d}{dt}$ means the derivative with respect to time t. The determinant $\boldsymbol{W}(t)$ is referred to as the **Wronskian**. A criteria for linear independence of a set of functions simply is $\boldsymbol{W}(t) \neq 0$. *If the set $X_i(t)$ are complementary functions, then $\boldsymbol{W}(t) \neq 0$ is a sufficient condition, that the set $\vec{X_1}(t), .., \vec{X_n}(t)$ constitutes a* **fundamental system of solutions for the homogeneous differential equation (199)** and hence constitute a complete basis of the function space of solutions of the homogeneous problem. We shall show later the statement: If we have found a fundamental system at time t with non-vanishing Wronskian $\boldsymbol{W}(t) \neq 0$ at certain time t, then the Wronskian remains nonzero for all times. This is important as it means, if a set of functions is a fundamental system to a certain time t, then it remains a fundamental system to all times.
So far, all our considerations hold also when the coefficients of the ODE in (194) are time dependent. However, for the sake of simplicity, we confine us now to linear differential equations with *constant*, i.e. time independent constants $a_k = const \neq a_k(t)$ in (194).
Linear ODEs with constant coefficients:
Example: Let us consider the second order differential equation

$$\hat{L}X(t) = b(t) \tag{205}$$

where

$$\hat{L} = D^2 + 2\alpha D + \beta^2 \tag{206}$$

and D denotes the time derivative operator. with constant (time independent) coefficients $\alpha, \beta = const \in R$. Let us first find the solution of the homogeneous differential equation (complementary function), (205), i.e. $b = 0$. Let us "try" a solution of the the form $e^{\lambda t}$ which is promising because of $De^{\lambda t} = \lambda e^{\lambda t}$, and $D^2e^{\lambda t} = \lambda^2 e^{\lambda t}$. We look now for which $\lambda\ e^{\lambda t}$ is a solution of the homogeneous problem which leads us to the condition

$$(D^2 + 2\alpha D + \beta^2)e^{\lambda t} = (\lambda^2 + 2\alpha\lambda + \beta^2)e^{\lambda t} = 0 \tag{207}$$

where we have to replace just D by λ in the Operator $\hat{L}$ in this equation. Since $e^{\lambda t} \neq 0 \forall t$, to make (207) equal to zero we have the condition that the so called **Characteristic Polynomial** $P(\lambda) = (\lambda^2 + 2\alpha\lambda + \beta^2)$ is vanishing

$$P(\lambda) = (\lambda^2 + 2\alpha\lambda + \beta^2) = 0 \tag{208}$$

(208) is referred to as *characteristic equation* of the differential equation. The characteristic polynomial $P(\lambda)$ is generated by replacing D in $\hat{L}$ by λ, i.e. we can say the operator $\hat{L}$ is the characteristic polynomial of the differential operator D, $\hat{L}(D) = P(D)$. Hence the characteristic polynomial of a differential equation of n^{th} order has a characteristic polynomial of n^{th} order and therefore has n zeros λ_i in our case $n = 2$, that is (208) has the roots

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \beta^2} \tag{209}$$

which are in general *complex quantities*. Hence the two functions $e^{\lambda_1 t}$, $e^{\lambda_1 t}$ solve the homogeneous problem. Now we need to check for which λ_i the set $e^{\lambda_1 t}$, $e^{\lambda_1 t}$ constitutes

a *Fundamental system* of solutions of the homogeneous problem. This set is only a fundamental system if the *Wronskian* of this differential equation is non-zero. The Wronskian in this case is given by

$$\mathcal{W} = \boldsymbol{D}et \begin{pmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{pmatrix} = \boldsymbol{D}et \begin{pmatrix} 1 & 1 \\ \\ \lambda_1 & \lambda_2 \end{pmatrix} e^{(\lambda_1+\lambda_2)t} = (\lambda_1 - \lambda_1) e^{(\lambda_1+\lambda_2)t} \tag{210}$$

that is $\mathcal{W} \neq 0$ if $\lambda_1 \neq \lambda_2$, i.e. the zeros λ_1, λ_2 must be *different to each other* that the set $e^{\lambda_1 t}, e^{\lambda_2 t}$ constitutes a *fundamental system* of solutions of the homogeneous problem. From (209) follows that the zeros are different if $\alpha^2 \neq \beta^2$. Let us first of all confine on the case that $\lambda_1 \neq \lambda_2$. Then the *complementary function* is given by

$$X_h(t) = C_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{211}$$

with arbitrary coefficients C_1, C_2. Let us now assume that $b(t) = e^{pt}$ in (205) where $P(p) \neq 0$, i.e. $p \neq \lambda_{1,2}$. Let us find a particular integral $X_p(t)$ for this case

$$\underbrace{(D^2 + 2\alpha D + \beta^2)}_{P(D)} X_p(t) = e^{pt} \tag{212}$$

Since we know that $D^k e^{pt} = p^k e^{pt}$, we can "try" a particular integral of the form (as we remember it is sufficient to find any particular integral) $X_p(t) = Ce^{pt}$ with a constant to be determined by inserting it into the differential equation $P(D)Ce^{pt} = e^{pt}$, namely

$$CP(p)e^{pt} = e^{pt} \tag{213}$$

or

$$C = \frac{1}{P(p)} = \frac{1}{(p^2 + 2\alpha p + \beta^2)} \tag{214}$$

where the denominator of this equation is non-zero as we have assumed above $P(p) \neq 0$. Thus a particular integral is

$$X_p(t) = \frac{e^{pt}}{(p^2 + 2\alpha p + \beta^2)} \tag{215}$$

and hence the *general solution of (205) with $b(t) = e^{pt}$ is a particular integral plus the complementary function, namely $X(t) = X_h(t) + X_p(t)$*, thus

$$X(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{e^{pt}}{(p^2 + 2\alpha p + \beta^2)} \tag{216}$$

where the complementary function in (216) holds only when $\lambda_1 \neq \lambda_2$.

Initial Value Problem: As we saw in this paragraph, a linear ordinary differential equation of n^{th} order has a complementary function with with n arbitrary constants C_i (see (201)). This means without any further condition the general solution (201) has n free parameters and is therefore non-unique. That is we can prescribe n conditions which can be satisfied by a certain choice of the C_i. In many cases nature defines such conditions in terms of *initial conditions*, that is the defection $X(t_0) = X_0$, the velocity $v(t_0) = DX(t_0)$ up to the $(n-1)^{th}$ time derivative $D^{n-1}X(t_0)$ have prescribed values to a certain "initial time" t_0 (without loss of generality, often one puts this time $t_0 = 0$ and the time evolution of the system then takes place for times $t > 0$). This consideration

leads us to the definition of a so referred to as **Initial Value Problem** (IVP): An IVP is characterized by a ordinary differential equation of n^{th} order of the form

$$\hat{L}X(t) = b(t) \tag{217}$$

with the (linear) operator

$$\hat{L} = D^n + a_{n-1}(t)D^{n-1} + .. + a_k(t)D^k + .. + a_1(t)D + a_0 \tag{218}$$

where $D = \frac{d}{dt}$ and with n initial conditions at $t = t_0$ (where the number n of initial conditions corresponds to the order of the differential equation), namely

$$\begin{array}{l} X(t_0) = A_0 \\ DX(t_0) = A_1 \\ .. \\ D^{n-1}X(t_0) = A_{n-1} \end{array} \tag{219}$$

with n **given** initial values (constants) $A_k = D^k X(t_0)$ $(k = 0, 1, .., n-1)$ (where we define $D^0 = 1$). From our previous considerations we know that the general solution of (217) has the form

$$X(t) = C_1 X_1(t) + C_n X_n(t) + X_p(t) \tag{220}$$

with the fundamental system $X_1, ..X_n$ constituting the complementary function and a particular integral $X_p(t)$. The n constants C_i are now not any more arbitrary, they are uniquely determined by the initial conditions (219), i.e. entering the initial conditions determine them uniquely as functions of the A_i. Inserting (220) into (219) leads to following equation system for the coefficients C_i

$$\begin{pmatrix} X_1(t_0), & .., & X_n(t_0) \\ DX_1(t_0), & .., & DX_n(t_0) \\ .. & .. & \\ .. & .. & .. \\ D^{n-1}X_1(t_0), & .., & D^{n-1}X_n(t_0) \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ .. \\ .. \\ C_n \end{pmatrix} + \begin{pmatrix} X_p(t_0) \\ DX_p(t_0) \\ .. \\ .. \\ D^{n-1}X_p(t_0) \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ .. \\ .. \\ A_n \end{pmatrix} \tag{221}$$

or

$$\begin{pmatrix} C_1 \\ C_2 \\ .. \\ .. \\ C_n \end{pmatrix} = \begin{pmatrix} X_1(t_0), & .., & X_n(t_0) \\ DX_1(t_0), & .., & DX_n(t_0) \\ .. & .. & \\ .. & .. & .. \\ D^{n-1}X_1(t_0), & .., & D^{n-1}X_n(t_0) \end{pmatrix}^{-1} \cdot \begin{pmatrix} A_0 - X_p(t_0) \\ A_1 - DX_p(t_0) \\ .. \\ .. \\ A_{n-1} - D^{n-1}X_p(t_0) \end{pmatrix} \tag{222}$$

We observe from (222) that the inverse of the matrix in (221) only exists if the Wronskian (which is the determinant of this matrix) $\mathcal{W}(t_0) \neq 0$ is non-zero. This is guaranteed when $X_1(t), .., X_n(t)$ are a *fundamental system* of solutions of the homogeneous problem.

In order to solve the IVP only the coefficients C_k of (222) need to be determined e.g. by application of *Cramer's rule* (100).

An alternative equivalent solution method which avoids this direct inversion is the solution with Laplace transforms. Therefore as a rule, for initial value problems, the Laplace transformation is more convenient, whereas the operator method is more convenient when we only look for the general solution of an ordinary differential equation without prescribed initial conditions. It is to be emphasized that both methods, the

Laplace transformation and the Operator method, which are to be discussed in the next sections, are equivalent and yield finally the same solutions $X(t)$ for an IVP which we will demonstrate by means of examples to follow. It is finally a matter of taste and depends on the problem to judge which method is more convenient.

2.4 Solution methods: Solution by using Operators

Let us now solve the IVP (217), by determination the coefficients C_k of (222) for the case that operator $\hat{L}$ of (218) has constant coefficients $a_k \neq a_k(t)$ and is of n^{th} order in D and has the form $\hat{L} = P(D) = \hat{L} = D^n + a_{n-1}D^{n-1} + .. + a_1 D + a_0 = (D - \lambda_1)..(D - \lambda_n)$ with $\lambda_i \neq \lambda_j \ \forall i, j$ are the zeros of the characteristic polynomial $P(\lambda)$[17] and $\lambda_i \neq b \ \forall i$, i.e. we assume here all roots λ_i have multiplicity one. Further we prescribe the n initial conditions

$$D^k X(t=0) = A_k, \qquad k = 0, 1, .., n-1 \tag{223}$$

where we define $D^0 = 1$. As the sequence of the factors $P(D) = (D - \lambda_1)..(D - \lambda_n)$ is arbitrary we can find a set of solutions $X_i(t)$ of the homogeneous problem $P(D) = 0$ where the $X_i(t)$ are solutions of

$$(D - \lambda_i)X_i = 0 \tag{224}$$

or $DX_i = \lambda_i X_i$ which has exponential solutions

$$X_i(t) = e^{\lambda_i t} \tag{225}$$

Hence the complementary function $X_h(t)$ writes

$$X_h(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t} \tag{226}$$

with constants C_i to be uniquely determined from the initial conditions (223) by employing (222). The Wronskian $\mathcal{W}(t=0)$ is in this case is determined by the **Vandermond determinant** which is evaluated in the appendix (eqs. (354) ff.) which indeed is non-zero when the λ_i are all different to each other, which also guarantees that the set $e^{\lambda_1 t}, e^{\lambda_2 t}, .., e^{\lambda_n t}$ constitutes a **Fundamental System of Solutions for the Homogeneous Problem if** $\lambda_i \neq \lambda_j$. Note: This is not anymore the case for multiple roots, as we shall see in later examples.
In order to fulfill the given initial conditions we need to determine the C_i of (226) by inverting (222). Then application of *Cramer's rule* (100) yields by employing the Vandermond Determinant $\mathcal{V}$ defined in (355)[18]

$$\begin{aligned} C_k &= \frac{\mathcal{V}(\lambda_1, .., \lambda_{k-1}, D, \lambda_{k+1}, .., \lambda_n)}{\mathcal{V}(\lambda_1, .., \lambda_{k-1}, \lambda_k, \lambda_{k+1}, ..\lambda_n)}[X(t=0) - X_p(t=0)] \\ &= \frac{(D - \lambda_1)..(D - \lambda_{k-1})(D - \lambda_{k+1})..(D - \lambda_n)}{(\lambda_k - \lambda_1)..(\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1})..(\lambda_k - \lambda_n)}[X(t=0) - X_p(t=0)] \\ &= \prod_{j \neq k}^{n} \frac{(D - \lambda_j)}{(\lambda_k - \lambda_j)}[X(t=0) - X_p(t=0)] \qquad (k = 1, .., n) \end{aligned} \tag{227}$$

[17] Just replace $D \to \lambda$ in $P(D)$ to obtain $P(\lambda)$.
[18] Where the product in $(227)_3$ is defined as $\prod_{i=1}^{n} \alpha_i = \alpha_1..\alpha_n$ by omitting factor α_k.

where the Vandermond determinants $\mathcal{V}(\lambda_1,..,\lambda_{k-1},D,\lambda_{k+1},..,\lambda_n)$ in the numerator is generated by replacing λ_k by D in $\mathcal{V}(\lambda_1,..,\lambda_k,..\lambda_n)$ which becomes then an operator polynomial of $(n-1)^{st}$ order in D. Hence the $C_k = C_k(A_0,..,A_{n-1})$ are expressed by the *given values for the initial conditions* $A_k = D^k X(t=0)$ by evaluating the operator polynomial by applying it on $X(t) - X_p(t)$ at $t = 0$.
With the coefficients C_k determined by (227) the complementary function (226) is determined in terms of the given initial data A_k given in (223) and the complete solution of the inhomogeneous IVP then is $X(t) = X_h(t) + X_p(t)$ where $X_p(t)$ the very particular integral employed in (227). Moreover, since $X_p(t) = 0$ for the homogeneous problem that by putting $X_p(t=0) = 0$ in (227) yields the coefficient C_k of the IVP of the homogeneous differential equation $P(D)X(t) = 0$ with the same given initial conditions (223), namely

$$\tilde{C}_k = \prod_{j\neq k}^{n} \frac{(D-\lambda_j)}{(\lambda_k-\lambda_j)} X(t=0) \qquad (k=1,..,n) \tag{228}$$

and we can extract from (227) the coefficients for the *inhomogeneous* problem $b(t) \neq 0$ with *homogeneous* initial conditions (223) $A_k = 0$, namely

$$\tilde{\tilde{C}}_k = -\prod_{j\neq k}^{n} \frac{(D-\lambda_j)}{(\lambda_k-\lambda_j)} X_p(t=0) \qquad (k=1,..,n) \tag{229}$$

Hence we get from (227) **(i)** the solution $\tilde{X}(t)$ of the *homogeneous IVP*

$$P(D)\tilde{X}(t) = 0 \tag{230}$$

with *inhomogeneous initial conditions* (223) $D^k\tilde{X}(t=0) = A_k \neq 0$ (for at least one A_k), namely

$$\tilde{X}(t) = \sum_{j=1}^{n} \tilde{C}_j e^{\lambda_j t} \tag{231}$$

with the $\tilde{C}_k$ given in (228). Thus $\tilde{X}(t)$ is a complementary function fulfilling the (non-zero) initial conditions.
Furthermore we get from (227) **(ii)** the solution $\tilde{\tilde{X}}(t)$ of the *inhomogeneous IVP*

$$P(D)\tilde{\tilde{X}}(t) = b(t) \tag{232}$$

with $b(t) \neq 0$ fulfilling *homogeneous initial conditions* (223) $D^k\tilde{\tilde{X}}(t=0) = A_k = 0$, namely

$$\tilde{\tilde{X}}(t) = X_p(t) + \sum_{j=1}^{n} \tilde{\tilde{C}}_j e^{\lambda_j t} \tag{233}$$

where $\tilde{\tilde{C}}_k$ given in (229) and $X_p(t)$ being *any* particular integral solving
$P(D)X_p(t) = b(t)$.
Eventually we get from from (227) **(iii)** the solution $X(t)$ of the *inhomogeneous IVP*

$$P(D)X(t) = b(t) \tag{234}$$

with $b(t) \neq 0$ with *inhomogeneous initial conditions* (223) $D^kX(t=0) = A_k \neq 0$ (at least for one A_k), namely

$$X(t) = \tilde{X}(t) + \tilde{\tilde{X}}(t) \tag{235}$$

with the *Complementary Function* $\tilde{X}(t)$ given in (231) and the *Particular Integral* $\tilde{\tilde{X}}(t)$ given in (233).
Later in section (2.5) we will see that the use of *Laplace transforms* always leads to a complementary function $\tilde{X}(t)$ which fulfils the initial conditions and to a particular integral $\tilde{\tilde{X}}(t)$ which fulfils zero-initial conditions.

Let us now consider the inhomogeneous IVP with inhomogeneous initial conditions (223) and specify $b(t)$, namely

$$P(D)X(t) = e^{bt} \tag{236}$$

As we need to find only any possibly simple particular integral, we easily can find one by trying $X_p(t) = Ce^{bt}$, with a constant C to be determined from the differential equation. Inserting $X_p(t) = Ce^{bt}$ into (236) yields a condition to determine the constant C, namely

$$P(b)C = 1 \tag{237}$$

or $(\lambda_i \neq b)$

$$C = \frac{1}{P(b)} \tag{238}$$

where $P(b) = b^n + a_{n-1}b^{n-1} + .. + a_1 b + a_0 = (b - \lambda_1)..(b - \lambda_n)$. For $\lambda_i \neq b$ the general solution writes then

$$X(t) = \underbrace{\sum_{i=1}^{n} C_i e^{\lambda_i t}}_{complementary\,function} + \underbrace{\frac{e^{bt}}{P(b)}}_{particular\,integral} \tag{239}$$

In the next paragraph we show how we can get this result by using **Laplace transforms** and the equivalence of both methods.
Let us consider now the case of *multiple roots*. Without loss of generality we consider one root λ_1 being multiple and the others having multiplicity one.
Given the ODE of n^{th} order

$$P(D)X(t) = e^{bt} \tag{240}$$

where

$$P(D) = (D - \lambda_1)^{n_1}(D - \lambda_2)..(D - \lambda_r) \tag{241}$$

and again we assume $b \neq \lambda_i, \forall i$. The general solution is to be determined by using the operator method: In this example we have the case of a n_1-fold multiple root λ_1 with $n_1 > 1$. To find a fundamental system we have to find solutions for

$$(D - \lambda_i)^{n_1} X_1 = 0 \tag{242}$$

and

$$(D - \lambda_i)X_i = 0, i \neq 1 \tag{243}$$

The last ones corresponding to $i \neq 1$ have the solutions $X_i(t) = e^{\lambda_i t}$, and also (242) has one solution $e^{\lambda_1 t}$. But as (242) is a differential equation of degree n_1 we need to find another $n_1 - 1$ linear independent solutions in order to get enough functions for a fundamental system. To this end let us try solutions $X_1 = e^{\lambda_1 t}C_1(t)$, i.e. finding further solutions by *variation of constants* $C(t)$[19]. Then we have

$$(D - \lambda_i)^{n_1} e^{\lambda_1 t} C_1(t) = 0 \tag{244}$$

[19]This method can be also found in many standard mathematics textbooks.

and because of $(D-\lambda_i)e^{\lambda_1 t}C_1(t) = e^{\lambda_1 t}DC(t)$, (244) assumes

$$e^{\lambda_1 t}D^{n_1}C_1(t) = 0 \tag{245}$$

or because of $e^{\lambda_1 t} \neq 0$

$$D^{n_1}C_1(t) = 0 \tag{246}$$

which has as solutions powers t^k with $0 \leq k < n_1$, which can be shown to be linear independent, thus

$$X_1(t) = C_1(t)e^{\lambda_1 t} = (C_1^{(0)} + C_1^{(1)}t + .. + C_1^{(n-1)}t^{(n-1)})e^{\lambda_1 t} \tag{247}$$

Hence the general solution of (240) is

$$X(t) = (C_1^{(0)} + C_1^{(1)}t + .. + C_1^{(n-1)}t^{(n-1)})e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + .. + C_n e^{\lambda_n t} + \frac{e^{bt}}{P(b)} \tag{248}$$

2.5 Solution Methods: Laplace transforms

In this paragraph we consider the same example (236) and solve it for given initial conditions by using Laplace transformation. The Laplace transformation of (236) writes[20]

$$\int_0^\infty e^{-st}P(D)X(t)\mathrm{d}t = \int_0^\infty e^{-(s-b)t}\mathrm{d}t = \frac{1}{(s-b)} \qquad Re\ \{s\} > \sigma_c = Re\ \{b\} \tag{249}$$

or

$$\int_0^\infty P(D+s)[e^{-st}X(t)]\mathrm{d}t = \frac{1}{b-s} \tag{250}$$

where $e^{-st}DX(t) = (D+s)[e^{-st}X(t)]$ is used. (250) can be also written as

$$\int_0^\infty (P(D+s) - P(s) + P(s))[e^{-st}X(t)]\mathrm{d}t = \frac{1}{b-s} \tag{251}$$

or

$$P(s)\tilde{X}(s) + \int_0^\infty (P(D+s) - P(s))[e^{-st}X(t)]\mathrm{d}t = \frac{1}{b-s} \tag{252}$$

where $P(s) = (s-\lambda_1)..(s-\lambda_n)$ (replace D by s in the characteristic operator $P(D)$). Note that $(P(D+s) - P(s))$ starts at least with the first order in D, namely

$$P(D+s) - P(s) = \sum_{k=1}^{n} a_k(s)D^k = D\sum_{k=1}^{n} a_k(s)D^{k-1} \tag{253}$$

Hence (251) assumes the form

$$\begin{aligned} &P(s)\tilde{X}(s) + \sum_{k=1}^{n} a_k(s)\int_0^\infty D^k[e^{-st}X(t)]\mathrm{d}t \\ &= P(s)\tilde{X}(s) - \sum_{k=1}^{n} a_k(s)D^{k-1}[e^{-st}X(t)]|_{t=0} = \frac{1}{b-s} \end{aligned} \tag{254}$$

[20]The Laplace transform of a (transformable) function $f(t)$ is defined as $\tilde{f}(s) = \int_0^\infty e^{-st}f(t)\mathrm{d}t$, $Res > \sigma_c$.

with $D^{k-1}[X(t)]|_{t=0} = e^{-st}(D-s)^{k-1}X(t)|t=0 = (D-s)^{k-1}X(t=0)$ (254) becomes

$$P(s)\tilde{X}(s) + \sum_{k=1}^{n} a_k(s) \int_0^\infty D^k[e^{-st}X(t)]\mathrm{d}t = P(s)\tilde{X}(s) - \sum_{k=1}^{n} a_k(s)(D-s)^{k-1} = \frac{1}{s-b} \tag{255}$$

Because of (253) we have $P(s+\lambda) - P(s) = \sum_{k=1}^{n} a_k(s)\lambda^k$ which becomes for $\lambda = D - s$ the expression

$$P(s+D-s) - P(s) = P(D) - P(s) = \sum_{k=1}^{n} a_k(s)(D-s)^k \tag{256}$$

or with

$$\sum_{k=1}^{n} a_k(s)(D-s)^{k-1} = \sum_{k=1}^{n} a_k(s)(D-s)^k(D-s)^{-1} = (P(D) - P(s))(D-s)^{-1} \tag{257}$$

is a polynomial of degree $n-1$ in s which becomes important below. Hence (255) becomes

$$P(s)\tilde{X}(s) - (P(D) - P(s))(D-s)^{-1}X(t=0) = \frac{1}{s-b} \tag{258}$$

or

$$\tilde{X}(s) = \frac{(P(D) - P(s))(D-s)^{-1}}{P(s)}X(t=0) + \frac{1}{(s-b)P(s)} \tag{259}$$

This equation (259) is the Laplace transform of the solution of the initial value problem. (259) is the crucial relation in order to solve IVP with Laplace transforms. All what remains is now to find the back-transformation of $\mathcal{L}^{-1}(\tilde{X}(s)) = X(t)$ which gives the time-domain solution of (236) for the IVP which obeys automatically the initial conditions (223) which is shown below together with some examples.

Since $(P(D) - P(s))(D-s)^{-1} = \sum_{k=0}^{n-1} b_k s^k$ is a polynomial of order $n-1$ in s there is partial decomposition of the form

$$\frac{(P(D) - P(s))(D-s)^{-1}}{P(s)}X(t=0) = \sum_{i}^{m} \frac{C_1}{(s-\lambda_i)} \tag{260}$$

where

$$C_i = (s-\lambda_i)\frac{(P(D) - P(s))(D-s)^{-1}}{P(s)}|_{s=\lambda_i}X(t=0) \tag{261}$$

and by using $P(s) = (s-\lambda_1)..(s-\lambda_n)$ and $P(s=\lambda_i) = 0$, we obtain

$$C_i = \frac{\prod_{j\neq i}^{n}(D-\lambda_j)}{\prod_{j\neq i}^{n}(\lambda_i - \lambda_j)}X(t=0) = \prod_{j\neq i}^{n}\frac{(D-\lambda_j)}{(\lambda_i - \lambda_j)}X(t=0) \tag{262}$$

We recognize that this equation coincides with the previously derived coefficients $\tilde{C}_k$ of eq. (228), i.e. when putting $X_p = 0$ thus the IVP is homogeneous thus $X_h(t) = \tilde{X}(t)$ coincides with eq. (231) and fulfills the initial conditions. That means both methods yield the same coefficients $C_k = \tilde{C}_k$ for the homogeneous IVP (i.e. when $b(t) = 0$). Moreover we show in the questions (question 26) that the particular integral derived from Laplace transforms coincides with $\tilde{\tilde{X}}(t)$ of(233) and fulfills zero-initial conditions. Both methods (Laplace transform and the solution with operators leading to (227)) are completely equivalent and yield the same solution $X(t) = X_p(t) + X_h(t) = \tilde{X}(t) + \tilde{\tilde{X}}(t)$ of (235) for the inhomogeneous IVP with nonzero initial conditions.

For the full partial decomposition of (259) we still need the decomposition of

$$\frac{1}{(s-b)P(s)} = \frac{1}{P(b)}\frac{1}{(s-b)} + \sum_{j=1}^{n} \frac{1}{(\lambda_j - b)\prod_{k\neq j}(\lambda_j - \lambda_k)}\frac{1}{(s-\lambda_j)} \tag{263}$$

Hence because of $\mathcal{L}^{-1}(\frac{1}{(s-a)}) = e^{at}$ we find for the inverse transform of (263) which gives a *particular integral (particular solution)* $X_p(t)$ fulfilling homogeneous initial conditions $D^k X_p(t=0) = 0$ whereas the back-transformation of (260) yields a solution for the homogeneous problem (complementary function $X_h(t)$ which fulfills the initial conditions (223), thus the complete solution $X(t) = X_h(t) + X_p(t)$ fulfills (223). Back-transformation into the time domain yields

$$X_p(t) = \frac{e^{bt}}{P(b)} + \sum_{j=1}^{n} \frac{e^{\lambda_j t}}{(\lambda_j - b)\prod_{k\neq j}(\lambda_j - \lambda_k)} \tag{264}$$

The back-transformation of (260) yields

$$X_h(t) = \sum_{i=1}^{m} C_i e^{\lambda_i t} \tag{265}$$

where the C_i are determined in (262) which guarantee $X_h(t)$ to fulfill the initial conditions (223). The solution of the initial value problem then writes

$$X(t) = X_h(t) + X_p(t) \tag{266}$$

where $X_p(t)$ is given by (264) and $X_h(t)$ is given by (265) with the constants C_i given in (262).

Example: Given the IVP

$$(D+1)(D+2)X(t) = e^{3t} \tag{267}$$

with the initial conditions

$$X(t=0) = 1, \quad DX(t=0) = 0 \tag{268}$$

Solution with Laplace transformation: $P(D) = (D+1)(D+2)$. The Laplace transform of the solution which fulfills the initial condition is given by (266)

$$X(t) = X_h(t) + X_p(t) \tag{269}$$

where the homogeneous is given in the form (265) with the coefficients C_i given by (261) and particular solution of the form (264). In our case we have $\lambda_1 = -1$ and $\lambda_2 = -2$. Then we find from (261) for C_1, C_2

$$C_1 = \frac{(D-\lambda_2)}{(\lambda_1 - \lambda_2)}X(t=0) = \frac{(D+2)}{(-1+2)}X(t=0) = DX(t=0) + 2X(t=0) \tag{270}$$

$$C_2 = \frac{(D-\lambda_1)}{(\lambda_2 - \lambda_1)}X(t=0) = \frac{(D+1)}{(-2+1)}X(t=0) = -DX(t=0) - X(t=0) \tag{271}$$

which yield with the initial values (268).

$$C_1 = 2,\ C_2 = -1 \tag{272}$$

Thus we find for the solution $X_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ of the homogeneous problem

$$X_h(t) = 2e^{-t} - e^{-2t} \tag{273}$$

which fulfills the initial conditions (268). *Again this example is consistent with our previous finding: When applying Laplace transformation, the decomposition (266) is always such, that $X_h(t) = \tilde{X}(t)$ (eq. (231)) fulfills the initial conditions and $X_p(t) = \tilde{\tilde{X}}(t)$ (eq. (233)) fulfills zero initial conditions, thus the solution $X(t) = X_h(t) + X_p(t) = \tilde{X}(t) + \tilde{\tilde{X}}(t)$ (eq. (235)) again fulfills the initial conditions.*

In order to complete the solution we still need to determine $X_p(t)$ by using (264) which assumes

$$X_p(t) = \frac{e^{bt}}{P(b)} + \frac{e^{\lambda_1 t}}{(\lambda_1 - b)(\lambda_1 - \lambda_2)} + \frac{e^{\lambda_2 t}}{(\lambda_2 - b)(\lambda_2 - \lambda_1)} \tag{274}$$

which yields with $b = 3$

$$X_p(t) = \frac{e^{3t}}{(3+1)(3+2)} + \frac{e^{-t}}{(-1-3)(-1+2)} + \frac{e^{-2t}}{(-2-3)(-2+1)} \tag{275}$$

or

$$X_p(t) = \frac{e^{3t}}{20} - \frac{e^{-t}}{4} + \frac{e^{-2t}}{5} \tag{276}$$

which fulfills as required zero initial conditions $X_p(t=0) = 0$ and $DX_p(t=0) = 0$ which can easily be checked. Thus the complete solution of the IVP is the sum of (273) and (276) which yields

$$\begin{aligned} X(t) &= 2e^{2t} - e^{-2t} + \frac{e^{3t}}{20} - \frac{e^{-t}}{4} + \frac{e^{-2t}}{5} \\ X(t) &= \frac{7}{4}e^{-t} - \frac{4}{5}e^{-2t} + \frac{1}{20}e^{3t} \end{aligned} \tag{277}$$

where the initial conditions (268) are easily be verified to be fulfilled by this result.

Example: Harmonic Oscillator
Here we consider the initial value problem

$$P(D)X(t) = e^{i\omega t}, \qquad P(D) = D^2 + pD + q = (D - \lambda_1)(D - \lambda_2) \tag{278}$$

with initial conditions (given displacement and velocity at $t = 0$)

$$X(t=0) = X_0, \qquad DX(t=0) = V_0 \tag{279}$$

where $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \neq i\omega$.

The solution with operator method (without Laplace transformation) is given by

$$X(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_1 t} + \frac{e^{i\omega t}}{P(i\omega)} \tag{280}$$

where the A_i are to be determined uniquely from the initial conditions and $P(i\omega) = (i\omega - \lambda_1)(i\omega - \lambda_2) = -\omega^2 + ip\omega + q$ (replace $D \to i\omega$ in $P(D)$). This leads to

$$\begin{aligned} A_1 + A_2 &= X_0 - \frac{1}{P(i\omega)} \\ \lambda_1 A_1 + \lambda_2 A_2 &= V_0 - \frac{i\omega}{P(i\omega)} \end{aligned} \tag{281}$$

which is invertible uniquely since the Wronskian $W(t) \neq 0$ and yields

$$A_1 = \frac{V_0 - \lambda_2}{(\lambda_1 - \lambda_2)} - \frac{(\lambda_2 - i\omega)}{(\lambda_1 - \lambda_2)P(i\omega)} \tag{282}$$

and

$$A_2 = \frac{V_0 - \lambda_1}{(\lambda_2 - \lambda_1)} - \frac{(i\omega - \lambda_1)}{(\lambda_2 - \lambda_1)P(i\omega)} \tag{283}$$

which we also can find by using Laplace transforms.

2.6 Systems of Linear Differential Equations

Linear Systems Of First Order. Many problems in dynamics are characterized by a set of N variables $\vec{X} = (X_1(t), .., X_N(t))^t$ and are defined by a a matrix differential equation of the form

$$\frac{d}{dt}\vec{X} = \mathbf{A} \cdot \vec{X} + \vec{f}(t) \tag{284}$$

where $\mathbf{A}$ denotes a constant (in time) $n \times n$ matrix $\vec{f} = (f_1(t), .., f_n(t))^t$ denotes an external force. Additionally we impose the initial condition

$$\vec{X}(t = 0) = \vec{X}_0 \tag{285}$$

where $\vec{X}_0$ contains n constant components. We can construct the solution by putting (we proceeded in full analogy with variation of constants just as in the case of one degree of freedom)

$$\vec{X}(t) = \exp(At) \cdot \vec{C}(t) \tag{286}$$

with a yet to be determined not constant vector $\vec{C}(t)$. Inserting into (284) yields

$$\frac{d}{dt}\vec{X} = \underbrace{\mathbf{A} \exp(At) \cdot \vec{C}}_{\mathbf{A} \cdot \vec{X}} + \underbrace{\exp(At) \cdot \frac{d}{dt}\vec{C}}_{\vec{f}} \tag{287}$$

thus $\frac{d}{dt}\vec{C} = exp(-\mathbf{A}t) \cdot \vec{f}$ or

$$\vec{C}(t) = \vec{C}(0) + \int_0^t exp(-\mathbf{A}\tau) \cdot \vec{f}(\tau) \mathrm{d}\tau \tag{288}$$

The integration constant (vector) can be chosen such that $\vec{C}(0) = \vec{X}_0$, thus our initial condition (285) is fulfilled. With (286) we can thus write for the solution of our initial value problem

$$\vec{X}(t) = \exp(\mathbf{A}t) \cdot \left(\vec{X}_0 + \int_0^t \exp(-\mathbf{A}\tau) \cdot \vec{f}(\tau) \mathrm{d}\tau \right) \tag{289}$$

In full analogy to our above considerations about IVPs the solution splits into a complementary function solving the homogeneous problem and fulfilling the initial condition and a particular solution (complementary function) solving the inhomogeneous problem and and fulfilling zero initial conditions, namely

$$\vec{X}(t) = \vec{X}_h(t) + \vec{X}_p(t) \tag{290}$$

where

$$\vec{X}_h(t) = \exp(\mathbf{A}t)\vec{X}_0 \tag{291}$$

which fulfills the initial condition and

$$\vec{X}_p(t) = \exp(\mathbf{A}t) \int_0^t \exp -(\mathbf{A}\tau) \cdot \vec{f}(\tau) \mathrm{d}\tau \tag{292}$$

which fulfills homogeneous initial conditions. For a matrix $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ having a diagonal-form $\mathbf{D} = diag(\lambda_1, .., \lambda_n)$ with eigenvalues λ_i and eigenvectors $\vec{v}_i$ equation (289) can be rewritten in explicit form (since $\exp \mathbf{A}t = \mathbf{V} \cdot diag(e^{\lambda_1 t}, .., e^{\lambda_n t}) \cdot \mathbf{V}^{-1}$)[21]

$$\vec{X}(t) = \sum_{j=1}^{n} \vec{v}_j \left(A_j^0 + e^{\lambda_j t} \int_0^t e^{-\lambda_j \tau} \tilde{f}_j(\tau) \mathrm{d}\tau \right) \tag{293}$$

where the A_j^0 are the initial conditions with respect to the eigenvector basis ($\vec{X}_0 = \sum_{j=1}^n \vec{v}_j A_j^0$), and $\tilde{f}_j(t)$ are the components of $\vec{f}(t)$ with respect to the eigenvector basis, namely

$$\begin{pmatrix} A_1^0 \\ A_2^0 \\ .. \\ A_n^0 \end{pmatrix} = V^{-1}\vec{X}_0 \quad , \begin{pmatrix} \tilde{f}_1^0(t) \\ \tilde{f}_2^0(t) \\ .. \\ \tilde{f}_n^0(t) \end{pmatrix} = V^{-1}\vec{f}(t) \tag{294}$$

Various examples are to be discussed below.

Linear Systems Of Second Order. As a rule engineering problems (such as the motion of structures, vibration of bridges etc) are described by second order systems which reflects the fact that these systems are representations of the Newtonian equation of motion for in general multi-degree of freedom systems. When we assume the state of the system is described by a vector of n variables $\vec{u} = (u_j(t))$ $(j = 1, .., N)$, each component representing one degree of freedom. The equation of motion then is written in the form

$$\mathbf{M} \cdot \frac{d^2}{dt^2}\vec{u} = -\mathbf{K} \cdot \vec{u} + \vec{f}(t) \tag{295}$$

where $\mathbf{M}$ is called the *mass matrix* (or matrix of inertia), $\vec{u}$ is the vector of (generalized) displacements and $\mathbf{K} \cdot \vec{u}$ denotes the elastic or spring force (driving the system back into its equilibrium $\vec{u} = 0$) and $\mathbf{K}$ is called the (generalized) *stiffness matrix*. The vector $\vec{f}(t)$ denotes external (applied) forces which are realized e.g. when wind affects a structure or by an earthquake etc. We assume throughout our considerations that

[21] where this representation can also be derived by 'variation of constants by putting $\vec{X}(t) = \sum_j^n \vec{v}_j C_j(t)$ where $C_j(t)$ are to be determined.

$\mathbf{M}, \mathbf{K}$ are constant time independent $n \times n$ matrices. We furthermore assume that both $\mathbf{M}$ and $\mathbf{K}$ are *positive definite invertible* $N \times N$ *matrices*. In most cases we can assume that $\mathbf{M} = diag(m_1, m_2, .., m_N)$ is a diagonal matrix with the masses m_j in the diagonal. The displacement vector $\vec{u}(t)$ are the unknowns to be determined. In order to make the solution $\vec{u}(t)$ unique we can impose $2N$ initial conditions, namely N initial displacements $\vec{u}(t=0) = \vec{U}_0$ and N initial velocities $\frac{d}{dt}u(t=0) = \vec{V}_0$, thus

$$\begin{aligned} u_j(t=0) &= U_j, \quad (j = 1, .., N) \\ \frac{d}{dt}u_j(t=0) &= V_j, \quad (j = 1, .., N) \end{aligned} \tag{296}$$

i.e. with the initial conditions (295) we impose our physical system N initial displacements (constants) U_j and N initial velocities (constants) V_j which *uniquely* determines the physical displacement vector $\vec{u}(t) = \vec{u}(\vec{U}, \vec{V}, t)$ for all times $t > 0$. As a rule in real-life engineering problems include often a huge number N of degrees of freedom. We mention that a N degree of freedom problem of second order can be transformed into a $2N$ degree of freedom problem of first order when introducing new variables $\frac{d}{dt}u_j(t) = v_j(t)$ and the method of 'variation of constants' as deduced in above Sec. 'Linear Systems Of First Order' can be applied.

Solution Method and Examples. Now we deduce a solution method of the second order initial value problem defined by (295) together with (296) which is straight-forward applicable to 'real-life' engineering problems. We derive the method by means of examples which are close to 'real-life' engineering problems. In order to construct the solution of (295) we multiply these eq. with $\mathbf{M}^{-1}$ and get

$$\frac{d^2}{dt^2}\vec{u} = -\mathbf{W}\vec{u} + \vec{b}(t) \tag{297}$$

where $\mathbf{W} = \mathbf{M}^{-1} \cdot \mathbf{K}$ and $\vec{b}(t) = \mathbf{M}^{-1}\vec{f}(t)$. The $-$-sign in front of $\mathbf{W}$ is introduced for our convenience that we can handle $\mathbf{W}$ being a constant *positive definite matrix*[22]. As in the above discussed examples we again assume that we can split our solution $\vec{u}(t)$ into a solution of the homogeneous problem (complementary function) (i.e. $\vec{b}(t) = 0$ and a particular integral (s special solution of the inhomogeneous problem $\vec{b} \neq 0$). To this end let us first of all start with *homogeneous problem*.

Solution of the homogeneous initial value problem (295) ($\vec{b} = 0$) with initial conditions (296). In view of what we derived about matrix functions, we can define the cos and sin of a matrix $\mathbf{W}t$ (where t denotes the time t) as

$$\begin{aligned} \cos(\mathbf{W}^{\frac{1}{2}}t) &= \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!}\sqrt{\mathbf{W}}^{2n} \\ \sin(\mathbf{W}^{\frac{1}{2}}t) &= \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!}\sqrt{\mathbf{W}}^{2n+1} \end{aligned} \tag{298}$$

where we easily verify that the matrices $\mathbf{Y}(t) = \sin(\mathbf{W}t), \cos(\mathbf{W}t)$ solve $\frac{d^2}{dt^2}\mathbf{Y} = -\mathbf{W} \cdot \mathbf{Y}$, thus we can write the solution of the homogeneous problem as

$$\vec{u}_h(t) = \cos(\mathbf{W}^{\frac{1}{2}}t) \cdot \vec{U} + \sin(\mathbf{W}^{\frac{1}{2}}t)\mathbf{W}^{-\frac{1}{2}} \cdot \vec{V} \tag{299}$$

[22] where positive definiteness of $\mathbf{W}$ means that all eigenvalues of $\mathbf{W}$ are positive.

where $\vec{U}, \vec{V}$ denote the constant vectors given as initial conditions ($2N$ constants) of (296). We observe that $\vec{u}_h(t)$ of (299) solves the homogeneous system (295) and fulfills the initial conditions (296). That (299) indeed is very handy in order to solve real-life becomes more obvious when we write it in eigenvector coordinates. For the sake of simplicity we confine on the (in most situations sufficient case) of a diagonalizable matrix $\mathbf{W} = \mathbf{V}\dot{\mathbf{D}} \cdot \mathbf{V}^{-1}$[23] where $\mathbf{V}$ is the matrix consisting of the N eigenvectors $\vec{v}_1, ..\vec{v}_N$ with $\mathbf{W}\vec{v}_j = \lambda_j \vec{v}_j$ where λ_j denote the eigenvalues. Then (299) can be conveniently re-written as

$$\vec{u}_h(t) = \sum_{j=1}^{N} \vec{v}_j \left(A_j \cos\left(t\sqrt{\lambda_j}\right) + \frac{B_j}{\sqrt{\lambda_j}} \sin\left(t\sqrt{\lambda_j}\right) \right) \tag{300}$$

where A_j, B_j are the initial conditions in eigenvector coordinates given by ($\vec{A} = (A_j), \vec{B} = (B_j)$)

$$\vec{A} = \mathbf{V}^{-1} \cdot \vec{U}, \quad \vec{B} = \mathbf{V}^{-1} \cdot \vec{V} \tag{301}$$

The inhomogeneous IVP ($b(t) \neq 0$) of (297) we then can solve by adding to (300) a particular solution (particular integral) $\vec{u}_p$ which satisfies *homogeneous initial conditions*, i.e. $\vec{u}_p(t=0) = 0$, and $\frac{d}{dt}\vec{u}_p(t=0) = 0$.

Especially in *computational mathematics - discrete mathematics* where *Partial Differential Equations* (PDEs) are approximated by systems of differential equations. This is done by approximating a continuous field quantity $u(x)$ by a finite set of values in certain point x_i, thus $u(x) \to u(x_n) = u_n$ that is a continuous field is mapped onto a discrete vector $\vec{u} = (u_n)$ with N components. Often one uses $x_n = nh$ where $n = 1, .., N$. For a constant x means $N \to \infty$ ($h \to 0$) that this approximation becomes more and more accurate. In the following section we discuss some examples.

In the following we discuss the discrete approximation of a free vibrating string which is discussed in its continuous formulation defined by a PDE (continuous wave equation) in question **24.)**.

[23] In physical and engineering application $\mathbf{W} = \mathbf{W}^t$ as a rule is is a *symmetric* matrix having real eigenvalues and mutual orthogonal eigenvectors.

3 Questions

3.1 Questions to part I

1.) Determine the limiting case $Y(x) = \lim_{n\to\infty}(1+\frac{x}{n})^n$ by using the *Binomial Theorem* (Relation (13)).

2.) Prove the well known relation $e^{t_1}\cdot e^{t_2} = e^{(t_1+t_2)}$ by multiplying their Mc-Laurin Series. Hint: Make use of the Binomial Theorem.
(i) By using (i), show that $e^{-t} = \frac{1}{e^t}$.
(ii) Determine the Taylor series of $f(t) = a^t$ around a.) $t = 0$, b.) $t = t_0$. Determine the radii of convergence in either cases.
(iii) Determine (by using $\exp(t) = \cosh(t) + \sinh(t)$) the expressions $\cosh(t_1+t_2)$ and $\sinh(t_1+t_2)$ in terms of $\cosh(t_i)$ and $\sinh(t_i)$.
(iv) Do the same for $\sin(t_1+t_2)$ and $\cos(t_1+t_2)$

3.) Show that $\ln(1+x) = \lim_{n\to\infty} n[(1+x)^{\frac{1}{n}} - 1]$. Hint: Assume $|x| < 1$ and expand $(1+x)^{\frac{1}{n}}$ into a Taylor series.

4.) Expand $(1+x)^{-1}$ into a Mac-Laurin series, i.e. into a Taylor with respect to x. This series is also referred to as *Geometrical Series*.
(i) Determine the radius of convergence.
(ii) What is the relation of this series to the series for $\ln(1+x)$ (relation (48))?

5.) (i) Show that $\cosh^2(t) - \sinh^2(t) = 1$ and that $\cos^2(t) + \sin^2(t) = 1$
(ii) Give the Mac-Laurin series of $\cosh(t)$ and $\sinh(t)$, respectively.
(iii) Let $x_1(t) = \cosh(t)$, $x_2(t) = \sinh(t)$ and $D = \frac{d}{dt}$ is referred to as the *Differential Operator*. Determine
a.) $Dx_1(t)$, $Dx_1(t)$
b.) $D^2x_1(t)$, $D^2x_1(t)$ and give the differential equation which obeys an arbitrary linear combination $y = c_1x_1(t) + c_2x_2(t)$.
c.) Let $D = \frac{d}{dt}$ and $D^{-1}f(t) = \int^t f(t')\mathrm{d}t'$ denotes the antiderivative (integral) Operator (unique only up to a constant) and $e^{\lambda t} = \exp(\lambda t)$. Determine
(i) $D^n\exp(\lambda t)$, and the antiderivatives $D^{-n}\exp(\lambda t)$, $n = 1, 2, ...$
(ii) By using Taylor series and with the definition of unity operator $D^0 = 1$, we can define the operator $\exp(hD) = 1 + hD + \frac{(hD)^2}{2!} + .. + \frac{(hD)^k}{k!} + ..$ $(D = \frac{d}{dt})$ where we can calculate with D as with numbers, by keeping in mind that D is defined only by acting on a function $Df(t) = \frac{df(t)}{dt}$ (e.g. $Dt^n = nt^{n-1}$). Determine $\exp(hD)\exp(\lambda t)$.

6.) Show that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n} \to$ is divergent.

7.) Determine
(i) $\lim_{n\to\infty} n^{\frac{1}{n}}$; **(ii)** the radius of convergence ρ_c and evaluate S for $|x| < \rho_c$:

a.) $S = \sum_{n=2}^{\infty}\frac{x^n}{10^n}$;

b) $S = \sum_{n=m}^{\infty}\frac{x^n}{b^n}$, where $b = b_1 = ib_2$.

c.) $S = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ln^n(a)$

d.) $S = \sum_{n=1}^{\infty} \frac{x^n}{n}$

e.) $S = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

f.) $S = \sum_{n=1}^{\infty} (1 + \frac{1}{n^2})^n$

8.) Find a shorter expression for
$S = \lim_{n\to\infty} 1 + n + n(n-1) + n(n-1)(n-2) + .. + \frac{n!}{k!} + .. + n!$

9.) Determine $\lim_{x\to\infty} \tanh(x)$

10.) Express by hyperbolic functions: $\cos(ix)$, $\sin(ix)$
(i) By using the identities (60)
(ii) By using their Mac-Laurin series and determine the radius of convergence of the Mac-Laurin series of $\cos(z)$ and $\sin(z)$ $(z = x + iy)$.

11.) Express by circular (trigonometric functions) $\cosh(it)$, $\sinh(it)$.

12.) Show that $\begin{pmatrix} n \\ k \end{pmatrix} + \begin{pmatrix} n \\ k-1 \end{pmatrix} = \begin{pmatrix} n+1 \\ k \end{pmatrix}$.

13.) Evaluate the Vandermond determinant (354) of appendix 4.

14.) Evaluate (i) $S = 1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + .. + \frac{1}{3^{2n}} + ..$, (ii) $S = 1 + \frac{2}{3^2} + \frac{3}{3^4} + \frac{4}{3^6} + .. + \frac{(n+1)}{3^{2n}} + ..$

15.) Show if the *geometrical multiplicity* of each eigenvalue of $\mathbf{A}$ is one, then exists a transformation $\mathcal{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ where $\mathcal{D}$ is the diagonal form of the matrix $\mathbf{A}$.

16.) Express **(i)** $\cos m\varphi$ and $\sin m\varphi$, respectively by $\cos\varphi$ and $\sin\varphi$ by using the *Binomial theorem.***(ii)** Do so for the corresponding hyperbolic functions.

17.) Given the recurrence $X_{n+1} = aX_n + bY_n,\ Y_{n+1} = bX_n + aY_n$, with general *Initial conditions* X_0, Y_0. Determine X_n, Y_n in dependence of a, b, X_0, Y_0.

18.) Show that for a matrix $\mathbf{A} = \mathbf{V} \cdot \mathcal{D}\mathbf{V}^{-1}$ (where $\mathcal{D} = diag(\lambda_1, .., \lambda_n)$ is its diagonal form) the relation $\boldsymbol{Det}[\exp(\mathbf{A}t)] = \exp[t\boldsymbol{Tr}(\mathbf{A})]$ where $Tr(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$.

19.) Given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \tag{302}$$

Calculate
(i) The eigenvectors and eigenvalues of $\mathbf{A}$; (ii) the expression

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{303}$$

and determine the limiting case $S = \lim_{n\to\infty} \frac{Y_2}{Y_1}$.
(iii) Determine the ratio $\frac{X_1}{X_2}$ for which holds

$$\lim_{n\to\infty} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{304}$$

Hint. Decompose the start vector $\vec{X} = C_1\vec{v_1} + C_2\vec{v_2}$ into a linear combination of eigenvectors.
(iv) Determine $\exp(\mathbf{A}t)$ where t is a scalar (real) parameter, e.g. the time and calculate

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \exp\left\{ \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \right\} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{305}$$

and determine $\lim_{t\to\infty} \frac{Z_2}{Z_1}$.
(v) Determine the matrix $\mathbf{B} = \frac{d}{dt} \exp(\mathbf{A}t)_{t=0}$.
(vi) Determine $\mathbf{C} = \sqrt{\mathbf{A}}$
(vii) Expand the matrix $\frac{1}{\mathbf{I}+t\mathbf{A}}$ into a Taylor series with respect to t around $t = 0$ and give the radius of convergence.

Further questions to part I:
Determine the complex zeros and write them also in polar form $z = \rho\, e^{i\varphi}$, $\rho = |z|, \varphi = arg(z)$, $-\pi < arg(z) < +\pi$ and prove the results by factorizing the given polynomials.

$$\begin{array}{l} x^2 + 2x + 10 = 0 \\ x^3 + 8 = 0 \\ x^4 - \frac{1}{16} = 0 \\ x^3 + 2x^2 - 5x - 6 = 0 \\ x^2 + 2ix - 1 = 0 x^4 + 2x^2 + 2 = 0 \\ x^3 - 2x^2 - 11x + 12 = 0 \\ x^3 - 2x^2 - 11x + 12 = 0 \\ x^4 + x^2 - 2 = 0 \\ 2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0 \\ x^4 - 5x + 2 = 0 \\ x^4 - 5x^2 - 36 = 0 \end{array} \tag{306}$$

Find the coefficients A, B, C, D, E such that

$$x^4 + 2x^3 + 2x^2 + 5x + 1 = A(x-1)^4 + B(x-1)^3 + C(x-1)^2 + D(x-1) + E \tag{307}$$

Find a quadratic function $y = ax^2 + bx + c$ which contains the three points $(x_i, y_i))$
1) $(0,0), (1,1), (5,2)$
2) $(-1,1), (0,0), (1,1)$
3) $(1,2), (3,5), (5,-10)$

Determine the real- and imaginary parts of the given complex numbers by writing them in the form $z = x + iy$, $i = \sqrt{-1}$

$$\frac{1}{(1+i)}, \qquad \frac{1}{(1-i\sqrt{3})}, \qquad (1+i)^2, \qquad (1+i)^4, \qquad e^{3\pi i}, \qquad \frac{3+i}{5-i},$$

$$\frac{2+i}{(1-i)(2-i)}, \qquad \sqrt{x+iy}\,, \qquad \sqrt{(4+3i)(4-3i)}$$

Decompose into partial fractions

$$\frac{(x+3)}{(x+1)(x-2)}, \qquad \frac{x}{x^2-1}, \qquad \frac{x^2+2x+1}{(x+1)^4}, \qquad \frac{3x+4}{(x^2-6x+9)(x-1)},$$

$$\frac{1}{x^2+2x+10}, \qquad \frac{50+10x}{x^3-8}, \qquad \frac{1}{(x^4+16)}, \qquad \frac{(x+2)}{(x-1)(x+4i)},$$

Determine an antiderivative of $\int \frac{\mathrm{dx}}{x(x-1)(x-2)}$ by finding the partial fractions.

Determine $\lim_{n\to\infty}(x+iy)^{\frac{1}{n}}$, and $\lim_{n\to\infty}\tanh(nx)$, for (i) $x>0$ and (ii) $x<0$, respectively

Express

$\tanh(2\phi)$ by $\tanh(\phi)$
$\cosh(2\phi)$ by (i) $\cosh(\phi)$, (ii) $\sinh(\phi)$, respectively
$\sinh(2\phi)$ by (i) $\cosh(\phi)$, (ii) $\sinh(\phi)$, respectively
$\cos(2\varphi)$ by (i) $\cos(\varphi)$, (ii) $\sin(\varphi)$, respectively
$\sin(2\varphi)$ by (i) $\cos(\varphi)$, (ii) $\sin(\varphi)$, respectively

Express in circular and hyperbolic functions, respectively: $\cos it$, $\sin it$, $\cos x+iy$, $\sin x+iy$, $\cos it$, $\cosh it$, $\sinh it$, $\cosh x+iy$, $\sinh x+iy$, $\tan it$, $\tanh it$

Determine all n **complex roots** z_i of $z^n=1$, i.e. all n complex values of $1^{\frac{1}{n}}$ by taking into account that $arg(z)\in(-\pi,\pi]$.

Determine the real and imaginary parts, the moduli and the arguments $\in(-\pi,\pi]$ of $\tanh(x+iy)$, $\tan(x+iy)$.

Given the recurrence (i) $c_{n+1}=\sqrt{\frac{(1+c_n)}{2}}$, initial value $0<c_0<1$.
Determine c_n in dependence of φ_0 and determine the limiting case $\lim_{n\to\infty}c_n$
Hint: Since $0<c_0<1$ put $c_0=\cos\varphi_0$ and $\cos(\frac{\varphi}{2})$ by $\cos(\varphi)$.
Solve the recurrence problem for an initial value $c_0>1$

Express $\tanh(2\phi)$ by $\tanh(\phi)$ and use it to determine the recurrence

$$X_{n+1}=\frac{2X_n}{(1+X_n^2)} \tag{308}$$

for a given initial value $X_0=\tanh\Phi_0$, $\Phi_0>0$.
Determine $X_\infty=\lim_{n\to\infty}X_n$

Evaluate the recurrence $x_{n+1}=ax_n+b$ for the initial value x_0. (i) Determine x_n in dependence from x_0. (ii) Now let $x_0=1$ and $a=e^{\frac{\pi i}{5}}$. Determine the smallest value

of n_0 for which holds $x_n = x_0$ and show that n_0 is independent of b.

3.2 Questions to part II

20.) Determine the Laplace transforms of
a.) $f(t) = 3e^{5t} + te^{-2t} + t^2$ and give the abscissa of convergence σ_c.
b.) Determine the Laplace transforms of the matrix $\exp(\mathbf{A}t)$ of **19.)** (iv) and give the abscissa of convergence.
c.) Determine the time domain representations of the Laplace transforms $\frac{1}{(s+1)}, \frac{1}{(s-5+3i)^2}, \frac{s^2+2s}{(s^3-1)}, \frac{1}{(s^2+1)}$ by using partial fractions.

21.) Evaluate the *Fibonacci numbers* defined by the recurrence (148) for $x_0 = 0$ and $x_1 = 1$ and determine the ratio $r = \lim n \to \infty \frac{X_{n+1}}{X_n}$.

22.) Determine the ratio $r = \lim n \to \infty \frac{X_{n+1}}{X_n}$ of the Fibonacci numbers directly from $X_{n+1} = X_n + X_{n-1}$ for the same initial values $x_0 = 0$ and $x_1 = 1$ by using for large n: $X_{n+1} = rX_n$with $r > 0$.

23.)
a) Single degree of freedom analysis
We consider the deflection $X(t)$ of a jetty structure responding to a periodic water wave impact. The (Newtonian) equation of motion for the deflection $X(t)$ is given by

$$M\ddot{X} + C\dot{X} + KX = f(t) \tag{309}$$

where M denotes the effective mass, $C > 0$ the damping coefficient and K the effective elastic module and $f(t)$ is the exciting force due to the wave impact. We assume a periodically repeating wave impact with period T. Then $f(t)$ has the Fourier representation

$$f(t) = \sum_{m=-\infty}^{\infty} b_m e^{j\omega_m t} \tag{310}$$

where $j = \sqrt{(-1)}$ denotes here the imaginary unit and with the angular frequency $\omega_m = 2\pi f_m = \frac{2\pi m}{T}$.

(i) Show that the series (310) fulfills the periodicity condition $f(t_0) = f(t_0 + nT)$, $n = 0, \pm 1, \pm 2, ...$

(ii) The steady state solution $X(t)$ can be written in a Fourier series with the same frequencies as (310), namely

$$X(t) = \sum_{m=-\infty}^{\infty} a_m\, e^{j\omega_m t} \tag{311}$$

Determine the Fourier components a_m of (311) by using the equation of motion (309) together with (310) and determine the *transfer function*

$$H(\omega_m) = |\frac{a_m}{b_m}| \tag{312}$$

(iii) For which values of the frequency ω_m takes the transfer function a maximum?

(iv) Deliver a schematic sketch of the function $H(\omega_m)$

b) Response on a periodic square-pule wave impact.
We now consider a wave impact which we idealize with a step function profile $f(t) = (H(t) - H(t - \frac{T}{2}))f_0$ ($f_0 = const$) where $\Theta(\tau)$ denotes the *Heaviside unit step function* being defined as $\Theta(\tau) = 1$ for $\tau > 0$ and $\Theta(\tau) = 0$ for $\tau < 0$.

We assume that the wave impact $f(t) = (H(t) - H(t - \frac{T}{2}))f_0$ ($f_0 = const$) repeats periodically with period T. Then $f(t)$ has a Fourier series of the form (310).

(i) Determine the Fourier components b_m by using the relation

$$b_m = \frac{1}{T}\int_0^T e^{-j\omega_m t} f(t)\mathrm{d}t \tag{313}$$

where $\omega_m = \frac{2\pi jm}{T}$. Write the result for even and odd m and consider $m = 0$ separately. (Hint: $e^{\pm j\pi} = -1$).

(ii) Determine

$$S = \sum_{m=-\infty}^{\infty} |b_m|^2 \tag{314}$$

by using the *Parseval Identity*

$$\int_0^T f^2(t)\mathrm{d}t = T\sum_{m=-\infty}^{\infty} |b_m|^2 \tag{315}$$

24.)
a) Vibrating String
We consider a string of length L the vibrations $u(x,t)$ of a string which is constraints at its ends by $u(0,t) = u(L,t) = 0$. The governing (partial) differential equation (wave equation) is

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})u(x,t) = 0 \tag{316}$$

(i) Determine the displacement field $u(x,t)$ for $f = 0$ (Homogeneous solution) for the given initial conditions[24]:

$$\begin{array}{l} u(x,t=0) = g(x) \\ \frac{\partial}{\partial t}u(x,t=0) = v(x) \end{array} \tag{317}$$

for given (with the boundary conditions compatible[25]) functions $g(x)$ and $v(x)$.

25.) Demonstrate the equivalence of (300) and (301) when $\mathbf{W} = \mathbf{VDV}^{-1}$ with the diagonal form $\mathbf{D} = diag(\lambda_1, ..\lambda_N)$ and $\mathbf{V} = (\vec{v}_1, .., \vec{v}_n)$.

26.) Proof the equivalence of the solution methods of IVP by employing operators and Laplace transform by means of the inhomogeneous IVP

$$P(D)X(t) = b(t)\,, \qquad P(D) = \prod_j^n (D - \lambda_j) \tag{318}$$

[24] An IVP with initial conditions (317) is referred to as **Cauchy problem**.
[25] i.e. $g(0) = g(L) = v(0) = v(L) = 0$.

with $\lambda_j \neq \lambda_j$ for $i \neq j$ and the n initial conditions

$$D^k X(t=0) = A_k \tag{319}$$

Hint: Make use when applying Laplace transformation that the decomposition (266) is always such, that $X_h(t) = \tilde{X}(t)$ (eq. (231)) fulfills the initial conditions and $X_p(t) = \tilde{\tilde{X}}(t)$ (eq. (233)) fulfills zero initial conditions and show that the solution $X(t) = X_h(t) + X_p(t) = \tilde{X}(t) + \tilde{\tilde{X}}(t)$ (eq. (235)) again fulfills the initial conditions.

Some suggested solutions

23.)

a.) Single degree of freedom analysis:

(i) Periodicity

$$f(t_0+nT)=\sum_{m=-\infty}^{\infty} b_m\, e^{\frac{2\pi jm}{T}(t_0+nT)}=\sum_{m=-\infty}^{\infty} b_m\, e^{j\omega_m t_0}\underbrace{e^{2\pi jmn}}_{=1,\,as\;\; nm=\pm integer}=F(t) \tag{320}$$

(and in the same way for $X(t)$).

(ii) By using the ansatz (311) we find by inserting it into (309)

$$\sum_{m=-\infty}^{\infty}(-M\omega_m^2+jC\omega_m+K)a_m\, e^{j\omega_m t}=\sum_{m=-\infty}^{\infty} b_m\, e^{j\omega_m t} \tag{321}$$

which leads (by using the orthogonality of the $e^{j\omega_m t}$) to the equation

$$(-M\omega_m^2+jC\omega_m+K)a_m=b_m \tag{322}$$

or

$$a_m=\frac{b_m}{(K-M\omega_m^2+jC\omega_m)} \tag{323}$$

which is a complex function. Hence the transfer function is

$$H(\omega_m)=|\frac{a_m}{b_m}|=\frac{1}{|(K-M\omega_m^2+jC\omega_m)|}=\frac{1}{\sqrt{(K-M\omega_m^2)^2+C^2\omega_m^2)}} \tag{324}$$

(iii) $H(\omega_m)$ assumes maxima when $y(x=\omega^2)=(K-Mx)^2+C^2x)$ which are determined by $\frac{d}{dx}y(x=\omega_{max}^2)=0$

$$\frac{d}{dx}y=2M(Mx_{max}-K)+C^2 \tag{325}$$

putting this equation to zero yields

$$x_{max}=\frac{C^2-2KM}{2M^2}=\omega_{max}^2 \tag{326}$$

and hence $H(\omega)$ assumes maxima for the two frequencies

$$\omega_{max}=\pm\sqrt{\frac{C^2-2KM}{2M^2}} \tag{327}$$

b) **Response of a square pulse periodic wave impact**:

(i) The Fourier coefficient are determined by

$$b_m = \frac{1}{T}\int_0^T e^{-j\omega_m t} f(t)\mathrm{d}t = \frac{1}{T}\int_0^{\frac{T}{2}} e^{-j\frac{2\pi m t}{T}}\mathrm{d}t = \frac{1}{(-j\omega_m T)} e^{-j\frac{2\pi m t}{T}}|_0^{\frac{T}{2}}$$

$$b_m = \frac{j}{\omega_m T}(e^{j\pi m} - 1) = \frac{j}{2\pi m}[(-1)^m - 1],\ m \neq 0 \tag{328}$$

$$b_0 = \frac{1}{2},\ m = 0$$

that is $b_0 = \frac{1}{2}$, $b_m = 0$ for m even, and $b_m = \frac{-2j}{2\pi m} = \frac{-j}{\pi m}$ for m odd. Hence $f(t)$ assumes

$$f(t) = \frac{1}{2} - j\sum_{n=-\infty}^{\infty} \frac{1}{\pi(2n+1)}\, e^{\frac{2\pi j(2n+1)t}{T}} \tag{329}$$

which yields by summarizing the terms for $\pm|(2n+1)|$

$$f(t) = \frac{1}{2} - j\sum_{n=0}^{\infty} \frac{1}{\pi(2n+1)}\, (e^{\frac{2\pi j(2n+1)t}{T}} - e^{\frac{-2\pi j(2n+1)t}{T}}) \tag{330}$$

or more beautifully written

$$f(t) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} \underbrace{\frac{(e^{\frac{2\pi j(2n+1)t}{T}} - e^{\frac{-2\pi j(2n+1)t}{T}})}{2j}}_{=\sin(\frac{2\pi j(2n+1)t}{T}} \tag{331}$$

or

$$f(t) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} \sin(\frac{2\pi j(2n+1)t}{T}) \tag{332}$$

which must be a real function ($f(t) - \frac{1}{2}$ is an *odd function with respect to* t which is reflected by the fact that in (332) only sin-terms occur).

(ii) Parseval identity

$$\int_0^T f^2(t)\mathrm{d}t = \int_0^{\frac{T}{2}} \mathrm{d}t = \frac{T}{2} = T\sum_{m=-\infty}^{\infty} |b_m|^2 = TS \tag{333}$$

Thus

$$S = \sum |b_m|^2 = \frac{1}{2} = \frac{1}{4} + \sum_{n=0}^{\infty} \frac{2}{\pi^2(2n+1)^2} \tag{334}$$

which yields the remarkable relation

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \tag{335}$$

24.) Vibrating string. (i) The solution of the *homogeneous problem* can be written as

$$u(x,t) = \sum_{n=-\infty}^{\infty} v_n(t)\sin(k_n x) + w_n(t)\cos(k_n x) \tag{336}$$

then the boundary conditions $u(0,t) = u(L,t) = 0$ leads to $u(0,t) = 0$ thus for any component $w_n = 0$. Further $u(L,t) = 0$ leads to $\sin(k_n L) = 0$ thus $k_n L = n\pi$, $(n = \pm 1, \pm 2, ..)$ thus $k_n = \frac{n\pi}{L}$. The functions $v_n(t)$ we can now determine from the differential equation which needs to hold for any Fourier component

$$sin(k_n x)\frac{d^2}{dt^2}v_n(t) = c^2 v_n(t)\frac{d^2}{dx^2}\sin(k_n x) \tag{337}$$

or

$$\frac{d^2}{dt^2}v_n(t) = -c^2 k_n^2 v_n(t) \tag{338}$$

hence

$$v_n(t) = A_n \cos(ck_n t) + B_n \sin(ck_n t) \tag{339}$$

thus the homogeneous solution of (316) assumes

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \cos(ct\frac{n\pi}{L}) + B_n \sin(ct\frac{n\pi}{L}))\sin(\frac{n\pi}{L}x) \tag{340}$$

where A_n, B_n are arbitrary constants[26] which can be determined uniquely by prescribing the initial displacements $u(x,0) = g(x)$ and the initial velocities $\frac{\partial}{\partial t}u(x,t)|_{t=0} = v(x)$. The $\sin(\frac{n\pi}{L}x)$-terms are called eigen-modes or simply modes (in analogy to eigenvectors). The mode with $n = 1$ is called *fundamental mode* with *fundamental frequency* $\omega_0 = \frac{c\pi}{L}$. The other modes with $n > 1$ have integer multiple frequencies $\omega_n = n\omega_0$ of the fundamental frequency and their sum is called *overtone series*. The model of the vibrating string is the basis of the acoustics of string music instruments such as violins, pianos etc.

In (340) the constants A_n, B_n occur as Fourier coefficients of g and v and are to be determined from

$$u(x, t = 0) = \sum_{n=0}^{\infty} A_n \sin(\frac{n\pi}{L}x) = g(x) \tag{341}$$

and

$$\frac{\partial}{\partial t}u(x,t)|_{t=0} = \sum_{n=0}^{\infty} c\frac{n\pi}{L}B_n \sin(\frac{n\pi}{L}x) = v(x) \tag{342}$$

where $g(x), v(x)$ must be given compatible with the boundary conditions, namely $g(0) = g(L) = 0$ and $v(0) = v(L) = 0$, thus

$$A_n = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi}{L}x)\mathrm{d}x \tag{343}$$

and

$$B_n = \frac{2}{L}(c\frac{n\pi}{L})^{-1}\int_0^L v(x)\sin(\frac{n\pi}{L}x)\mathrm{d}x = \frac{2}{n\pi c}\int_0^L v(x)\sin(\frac{n\pi}{L}x)\mathrm{d}x \tag{344}$$

Thus we have with (340) together with (343), (344) solved the homogeneous initial value problem of a constraint string for the given initial conditions (initial displacements and velocities) $g(x)$ and $v(x)$. However to get the solution explicitly in terms of $g(x), v(x)$ we can make use of

$$\sin(\frac{n\pi}{L}ct)\sin(\frac{n\pi}{L}x) = \frac{1}{2}(\cos(\frac{n\pi}{L}(x - ct)) - \cos(\frac{n\pi}{L}(x + ct))) \tag{345}$$

$$\cos(\frac{n\pi}{L}ct)\sin(\frac{n\pi}{L}x) = \frac{1}{2}(\sin(\frac{n\pi}{L}(x - ct)) + \sin(\frac{n\pi}{L}(x + ct))) \tag{346}$$

[26] which need to fulfil $\sum_{n=0}^{\infty}|A_n|^2 + |B_n|^2 < \infty$.

Then we have for the first product sum in (340) with (341) with (346)

$$\sum_{n=1}^{\infty} A_n \cos(ct\frac{n\pi}{L}) \sin(\frac{n\pi}{L}x) = \frac{1}{2}(g(x-ct) + g(x+ct)) \tag{347}$$

and for the second product sum in (340) with (342) and (345)

$$\sum_{n=1}^{\infty} B_n \cos(ct\frac{n\pi}{L}) \sin(\frac{n\pi}{L}x) = \frac{1}{2}\sum_{n=1}^{\infty} B_n(\cos(\frac{n\pi}{L}(x-ct)) - \cos(\frac{n\pi}{L}(x+ct))) \tag{348}$$

the last equation can be re-written by integrating (342)

$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\mathrm{d}s = \frac{1}{2c}\sum_{n=0}^{\infty} c\frac{n\pi}{L}B_n \int_{x-ct}^{x+ct} \sin(\frac{n\pi}{L}s)\mathrm{d}s \tag{349}$$

With (347) and (349) we just have expressed the displacement $u(x,t)$ of (340) in terms of the given initial conditions $u(x,0) = g(x)$ and $\frac{\partial u}{\partial t}(x,0) = v(x)$, namely

$$u(x,t) = \frac{1}{2}(g(x-ct) + g(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\mathrm{d}s \tag{350}$$

Solution (350) is know in the literature as **D'Alembert solution**. It is easily verified that the D'Alembert solution indeed solve the wave equation and fulfills the initial conditions. The D'Alembert solution (350) applies not only for the vibrating string, *generally* solves **Cauchy problems** which are defined a homogeneous wave equation (316) together with the initial conditions (317).

4 Appendix

Here we give the proof for (108) which corresponds to the case that the set of n eigenvectors are linear independent if the eigenvalues are all different to each other: To this end we assume the set of eigenvalues $\{\vec{v_i}\}$ are linear dependent. Then the $n \times n$-equation system

$$\sum_{i=1}^{n} C_i \vec{v_i} = \mathbf{V} \cdot \vec{C} = 0 \tag{351}$$

would have nontrivial (non-zero) solution-vector $\vec{C} = (C_i)$. Successive Multiplication with matrix $\mathbf{A}, \mathbf{A}^2, ..\mathbf{A}^{n-1}$ yields[27], taking into account $\mathbf{A}^k \vec{v_i} = \lambda_i^k \vec{v_i}$

$$\mathbf{A}^k \mathbf{V} \cdot \vec{C} = \sum_{i=1}^{n} C_i \vec{v_i} \lambda_i^k = 0 \tag{352}$$

where we generate n such equation systems ($k = 0, .., n-1$). When we look at the p^{th} component of these n equation systems we obtain n equations of the form where $(\vec{v_i})_p$ denotes the p^{th} component of eigenvector $\vec{v_i}$

$$\sum_{i=1}^{n} C_i \vec{v}_{ip} \lambda_i^k = \mathbf{P} \vec{\tilde{C}} = 0 \tag{353}$$

with $\vec{\tilde{C}} = (C_i (v_i)_p)$

$$\mathbf{P} = \begin{pmatrix} 1, & .., & 1 \\ \lambda_1, & .., & \lambda_n \\ .. & & \\ \lambda_1^{n-1}, & .., & \lambda_n^{n-1} \end{pmatrix} \tag{354}$$

The determinant $\mathcal{V} = \boldsymbol{Det}\, \mathbf{P}$ is known as the **Vandermond Determinant** playing an important rule in many cases. In the questions (Question **13.)**) we show that the Vandermond determinant indeed is non-vanishing for $\lambda_i \neq \lambda_j$ $(i \neq j)$ and yields

$$\mathcal{V}(\lambda_1, \lambda_2, .., \lambda_n) =$$

$$\boldsymbol{Det}\, \mathbf{P} = \underbrace{(\lambda_2 - \lambda_1)..(\lambda_n - \lambda_1)(\lambda_3 - \lambda_2)..(\lambda_n - \lambda_2)..(\lambda_n - \lambda_{n-1})}_{\frac{n(n-1)}{2}\ factors} = \prod_{i<j} (\lambda_j - \lambda_i) \tag{355}$$

Because of $\mathcal{V}(\lambda_1, \lambda_2, .., \lambda_n) \neq 0$, i.e. the column vectors of $\mathbf{P}$ are **linear independent** (because the $\lambda_i \neq \lambda_j$ are all different) it follows that (352) has only the trivial solution vector $\vec{\tilde{C}} = 0$ and hence $\vec{C} = (C_i) = 0$ which means that (351) has only the trivial solution. Hence the set of n eigenvectors $\vec{v_1}, .., \vec{v_n}$ are indeed **linear independent**, which proves (108).

5 Literature

There is a vast amount of textbooks which cover the contents of this lecture. The two following books are a *suggested* as they cover most of the contents of the lecture. There are plenty of other standard textbooks which cover this materials as well. Further reading

[27] $\mathbf{A}^k = \underbrace{\mathbf{A} \cdot .. \cdot \mathbf{A}}_{k\ times}$

is explicitly encouraged (also in other than the below referenced textbooks). However, as the lecture does not strictly follow any textbook, these notes should be sufficient for the comprehensive understanding of the topics addressed in this lecture.
Part I:

[1] Glyn James (2004) Advanced Modern Engineering Mathematics, Pearson Education 2004 (Third Edition).
ISBN: 0 130 45425 7
[2] Glynn James (2004) Modern Engineering Mathematics, Pierson Education.
ISBN 0 130 45425 7 18319 9
Part II:

[2] Bender, C. M.; Orszag, S. A., Advanced Mathematical Methods for Scientists and Engineers, New York: Springer, 1999
ISBN 978-0-387-98931-0

6 Mock Exam

The following questions which comprise the complete module CIV6400 "Mathematics for Dynamicists" are similar to those of previous exams. Answering of these questions thus is highly recommended.

Question 1

$$(\frac{d^2}{dt^2} + 2a\frac{d}{dt} + b^2)X = f(t) \tag{356}$$

for $a, b \in R$, $b \neq 0$, $a > 0$.
1.) Give a fundamental solution system of the homogeneous problem (i.e. put $f(t) = 0$).

2.) Show by using the Wronskian determinant that the system of solutions of the homogeneous problem you gave in 1.) indeed is a fundamental system.

3.) Determine the general solution $X(t)$ for $f(t) = e^{\alpha t}$ for $\alpha^2 + 2a\alpha + b^2 \neq 0$.

4.) Solve the initial value problem

$$(\frac{d^2}{dt^2} + 2a\frac{d}{dt} + b^2)X = 2\cos t \tag{357}$$

with initial conditions
$X(t = 0) = A_0$,
$\frac{dX}{dt}(t = 0) = A_1$.

Hint: make use of $2\cos t = e^{it} + e^{-it}$, $i = \sqrt{-1}$ by using:
(i) Operator method
(ii) Laplace transformation

5.) Consider now (356) for $\alpha = i\omega$, i.e. a driving external force $f(t) = e^{i\omega t}$ where $\omega \in R$ is the frequency of this force. Determine for this case the so called enhancement factor V which is defined by

$$V =: \lim_{t\to\infty} \frac{|x_p(t)|}{|f(t)|} \tag{358}$$

where $x_p(t)$ denotes any particular solution of the non-homogeneous problem. For this limiting case which leads to the 'steady state' solution the fact that $a > 0$ (which guarantees exponential damping of the homogeneous solution) is important.

(i) Why is definition (358) unique, i.e. independent of given initial conditions?

(ii) For which values of the exciting frequency ω the enhancement factor V takes a maximum?

(iii) Determine the maximum value of V.

(iv) Sketch $V(\omega)$ vs ω schematically.

6.) Determine the so referred to as phase shift of the solution **3.)** which is defined by

$$\phi =: \lim_{t\to\infty}\{arg(X_p(t)) - arg(f(t))\} = \lim_{t\to\infty} arg[\frac{X_p(t)}{f(t)}] \tag{359}$$

Question 2
Given the recursion

$$X_{n+1} = aX_n + bY_n; \qquad Y_{n+1} = bX_n + aY_n; \quad n = 0, 1, 2, .. \tag{360}$$

initial values $X_0 = 2$, $Y_0 = 1$. Hint: It is convenient to consider the given recursion in matrix form, i.e. $\vec{x}_{n+1} = \mathbf{M}\vec{x}_n$ with start vector $\vec{x_0}$.
1.) Determine $\vec{x}_n = (X_n, Y_n)$ in dependence of $\vec{x_0} = (X_0, Y_0)$.
2.) Determine the limiting case $A = \lim_{n\to\infty}(\frac{X_n}{Y_n})$ for $b > 0$.

(i) Does the result depend on c for initial conditions $X_0 = c$, $Y_0 = c$?

3.) Consider now the modified recursion

$$X_{n+1} = \alpha(aX_n + bY_n); \qquad Y_{n+1} = \alpha(bX_n + aY_n); n = 0, 1, 2, .. \tag{361}$$

for $a > b > 0$.

(i) For which values $\alpha > 0$ exist the sums $S_x = \sum_{n=0}^{\infty} X_n$, $S_y = \sum_{n=0}^{\infty} Y_n$? Hint: Reduce the problem to the consideration of geometrical series of the form $\sum_{n=0}^{\infty}[\alpha(a \pm b)]^n$ and consider for what values of $\alpha > 0$ this series is convergent.

(ii) In the case of convergence, determine the values of these sums for the initial condition ($X_0 = 1$, $Y_0 = -1$).

Question 3
Given the recursion

$$X_{n+1} = X_n\rho\cos\varphi - Y_n\rho\sin\varphi; \quad Y_{n+1} = X_n\rho\sin\varphi + Y_n\rho\cos\varphi; n = 0, 1, 2, .. \tag{362}$$

with $\varphi \in R$ and $\rho \in R$ with initial conditions $X_0 = A$, $Y_0 = B$. For which ρ hold

(i) $\lim_{n\to\infty} X_n^2 + Y_n^2 = 0$?

(ii) $\lim_{n\to\infty} X_n^2 + Y_n^2 = A^2 + B^2$?

(iii) $\lim_{n\to\infty} X_n^2 + Y_n^2 = \infty$?

Question 4
Determine the solution for the initial value problem ($D = \frac{d}{dt}$)

$$P(D)X = e^{-3t}; \quad P(D) = D^2 + 6D + 9 \tag{363}$$

with initial conditions $X(t = 0) = 1$, $\frac{d}{dt}X(t = 0) = 0$ by using
(i) Operator method
(ii) Laplace transformation

Question 5

Given the differential equation $L[u] = f(t)$ where $L[u] = u'' - 6u' + 25u$.

a) Determine the general solution $u_h(t)$ of the homogeneous differential equation ($f(t) = 0$).

b) Determine that special solution (particular integral) $u_p(t)$ for $f(t) = e^{i\omega t}$ which is periodic.

c) Determine the frequency of resonance, i.e. the frequency ω_r for which $|u_p|$ is a maximum.

d) Determine the times $\tau_n > 0$ when the periodic solution $u_p(t)$ is purely real?

e) Determine the special solution of the non-homogeneous differential equation for $f(t) = e^{i\omega t}$ which fulfills the initial conditions
$u(t = 0) = A$
$u'(t = 0) = B$

Question 6

Given the differential equation $L[u] = f(t)$ where
$L[u] = u''' - 3u''\lambda + 3u'\lambda^2 - \lambda^3$.

a) Determine the special solution (complementary function) $u_h(t)$ of the homogeneous differential equation ($f(t) = 0$) which fulfills the conditions
$u(t = 0) = A$
$u'(t = 0) = 0$
$u''(t = 0) = 0$
b) For which value of A holds $u_h(1) = e$?

c) Determine a particular solution $u_p(t)$ for $f(t) = e^{bt}$
(i) for $b \neq \lambda$
(ii) for the case $b = \lambda$

d) Consider now the case $b = \lambda$, i.e. $f(t) = e^{\lambda t}$:
(i) Give the general solution $u(t) = u_h(t) + u_p(t)$.

Question 7

Given the set of functions: $X_1 = e^{-t}$, $X_2 = e^{t}$, $X_3 = e^{(2+i)t}$, $X_4 = e^{(2-i)t}$.
Determine the homogeneous differential equation $P(D)X(t) = 0$ from which this set of functions constitutes a fundamental system.