Second conference of the International Network for Didactic Research in University Mathematics

INDRUM 2018

April 5-7

KRISTIANSAND

University of Agder

PROCEEDINGS

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INDRUM

UNIVERSITY OF AGDER

MatRIC Centre for Research, Innovation and Coordination of Mathematics Teaching

PROCEEDINGS of INDRUM 2018
Second conference of the International Network for Didactic Research in University Mathematics

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Editorial

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The INDRUM 2018 conference was held in Kristiansand, 5-8 April 2018. The INDRUM conferences fall within the activities of INDRUM (International Network for Didactic Research in University Mathematics), which has been initiated by an international team of researchers in didactics of mathematics at university level. This network aims to contribute to the development of research in didactics of mathematics at all levels of tertiary education, with particular focus on support for young researchers in the field and for dialogue with mathematicians.

The idea for the network and biennial conferences was first discussed in Paris (France), November 2014 and then Oberwolfach (Germany), December 2014. Following these discussions, a scientific committee with 19 scholars from 12 countries was established. The decision for organising the first conference in Montpellier (France), 31 March - 2 April 2016, with Elena Nardi and Carl Winslow as chair and co-chair was taken during CERME 9 in Prague, in February 2015.

Following the success of INDRUM 2016, the decision for organising the second INDRUM conference in Kristiansand in April 2018 with Simon Goodchild as Chair of the local committee, was taken during INDRUM 2016 in Montpellier. The International Scientific Committee held a meeting in Dublin during CERME 10, and nominated the INDRUM 2018 International Programme Committee: Viviane Durand-Guerrier (Montpellier, France) Chair; Reinhard Hochmuth (Hannover, Germany) Co-chair ; Marianna Bosch (Barcelona, Spain); Simon Goodchild (Kristiansand, Norway) ; Thomas Hausberger (Montpellier, France) ; Ninni Marie Hogstad (Kristiansand, Norway) ; Elena Nardi (Norwich, United Kingdom) ; Chris Rasmussen (San Diego, United States) ; Carl Winsløw (Copenhagen, Denmark). The Local Organising Committee was composed of Simon Goodchild (Kristiansand, Norway) Chair; Lillian Egelandsaa (Kristiansand, Norway); Ninni Marie Hogstad (Kristiansand, Norway); Thomas Hausberger (Montpellier, France); Elisabeth Rasmussen (Kristiansand, Norway). As for INDRUM 2016, INDRUM 2018 was an ERME Topic Conference.

A total of 53 papers and 14 posters was accepted for presentation. The final number of papers and posters presented at the conference and included in these proceedings (51 full papers and 14 posters, with the latter represented in the Proceedings as two-page short papers) varied slightly as a small number of delegates withdrew submissions or cancelled attendance for personal reasons. Discussion of the accepted papers and posters was organised in six thematic working groups (TWG1-TWG6), based on a classification of contents. Two members of the INDRUM Scientific Committee were invited to lead each of the five TWGs:

TWG1: Calculus and Analysis (María Trigueiros, Fabrice Vandebrouck)

TWG2: Mathematics for engineers; Mathematical Modelling; Mathematics and other disciplines (Alejandro S. Gonzáles Martín, Ghislaine Gueudet)

TWG3: Number, Algebra, Logic (Faïza Chellougui, Viviane Durand-Guerrier)

TWG4: Students' practices (Chris Rasmussen, Elena Nardi)

TWG5: Teachers' practices (Marianna Bosch, Simon Goodchild)

TWG6: Transition to and across university (Thomas Hausberger, Reinhard Hochmuth)

The scientific programme comprised: A plenary talk by Duncan Lawson (United Kingdom): Lessons for mathematics higher education from 25 years of mathematics support; a presentation of posters and of thematic working groups; a plenary panel chaired by Carl Winsløw: Preparation and training of university mathematics teachers: Panelists: Rolf Biehler (Germany), Barbara Jaworski (United Kingdom), Frode Rønning (Norway), Megan Wawro (United States). The accepted papers were presented in two parallel sessions and discussed in four thematic working group (TWG) sessions. A report was prepared and presented in plenary on April 8th.

The conference was attended by a total of about 120 registered participants. In the light of the volume and quality of submissions, and substance of exchanges during the sessions, we are happy to conclude that the second INDRUM conference turned out as a further eminent success.

Papers appear in these Proceedings in a version chosen by the authors following the (optional) possibility to upload a final version of their paper soon after the conference.

Very special thanks are due to the organising committee, chaired by Simon Goodchild and co-chaired by Ninni Marie Hogstad, for their tireless work over many months towards this event. Ninni Marie was responsible for the website, and received continuous support from Thomas Hausberger. Administrative support was offered by Lillian Egelandsaa and Elisabeth Rasmussen. These colleagues worked unstintingly before, during and after the conference to ensure that every participant had a smooth, productive and enjoyable INDRUM experience. They have set the bar high for the conferences to follow and we are indebted to them all.

The organizers are grateful to MatRIC, Centre for Research, Innovation and Coordination of Mathematics Teaching for financial support covering the work of the local organizing committee and some other conference arrangements, also the University of Agder for technical and domestic services and conference accommodation.

INDRUM follow-up

Strengthening the Network through publications is an important goal of the INDRUM network. Apart the INDRUM conferences proceedings, two publications were planed after INDRUM 2016.

First, an International Journal for Research in Undergraduate Mathematics (IJRUME) Special Issue has been guest-edited by Elena Nardi and Carl Winsløw, with support from IJRUME Editor Chris Rasmussen and reviewers including members of the INDRUM2016 Scientific Committee, was published in time for the 2018 conference and participants were able to download a copy of the issue without charge.

Second, a book reporting from INDRUM 2016 and INDRUM 2018 will be published in the Routledge ERME Series. It will be based on the scientific work developed in the TWGs during both conferences. A TWGs session of INDRUM 2018 was devoted to provide input to the book, in addition to input provided during INDRUM2016. Carl Winslow, Viviane Durand-Guerrier, Elena Nardi and Reinhard Hochmuth will be the editors.

Third, during INDRUM 2018, 13 colleagues from 10 countries have accepted an invitation to be members of the INDRUM International Scientific Committee, that now comprises 31 colleagues form 15 countries.

Finally, we are very happy to be able to announce that the Faculty of Sciences of Bizerte (Tunisia) will host INDRUM 2020; 27-29 March 2020, with Faïza Chellougui as chair of the Local Organising Committee and Rahim Kouki as co-chair. The conference will be chaired by Thomas Hausberger and Marianna Bosch.

We now invite you to carry on reading this volume and we hope that the promise of its contents will encourage you to join, or continue to be part of, the ambitious, bold enterprise that is INDRUM!

Plenary talk

Lessons for mathematics higher education from 25 years of mathematics support

Duncan Lawson[1] and Tony Croft[2]

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INDRUM KEYNOTE PRESENTATION

The scale and scope of mathematics support within UK universities have grown significantly since the 1990s. Mathematics support has evolved from a 'cottage industry' initiated by enthusiasts into a main-line student support provision overseen by institutional senior managers. Over this 25+ year period, the importance of the mathematical sciences in other disciplines has similarly boomed. No longer is it just engineering and physics undergraduates who need to acquire highly developed mathematical skills. Today geographers, bioscientists, sociologists and political scientists (to name but a few) have to be more skilled than ever before with understanding mathematical and statistical models and methods, particularly if they are to be able to access the international research literature and compete in the international employment market. Just as in the 1980s and 1990s, the Engineering Council produced reports warning of 'the mathematics problem', so in the 2000s and 2010s, the British Council and Royal Society of Arts have done the same. This presentation will outline how mathematics support has developed throughout the UK to meet this increasing demand.

Whilst the initial impetus for mathematics support came from a desire to improve the mathematical learning of students from other disciplines, it is an indisputable fact that a significant proportion of the users of mathematics support has been, and remains, mathematics undergraduates. This gives us cause to reflect: why is mathematics support so attractive to mathematics undergraduates? To answer this question, we explore the views of mathematics undergraduate students as expressed through the National Student Survey and in focus groups and individual interviews. The views the students express shed light on the reasons why many of them find mathematics support to be an attractive resource to support their learning.

Panel

Education and professional development of University Mathematics Teachers

Panellists: Rolf Biehler (Paderborn University, Germany)

Barbara Jaworski (Loughborough University, United Kingdom)

Frode Rønning (Norwegian University of Science and Technology, Norway)

Megan Wawro (Virginia Tech, United States)

Chair: Carl Winsløw (University of Copenhagen, Denmark)

ABSTRACT

The theme of this panel may surprise some, as university teachers of mathematics typically hold a PhD in mathematics or some adjacent field, and in many places some "pedagogical training" is also foreseen. However, university teaching presents still more challenges (in many places: more inhomogeneous or different student groups to teach), and opportunities (including new technology, and – we hope – useful resources from research on UME). For all of these reasons, the panel will address the following questions:

1. What is the current, typical preparation of University Mathematics Teachers for their function as teachers? What "in-service" opportunities for teacher development exist? - naturally, answers will depend both on countries and institutions, but sharing experiences could help to provide an updated picture of how the "professional knowledge of UME teachers" is currently built and sustained.
2. Do the current preparation and opportunities for development meet the demands that exist or can be foreseen? Could the preparation and development opportunities be improved, for instance by giving university teachers (more) access to selected parts of current research on UME, and possibly also participate in research and development projects? What initiatives exist, and which could be imagined as beneficial – both to increase the impact and quality of research on UME, and of UME itself?

TWG 1: Calculus and Analysis

Are all denumerable sets of numbers order-isomorphic?

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In this paper we study cognitive conflicts on the issue of number sets being dense, ordered and denumerable. We first provide historical-epistemological background related to these notions. Then we consider the cognitive conflicts under the lenses of concept image and concept definition, which we use to analyse empirical data collected in order to understand better the didactical and cognitive issues at stake.

Keywords: density, ordered set, denumerable set, concept image versus definition.

INTRODUCTION

Our interest in the title question comes from our teaching experiences in the first year of the Master degree in Mathematics course in Italy for the first author, and in first-year university courses in France for the second author. In both cases, the focus was on the distinction between density and continuity for an ordered set of numbers. Both authors were surprised by the following students' questions concerning the denumerable sets N and Q.

> Q1. How is it possible to find an order in Q if Q is dense, i.e. when the consecutive number of a rational does not exist?
>
> Q2. How is it possible that there is a bijection between N and Q, but N is discrete and Q is dense?

The students' questions highlighted potential conflicts which emerge when making explicit the properties of density[1] and denumerability of the set of rational numbers at the same time. This motivates a research investigation into the didactical transposition of the objects that underlie the two properties, namely order on a set, properties of both discrete[2] and denumerable sets, bijection, ordered isomorphism, difference between cardinal and ordinal numbers, and enumeration[3]. Our general research question is: how should we deal with these questions in classroom activities in order to help students overcome the apparent contradictions? In this paper, we will focus on a less ambitious sub-question concerning the first question posed by students (Q1).

[1] From the point of view of order, an order dense set is a linearly ordered set (X,<) with the property that if x<y then there exists z∈X with x<z<y (Jech, 2003). Here with the term "density" we refer to order density.

[2] A set S is discrete in a topological space X if every point x∈X has a neighbourhood U such that $S \cap U = \{x\}$ (points are said to be isolated) (Krantz, 1999, p.63).

[3] An enumeration is a complete, ordered listing of all the items of a set. An enumeration for an infinite set is a one-to-one correspondence between this set and the set of positive integers.

RQ: What are university students' and teachers' concept images and concept definitions of *dense, ordered* and *denumerable* set? How do they connect them?

We analysed students' and teachers' answers from two perspectives: i. a historical-epistemological analysis of the topics which emerged in the students' questions: ii. *concept image* and *concept definition* (Tall & Vinner, 1981). The first point is addressed in the first section: we refer to historical works in which infinite sets were studied from the point of view of cardinality, ordering and enumeration (Galilei, 1638; Lolli, 2013; Peano, 1889). The second point is addressed in the second section; we use our historical-epistemological analysis, together with results in mathematics education (Tirosh & Tsamir, 1996; Bergé, 2010; Durand-Guerrier, 2016; Branchetti, 2016), as resources to identify *a priori* possible students' concept images that could conflict with each other. In the third section, we describe the contexts and methodology of data collection and analysis, carried out in parallel in France and Italy which involved university teachers and Master degree students. Finally, we provide a brief overview of the way the concepts are introduced in scholastic stages prior to university studies in both countries, as a relevant background for our conclusions and starting point for further developments.

HISTORICAL EPISTEMOLOGICAL ISSUES

Galileo's view on numbers and their squares: an issue about cardinality

In one of the most famous books by Galileo Galilei (1564 - 1642), *Dialogues concerning Two New Sciences* (1638), the Italian physician, mathematician and philosopher introduced the one-to-one correspondence between natural numbers and their squares. We report here a brief summary of the main ideas. The dialogue concerns the difficulty that appears when trying to compare the number of points contained in two segments, one being longer than the other. Salviati, the voice of Galilei, states; "This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited" (English translation, 1914, p.31). He then moves to numbers and develops an argument on the impossibility of comparing the totality of all numbers with the numbers of squares since they are both infinite:

> "neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and, finally, the attributes "equal," "greater," and "less" are not applicable to infinite, but only to finite, quantities." (p. 32-33)

Density of *Q*, cardinal and ordinal numbers: Cantor's contribution

Cantor (1845 - 1918), working on trigonometric series and their convergence, moved on to the creation of a new theory of transfinite numbers, and the perspectives of Number Theory and Set Theory. Cantor came firstly to the definitions of a derived set – the set of limit points – and of a dense set and then of a dense-in-itself set, like the rational numbers set (Lolli, 2013). Studying infinite sets, he introduced the diagonal argument to prove that not only do the squares have the same cardinality as natural numbers, but so also does the Cartesian product of the set of natural numbers by itself, and hence the set of rational numbers, thanks to the existence of a surjection from the Cartesian product to this latter set. In other words, he had to find a way to enumerate the ordered pairs of natural numbers, being sure to consider every pair once and once only, following an ordering principle. To do this, he moved from the usual linear image of order to a 2-dimensional image. A crucial distinction, introduced by Cantor when he was facing such problems, is between cardinal numbers and ordinal numbers (Lolli, 2013). While in the set of natural numbers with its standard structure (formalized by Peano in 1889) the relation of order is strictly connected to the problem of ordering and with the induction principle, this is no longer the case in *Q*. Indeed, the standard order on *Q* (i.e. that consistent with measurement of magnitudes onto the line) is not consistent with Cantor's diagonal ordering principle, i.e. in the resulting order, $\frac{3}{4}$ is listed before $\frac{5}{4}$, but after $\frac{3}{2}$.

Peano's formalization of Arithmetic and the issue of order in natural numbers

In Peano's Arithmetic, the ideas developed by Cantor were formalized and used as principles to grasp the "essence" of natural numbers: the injective function that establishes Cantor's "first generation principle" (Lolli, 2013) for the consecutive element of a natural number is strictly linked to the operation of addition (the consecutive of *n*, being S(*n*), is equal to *n+1*) and to the comparison between natural numbers, if we consider the standard order. In this structure, the consecutive element is always greater than its precedent in respect to the standard order.

RESEARCH FRAMEWORK

According to Tall & Vinner (1981) every concept, from a cognitive point of view, is associated to different *concept images*. A relevant cognitive feature of conceptualization concerns the introduction of formal definitions: it often happens that the *concept definition* is not introduced appropriately by teachers in relation to the concept images. According to the authors, "a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure" (p. 3). Some students' concept images may be recognized as conflicting and inconsistent from an expert point of view, but they can coexist in their mind until a conflict is shown evoking them together simultaneously (p. 2). Such *cognitive conflicts* are occasions for learning and advancing in the process of conceptualization, but if not recognized and suitably overcome, they can become obstacles in the learning processes. We hypothesised that the students who asked our two questions (Q1 and Q2) were facing cognitive conflicts and trying to manage the following apparent conflicting images of

Q: i. a dense set; ii. an ordered set; iii. a set with the same cardinality as a discrete set, like *N*.

A priori identification of concept images

The concept definitions relevant to our topic are the following: *linear ordering*; *enumeration*; *dense* set; *discrete* set; *denumerable* set; *cardinality*; *bijection*; *isomorphism*. Relying on literature review and historical-epistemological analysis, we identified possible features of *concept images* that may cause cognitive conflicts:

CI1) ***Consecutive element is greater***: generalizing an association that is typical of N, reinforced by the spatial image of the oriented line (a greater element is on the right as the consecutive element). Students may think that a consecutive element in a list must be greater than the previous one (considering the standard order).

CI2) ***A dense set cannot be enumerated***: Students might have a concept image of ordered sets as sets in which the elements are "one after the other" on the line: an enumeration must move from left to right consistently with the standard order.

CI3) ***Dense not discrete***: *Q* might be said to be dense as opposed to *N*, which is not. The difference is "shown" either on the line or with numerical examples as an absolute difference (to have elements in "between" or not), independent of the particular order. Density may thus be considered an absolute property of a set and the "visual contrary" of discrete.

CI4) ***Linear or bidimensional representation***: the cardinalities of *Q* and *N* are shown to be equal, "re-ordering" *Q* and constructing a bijection between the two sets. The ordering procedure is usually represented in two dimensions (the "dovetail" counting method) while the standard order is represented using the line.

CI5) ***Bijection is identification***: students may associate the term "bijection" with a total identification between the structures (A=B), not just in terms of cardinality.

CI6) ***Finite versus infinite***: we represent indeed finite quantities of corresponding integer and rational numbers. This may lead the students to implicitly compare images of finite subsets of *N* and *Q* (maybe in the same graphic representation of intervals), concluding that *N* and *Q* cannot be composed by the same quantity of elements, since "rationals are more than integers" (see Galilei, 1638; finite reasoning applied to infinite sets, Tirosh & Tsamir, 1996).

METHODOLOGY AND PRELIMINARY DATA ANALYSIS

To answer our research question, we collected the following data: i. the Italian student's explanation of her doubt in written form; ii. answers to a similar questionnaire by an Italian university teacher and Master degree students in France preparing the selection procedure exam to become mathematics secondary teachers. The questionnaire was written in English and then translated into Italian and French; it is based on the historical-epistemological analysis and on the hypothesized student concept images. We created different versions of the same questionnaire, modifying them according to the two different countries' curricula for high school and university syllabi, and to the target of the questionnaire (university teachers or Master students).

At the beginning of both questionnaires, we present a realistic didactical situation, and we pose appropriate questions (see below).

We analyse one excerpt from a written interview to the Italian student, then we comment on excerpts of the answers from an Italian university teacher. Finally, we provide a brief summary of the answers to a version of the questionnaire submitted to Master students in France. To carry out the analysis of the students' interviews, consistently with our research question, we searched for their concept images and concept definitions before comparing the two source groups for connections and potential conflicts: in the case of the students, we consider personal concept images, while in the case of the teacher we look for examples that can reveal the hypothesized student concept images and concept definitions.

Analysis of the first student's comment (Italy)

The student who asked the first question explained her doubt as follows:

> "The doubt arose when [1] looking at the schemata that is used to [2] find a bijection between *N* and *Q*. [image] *N* is [3] ordered by definition, because, starting from 0, every element has a consecutive element, while [4] *Q* is not, since it is dense. But what prevents me from saying that 3/2 is the consecutive of 4/1? According to Peano's axiomatization, *N* is an [5] abstract structure that we can apply to natural numbers but also to *Q*, thanks to the bijection. I could thus say: [6] there are infinite rational numbers between 1/3 and ½, so there are infinite elements between the natural numbers associated with 1/3 and ½, using the [7] bijection in the reverse way. Reflecting more deeply, I realize that the problem is caused by the fact that [8] the bijection between *N* and *Q* is not "ordered" like that between *N* and *P* (pair numbers): $2 < 3$, $2/1 > ½$. I still do not understand why *Q* is dense and *N* is not, since we said that Peano's axioms can be used for several models and not just the natural numbers we already knew, but [9] if I enumerate *Q* with the natural numbers, it no longer has any sense to say < or > in *Q*."

The student was reasoning according to images (1,2) rather than definition and in mentioning the definition of order (3) she said: "every element has a consecutive element", revealing how she is not really using a formal definition but an image of ordering where each one is set after the other [CI2]; she was surprised that the consecutive could not be greater [CI1], so much so as to claim that in *Q* the meaning of < and > disappears. Also, she uses a representation of the bijection that she herself mentions as an identification (5, 7), reasoning on the schemata; indeed, she said that we can use it in both directions, identifying completely a couple of elements, one in the first and one in the second set [CI5]. She also used the image of "infinite elements in between" to say that *N* must be dense since we can image infinite elements between the natural numbers associated to a couple of rational numbers [CI3]. The student tried to connect different concept images and conflicts which emerged. The conflict that is expressed in the question, as she said, is caused by the use of images according to which [CI2] natural numbers and rational numbers are completely identified by means of a bijection, resulting in contradictions. It is impossible for the following to be simultaneously true: 1. *Q* is dense and *N* is not dense and *N* and *Q* are "the same structure"; 2. in the identification, *Q* loses the property that the consecutive element is greater, while *N* conserves this same property.

Analysis of one university teacher's questionnaire (Italy)

For this instance of the questionnaire, we provided the following realistic situation: during the lesson, a teacher is interrupted by a student asking the first question. The university teachers were explained what students are expected to have been taught before on *density*, *infinite cardinality* and the *problem of "consecutive numbers" in Q*. We asked the teachers to interpret the students' doubts and to propose how to deal with said situation in the classroom. We report and comment some excerpts from the answers to the questions (1 refers to Q1, and 2 refers to Q2):

1a. How would you answer the questions? How would you explain it to the whole class?

I. "The [1] order is not linked to the state of consecutive-ness: when she speaks of consecutive numbers we are in the domain of the [2] induction principle, which is only valid in *N*. In *N* we have much more: it is [3] well-ordered so there are no lower-unlimited subsets. Once the properties of N had been observed, i.e. the [4] "smallness" of the set that must satisfy all these features, I would move to the [5] differences between *N* and *Q*. Finally, I would observe that everything results from the fact that we are able to say that [6] one number is greater than another and this definition is also valid in *N*, as *N* is a subset of *Q*. I would say that [7] we cannot compare two properties linked to different definitions. Also, I would say that [8] in *N* we can give a definition of order that is linked to the induction principle but NOT generalized to bigger sets, but I would avoid going indepth into this issue, so as not to create confusion for the weaker students. Here, [9] the problem is that the sets are infinite".

II. "To be [10] dense is very different from being denumerable, I would [11] remind them of the definitions."

2. *Do you find a possible connection between the two questions, whereby these chance episodes could be used to help deal with some important topics in mathematics? What further examples or explanations would you propose to the student or what activities would you design in order to deal with such a question?*

"I would [12] show that {1/n} becomes more and more dense as it approaches 0. Then I would point out that the student should not be surprised when one proves that there is a [13] bijection between pair numbers and N, as well as with odds, the multiples of 3, the prime numbers, the negative integers. This is to [14] see (and prove, showing the application) examples of sets that are in a bijection with *N*."

3. Would you take the opportunity to explain something to students that could help them in trying to answer these kinds of questions on their own in the future?

"What we learn from these questions is that science proceeds with [15] analogies and differences, that we [16] must always consider the definitions, and that it's important that these are very precise."

On several occasions the teacher mentions definitions (6, 7, 8, 10, 11, 16), with different goals: we can't compare properties linked to different definitions; to be dense and to be denumerable are not linked because they concern different definitions (6, 7, 8, 10); we must use the definitions (11, 16).

When the teacher proposes examples, though, he uses words like show and see, and he uses images that may cause conflicts: he refers to "greater" and "smaller" related to infinite sets [CI5] and uses cardinality to compare *N* and *Q*. He mentions the bijection between *N* and its subsets and *Q* by showing the application, identifying it thus without stressing the issue of ordering [CI5]. In one case [CI2], the ordering is consistent with the linear order and in the other not, but it is not stressed. He shows that {1/n} becomes increasingly denser near 0, encouraging the use of images of density [CI3, CI5]. If we think about the student's comments, these answers would not have clarified the point she was "struggling" with: what she thought of as identification did not identify *N* and *Q* exactly. He mentions the definitions but, in the examples, he uses images and never seems to connect the definition to the images.

Preliminary analysis of the data collected in France

In France, an adapted version of the questionnaire was submitted to 30 first-year Master's students on October 12th, 2017, in the first half-hour of a teaching session on didactic and epistemology of mathematics. No epistemological or didactic work on this topic had been done before with these students. The answers were then discussed later in the fall as a starting point in a session devoted to epistemological and didactical aspects of numbers construction. For both Q1 and Q2, we asked students (in French): 1. *Have you ever asked yourself this question? If so, in what context and how did you answer this question yourself? 2. Imagine that a student of a lyceum or of a preparatory class for the "grandes écoles" is asking you this question. How would you answer?* Finally, the last question was: *Do you find a possible connection between the two questions, whereby these chance episodes could be used to help deal with some important topics in mathematics?* The questionnaire was anonymous. The students were asked to indicate their previous university studies.

To the first question, 6 students answered "yes" and commented on their answer; 8 answered "no" and commented on their answers; 16 answered "no" without comments. Some Master's students claimed that the notion of density was still unclear for them. In answering the second question, some of these students proposed an incorrect explanation for hypothetical younger students, which relied on *concept images* of successor without any reference to definition:

> M1-18 "the successor of an integer exists, so in *Q* there are elements having successors. Thus, we can say that the successor of a rational does not 'always' exist rather than 'does not exist'".

Several students explained that thanks to the definition of density, it is possible to define an order, while in order to define density-in-itself, it is necessary to already have an order, as in the example below:

> M1-15 - in first year university when the set theory was introduced - I told myself that in a dense set like *Q*, the notion of successor as may be imagined on *N* [1] does not exist, but thanks to the definition of density [2] of a set, it was possible to define an ordering [3] of this set and consequently order this set [4]

The student begins with reference to *an image* [1], then refers to *the definition* [2], and concludes with the possibility *to define an ordering* [3], which is an inversion of the definitions between density and order. It is noticeable that the student makes a distinction between "ordering" and "order", while there is no reference to the already-known order of *Q*. Among the 30 answers, only one student relies on the existence of the standard order in *Q* to justify that it is possible to define an order on *Q*. This brief summary of the students' answers accounts for the weakness of their knowledge of the concept of density, and of its link with the concept of order.

Insights into the Italian Curricula and traditional didactical practices

In the Italian high school curricula from grade 9 to 13, order and density are never mentioned explicitly; teachers are, however, advised to introduce the concept of infinite, showing the connection between mathematics and philosophy, in grades 11 or 12 while introducing transcendental numbers. Natural, integer and rational numbers are mentioned, but attention is focused on computation techniques, representations of numbers (fractions, decimal numbers, points of a line) and approximation. In the curriculum for primary school, both "sequential" and "cardinal" sense of numbers appears. In middle and high school, students are taught that between two rational numbers you can always find a rational number, and that it follows that a rational number has no consecutive element. What is generally not made explicit by high school teachers is that this is not an absolute property of *Q*, but depends on the order chosen in *Q*. Also, numbers are usually represented in Italy on a number line, so the discrete and the dense are distinguished using more visual than theoretical considerations (the existence or not of "something in between"). Discrete is often counter-posed to continuous. At the end of high school and/or in the first year of university, the concept of accumulation point is introduced for dealing with limits and discontinuity of real functions with real variables. The existence of an infinite quantity of real numbers "between" two real numbers is said to be due to the density property of *R*. This kind of practice is likely to reinforce the concept images we identified and to reinforce the habit of reasoning in absolute terms while referring implicitly to the standard order, without stressing the dependence of the properties on the choice of order relation.

The French context

In France, the situation is not very different. Durand-Guerrier (2016, p.341) presented it briefly, and provides evidence of the weakness of fresh university students' knowledge about numbers, which could be related to the curriculum. Briefly, high school students deal with approximations, mainly with the use of calculators. In grade 12, they learn the *mean value theorem for derivatives* without a proof, and without discussion about the fact that this theorem holds in the set of real numbers but no longer applies in the sets of decimal or rational numbers. Consequently, students beginning university generally have no idea of the differences and interplay between finite decimal numbers, rational numbers and non-terminating decimal expansions, and thus are not prepared for what they will be taught at university. Indeed, in many French universities, in first-year mathematical courses, an axiomatic definition of the set of real numbers is given, most often via "the supremum property", without any explicit construction. In some cases, the representation of real numbers as non-terminating decimal expansions and the corresponding characterization of the type of numbers are introduced, and improper expansions such as $0.\underline{9}$ are discussed with students (Durand-Guerrier, 2016, p.341). A topological course is generally offered, but it is mainly theoretical, and students have very few opportunities to connect the theoretical concepts with their interpretation in the ordered field of real numbers.

CONCLUSIONS AND DEVELOPMENTS

A first relevant result is that the framework is suitable to interpret our data. Our epistemological investigation and empirical data analysis do indeed help to formulate interpretations of the conflicts which appeared in the first question, as confirmed by the Italian student's interview analysis. Also, we observed a total identification between structures due to the constructions of bijections between elements of the sets, which is implicitly present in the high school practices. University teachers mentioned merely definitions, but, as Tall & Vinner (1981) showed in the case of discontinuity of functions, concept definition may be largely inactive in the cognitive structure and concept images may be used instead of the definition in order to grasp better its meaning. In this case, in the first question (that we analysed in depth here), the conflict emerged at the level of concept images, so definitions would not have been sufficient to solve the students' doubts. For the French Master students who answered our questionnaire, this lack of awareness of the links between the concepts of density-in-itself and order of Q might prevent them from designing appropriate learning situations, once they pass the selection procedure exam and become teachers. We hypothesise that, even if such questions emerge in university courses, the reasoning and consequent conflicts can be due to the lack of explicit reference to the dependence of Q properties on the order relation which still exists in the high school, a lack which calls for epistemological and didactic clarification in teacher training. As developments, we consider it crucial to identify the curricula issues where non-recognition could generate such conflicts, and to look for suitable teaching strategies in high school and university to deal appropriately with these concepts. “When the teacher is aware of the possible concept images, it may be possible to bring incorrect images to the surface and, by discussion, rationalise the problem” (Tall & Vinner, 1981, p. 17).

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An empirical study of the understanding of formal propositions about sequences, with a focus on infinite limits

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In this paper, we analyze the answers of one group of high-school students and two groups of first-year University students to a questionnaire designed to test their level of recognition and understanding of the formal definition of the concept of infinite limit. Although this empirical study is ancillary to a larger project centered on didactic engineering, its analysis sheds light on the key issue of the logical prerequisites for the learning of the fundamental concepts of analysis. It also provides a new tool to investigate students' concept-image of limits, and assess the impact of teaching contexts and teaching paths.

Keywords: Teaching and learning of analysis and calculus, teaching and learning of logic, reasoning and proof, definitions, limits.

CONTEXT AND RATIONALE

At the INDRUM 2016 conference, Cécile Ouvrier-Buffet and Renaud Chorlay presented a poster outlining a medium-scale project on definitions in analysis (Chorlay & Ouvrier-Buffet, 2016), with a focus on the formal definition of the limit of a numerical sequence. This topic lied at the intersection of the research interests of the two researchers: Cécile Ouvrier-Buffet is a maths-education researcher with a strong epistemological background, whose work bears mainly on *definitions*, their use, and the conditions for their genesis in teaching-contexts (Ouvrier-Buffet 2011). Since most of her former work bore on discrete mathematics, she wanted to investigate the extent to which the theoretical tools she had developed in this context had to be adapted to deal with a teaching context with very different mathematical (continuous *vs* discrete) and didactical (transition from *calculus* to *analysis*) features. Renaud Chorlay is a historian of mathematics and teacher educator with a long-standing interest in the history (Chorlay, 2011) and didactics of analysis.

We selected the topic of limits because we felt many years of didactical investigations had made it a *mature* topic; a topic about which knowledge has accumulated to form a sound and coherent body of knowledge. Indeed, we know a lot about limits in terms of conceptions and misconceptions (Robert, 1982); also in terms of obstacles (Sierpinska, 1985). As far as the genesis or rediscovery of the (or *a*) definition is concerned, many attempts have been made and reported upon in details, whether in the framework of didactic engineering[1] (Robert, 1983) (Bloch & Gibel, 2011) or with other research tool-boxes (Mamona-Downs, 2001) (Przenioslo, 2005) (Swinyard, 2011) (Lecorre, 2016)

(Roh & Lee, 2017). The tricky logical aspects were studied, in particular, in (Arsac & Durand-Guerrier, 2005).

On this solid basis, our work on the genesis and use of definitions has so far been engaged along three different lines of investigation; we will distinguish between *ex-ante* studies – *before* students' first encounter with formal definitions of limits – and *ex-post* studies.

- *Ex-ante* 1: For year 12 (final year of secondary education), the French curriculum requires that students majoring in mathematics and the sciences study a definition of limits (finite or infinite) of numerical sequences. Students are not really expected to use this definition on their own; rather, the teacher is expected to use these definitions on a few occasions, to show that some properties of limits can actually be proved mathematically (in particular: any unbounded and increasing sequence tends to $+\infty$). The underlying idea is that early encounter with a few rigorous definitions and proofs should ease the transition between high-school *calculus* – with its combination of algorithmic procedures and graphical intuition – and university *analysis*. This classroom work on the formal definition of limits is connected to another requirement of the current curriculum, namely that throughout high-school, the basic notions and the standard notations of mathematical logic be gradually made explicit. In this context, the discovery of a definition for limit, with its specific sequence of nested quantifiers, is supposed to be the culmination of this gradual process. In 2016, one of us (Chorlay) designed a teaching-session in the spirit of didactic engineering, for students to gradually formulate a formal definition of the infinite limit. We will report on this in detail in another context.
- *Ex-post* 1: in 2015-2016 we studied how – if at all – prospective maths-teachers made use of the definition of limits in order to identify and analyze vague, informal or erroneous statements regarding limits. We reported on this in a poster presented at the INDRUM 2016 conference.
- *Ex-post* 2: in 2016-2017 we designed a questionnaire in order to assess the level of recognition and understanding of the formal definition of the infinite limit. This questionnaire, and the answers collected with three groups of students are be the topic of this paper.

QUESTIONNAIRE - DATA COLLECTION

The questionnaire was of the True/False type, divided in two parts. We give below an English translation, along with indications on the correct answers.

Part I. For each one of the implications below, circle either "True" or "False". If you circle "False", justify your answer.

#1	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R}$ $\exists n_A \in \mathbf{N}$ such that $n_A \geq A$
T - F	Justification (if "False"):
#2	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R}$ $\forall n \in \mathbf{N}$ $u_n \geq A$
T - F	
#3	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R}$ $\exists n_A \in \mathbf{N}$ such that $n_A \geq A$
T - F	
#4	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R}$ $\forall n \in \mathbf{N}$, $u_n \geq A$
T - F	
#5	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R}$ $\exists n_A \in \mathbf{N}$ such that for any integer n greater than n_A $u_n \geq A$
T - F	

Correct answers:

#1 True: Here the consequent means "not bounded above".
#2 False: Here the consequent is a property which never holds; hence the implication is always invalid.
#3 True: Here the consequent is always valid, hence the implication is always valid.
#4 True: Here the consequent means "bounded below".
#5 True: Here the consequent is the definition, worded semi-formally.

Part II. The four implications below are taken from part I. For each one of them, first state its converse, then circle "True" or "False" regarding the converse. Justify if "False".

#1	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R}$ $\exists n_A \in \mathbf{N}$ such that $n_A \geq A$ Converse :
T - F	Justification (if "False"):
#3	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R}$ $\exists n_A \in \mathbf{N}$ such that $n_A \geq A$ Converse :
T - F	
#4	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R}$ $\forall n \in \mathbf{N}$, $u_n \geq A$ Converse :
T - F	
#5	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R}$ $\exists n_A \in \mathbf{N}$ such that for any integer n greater than n_A $u_n \geq A$ Converse:
T - F	

Conv. of #1 False: standard counter-examples are $(-2)^n$, $(-1)^n \times n$ …
C of #3 False: the antecedent being always true while the consequent can be false, the implication is invalid.
C of #4 False: being bounded below does not imply lim = +∞.

C of #5 True: definition.

The specific form of the questionnaire derives from its original intended use. It was first designed to assess the didactic engineering, which focused on the formal definition of the infinite limit. Other forms of assessment of the ability to recognize, and of the level of understanding of the formal definition were ruled out, in particular interviews (as in (Robert, 1982)) or proof-writing (as in (Roh & Lee, 2017)). We felt this questionnaire would give us feedback regarding two key features of the engineering, namely (1) the role of logic, hence the flood of formulae with nested quantifiers in this questionnaire; (2) the fact that "not bounded above" is a necessary condition for $\lim u_n = +\infty$ but not a sufficient condition, hence the importance of item #1 and its converse.

We did not ask for justifications when the item was deemed "True" by the students, mainly to save time and keep the questionnaire feasible in about 20 minutes. In addition, the justificatory task for True statements could vary a lot across teaching-contexts and would not easily lend itself to comparison. For instance, considering item #4 (if $\lim u_n = +\infty$ then the sequence is bounded below): in some contexts citing a theorem studied in class would suffice whereas in other contexts students would have to devise and write a non trivial proof. We also chose to drop the converse of item #2, since the fact that an implication whose antecedent is False is considered valid is a purely logical matter.

In the spring of 2017, the questionnaire was administered to three groups of students: Group 1 is one of the two French Year-12 classes which had experienced the engineering; Group 2 and 3 are first-year university students in Mons University (Belgium), with high-achieving maths majors in Group 2 and medium-achieving[2] computer science majors in Group 3. In all three cases, the questionnaire was given several months after the course on limits had been taught, and students had not been asked to revise anything in particular. They were told the questionnaire was given for research purposes, and would not be graded. They were given between 20 and 30 minutes. The number of students was: 31 (group 1), 50 (group 2), and 17 (group 3).

We originally hoped a comparison between the three groups would enable us to study the effects of three teaching units: our engineering (group 1), a "standard" maths-lecturer course (group 2), and Robert's engineering (group 3, as reported upon in (Bridoux, 2016)). Unfortunately, we were not able to do that, since other factors seemed to have had a more significant impact.

FINDINGS

Result #1

A first result is that this questionnaire is *not* unfeasible. In group 2, 14 of the 50 questionnaires were answered perfectly correctly, with relevant counter-examples for the False statements. Some of these counter-examples had been studied in class (such as $(-1)^n \times n$ for the converse of #1); in these cases, students managed to interpret "$\forall$ A $\in$ **R** $\exists$ n_A $\in$ **N** $u_{n_A} \geq$ A" as "not bounded above" and selected a relevant counter-example in a memorized repertoire. In other cases, counter-examples had not been studied in the course on limits – because they had nothing to do with limits – and students crafted *ad-hoc* counter-examples, displaying some command of logic (for instance, to prove that the negation of "$\forall$ A $\in$ **R** $\forall$ n_A $\in$ **N** $u_{n_A} \geq$ A" always holds).

Result #2

A second set of results sheds light on the role of an explicit teaching of logic. When we collected the data we first engaged in quantitative analysis, and were pretty unhappy about the following result: in group 1 (our engineering), only 26% of the students considered #4 to be "True", compared to 86% in group 2 and 71% in group 3. A closer look at the answers showed that in group 1, a significant number of students had actually engaged in another task than the prescribed task. In Fig. 1 and 2 we translated extracts of answer-sheets from group 1:

#2	If $\lim u_n = +\infty$ then $\forall$ A $\in$ **R** $\forall$ n $\in$ **N** $u_n \geq$ A
T - F	$\exists$ n $\in$ N such that u(n) $\geq$ A
#3	If $\lim u_n = +\infty$ then $\exists$ A $\in$ **R** $\exists$ n_A $\in$ **N** such that $u_{n_A} \geq$ A
T F	$\forall$ A $\in$ **R**
#4	If $\lim u_n = +\infty$ then $\exists$ A $\in$ **R** $\forall$ n $\in$ **N** , $u_n \geq$ A
T F	It's beyond some rank n

Figure 1. Student 29 of group 1

#1	If $\lim u_n = +\infty$ then $\forall$ A $\in$ **R** $\exists$ n_A $\in$ **N** such that $u_{n_A} \geq$ A
True False	Justification (if "False"): One forgot to specify $\forall$ $n \geq n_A$ and $\exists$ A This implication proves that there exists a term greater than $\forall$ A $\in$ **R**

Figure 2. Student 3 of group 1

In these answer-sheets, the students did not engage in an assessment of the logical implications but in a comparison between the formal statements given as consequents (in part I) and the definition of lim u_n = +∞. In these examples the comparison can be clumsy (as for #2 for student 29, or the "and $\exists$ A" for student 3). Nevertheless, it rests on the fact that the definition is known (correct answers for #5 and its converse), and is seen as the relevant template against which other quantified formulae ought to be contrasted. Moreover, the comparison is not purely syntactical: in her assessment of #4, student 29 did not only spot that "$\exists$

A $\in$ **R** $\forall$ $n \in$ **N** , $u_n \geq$ A" is *not* the definition, but also elicited in her own words *why* it could not be, namely "$\forall$ $n \in$ **N** $u_n \geq$ A" does not capture "beyond a certain rank", which is a key element of the definition. The reinterpretation of the prescribed task is typical of at least one third of the questionnaires from group 1.

By contrast, only one of the 67 students from Mons University reinterpreted the implication-assessment task as a comparison-with-the-definition task. A key difference between group 1, on the one hand, and groups 2 and 3, on the other hand, is that at Mons University students had studied logic in the first term, whereas the French high-school students had only occasionally been exposed to logic. The French students were familiar with the notion of converse, and had some knowledge of the meaning of quantifiers $\forall$ and $\exists$, but were not familiar with sequences of quantifiers; much less with the negation of such sequences. These formal aspects were not problematic for a large majority of the Mons students. This does not mean that all the logical aspects were mastered by the Mons students. In particular, when it came to proving that some formal statement was valid, many answer-sheets showed misconceptions regarding the use of $\forall$ and $\exists$.

This sheds some light on the standard but thorny issue of prerequisites: since the formal definition of limits involves a sequence of nested quantifiers, how much logic should be taught (either beforehand or along the way) for students to be able to *do* anything with it? Our results suggest that the answer depends on how "do" something with a definition is construed. *Using* the formal definition to design and write proofs probably requires some know-how regarding the interpretation of hitherto unknown sequences of quantifiers, and the negation of such sequences; for a significant proportion of the French student, their occasional and in-context encounters with logical notations did not allow them to acquire such know-how. However, if "do" is taken to mean "remember the definition" and even "understand the definition", then for a large majority of the French students, their command of logic was adequate. For instance, we consider the work of student 29 of group 1 (fig.1) to display some degree of conceptual understanding of definition, namely some understanding of the specific role of each of the three quantifiers. Student 3 is clearly able to interpret "$\exists$ A $\in$ **R** $\forall$ $n \in$ **N** $u_n \geq$ A". This understanding does not rest on a general ability to make sense of and formally manipulate logical formulae, but is limited to the context of the definition of limits. Since it relies on the specific connections between the concept-image and concept-definition of "limit" targeted (and, apparently, stabilized) in the didactically engineered teaching-session, this understanding is probably not only context-dependent but also path-dependent.

Result #3.

In the *a priori* analysis for the engineering, we studied the relations between three mathematical properties of numerical sequences:

(1) $\lim u_n = +\infty$;

(2) (u_n) is not bounded above;

(3) (u_n) is strictly increasing, at least from a certain rank onward.

Our hypothesis was that properties (2) and (3) were part of the concept-image of (1) for most students; of a concept-image[2] in which all three properties are considered to "go together", without any specific and explicit logical connections being part of the cognitive structure. This hypothesis was based on the didactical literature (Robert 1982) (Mamona-Downs 2001) (Swinyard 2011), and was perfectly confirmed during the two implementations of the engineering. For this reason, our design aimed for conceptual differentiation, to be achieved first through the study a few well-chosen sequences, and then through the formal explicitation of the logical connections between (1), (2), and (3). Consequently, we wanted our post-experiment questionnaire to help us assess to what extent students knew that (1) $\Rightarrow$ (2) is valid, while its converse is not.

Due to the significant level of reinterpretation of the prescribed task in group 1, the data gathered do not easily lend themselves to quantitative comparison. However, the fact that "not bounded above" (2) is a key component of the concept image of $\lim u_n = +\infty$ (1) is again confirmed beyond doubt. Let us first compare groups 2 and 3. In group 3 – the medium-achieving computer science majors – all 17 students deemed the converse of #1 to be True. Leaving out 3 students whose answer-sheets show an inadequate command of the logical aspects, it seems that Aline Robert's engineering (which targeted the definition of finite limits) had no impact on the belief that if a sequence (u_n) takes on arbitrarily large values, then $\lim u_n = +\infty$. In group 2, that of high-achieving maths majors, the results were not as striking; they were telling just as well. Among the 50 answer-sheets, let us focus on the subpopulation of those for which all the answers to part I were correct (including relevant counter-examples for #2), and all the converses were stated correctly. Among these 33 students, 17 deemed the converse of #1 to be False – which is the correct answer – and all but one provided a relevant counter-example (usually $(-1)^n \times n$ which – as the lecturer confirmed – had been studied in detail). Student 25 even wrote: "$\forall A \in \boldsymbol{R} \quad \exists n_A \in \boldsymbol{N}$ *such that* $u_{n_A} \geq A$ means that the sequence is not bounded above, but it doesn't mean it tends to +∞, it may oscillate. Let's consider $(-1)^n.n$ (…)". However, the other 16 students ticked "True" for the converse of #1. The resistance of this belief, even among students with a reasonable command of logic, who know the definition of $\lim u_n = +\infty$ (item #5 and its converse), and who had been exposed to a teaching which explicitly tackled this issue suggest that the conflation of (1) and (2) is an epistemological obstacle (Chorlay & de Hosson, 2016). It is probably not independent from the belief that all sequences are monotonous, at least after a certain rank (Robert 1982), but our questionnaire offers no new insight as to this.

This confirms – in hindsight – that we were justified to take (2) into account when designing a teaching-session on the formal definition of (1). However, it does not tell us whether targeting the formulation of the definition of (1) through a process fostering the conceptual differentiation between (1) and (2) was didactically relevant – as standard constructivist tenets suggest – or just foolhardy.

The results of group 1 allow us to be cautiously optimistic. From a purely quantitative viewpoint, 58% of the students of group 1 deemed the converse of #1 to be "False" – which is the correct answer – but no conclusions can be drawn from this fact beyond that this 58% stands in sharp contrast with the 0% of "False" on the subpopulation of OK-answer-sheets of group 3. In group 1, for instance, the third of the students who clearly reinterpreted the task as "compare with the definition" ticked "False", but this does not indicate that they are aware of the connections between properties (1) and (2), or that they were able to reformulate "$\forall\ A \in \mathbf{R} \quad \exists\ n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$" as "not bounded above". Answer-sheet 30 of group 1 shows, again, that some conceptual understanding can be achieved in a formal context in spite of a poor level of command of symbolic logic. This student systematically stated ¬B⇒A as converse of A⇒B; hence one has to study her assessment of the converse of #4 – instead of #1 – to see if she mistakes (2) for (1); which she does *not*, actually.
Of the 31 students of group 1, only two interpreted the task correctly *and* provided relevant correct answers for the converse of #1, either with a formulaic counter-example $(-5)^n$ or with a graphical counter-example (of the $y = x.\sin x$ type). However, about one fourth of the students deemed the converse of #1 to be false, interpreted the task as "assess the implications" and provided arguments which we could be indicative of some conceptual understanding. In these cases, they justified their assessment not by displaying a counter-example, but by explaining why the antecedent was not strong enough to warrant the consequent: under the hypothesis "$\forall\ A \in \mathbf{R} \quad \exists\ n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$", the sequence can oscillate; or: the antecedent does not imply that the sequence is increasing. Our empirical data does not enable us to tell which of the following is the case: either, students argue on the basis of the fact that if a sequence is increasing and not-bounded above then it tends toward $+\infty$ (a theorem they are familiar with); or, students conflate (1) and (3).

CONCLUSIONS AND RESEARCH PERSPECTIVES

While the questionnaire studied in this paper was originally designed to compare the effectiveness of three teaching-modules on the definition of limits of sequences, it turned out that they could not serve that purpose due to the decisive impact of another factor, namely the level of familiarity with predicate calculus – both in terms of syntactic command, and in terms of ability to make sense of logical formulae involving nested quantifiers. Nevertheless, we claim that meaningful conclusions or insights can be gained from the analysis of our empirical results.

For students with some command of logic – a command which cannot be gained through an occasional and in-context use of logical formalism – this questionnaire does provide insight into the connections between concept-image and concept-definition for limits, thus providing a new investigative tool to study this issue; a tool which does not involve conducting interviews or studying students ability to use the definition in proofs. As far as students are concerned, the comparison between group 2 and group 3 suggest that not all teachings on limits are equivalent in this respect; the case of group 2 shows that – under circumstances which call for further investigation – first-year university students can display a reasonable command of the concept of limit.

As far as group 1 is concerned, the result show that the prerequisites in logic may not be as high as one might expect, if what is targeted is the ability to memorize the formal definition, and the ability to display understanding of some key features of the concept. As far as our didactic engineering is concerned, these results show that (1) it was not a complete failure, (2) some of its guiding principles – such as the importance of the conceptual differentiation between infinite-limit and not-bounded-above, or the use of logical formalism – seem relevant. However, in this context, this questionnaire is probably not the best tool for a fine-grained assessment of what the actual impact of this engineering is.

1. For introduction to didactic engineering as task-design oriented research method, see (Bosch & Barquero 2015).

2. This assessment of the overall level of the groups is that of the team of maths lecturers at Mons University, as communicated to us by Stéphanie Bridoux, who is both a member of that team and a mathematics education researcher (LDAR). Many thanks to her for her collaboration on this project.

3. D. Tall and S. Vinner introduced the distinction between the image and the definition of a concept to stress the difference between mathematics as a mental activity and as a formal system. “We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. (…) it needs not be coherent (…).” (Quoted in (Tall 1991, 7)).

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Students’ understanding of ε-statements involving equality and limit

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The purpose of this paper is to explore how students may understand the link between the formalisations through ε-statements of infinite processes in the Archimedean continuum. These processes illustrate either equality or limit. In particular, we focus on the extent to which students perceive the formalisation of the infinite closeness notion in the two processes. The data is collected from an extensive design research carried out at the transition between Calculus course and Analysis course. TDS construct of milieu is deployed to build and to analyse exploratory teaching-experiments. The results put forward how ε-statements may assist students to reconsider their informal understanding of limit.

Keywords: ε-statement, process, limit, equality, milieu.

INTRODUCTION

In the transition between Calculus and Analysis courses, formal definition of the limit is needed not only to establish precise definitions of fundamental notions such as differential, integral, and series, but more importantly, to build up and use formal statements for making formal proofs. Yet, the key question of how to create a rigorous understanding of infinite processes and initiate the use of formal statements remains a challenging issue for researchers in the field of Calculus education.

Considerable research has been conducted on students’ difficulties to encapsulate the infinite processes of limit into the formal limit (Tall & Vinner, 1981; Przenioslo, 2004; Roh, 2008; Oehrtman, 2009). Most of this research highlights the impact of students’ previous use of informal statements to represent infinite processes both graphically and numerically. Those statements usually involve expressions related to successive computation of terms and closeness such as: *the more is x close to infinity, the more is f(x) close to l*, and inversely. Several other studies have explored the complex structure of the formal limit and have shown multiple aspects that may not help students develop efficient interpretations of formal statements (Cottrill et al., 1996; Durand-Guerrier & Arsac, 2005; Mamona-Downs, 2001; Roh, 2010; Oehrtman et al., 2014). Those aspects fundamentally refer to the role of quantifiers and their order, the arbitrariness of ε and its relation to the other parameter, and the connection between the statements expressing changes in the variables. Some other research have designed tasks to assist students connecting informal and formal statements related to limit (for an overview of the concerned literature, see Bressoud et al, 2016). Specifically, Swinyard (2011) has demonstrated that students are able to reinvent limit using formal statements. Drawing on this study and on the *genetic decomposition of limit* of Cottrill et al. (1996), Swinyard & Larsen (2012) develop a six steps model of how students come to understand the formal definition of limit. This model provides consistent arguments of how students reason about two infinite processes: 1) the process of *finding limit* which

is associated to the first three steps (*as x gets closer to a, f (x) gets closer to L)*, and 2) the process of *validating limit* which is described by the formal limit and encapsulated in the last step through the formalisation of the infinite closeness notion (*gets closer to*) via the concept of arbitrary closeness. Swinyard & Larsen hypothesized that limit at infinity may assist students to focus first on variation of the dependent variable and to shift to the validating process. In addition, the focus on the variation of a single variable may improve students' reasoning on the infinite closeness notion. Although we agree with those hypotheses, the empirical data does not outline how students connect informal statements of the first process to the formal statement of the second process. These processes encompass the cornerstone notion of infinite closeness, so why students do not feel the need to formalise the finding process and to emphasize its difference with the validating process and by the way, to understand why quantifiers should be described in such a way? However, Swinyard & Larsen call for research that could investigate students' formalisation of infinite processes in the context of whole-class teaching experiments and beyond the context of reinvention (p.492).
In this paper, we focus on the formalisation of two infinite processes in the Archimedean continuum by using formal statements that we call ε-statements. These processes involve the formalisation of infinite closeness notion and illustrate either equality or limit of function at infinity[1]. The research of Swinyard & Larsen has served to structure our thought and to rigorously address our central question: to what extent the formalizations of the infinite closeness involved in these two processes and their link may assist students' understanding of formal limit? The aim of this paper is to give some insights on the potency of this link in way that somewhat guarantee students' making sense of formal limit beyond restrictive contexts; this is why we deploy the Theory of Didactic Situations (TDS) constructs to conduct exploratory whole-class teaching experiments in the transition between Calculus and Analysis courses.

THEORETICAL FRAME

The TDS is a model of learning mathematical notions founded on an optimization of the interactions taking place within the system of relationships between students, a teacher, and a mathematical milieu which includes mathematical knowledge (preconstructed tasks, tools, graphs, symbols, etc.), students' prior knowledge, and students' informal understanding. The situation refers to the actual implementation in a classroom of this ideal model (noted Situation with capital S) in accordance with a targeted mathematical notion. The students' work and the teacher management are modelized at several levels according to the nature of the milieu. The expected interactions are materialized through the role of both students and teacher specifically in three particular levels: milieu for action, milieu for formulation and milieu for validation. The efficiency of the interactions among peers is ensured by teacher's enrichment of the mathematical milieu. Depending on the complexity of the targeted

[1] In this paper, the formal statement related to limit of function at infinity is: *The limit of a function f is L at infinity if for all $\varepsilon > 0$, there exists $A > 0$ such that for all $x > A$, $L - \varepsilon < f(x) < L + \varepsilon$.*

notion, the teacher may ask questions and provide some others auxiliary mathematical knowledge without minimizing students' responsibility in producing knowledge (this is the case for example of the formal limit). As mentioned by González-Martin et al. (2014): "*It is important to stress that the central object of TDS is not the **cognizing individual**, but the **Situation**, which shapes and constrain the adaptive processes students can develop, and thus the mathematical knowing which can be constructed.*" (p.118). However, the robustness of a Situation depends fundamentally on the mathematical milieu. The elaboration of this milieu is based upon a consistent epistemological analysis of the targeted notion; this analysis should allow students to experiment motivating questions and to reconsider their informal understanding – test and make conjectures, provide examples and non-examples, and refute formulations.

The aim of this research is to design situations to explore how students understand the formalization of two infinite processes that are strongly connected to the natural root of limit idea, and the extent to which the link between those processes may provide some insights on the formal limit. The starting point for the building of the mathematical milieu is the fundamental historical idea of validating equalities by means of infinite processes. The use of those processes provides results (for example, the area of parabolic segments, the sum of infinite geometric progressions, etc.) that would now be dealt with by means of limits and initiates the shift towards the formal limit. If we look to the nowadays structure of equality: *$a = b$ if for all $\varepsilon > 0$, $-\varepsilon < a - b < \varepsilon$*, we may notice that the link between this structure and formal limit is modest. But, this equality can be applied to a function f using the property P: *There exists $A > 0$, for all $\varepsilon > 0$ such that for all $x > A$, $L - \varepsilon < f(x) < L + \varepsilon$*. If f verifies P then *for all $x > A$, $f(x) = L$*. However, if we exchange the quantifiers in P, we obtain Q: *For all $\varepsilon > 0$, there exists $A > 0$ such that for all $x > A$, $L - \varepsilon < f(x) < L + \varepsilon$*. Yet, if f verifies Q then the limit of f in plus infinity is L. In this Situation, we focus on the role of ε in ε-statements: it leads "*at most*" to equality and "*at least*" to limit depending whether the involved statement contains *there exists $A > 0$ for all* $\varepsilon > 0$ or *for all $\varepsilon > 0$ there exists $A > 0$*. In the following, we explain how the constructed milieu concentrates on the formalization of infinite closeness in order to help students to recognize the utilities of formalizing infinite processes through ε–statements. In this milieu, the use of finite limit at infinity helps students to focus on the dependent variable and on the specific role of quantifiers in each ε–statement.

METHODOLOGY

Whole-class teaching experiments

This study is based on extensive design research carried out from 2013 to 2015 at the last year of secondary school in France involving a succession of eight situations related to the limit notion (Lecorre, 2016). The teaching experiments were conducted by one of the two authors serving both as the classes teacher (this author was the official mathematics instructor of those classes) and as a researcher. The teaching experiments took place inside classes' allowed time; each class contains about thirty 17-18 years old students. The teacher-researcher provided the whole-class with preconstructed tasks and gradually enriched the mathematical milieu asking questions to assist the progression of students' work and giving tools that would help students to address problems. Data consisted of audiotape recordings and copies of students' written work. In this paper, we focus on the transcripts of four successive class sessions that are related to the fifth and sixth situations of the whole design; each class session lasted two hours. The fifth situation is based upon the graph of *the monster* (fig.1) and it is supposed to destabilize students' informal understanding of the infinite closeness notion in the limit process and to trigger the need to formalize this process using ε–statements. The sixth situation is designed in way that students face: 1) the problems of validating equalities and limits candidates using two infinite processes; and 2) the individual subtle formalization via ε–statement of each process depending on whether the statement contains *there exists…for all* or **contains** *for all…there exists*. Prior to taking part in the selected class sessions, the students had participated to the preceding teaching experiments concerning the first four situations of the whole design, and they were already familiar with whole-class discussions. Specifically, the students had constructed informal understanding of limit of function at infinity. Building on the graphs of paradigmatic functions (for example *1/x)*, they had investigated limits at infinity by using expressions such as *close to infinity* and *gets closer to*. In addition, they had explored double quantified statements and that double quantifications should be differenced according to the order of the quantifiers and to the convention of interpretation. However, the formal definition of limit is still not introduced to them.

A priori analysis of the monster situation

Students' previous work on the statement *f(x) is upper bounded by g(x) "in infinity"* led the teacher-researcher to formalize "in infinity" by means of *There exists A > 0 such that for all x > A*. This formalization which is one of the fundamental elements of the mathematical milieu of this situation is part of prior students' knowledge. The central element of the mathematical milieu is *the monster* (fig.1).

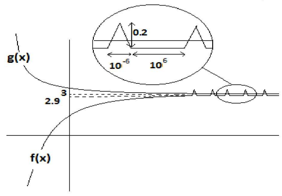

Figure 1: ***The monster***

The monster is the graphical representations of two functions *f* and *g* such that *f* remains below *g* (*g* which "soon" becomes a constant) except in rare but regular peaks (every 10^6) where *f* is over *g* on small intervals (less than 10^{-6}). In addition, this milieu contains the conjecture C3: *Given two functions f and g having no infinite limits in infinity. If the limit of f is strictly less than the limit of g in plus infinity then there exists A >0 such that for all x > A, f(x) <g(x)*. The students are asked to answer the core question of this situation: *The monster is an example, a counter-example, a non-example of C3?* The use of graphs allows students to create ideas about the limit process; it also provides them with helpful feedbacks -even if not formal- that may

contribute to reconsider their informal understanding (p.124). The graphs of paradigmatic functions are given by the teacher-researcher during the debate to reinforce the doubt about the meanings that students give to the infinite closeness in limit process. Students' formulations are based on their informal understanding of this closeness. The class discussion may lead to broad agreement about the validity of C3 but some students may remain uncertain considering that it has not been proven yet. This inquiry is not exactly a request for the formalization of the infinite closeness in limit process, but it is the beginning of the awareness that prior understanding about limit process are fragile and have to be formally structured.

A priori analysis of the ε-statements situation

The arbitrariness of ε is the keystone idea of the ε-statements; it founds the equality process and the limit process through the decreasing of ε towards 0. This situation contains three phases, they are planned in way that: the first phase focuses on the formalization of infinite closeness using statements with the only ε; the second phase deals with the formalization of infinite closeness using statements with ε and other variables; and the third phase highlights the formalization of infinite closeness involved in the limit process by emphasizing the role of quantifiers.

- 1st phase: The mathematical milieu contains the property P for $A = 50$ and $L = 2$ (P1: *For all ε>0, for all x > 50, 2 - ε < f(x) < 2 + ε*), the conjecture C4-1: *If f* verifies P1 then *for all x > 50, f(x) = 2* and the conjecture C4-2: *If f verifies P1 then* $\lim_{x\to+\infty} f(x) = 2$. The students are firstly asked to say what can be concluded if f verifies P1. Then, the teacher-researcher has the responsibility to enrich the milieu by asking the students whether or not: The function *f(x) = 2 + 1/x* is an example, a counter-example, a non-example of C4-1? Depending on the evolution of the debate among the class students, the use of the same function should permit to study C4-2. More precisely, students are familiar with the use of graphs to give examples in order to make or to verify conjectures. Graphical representations of functions may lead to the visualization of the closeness of *f(x)* to *2* by using several values of ε. It is expected that students' validation of C4-1 via a reductio ad absurdum reasoning permits to focus on the arbitrariness of ε as formalizing infinite closeness involved in P1. The study of the function *f(x) = 2 + 1/x* which does not fit the hypothesis of C4-1 -instead of verifying *there exists A* (*A=50*)*, for all ε>0 [...]*, this function verifies *for all ε>0, there exists A > 0 [...]*- should support students' formulations about infinite closeness involved in limit process. The discussion of C4-2 should reinforce those formulations and assists students on thinking about the link between the two processes through the notion of infinite closeness. It is rather probable that the choice of *A* (*50* in P1) will be questioned: does any other *A>0* and *A ≠50* exist in way that the *function f(x) = 2+1/x* fits the hypothesis? The issue related to the values of *A* will be discussed in the following phase.

- 2nd phase: The mathematical milieu contains the property P which is given for unknown *A* (P2: *There exists A >0, for all ε>0 such that for all x > A, 2 - ε < f(x) < 2 + ε*), the conjecture C4-3: *If f verifies P2 then for all x >A, f(x) = 2* and the conjecture

C4-4: *If f verifies P2 then* $\lim_{x\to+\infty} f(x) = 2$. The students are firstly asked to give some properties of functions verifying P2. Students' formulations may lead to the establishment and the discussion of C4-3 and C4-4. A validation of C4-3 based on reductio ad absurdum proof is not expected; however, the validation will inevitably highlight the formalization of infinite closeness within statement containing ε and another variable (A in P2). The study of C4-4 is supposed to improve students' formulations about infinite closeness in the limit process using the closeness involved in P2. It isn't expected that at this stage students will feel the need to talk about the role of quantifiers; but, when we inverse P2 into P3: *For all ε>0, there exists A >0 such that for all x > A, 2 - ε < f(x) < 2 + ε*, fruitful discussions about the double quantification statements may arise among students; the third phase deals with this inversion.

- 3rd phase: The milieus of the above phases are planned to bring into focus the use of ε-statement of equality to validate a limit candidate and so to stimulate students thinking about the formalization of limit process. In this phase, the milieu focuses on the ε-statement of limit to validate a limit candidate. This milieu contains P3, the conjecture C5-1: *If f verifies P3 then there exists B>0 for all x>B such that f(x) = 2* and the conjecture C5-2: *If f verifies P3 then* $\lim_{x\to+\infty} f(x) = 2$. The students are firstly asked to say whether *f(x) = 2+1/x* is an example, a counter-example, neither an example nor a counter-example of C5-1. Then the teacher-researcher has in charge to add C5-2 and to ask the following question: what do you think about this conjecture? It is expected that the starting point of class discussion concerns the question related to whether *f(x) = 2+1/x* verifies or not the hypothesis of C5-1. Students' formulations may concentrate on the finding of the target A given a specific value of ε; the validation emerges from the necessity to generalize this argument for each ε. The discussion of C5-1 emphasizes the need to elucidate the link between the formalizations of infinite closeness in both P2 and P3. It is expected that the use of *f(x) = 2+1/x* helps students to catch the subtleties of this link through the inversion of quantifiers. Students' formulations about C5-2 are supposed to concentrate on the formalization of infinite closeness involved in limit process.

Brief description of data analysis method

In the TDS frame, the a priori analysis is important not only to control the data analysis of the experimental situations but mainly to highlight what does not happen as expected specifically by focusing on how students' understanding assist them to progress or not as planned by the situation. In the case of this research, the data analysis is conducted in this spirit and it is organized through two major levels. The global level of the data analysis involved reviewing transcripts paying attention to the potency of the situations to tackle the research question. The global data analysis shows that the situations give students the opportunity to enter on the problem and to test their understanding through actions, formulations and even validations. The planned milieus incite students to express and share their understanding of limit process and to progress towards formal understanding. The social dimension of these situations succeeds to stimulate valuable discussions among students who acted to convince their peers or to be convinced by them[2]. We take advantage of those discussions to engage on the local level of analysis which focus on students' understanding of infinite closeness and its formalizations in the equality and limit processes. This study is based on the evolution of students' work through the three levels of the milieu (action, formulation and validation) and on the arguments they used to explain their work. In the following section, some results of the local level are exposed and exemplified by generic[3] students' utterances that are translated verbatim from French. We mainly focus on students' shared understanding; however, the individual student's understanding is underlined when it is awkward and deep.

RESULTS

Students' understanding of infinite closeness in the limit process

As expected, class's discussion about *the monster* put forward the diversity of students' informal understanding of limit process. Students' actions involve the use of expressions related to closeness such that *approaching more and more, from below, from above, gets closer to*. Yet, Students' argumentations strengthen the need to give more precisions about those expressions. Building on the graphics of prototypical functions (fig.2), the following formulation gains broad agreement about how the infinite closeness should be stated in the limit process: *For this kind of sinus curve no limit, the second, it is sometimes above and sometimes under […] always going closer to the limit […] the third function […] the peaks are shrinking and the values of the function are getting closer to the limit each time*.

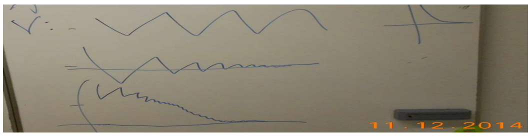

Figure 2: ***Infinite closeness through 3 graphics***

The visualization of infinite closeness through graphical representations helps students not only to share the same meanings but also to get aware of the fragility of their informal understanding. Of course, this is not enough to ensure their engagement in the formalization of closeness in limit process mostly because they have not yet felt the need to validate their limit candidates.

Students' understanding of the formalization of infinite closeness in equality process

To examine students' understanding of the formalization of infinite closeness in equality, we mainly focus on the data analysis related to students' work on C4-1 and C4-3. In the following, the results are organized into two steps depending on whether the statements used refer to the only ε (P1) or not (P2). In both cases, students' work concentrates simultaneously on the process involved in the statement as a way to verify equality as well as a way to validate equality.

[2] Due to space constraints, the results of the global level of analysis are limited to this description.

[3] By generic we mean that it is representative of whole class utterances.

Students' work involving ε-statement with the only ε

The use of several graphical representations assists students' formulations about the role of ε in the statement: *little epsilon*, *change the value of epsilon*, etc. At this stage, the arbitrariness of ε as formalizing infinite closeness is strongly highlighted and it constitutes the starting point for the shift towards validating the equality. The validation is based upon a graphical reductio ad absurdum starting naturally from $x > 50$ (fig.3): *To show that it's true… show that f(x) can't be different from two [...].*

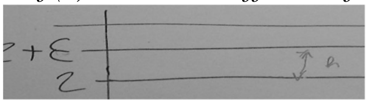

Figure 3: *Graphical reductio ad absurdum*

Students' understanding of the formalization of infinite closeness involved in P1 (fixed $A = 50$) is aided by graphical arguments and emerges from the necessity of both verifying and validating equality.

Students' work involving ε-statement with ε and A

Students' work on P2 (any A) leads to the discussion of C4-3. The students argue on the validity of this conjecture on the basis of the graphical reduction ad absurdum specified for a fixed *A* (*50*): *It is exactly the same statement with A instead of 50*. This generalization is not yet a proof that students' understanding of the formalization of infinite closeness in equality takes account of quantifiers in P2 statement.

Students' understanding of the infinite closeness in limit process through its formalization in equality process

To study students' understanding of infinite closeness in limit through its formalization in equality, we mainly focus on the data analysis related to students' work on C4-2, C4-4 and the case of $f(x) = 2 + 1/x$. Students' work about whether this function fits or not P1 and P2 is supposed to pave the way for linking equality and limit processes as well as to underline the quantifiers and their order in P2 statement.

Students' work involving ε-statement with the only ε

Students' formulations about whether $f(x) = 2 + 1/x$ fits or not P1 are based on numerical computations and lead soon to the necessity to invalid this example by using the case of $x = 51$ and $\varepsilon = 0.001$. Students' actions on C4-2 are mostly based on the already stated validation of C4-1: *f equals 2 and this result does not give information about the limit of f*. The use of $f(x) = 2 + 1/x$ reinforces the doubt on the validity of C4-2 and some students' formulations about this case permit progressively to highlight the specificities of the relationship between ε and $A = 50$ in P1: *here for all ε there is the same A equal to 50 from which f(x) equal 2 thus f(x) is between 2 – ε and 2+ ε [...] and so the limit is two.* Yet, the involved argument does not provide successful feedbacks among peers. However, the necessity to validate a limit candidate through the use of the equality ε-statement (P1) compels students to reorganize their understanding of the infinite closeness in limit process by taking into account the arbitrariness of ε.

Students' work involving ε-statement with ε and A

Students' validation of the statement *f(x) = 2 + 1/x is a non-example of C4-4* is based on a numerical argumentation which is expressed as follows: *I would like to ask those* *who think it is true, to choose an A, any A, and I will be able each time to find a counter example (an epsilon in fact).* This argument emphasizes the order of quantifiers in the equality process but students' work on C4-4 is inconclusive mostly because they do not succeed to draw upon the arbitrariness of *ε* to formalize the limit process. However, their understanding of the infinite closeness is enhanced by the use of P2 as an ε-statement firmly consent with the limit process.

Students' understanding of the role of quantifiers in the formalization of infinite closeness in limit process

These results are mainly based on students' work on C5-1, C5-2 and the case of $f(x) = 2 + 1/x$. They are splitted into two sections: 1) students' interpretations of quantifiers' orders; and 2) the potential sum up of limit process into the formalized ε–statement P3.

<u>Interpretations of quantifiers' orders</u>

Students' work on whether $f(x) = 2+ 1/x$ verifies P3 or not highlights their difficulties to perceive the distinction between P2 and P3 and progressively emphasizes the necessity to take care of quantifiers' orders. Students' firstly act as for P2 to interpret the quantifiers in P3 before focusing on a peer intervention: *[…] the question is written as for all there exists he must give us an epsilon and we have to find an A*. Students' discussions highlight the inversion of quantifiers issue and the need for convention of interpretations. The teacher-researcher intervenes in order to help students finding the targeted *A* for every given ε and to confirm the invalidity of C5-1.

<u>Sum up of limit process via ε-statement</u>

During the debate concerning quantifiers, students' work with ε-statements is strongly connected to the necessity to answer those questions: given epsilon, how to give A? Given A, how to give epsilon? In addition, students' work on the validation of the limit candidate involved in C5-2 leads to the formalization of the infinite closeness in limit process. Students' further formulations put forward the need to explore additional question: to what extent this formalization is sufficient to sum up the formal limit?

CONCLUSION

This study examines students' understanding of the formalisations of two infinite processes in the Archimedean continuum by using ε-statements. These formalizations concern with the infinite closeness notion and refer to equality and limit of function at infinity. The aim of this paper is to give data on how these formalizations and their link may support students' understanding of formal limit. We deploy TDS constructs to design situations in which the milieu is built on students' informal understanding of limit and concentrates on the role of quantifiers to differentiate the formalizations given to the infinite closeness in the process of equality and limit, respectively. This study highlights three main results concerning students' understanding of ε-statements: 1) It is possible to organize a milieu that leads students to question their informal understanding of limit process: in this study, the doubt emerges through several ways used by students to perceive the infinite closeness in the limit process; 2) The only formalization of the infinite closeness in the equality process cannot provide insights on its formalization in the limit process. The focus on the quantifiers' orders is crucial to achieve this formalization; 3) The formalization of the infinite closeness in limit process does not ensure students' sum up of limit process into the formalized ε–statement, this issue needs further investigation. The social dimension of TDS helps students to progressively construct meanings that will constitute the bricks of the meaningful argument which tends to be collectively adopted. In the end, students can admit the irrefutability of the reasoning when all their reluctances are taken into account by their peers.

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“A function is continuous if and only if you can draw its graph without lifting the pen from the paper” – Concept usage in proofs by students in a topology course

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Many students enter university having learned that the graph of a continuous function is “in one piece” and “can be drawn without lifting the pen from the paper.” Rigorously, a function $\mathbb{R} \to \mathbb{R}$ is continuous if and only if its graph is path-connected. In this article, I examine proofs of this fact by students in a topology course. Based on Moore (1994), concept usage of continuity and path-connectedness is analysed through recognition and building-with of the RBC-model of epistemic actions (Dreyfus & Kidron, 2014) in combination with a refinement of Oerter's (1982) contextual layers of objects. A “propositional” layer to describe relationships between objects used in proofs is introduced and used to perform case studies of students' solutions.

Keywords: Teaching and learning of specific topics in university mathematics, teaching and learning of analysis and calculus, topology, continuity, epistemic actions.

INTRODUCTION

The concept image of a continuous real function of one real variable as one whose graph is “in one piece” or “can be drawn without lifting the pen from the paper” (provided that the function is defined on an interval) is held by many students in school or university (Tall & Vinner, 1981; Hanke & Schäfer, 2017).

This piece of research is intended to expand the viewpoint from students' concept usage of continuity from first-year analysis to higher courses and sensitise for some students' thinking processes. To a large extent, this paper is of philosophical nature and suggests a refinement of Schäfer's (2010) approach to describe epistemic actions with the RBC-model, namely by introducing a new specific layer called *propositional layer*. This is relevant for the specifically mathematical procedure of deduction from theorems about relationships between objects. This refinement is then applied to students' (partial) proofs of the fact that a real function defined on $\mathbb{R}$ is continuous if and only if it has a path-connected graph. Thus, this study begins to fill a gap in the literature on this highly recurring concept image and the difficulties in finding a rigorous proof which requires some topological knowledge.

The general research question before starting this investigation was which mental images students use in a proof to a prevalently, vividly acceptable theorem. Here, the aim of this article is to display students' proofs and moot a way to dissect these according to the levels of concreteness of the objects the students used.

The task

The exercise in question was (translation E.H.; see Ross (2013, p. 182)):

> "In school, one often says 'A function is continuous if you can draw its graph without lifting the pen.' Prove the following exact version of this proposition: A function $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if its graph $\Gamma_f = \{(x, f(x)): x \in \mathbb{R}\} \subset \mathbb{R}^2$ is path-connected."

The intuition of "without lifting the pen" has to be translated into a valid statement carefully. Path-connectedness really is required instead of connectedness, and the theorem is no longer valid for functions from an arbitrary path-connected space to the real numbers. It is tacitly assumed in this task that the topologies for $\mathbb{R}$ and $\mathbb{R}^2$ are Euclidean and Γ_f inherits the induced topology. Note that the graph Γ_f is the image of the function $\mathrm{id} \times f: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, f(x))$. Thus, the forward implication in the exercise follows from the facts that functions into products of topological spaces are continuous (with respect to the product topology) if their components are continuous, and continuous images of path-connected sets are path-connected. For the reverse direction, continuity of f at $p \in \mathbb{R}$ can be proven by contraposition via the ε-δ-definition using the existence of a path of the form $\gamma = (\varphi, f \circ \varphi)$ between $(u, f(u))$ and $(v, f(v))$ in Γ_f for some $u < p < v$.

Since this paper has theoretical aims next to the empirical investigation and due to page restrictions, I omit an à-priori-analysis of different possibilities of proving this theorem and prerequisites.

FRAMEWORK: THEORY AND METHODOLOGY

Concept usage, object layers and the model of nested epistemic actions

The *concept image* of a learner for a mathematical object, class of objects or procedures is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). In contrast, *(personal) concept definitions* are students' attempts to specify a concept. Moore (1994) claims that besides "mathematical language and notation" and "getting started" one of the major difficulties in proving is "concept understanding" (mix of concept images, concept definition and concept usage) (p. 249): This means, many students who fail in a proof "lack intuitive understanding of concepts", "cannot use concept images to write a proof", "cannot state the definitions [properly, E.H.]" or "do not know how to structure a proof from a definition" (Moore, 1994, p. 253). The term *concept usage* "refers to the ways one operates with the concept in generating or using examples or in doing proofs" (Moore, 1994, p. 252).

The *Abstraction in Context Methodology (AiC)* (Dreyfus & Kidron, 2014) offers a theory about learning, originating in the need for a new mathematical construct, its construction of knowledge and its consolidation taking into account a specific form of context. In this setting, construction of knowledge means performing *nested epistemic actions (RBC-model)*: *Recognising*, *building-with* and *constructing*. Recognising is “seeing the relevance of a specific previous knowledge construct to the problem at hand”, building-with “comprises the combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification, or the solution of a problem” and construction “consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct” (Dreyfus & Kidron, 2014, p. 89). The actions are nested in the sense that building-with something requires its recognition and construction requires building-with. Hence, the word construction is meant globally within the AiC methodology and locally in the RBC-model. It is noteworthy that flexibility and availability of a construct does not stem from construction itself but *consolidation*.

Based on Oerter’s (1982) theory of activity, Schäfer (2010) differentiated between three layers which help to describe recognition processes of objects more precisely. Next to objectification [orig. Vergegenständlichung] which is the creation of objects, objectual concern [orig. Gegenstandsbezug] towards previously constructed objects can be classified on three layers (Oerter, 1982): On the *singular layer* objects are not distinguishable from the action of an individual itself (e. g. recognising numbers in a table); for the actor, the objects are not yet seen as objects and do not need to have names. On the *contextual layer* objects are characterised by their usage – not anymore restricted to an individual but shared within a community – and the usage is performed within a specific contentual context and similarity of situations; the objects gain persistence beyond singular action. Finally, on the *formal layer* objects are disengaged from specific actions or context (Oerter, 1982; Schäfer, 2010). The objectual concern of previously constructed objects (of a learning process) is reflected in the way someone can use this object. In this article, this activity theory oriented standpoint is specified to mathematics practice in students’ attempts to prove a topological fact.

The approach to consider object layers and actions on them seems related to Sfard’s (1991) idea of the duality of operational and structural conceptions (views of an individual of a concept) within the process of concept formation; concept here means a “mathematical idea […] in its ‘official’ form” (p. 5). Structural conception considers concepts as “abstract *objects*” and its dual form, the operational conception, is about “*processes, algorithms and actions*”, not the notion as an object itself (Sfard, 1991, p. 4; emph. orig.). Both of these views of conception can be seen in mathematical practice: In the action of formal recognition as well as propositional recognition and building-with (see below) objects are conceptualised as structural and through operational conceptions a conclusion is achieved. Propositions themselves are structural, and their scope is reflected in their use, allowing a deduction or justification (not necessarily mathematically correct though).

Founded in the Anthropological Theory of the Didactic, Hausberger (2018) described “structuralist praxeologies” which characterise mathematical justification practices that are oriented towards replacing a statement about the particular with one about the general: “*Structuralist thinking* is characterized by reasoning in terms of *classes of objects, relationships between these classes and (structural) stability of properties* *under operations on structures*” (pp. 82f.; emph. orig.). What I describe here as propositional building-with action would fall most likely under his levels 2 and 3 of “*structuralist dimension*” of a proof (Hausberger, 2018, p. 81; emph. orig.) which describe the application of theorems to a task at hand, therefore reasoning on structuralist rather concrete object level. For example, the task of showing that an object O (e. g. $\mathbb{Z}$) possesses property A (e. g. unique factorisation domain) can be changed to the task of showing that O belongs to a class of objects C (e. g. Euclidean rings) in order to apply a theorem which states that each member of C possesses A (e. g. every Euclidean ring is a unique factorisation domain). The identification of O’s membership to C resembles in the proofs in this paper to recognition actions (of different layers) that O possesses property C. This procedure is “illuminating as to the ‘root causes’ behind the result” (Hausberger, 2018, p. 83).

Mathematical notions such as “continuous real function” are on the formal layer for experienced students and mathematicians, and instances thereof can be recognised as having the general properties of elements of the class they belong to. However, mathematical theorems can also be seen as objects on the formal layer. If they come into use, e. g. by specialisation to a concrete situation in an exercise, they become an object one builds-with. In fact, since the usage of objects is of particular interest here, I claim that a new object layer should be included, the *propositional layer*: On the propositional layer we find theorems as objects that describe properties of objects on the formal layer. Thus, it does not merely contain abstract mathematical objects but relationships between objects as own objects. These relationships can then be applied in a propositional building-with action towards objects on the formal or the contextual layer. [1]

In practice, the decision for which object layer occurs at which place can be guided by the instantiation of objects (“let f be given by $f(x) = x^2 + 1$”), which mostly suggests the singular or contextual layer, or their declaration (“let f be a function such that …”), which highlights an object rather on formal layer. In proofs, the identification of propositional recognition and building-with actions may be assisted by the writer, e. g. referencing the lecture, the number of a theorem etc. However, somebody’s “personal mathematical toolkit” may determine which layer really is in use. The analyses below reflect an interpretation of the written product, not the way of finding the proof.

Examples for the identification of the object layers

If one wants to show that the unit circle S^1 is compact, one can directly use the definition of compactness by taking any open cover of the circle, assuming there was no finite subcover and, using concrete, contextual properties of S^1, evoking a contradiction. This way, propositional objects are not necessarily involved (except for the definition of course, and depending on the particular argumentation). On the other hand, identifying S^1 as continuous image of $[0,1] \to \mathbb{R}^2$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$, is a contextual recognition of S^1 to the context of the map, and together with the propositional recognition of the fact that continuous images of compact spaces are compact the propositional building-with action of this fact to the situation at hand yields the compactness of S^1. Structurally, these two proofs are completely different.

Next, consider the above proof that the graph of a continuous function is path-connected. First, the graph is recognised as the image of a certain map $\mathrm{id} \times f$ (contextual layer), and this map is recognised as a product of continuous maps (contextual layer). The theorem that products of continuous maps are continuous is recognised on the propositional layer and used to deduce the continuity of the map written down in the proof (building-with on the propositional layer to further recognise the continuity of $\mathrm{id} \times f$ on formal layer: The concrete map is no longer important, simply its continuity). Finally, the theorem that continuous images of path-connected spaces are path-connected (recognition on the propositional layer) is used in a building-with action on the propositional layer yielding that the graph of f is path-connected by recognising that the theorem is applicable to the situation at hand. Even though this proof is very short, many recognition and building-with actions had to be completed, brought into order and were compressed in only two sentences.

The “if”-direction of the given task can be proved in the formal/propositional manner as described and is a special case of the theorem “If $F: X \to Y$ is any function between a path-connected topological space X and any topological space Y, then the graph of F is path-connected if F is continuous.” The main ingredient is the fact that for any path-connected space U and any function $g: U \to V$ between topological spaces, continuity of g implies the path-connectedness of $g(U)$. Since the reverse directions of these two statements are not true [2], the reverse direction of the students’ task cannot be proved (completely) in the formal/propositional manner considering solely (path-connected) topological spaces, continuous maps and their properties as above. Nevertheless, it is surely possible to argue with propositional objects, e. g. using the intermediate value theorem or the intermediate value property of functions (Ross, 2013, p. 182f.).

DATA COLLECTION

The data collection for this study took place during the spring semester 2017 at the University of Bremen within the topology class for Bachelor students in pure mathematics. I was not involved in this class but was informed by the lecturer about the contents. During the third week of the semester the students had to solve (besides others) the task presented above. Construction is not directly observable in the analyses below because all notions involved are not new to the students. The context of the contextual layer is understood very locally, depending on the objects already available in the particular solution, for instance those that have been declared before.

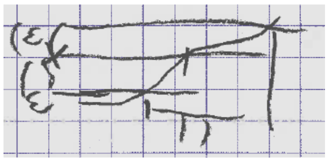
RESULTS

The following cases are supposed to illustrate the work with the above framework (for groups 2 and 3 only one implication is shown). Overall, the solutions were very different regarding the approaches used. More details cannot be included here. All of the following transcripts were translated from German respecting (unusual) syntax.

Notational and language errors are mostly ignored. Abbreviations of German words were often not abbreviated. Small notational errors like forgetting a closing bracket were corrected. An open ball of radius ω centred at a is denoted by $U_\omega(a)$.

Case study: Group 1

The following is a transcript of the solution of group 1 with a sketch redrawn by myself.



1 “⇐” Let $\varepsilon > 0$

2 Since Γ_f is path-connected, there exists $\gamma: [0,1] \to \Gamma_f$

3 with $\gamma(0) = (x - \delta_1, x - \varepsilon)^T$ and $\gamma(1) = (x + \delta_2, x + \varepsilon)^T$

4 Choose $\delta = \max\{\delta_1, \delta_2\}$, then it holds that

5 $|x - y| < \delta: |f(x) - f(y)| < \varepsilon \quad y \in \mathbb{R}$

6 and thus f is continuous because ε arbitrary.

As a first step, the students declare a positive ε (line 1) which indicates that they would like to test the ε-δ-definition of continuity. It looks like a recognition on the formal layer, but it is not stated explicitly at which point they want to check continuity; most probably it is “x”. Afterwards, the students assume that there exists a function (most likely a path, even though not stated) $[0,1] \to \Gamma_f$ connecting the points $(x - \delta_1, x - \varepsilon)^T$ and $(x + \delta_2, x + \varepsilon)^T$. It is not clear why these points should lie on the graph of f but this should be the case since the path lies in Γ_f by their assumption (recognition on contextual layer) (lines 2-3). Interpreting the students’ sketch of the graph, which they do not refer to, I hypothesise that they actually mean the points $(x - \delta_1, f(x) - \varepsilon)^T$ and $(x + \delta_2, f(x) + \varepsilon)^T$, and δ_1 and δ_2 are chosen such that $x - \delta_1$ and $x + \delta_2$ are preimages of the corresponding second components of the points on the graph under f. Thus, the path is recognised on contextual layer using false assumptions. Then, the students choose the minimal of these δs to implicate that f fulfils the ε-δ-definition of continuity in line 5 (lines 4-6). This is a building-with action on contextual layer since taking the minimum of the δs is quite a standard technique in analysis. However, if the function is not “nice enough” between $x - \delta_1$ and $x + \delta_2$, for example monotonically increasing on $[x - \delta_1, x + \delta_2]$ as indicated by the students’ figure, then the interval $(x - \delta, x + \delta)$ is not necessarily mapped into the interval $(f(x) - \varepsilon, f(x) + \varepsilon)$. Hence, the building-with action does not lead to the needed conclusion in line 5. Even if there were preimages of $f(x) \pm \varepsilon$ (only to the left or right of x according to the sign of ε), one had to choose $\delta_1 = \inf\{\delta > 0: f(x - \delta) = f(x) - \varepsilon\}$ and $\delta_2 = \inf\{\delta > 0: f(x + \delta) = f(x) + \varepsilon\}$, and would need to show that these are different from 0. Since functions often encountered are “nice enough” or monotonically increasing, this wrong argumentation might originate in students’ “met-befores” (McGowen & Tall, 2010).

7 “⇒” Let $\Gamma_f = \{(x, f(x)): x \in \mathbb{R}\} \subset \mathbb{R}^2$ be the graph of the continuous function f.

8 Let $(x, f(x))^T, (y, f(y))^T \in \Gamma_f$ (wlog $x < y$)

9 To show $\exists \gamma: [0,1] \to \Gamma_f$ path.

10 Let $z \in [x, y]$. Since f is continuous, it holds that

11 $\forall \varepsilon > 0 \; \exists \delta > 0 : |z - a| < \delta: |f(z) - f(a)| < \varepsilon \quad (a \in \mathbb{R})$

12 No matter how small the neighbourhood $U_\varepsilon\big(f(z)\big)$ is chosen, the values of $f(a) \in U_\varepsilon\big(f(z)\big)$ for $a \in U_\delta(z)$.

13 $\Rightarrow \Gamma_f' = \big\{\big(a, f(a)\big): a \in (z-\delta, z+\delta)\big\} \subset U_\delta(z) \times U_\varepsilon\big(f(z)\big)$

14 Since $\mathbb{R}$ is connected, the neighbourhoods are also connected.

15 $\Rightarrow$ It exists a path $\gamma: [0,1] \to \Gamma_f$

16 with $\gamma(0) = (x, f(x))^T \quad \gamma(1) = (y, f(y))^T$

The solution starts with the definition of the graph of f and the students choose two points on the graph (lines 7-8). They want to show the existence of a path in Γ_f (line 9), presumably that links the two points, although they do not state it. This is contextual recognition since the path is adjusted to the concrete setting and formal recognition would be hypothetical because the definition of path-connectedness is not completely adapted correctly to the given problem. Afterwards, they recognise the definition of continuity of f at some point z between the first components of the given points on the graph (lines 10-11) on the formal layer (independent of a concrete f). Next, they recognise on the formal layer a topological version of continuity via neighbourhoods (line 12) and built-with on the contextual layer the implication that a part of the graph lies inside the product of the neighbourhoods $U_\delta(z)$ and $U_\varepsilon\big(f(x)\big)$ (nevertheless, one has to mention that the neighbourhoods have never been instantiated because ε and δ only appear within quantifiers) (line 13). Afterwards, the recognition of $\mathbb{R}$ as a connected space is formal (proven property in the lecture) but the deduction of the connectedness of the neighbourhoods (likely those in lines 12-13), or their product, is not justified (line 14) (possibly, the students also mixed up connectedness with path-connectedness here). As a last step, the group now directly concludes that there exists a path in Γ_f joining the points on the graph chosen in the beginning (lines 15-16). It can be interpreted that the students believe that subspaces of path-connected sets are path-connected and apply this to $\Gamma_f' \subset U_\delta(z) \times U_\varepsilon\big(f(z)\big)$ without making clear that the neighbourhoods are path-connected, not only connected. Under this interpretation, the building-with action would be propositional, but since the group's claimed implication does not seem to be logically connected to their previous proof steps, their thinking cannot be ascertained.

Case study: Group 2

17 "$\Rightarrow$" Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Note that $\mathbb{R}$ is path-connected.

18 Now let $\big(a, f(a)\big)$ and $\big(b, f(b)\big) \in \Gamma_f$.

19 Since $\mathbb{R}$ is connected, there is a path γ such that $\gamma(0) = a, \gamma(1) = b$.

20 Then, $g = (\gamma, f \circ \gamma)$ is continuous because the composition of continuous maps is continuous.

21 g is also a path from $\big(a, f(a)\big)$ to $\big(b, f(b)\big)$ because:

22 $g(0) = \big(\gamma(0), (f \circ \gamma)(0)\big) = \big(a, f(a)\big)$

23 $g(1) = \big(\gamma(1), (f \circ \gamma)(1)\big) = \big(b, f(b)\big)$

24 thus, Γ_f is path-connected since g is continuous.

On the formal layer, the students recognise that $\mathbb{R}$ is path-connected (note that no additional information of $\mathbb{R}$ is used) (line 17) and that they have to find a path between two arbitrary points in the graph which is indicated by the declaration of two points in Γ_f (line 18). On the propositional layer, they build (which means they state its existence) a path γ between the first components of the chosen points using that $\mathbb{R}$ is connected (in fact, they should have used that it is path-connected; it is not clear whether this is just a notational error since they recognised before that $\mathbb{R}$ is path-connected) (line 19). Next, the students recognise on the propositional layer that compositions of continuous functions are continuous and build-with this fact on the propositional layer that the product function $g = (\gamma, f \circ \gamma)$, contextually recognised as a product of continuous maps, is continuous (their argument lacks the fact that the continuity of the components implies the continuity of the product map) (line 20). Recognising that they have to plug in 0 and 1 for verification (formal layer of a part of the definition of path-connectedness), they conclude that g is in fact a path between the initial points (lines 21-23) (singular/contextual layer in lines 22-23 because the concrete form of g is used). The last line contains again a propositional building-with action since the graph of f is (presumably) shown to have the property that any two of its points can be linked with a path (even though not explicitly stated).

Case study: Group 3

25 "$\Leftarrow$" Approach: If one can show, let it be supposed that there exists a path between two points of the graph of a function, then there also exists an injective path, it follows (*)

26 Suppose, f not continuous at $x \in \mathbb{R} \Rightarrow \exists \varepsilon > 0: \forall \delta > 0\ \exists \tilde{x} \in \mathbb{R}: \tilde{x} \in U_\delta(x), f(\tilde{x}) \notin U_\varepsilon\big(f(x)\big)$

27 By assumption there is a path $\mu: [0,1] \to \Gamma_f$ with $\mu(0) = \big(\tilde{x}, f(\tilde{x})\big)$, $\mu(1) = \big(x', f(x')\big)$.

28 Wlog $\tilde{x} < x < x'$.

29 By (*) there exists an injective path $\tilde{\mu}$. Now let $\tau: [a,b] \to [0,1]$, $\tau(x) := (x - a)/(b-a)$, is continuous!

30 $\Rightarrow\ \pi_2 \circ \tilde{\mu} \circ \tau = f_{|[a,b]}$, where π_2 is the projection of the second component.

31 Namely, for $x \in [a,b]$ it holds:

32 $\pi_2\left(\tilde{\mu}\big(\tau(x)\big)\right) = \pi_2(\tilde{\mu}((x-a)/(b-a)) = \pi_2(x, f(x))$, since there is only one possibility for an injective continuous map in the first component of $\tilde{\mu}$ in

33 question. $\pi_2\big(x, f(x)\big) = f(x)$. // ↯ to f not continuous

The group begins a proof by contradiction in line 26 and therefore the students recognise the negation of the ε-δ-definition of continuity of f at some x on the formal layer, relying on the notation with neighbourhoods from the lecture. The students built-with on the contextual layer a path from two points on the graph whose first coordinates $\tilde{x}$ and x' surround x, the point where the function is assumed to be discontinuous (lines 26-28) (however, x' is not explicitly declared). Taken for granted that one can construct an injective path linking two points given any path between these two (line 25) – admittedly, the group does neither argue how this may work nor state explicitly that this injective path has to have the same start and end point or domain – the students use such a path $\tilde{\mu}$ and compose it with the above τ which is recognised on contextual level as continuous (line 29), to perform a building-with action deducing that $\pi_2 \circ \tilde{\mu} \circ \tau$ is equal to f restricted to the interval $[a, b]$ (lines 29-30). The map τ is used here to transform the path defined on the unit interval to a path $\tilde{\mu} \circ \tau$ in Γ_f defined on $[a, b]$ which shall function as the domain where f can be applied (likely, the students actually meant $a = \tilde{x}$ and $b = x'$). In lines 30ff., the students try to justify that the second component of $\tilde{\mu} \circ \tau$ is $f_{|[a,b]}$. Here, the students believe to recognise the first component of $\tilde{\mu} \circ \tau$ as the identity on $[a, b]$ because there shall be only one possibility for an injective path between two real numbers (lines 32-33). This is wrong. However, this mistake could formally be resolved by performing a "velocity change of paths" which makes the first component of $\tilde{\mu} \circ \tau$ equal to the identity on $[a, b]$. The second equal sign in line 32 is then only justified by this erroneous recognition of the only injection $[a, b] \to [a, b]$ being the identity. Obtaining $\pi_2\big(x, f(x)\big)$ as solution of the calculation in line 32 is a building-with action on the singular/contextual layer; singular here refers to the special situation – the assumption in line 25 – the students find themselves in. As a last step of the proof, I hypothesise that the students recognise $f_{|[a,b]}$ (they write f in line 33 though, which clearly agrees with f on $[a, b]$) to be continuous (at x) (a contradiction to their assumption in line 26). Their justification is however not directly observable; they may have used the composition of the continuous maps τ, $\tilde{\mu}$ and π_2. This is recognised as a contradiction to the assumption of discontinuity of f at x on formal layer (line 33). Finally, the end of the proof is obtained as the result of the propositional building-with action that finding a contradiction to the hypothesis of the contraposition of the statement to prove is equivalent to the original statement.

DISCUSSION & CONCLUSION

The notion of *propositional layer of objects* refines the three layers of objects Schäfer (2010) used to analyse the epistemic action of recognising. In particular, this new layer describes the building-with action of applying a proposition about relationships between objects on the formal layer. In the case studies, it turned out that the recognition and building-with actions usually succeeded when the prerequisites of a definition or theorem in use had been successfully recognised. However, the recognition of the non-satisfaction of necessary conditions for the application of a theorem failed several times because of insufficient mental imagery of continuity (e. g. "local niceness" in the "only if"-proof of Group 1) or paths (e. g. injectivity of paths with group 3) and wrong properties attributed to the objects to be acted on. Nevertheless, subsequent building-with actions on the propositional layer were often carried out coherently based on these wrong assumptions.

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NOTES

1. Of course, in definitions previously defined concepts are usually specialised, thus definitions may also be seen as objects on propositional layer. Seeing what makes up a definition or whether something satisfies a definition is regarded as a recognition action here, and objects which are recognised to satisfy a definition will be on formal layer.

2. The graph and the image of arg: $S^1 \to \mathbb{R}$, $e^{\theta i} \mapsto \theta$ (with $0 \leq \theta < 2\pi$) are path-connected but arg is not continuous.

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Learning complex analysis in different branches – Project Spotlight-Y for future teachers

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At the University of Bremen in teaching complex analysis, we split the last part of the lecture into two branches according to profession: While future mathematicians deepen their understanding in a branch for them, future teachers take a branch of the lecture where they prepare a task with dynamical geometry software for pupils which is based on phenomena of complex analysis. Here, we describe the design of the course, some general aims and first results obtained from the branch for future teachers.

Keywords: Novel approaches to teaching, teaching and learning of specific topics in university mathematics, specialised content knowledge, teacher training, complex analysis.

INTRODUCTION

More than one hundred years ago, Felix Klein (2016) acknowledged that there is a "double discontinuity" in teacher education in mathematics. When a pupil enters university he or she does not see the connection of elementary mathematics taught in school with the formal mathematics in university, and when teaching in school he or she does not see how the university maths informs his or her practice (Hefendehl-Hebeker, 2013; Vollstedt, Heinze, Gojdka, & Rach, 2014). Within mathematics courses at university, it is thus necessary to highlight connections between the mathematics future teachers are taught in lectures at university and the mathematics they will teach in schools (Prediger, 2013).

The project "Spotlight-Y" aims for an institutional link of mathematics and mathematics education and at providing students with the experience to relate content from the lecture in complex analysis to their future teaching practice. By helping them to design tasks for pupils with mathematics from a "higher standpoint" (Klein, 2016), the students get the opportunity to try out these tasks with pupils from local schools.

THEORETICAL CONSIDERATIONS

Ball, Thames, and Phelps (2008) proposed to distinguish the mathematical content knowledge (CK) of teachers further into common content knowledge (CCK) and specialised content knowledge (SCK). The first is mathematical knowledge that does not depend on a specific profession and is known to pure mathematicians as well as teachers and others, and the latter is mostly useful only for teaching mathematics. So for a teacher, not only pedagogical content knowledge (PCK) is important and unique for the profession but also SCK.

The international study TEDS-FU established that PCK is the most important factor to judge the didactical problems and opportunities of a classroom situation and that CK is the most important factor to assess the utterances of pupils (Blömeke et al., 2014).

Thus, teaching mathematics requires an interplay of different kinds of knowledge for the teacher and it seems worthwhile to try to specifically address SCK when doing teacher education programs at university.

Winsløw and Grønbæk (2014) proposed to study the phenomenon of the second discontinuity by the study of different mathematical praxeologies between university and high school. In their paper, they are able to identify certain challenges for this transposition. For university students, one of these challenges is the establishment of a school praxeology for a problem typically solved with a university maths praxeology (their example is the method of least squares for simple linear regression). The authors claim that the main related difficulties lie in spawning capacities for students' autonomous research and handling (ir-) relevant literature. This is where Spotlight-Y is situated. Our approach can be seen as complementary: Instead of focusing on how the mathematical praxeologies may change and what challenges may hinder, we try to focus on how the didactical praxeologies are combined with the mathematical praxeologies from university.

Research question

In this paper, we want to foster the understanding of the combination of PCK and SCK by the Y-model that structured the lecture (see next section). We try to address the following question: "In what ways do students combine specialised and pedagogical content knowledge within the preparation and implementation of their learning environments?" For this purpose, we will describe example projects.

OVERVIEW OF THE PROJECT

At the University of Bremen the lecture in complex analysis is the final course in mathematics for future teachers. Historically, many teacher students in Bremen are known to view this as a course you have to pass to get through your studies, but which is thought of having no direct connection to what is taught in school. From the standpoint of the designer of the curriculum of teacher training in mathematics, complex analysis was seen as a course which brings different aspects of mathematics together and may help to get a more holistic view on various areas of the subject, e. g. polynomials, trigonometric functions, power series or conformal mappings. As such, it should be a good starting point to get a "higher standpoint" as Klein (2016) put it.

"Spotlight-Y" is a design research project within the project "Schnittstellen gestalten" at the University of Bremen, funded by the German Federal Ministry of Education and Research (BMBF) that started in fall 2016. Its general aim is the interlock of the scientific disciplines of mathematics and mathematics education. Our students shall see that elements of school mathematics, in particular all relevant classes of functions, can be understood from the complex setting which explains more than the real picture. In three design cycles we develop

1) the structure of the lecture on complex analysis,
2) the specific branch for future teachers with a focus on identifying phenomena of complex analysis relevant for teaching mathematics in upper classes of secondary schools and creating exploratory learning environments, and
3) a day for eXperimental Mathematics (XMaSII) which local pupils from upper classes of secondary school (Sekundarstufe II) attend to work on the learning environments by the teacher students.

It was mandatory for our students to use the free dynamical geometry software (DGS) GeoGebra (https://www.geogebra.org/) for implementation because we assumed that this would make difficult concepts easier to present and provide a suitable technology to really be able to explore a phenomenon. Besides, our students get an opportunity to create authentic materials, plan and structure group work with pupils from secondary schools in 11th or 12th grade and gain experience in working with them. In particular, they get experience with pupils outside internships and their practical semester teaching mathematics for rather gifted learners.

Some of the general research questions in the project are: How do students combine mathematics from the complex analysis course, their general knowledge on mathematics education taught earlier in their studies and simultaneously to the course on complex analysis (a seminar on task design)? In which ways do they handle their learning environments and the material for the pupils? How do the future teachers see mathematics education, do they consider it as a scientific discipline or rather a collection of methods to teach school mathematics?

After about two years, we will report on our design principles, experiences and empirical data we collect during the design project to establish a transfer package for other mathematics lectures. In the second year, we perform a first transfer to the stochastics lecture in the Bachelor for future mathematicians and mathematics teachers.

Structuring a mathematics lecture: The Y-model

In “Spotlight-Y” we adopt a Y-shaped model for the course in complex analysis. The course takes place in the fall semester and is attended by pure/applied mathematics students in the Bachelor as well as teacher students in their first year of the Master programme. The course takes one semester and the participants are split up around Christmas after two thirds of the semester (see Figure 1a). The part for everyone covers the introduction of holomorphic and conformal mappings, line integration and Cauchy’s theorem up to the residue theorem. The specific branch for future mathematicians continues with more advanced topics like invariance of integrals with respect to homotopy or homology of paths, analytic continuation, the Riemann mapping theorem or the prime number theorem etc., depending on the lecturer. In the branch for teachers we cover geometric properties of complex functions and an introduction to Riemann surfaces. Furthermore, we give the students time to work on their learning environments. The course is finished with a written or an oral exam.

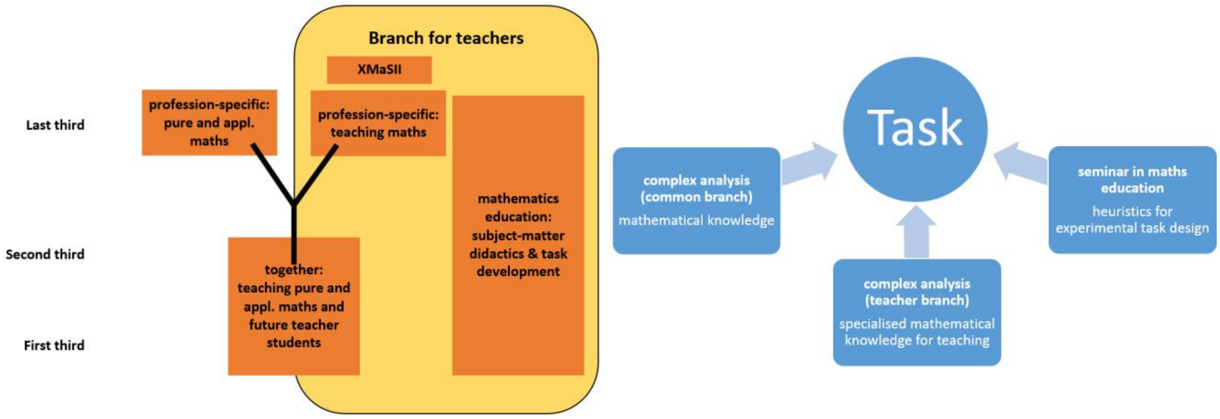

Figure 1: a. Lecture with Y-model and seminar on task design (left), b. The different components that support the task design process (right)

In fall 2016/2017, the first design cycle in the course on complex analysis started and the next semester was used to start analysing data we gathered during the first implementation. In fall 2017/2018, the second design cycle lead to improvements with respect to the profession-specific branch for the future teachers. In fall 2018/2019, the last cycle will start: A third run during the course on complex analysis will be devoted to the final answers to our research questions and in summer 2019 a second transfer to the course on stochastics will be performed.

Elements of the task design process

In order to prepare the tasks for the pupils, the students have to bring input from three different components together (cf. Figure 1b). They make use of the mathematical CK from the general branch of the lecture, they activate their SCK from the branch for future teachers and bring those branches together with their PCK, i. e. heuristics on task design from their seminar on mathematics education.

Methodology

For the whole project we collected several data. First, the students had to write a “preflection”, a short reflection which is not after an action on an action but rather the anticipation of actions, aims and sequences: They should fix which phenomenon they want to discuss in their learning environment, what the pupils are supposed to discover, the planned sequence of events, expected difficulties and the mathematics from the lecture that is directly used. Immediately after the implementation of XMaSII, the students filled in a questionnaire named “Ad-hoc notes” to write down own executed actions, recapitulate the execution and match it with expected or anticipated occurrences. In order to pass the course, the students also needed to write a reflection on their project and their learning. They had to discuss their topic and how they found it, describe detailed the phenomenon to be explored by the pupils, the organisation of their tasks, the schedule of the implementation etc. In a final section, we explicitly asked whether there were certain elements of the course on complex analysis that changed the relationship of the students to mathematics as a science and mathematics as a school subject. In total, 19 students participated in the study. We also conducted guided interviews with two students of four groups each to get more insight to the aspects above (and some more). We omit details here since this paper deals mainly with the learning environments the student groups created.

EXAMPLE PROJECTS

We describe two case examples. Group A created a learning environment called "Differentiation as linearisation" and Group B worked on "Polynomials of infinite degree". Two members of Group A have also been interviewed. Two other groups created tasks for spherical geometry, another one introduced complex numbers and a sixth group also worked on power series. As mentioned above, the tasks themselves focus on phenomena that are rooted in complex analysis and can be explored by the pupils from high school by the means of a DGS.

"Differentiation as linearisation"

Group A looks at real differentiation, i. e. differentiable functions defined on (subsets) of the real numbers to the real numbers. The students concentrate on the image that a differentiable function locally looks like a straight line by magnifying the graph around a given point, say $(p, f(p))$. The students emphasise that the derivative at a point can be interpreted as the slope of the function, respectively of the tangent at the graph of the function, at this point (which is CCK). They create a sequence of tasks and subtasks within their learning environment that aim at magnifying the graph of a function (use of PCK) and lead to answer the question which exponential function has itself as a derivative. However, the students do not explicitly clarify what this has to do with linearisation besides the optical appearance. These two "Grundvorstellungen" ("basic mental models", Greefrath, Oldenburg, Siller, Ulm, & Weigand (2016, p. 101)) of the derivative at a point, "tangent slope" and "local linearity", are usually distinguished in mathematics education literature (see e. g. Greefrath et al. (2016) who also discuss "local rate of change" and "amplification factor"). However, we do not see this as a lack of understanding of different Grundvorstellungen with the students. Rather, group A focuses on a geometric point of view and many of their tasks focus on the idea of zooming into the graph of a function to get a more and more straight line (PCK, SCK) on the computer screen (e. g. "Plot with GeoGebra the function $f(x) = x^3 - 0.75x^2 - 9x + 1$. Then draw the point $A(1, f(1))$ […] Zoom in the neighbourhood of point A into the drawing. What do you notice? Try to describe your observation." In a later task: "Describe with this magnified image how to calculate the slope in point A approximately (Hint:

Remember the slope of a secant).” (own translation)). In other subtasks, group A provides the pupils with GeoGebra worksheets they have prepared before, where they had plotted several functions and their “slope function”, and wants the pupils to make observations about these. In another task at the end of the task sequence, they display an exponential function $f(x) = a^x$ and its slope function with a parameter a which can be changed with a slider and ask for which a these functions coincide.

The definition of derivative at a point with limits of difference quotients is not stressed in detail but the GeoGebra worksheets nicely implements that when zooming into the graph two points on it approach the point $(p, f(p))$ (one from left, another from right) to show that the slopes of the secants through these points eventually yield the slope of the tangent, no matter if one approaches from left or right. However, since the students do not provide counterexamples, it does not seem very likely that the concept of derivative can be fully understood, since the class of functions the students used are pretty standard and “friendly” in the way that they all really look locally straight (e. g. s given by $s(x) = x^2 \cdot sin(1/x)$, $s(0) = 0$, is differentiable at 0 but oscillates very much around the origin and therefore hardly is locally straight).

An additional task in the learning environment is about rotation-dilation, which is an interpretation of the complex derivative discussed in detail in the first third of the lecture and exercise classes (SCK). A hexagon and two points are displayed in the Cartesian plane and pushing around the points changes the hexagon according to the multiplication by the complex number associated to the points (see Figure 2a, SCK). However, such a multiplication is not made explicit in the task. Rather, the pupils shall measure distances and angles, and the task seems rather unrelated to differentiation. Not even a function is in play. The complex interpretation of derivative is transformed into a task about rotation-dilation, but not related to the previous tasks on differentiation. This is not surprising since the idea of magnifying pursued in the real approach does not shed any light on dilation-rotation. In the interview with two of the three group members we asked for an intuitive, vivid meaning of a holomorphic function. The students were unable to argue and did not even respond with the keyword of (local) rotation-dilation.

Thus, in terms of our framework, the connection of SCK and PCK did not happen as was hoped for. One problem seems to be that the necessary SCK for the complex derivate could not be utilised by the students, while they argue very well about prerequisites the pupils should have for their learning environment such as secant, tangent, linear functions etc. (SCK for the real derivative). The students of group A did not take part in the parallel seminar on task construction. Therefore, we hypothesise that they followed an alternative strategy to construct tasks: They scrolled through mathematics education literature to find a suggestion on how to fill the aspect of local linearity with life. In the interview, one of the students said, roughly speaking, “From the earlier lectures [in maths education] one knew at least where to research, because it is clear that one cannot have everything stored [in one's mind] and fetch it up immediately, so one needs to do research" (own translation).

From a real point of view in terms of concept formation, the learning environment seems rather unproblematic. We believe this is the case since didactic aspects of real functions and derivatives are very present in the courses on mathematics education the students in Bremen are required to take. However, there does not seem a thorough idea on how to relate the complex interpretation of the derivative of a complex function to the real setting. Thus, intuition of differentiation is not coherently transported from one to the other setting. Also, group A does not argue whether their real images of derivatives find their counterpart in the complex setting.

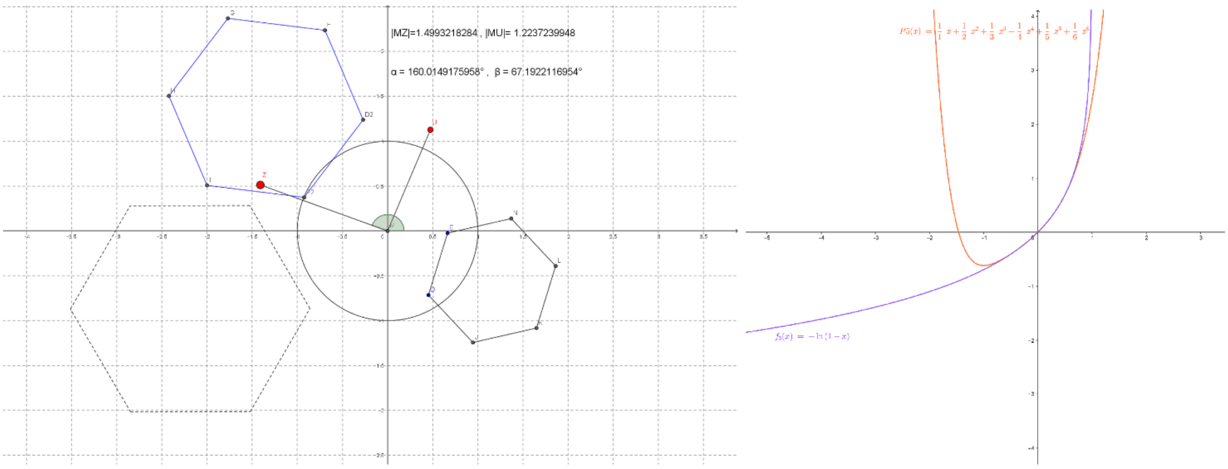

Figure 2: a. Dilation-rotation of a hexagon (left, from group A's GeoGebra worksheet), b. *-ln(1-x)* and its sixth Taylor polynomial at 0 (right, from group B's GeoGebra worksheet).

"Polynomials of infinite degree"

Group B has two general aims with its learning environment about power series which they steadily call "polynomials of infinite degree". Firstly, in school polynomial functions form a very frequently used class of functions (PCK) and power series seem a quite direct generalisation of it (CCK). Secondly, nearly all functions that appear in school can be represented by power series (even though this is usually not made explicit in schools, PCK). For a motivation why power series can be useful for pupils, group B argues that using power series one may be able to explain easily why the derivative of the exponential function is itself and the derivative of the sine function is the cosine function etc. (PCK, SCK).

As necessary mathematical background from the complex analysis course group B recognises holomorphic functions, power series and their derivatives, entire and non-entire functions, radii of convergence of power series and the power series expansion theorem. They relate all these ideas in a correct manner to each other. Nevertheless, the language the students use in their written reflection and on the task sheets for the pupils is quite tenuous: The words polynomial and power series are not clearly differentiated. It may be possible that the students wanted to use the same word "polynomial" with the addition of "finite degree" or "infinite degree" because it is known to the pupils. Group B recognises that the distinction of entire and non-entire functions is relevant for school and illustrates this with examples like $-ln(1-x)$ and $(x+1)^{1/2}$ (see Figure 2b, PCK and SCK).

In their preparation and reflection the students raise the question to what extend convergence issues of power series or why Taylor approximation works (CCK) should be covered (PCK), but they consider this too difficult and not anymore relevant for the discovery of the phenomenon for the pupils that previously known functions – and basically all encountered in school – can be expressed differently. In fact, the group recognises that arguing in whatever way that Taylor expansion requires differentiation and defining elementary functions by their series expansions may lead to circular reasoning (CCK).

The mathematics background is well explained with a few mistakes mostly in language and notation. Group B's learning environment really allows to discover one of the fundamental phenomena of complex analysis, namely that every holomorphic is at least locally a power series, in the real and school-related setting. In the GeoGebra environment, several functions and their Taylor polynomials of varying degree, adjustable with a slider, are presented. This shows a creative part in the work of our student group, since it visualises very nicely the approximation of a function by its Taylor polynomials. However, the distinction of entire and non-entire functions is mathematically deep and relevant for school (SCK). The question of why the representation of a function by a power series is relevant for pupils is answered with a deepening of their insights into higher mathematics and the application to calculate derivatives similar to the procedure with polynomials. Unsurprisingly, as described above, the problem of convergence is left aside.

Nevertheless, the task sheets for the pupils lack rigor and do not provide consistent language at certain places. For example, in a "remember box" group B writes "A function $f(x)$ is an entire function if it is a polynomial of degree n or is a function which can be written as a polynomial of infinite degree $P(x)$ for every $x \in \mathbb{R}$ [grammatical errors deleted, E.H.]", and in another box they write "A function $f(x)$ is a non-entire function if it is NO polynomial of degree n but can be approximated by a polynomial of infinite degree. However, this polynomial of infinite degree approaches the function $f(x)$ only in some interval. It [the function] is not defined for all $x \in \mathbb{R}$ or does not have a tangent everywhere [slight grammatical adaptions, E.H.]". The absence of the domain of f is usual in school and probably left out for this reason. It could have been stressed more detailed that a power series expression for an entire function is *globally* valid, i. e. that P equals the given function on the whole of $\mathbb{R}$. In the definition of non-entire functions, on the other hand, the *local* approximation by a power series is mentioned correctly (assuming that the group meant that the sequence of Taylor polynomials approximates the original function) but it is not clarified that this power series actually equals the function in some interval containing more than one point. Also, it should have been stressed that this local phenomenon is valid for every point in the domain of the non-entire function but the corresponding power series may change. Another problem is the mentioning of tangent. Tangents do never show up in the learning environment nor in the students' reflection again and it is very unclear what the students wanted to say.

In this case the connection between SCK and PCK seems to be working much better, although there are some flaws in the formulation of the definition. These problems may be rooted in misconceptions in the CCK.

FINDINGS

The learning environments differ heavily in mathematical rigour, creativity and recognisability of complex phenomena in either a real setting or else. In the section on the example projects we described that Group A showed little understanding of the geometric meaning of the derivative of holomorphic functions (SCK) (or, at least, they did not implement this well) but otherwise provided a nice example driven and exploratory environment. Group B, on the other hand, translated the notion of power series representation of entire and non-entire functions to real examples and made a complicated phenomenon accessible to pupils (PCK and SCK). Due to the complexity, arguments on why and where power series converge and how to find Taylor series expansions have not been dealt with (CCK). The written report is mostly correct, however the material for the pupils needs improvement with respect to consistency in language, mathematical exactness on the level of pupils and correctness from a formal point of view.

Still in evaluation phase, i. e. coding the interviews, we are nevertheless already able to state first results regarding our design process:

1. Most students seemed more engaged in creating the tasks than in the lecture.
2. One has to make connections between SCK and PCK explicit, many students fail to see them on their own.
3. It has been evident from the interviews that basic mathematical content knowledge in complex analysis was not available two months after the exam, but the content of the tasks they created was.
4. One needs to clarify and guide how to implement a topic from complex analysis when complex numbers are not available for pupils or shall not be introduced.
5. A possible improvement for XMaSII could be to restructure the day into different sessions or workshops: 1) Introduction to complex numbers and geometry of the complex plane either by the lecturers or some student groups and then 2) working groups dealing with different phenomena from complex analysis.

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On students' understanding of Riemann sums of integrals of functions of two variables

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APOS (Action-Process-Object-Schema) Theory is used to pose and test a conjecture of mental constructions that may be used to understand the relation between integrals of two variable functions over rectangles and corresponding Riemann sums. Interviews with ten students who had just finished a multivariable calculus course showed that the conjectured mental constructions are necessary.

Keywords: Functions of two variables, APOS, integral calculus

Multivariable functions and multivariable Calculus are important in engineering and the natural sciences as a tool for modelling. Their learning has received more attention lately from the Mathematics Education community. Starting with the analysis of students' understanding about two-variable functions (for example: Trigueros and Martínez-Planell, 2010; Martínez-Planell and Trigueros, 2012) researchers have documented students' difficulties and have shown that the transition from one-variable Calculus to multivariable Calculus is far from being smooth. There are few studies in the literature that deal with students' difficulties and understanding of the integral multivariable Calculus (Jones and Dorko, 2015; Martínez-Planell and Trigueros, 2017). In one of these few studies, McGee and Martínez-Planell (2014) showed that a course based on lectures did not promote students' understanding, while activities introducing the use of semiotic chains and the development of synergy among representations helped students understand this concept. The research questions are:

What constructions relating double integrals and Riemann sums are evidenced by students who finished a Multivariable Calculus course based on lectures?

What constructions may be needed to relate double integrals and Riemann sums?

THEORETICAL FRAMEWORK

APOS theory (Arnon et al. 2014) is used in this study to analyse possible mental constructions by students who have already taken a course on multivariable calculus. We only summarize the main structures of this theory. An Action in APOS Theory is a transformation of a previously constructed mathematical object that the individual perceives as external in the sense that students need some guidance and are not able to justify what they do. When an Action is repeated, and the individual reflects on what he or she does, it may be interiorized into a Process. A Process is perceived as internal in the sense that it has meaningful connections to other mathematical knowledge of the individual. A Process allows the individual to imagine doing the Actions without actually doing them, to omit steps, anticipate results, and to justify the Process. Different Processes may be coordinated to form new Processes. When the individual needs to apply Actions on a Process, it can be encapsulated into an Object. When an individual shows a Process or Object conception of a mathematical notion we say that the individual "understands" the notion. A Schema is a coherent collection of Actions, Processes, Objects, and other Schemas, that the individual uses to work with problems related to some mathematical notions. Schemas are not used in this paper.

To analyse students' work using APOS Theory, a conjecture of those constructions that may be used to understand a specific mathematical notion is designed. This model, called a genetic decomposition (GD), does not pretend to be unique and needs to be tested with research data. The GD may be revised and expanded in successive cycles of research, teaching material development, and implementation. This research cycle makes it possible to use APOS theory to be better suited to future research needs to study the multivariable integral calculus.

GENETIC DECOMPOSITION

We only present a portion of the GD of integrals of functions of two variables over rectangles. Its development is based on mathematics, on the researchers' teaching experience, and data from the research literature: mainly, ideas about representation registers (Duval, 2006) described in the study by McGee and Martínez-Planell (2014) and the ideas of "orienting pre-layer" and "product layer" described by Sealy (2014), which stress the need of attending to the individual meaning of the product $f(x_i)\Delta x$ and its components in the construction of integrals of one-variable functions.

The GD starts with pre-requisite constructions which include: a Process conception of two-variable functions and volume of prisms as Object.

Actions are performed on a given two-variable function in any representation with domain restricted to a rectangle, to produce the geometric representation of the restricted domain as a subset of 3D space. Actions are performed on the same function to obtain values of the function on the given domain and to represent them in the space as points and/or curves in the graph of the function. These Actions are interiorized into a treatment or conversion Process to represent the graph of the function over the given rectangle together with the rectangle so that the student can imagine the relation between function and rectangular domain as a graph in space.

Actions of evaluating the given function of two variables at a specific point of a given sub-rectangle of its domain, multiplying it by the length and width of the rectangle to form a product of the form $f(a,b)\Delta x\Delta y$ are done. These Actions are interiorized into a Process which can be coordinated with conversion Processes between different representations of function, rectangle, and given point, to imagine the product as the volume of a rectangular prism in space.

Given a continuous function in different representations defined on a rectangle, with the function simple enough so that its maximum and minimum values on the rectangle may be recognized without doing any explicit computation, the Action of obtaining an overestimate and an underestimate of the product $f(a,b)\Delta x\Delta y$ is taken. These Actions may be interiorized into a Process that enables to imagine the existence of points (a,b) where underestimate and/or overestimate of the product $f(a,b)\Delta x\Delta y$ are attained. This Process is coordinated with a treatment or conversion Process to draw a rectangular prism corresponding to over and/or underestimate in space. Actions are performed to change the chosen point to construct a prism that better approximates a given exact value of the integral. These Actions are interiorized into a Process that enables the recognition that for such continuous function, there is a point somewhere on the rectangle that will produce the exact value of the volume between the graph of the function and its rectangular domain.

Given two small specific positive integer numbers, *n* and *m*, the Action of subdividing given intervals [a,b] and [c,d] into subintervals of equal length both numerically and geometrically is done to obtain a subdivision of the rectangle $[a,b]\times[c,d]$. These Actions are interiorized into the corresponding Process. Given a continuous function *f* defined on the rectangle, the Action of choosing a prescribed point (x_i, y_j) on each sub-rectangle of the given partition and producing the products $f(x_i, y_j)\Delta x\Delta y$, and the corresponding sum, interpreting this sum geometrically, numerically, symbolically as an extended sum, symbolically using sigma notation, and verbally, may be interiorized into a Process that enables imagining forming such sums in different representations for the collection of sub-rectangles in any partition of any given rectangle.

METHOD

Ten students were chosen by their professor to be interviewed at the end of a multivariable calculus course selecting four over-average, three average, and three under-average. The course was completely based on lectures. The interviews lasted 46 minutes on average. Students answered a set of questions designed in terms of the GD and also related to what was covered during the course, and produced a written response while sharing their thoughts out-loud. The interviews were recorded, transcribed, individually analysed, and results were negotiated by the two researchers. Students' responses were analysed according to the GD, while keeping notes on unexpected responses and other difficult to classify observations. These were the questions used:

1a. The following is the complete graph of function $z = f(x, y)$. Represent the domain of *f* in the figure (Figure 1).

1b. Let $g(x, y) = x^2 + y$ be a function with domain restricted to $0 \le x \le 2$ and $1 \le y \le 2$. Use the coordinate system given in the following figure to represent the domain in three-dimensional space [An empty drawing of the first quadrant was given].

1c. The above functions *f* and *g* are the same [Figure 1 was given again here and in the rest of the problems]. If $\Delta x = 2$ and $\Delta y = 1$, what is the numerical value of $f(0,1)\Delta x\Delta y$? What does it represent geometrically?

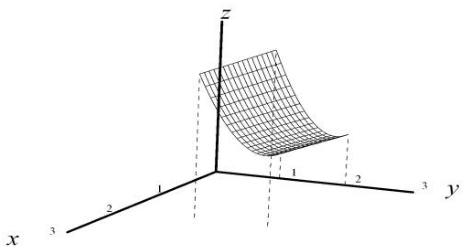

Figure 1: (repeated in each part of problem 1 except 1b)

1d. Let $\Delta x = 2$ and $\Delta y = 1$. How does $f(0,1)\Delta x\Delta y$ compare with $\iint_D f(x,y)dA$? [No numerical computations are needed in parts d, e, f, and g.]

1e. How does $f(2,2)\Delta x\Delta y$ compare with $\iint_D f(x,y)dA$?

1f. Is there any point (*a*,*b*) in the domain *D* of *f* such that $f(a,b)\Delta x\Delta y$ is equal to $\iint_D f(x,y)dA$?

1g. Let $\Delta x = 1$ and $\Delta y = 1/2$. Consider the Riemann sum $f(0,1)\Delta x\Delta y + f(0,1.5)\Delta x\Delta y + f(1,1)\Delta x\Delta y + f(1,1.5)\Delta x\Delta y$ of the integral $\iint_D f(x,y)dA$. What does the Riemann sum represent geometrically and how does its value compare to that of $\iint_D f(x,y)dA$?

Note that problems 1a and 1b are essentially the same in different representations. They both test the portion of the GD dealing with recognition of rectangle and function. Problem 1c gives information on the portion of the GD dealing with forming one term of a Riemann sum. Problems 1d, 1e, and 1f relate to the portions of the GD dealing with underestimate, overestimate, and exact value. Problem 1g gives information on the portion of the GD dealing with a partition and Riemann sum.

RESULTS

On function and domain of a function

Many students showed they had not constructed the concept of two-variable function. They gave evidence of considering these functions in terms of a correspondence rule, and showed difficulty interpreting functions given graphically. Moreover, these students also showed not to have constructed the concept of domain of the function. Most of them considered that the domain of a two-variable function should include information about the function, since it had to be represented in 3D space. Eight students showed difficulties similar to those of Luis, as exemplified in the following discussion with the interviewer (in Problem 1a):

Luis: I can tell you what the domain is but if I don't have a function I don't think I can tell you the exact point where each of the points in the graph is…

Interviewer: So, is the graph part of the domain?

Luis: No, the domain is obtained from the graph. I can obtain the domain having the function but to do so I have to define the function.

After some discussion:

Interviewer: So the domain, is it only *x* and *y* or may it also include *z*?

Luis: The domain may include the *z*.

This example shows how Luis needs a correspondence rule to determine the domain of the function. It also evidences that he considers the function itself should be part of the domain of the function. Other students showed this difficulty.

Students' responses pointed to a need to pay attention to the different representations of functions in 3D space and to have students do treatments and conversions between representations. In Problem 1a, some of them quickly represented the rectangular domain as part of the given figure in 3D space. However, when the function was given symbolically their notion of domain seemed to change. This shows that recognizing the domain of a two-variable function is a construction that needs the interiorization of Actions on functions given in different representations. These difficulties as well as the counterfactual belief of teachers that students may easily generalize concepts for one-variable functions to multivariable functions have been reported before (Martinez-Planell and Trigueros, 2012).

Area and volume

Students also showed an unexpected confusion between area and volume when they described graphs of functions in 3D-space. This difficulty surfaced in Problem 1c. All students were able to calculate the value of that product; however, they were in trouble when explaining its geometrical meaning. Brian, for example, explained:

Brian: … this part, *f*(0,1) would be a point in this graph here. Change in *x*, change in *y*, I am not a hundred percent sure… that would be an area then, of the surface, or the entire function…

And later:

Interviewer: Can you tell me what does the double integral of *f*(x,y)*dA* represents?

Brian: *dA* is the area of the function, the area of this figure,

Interviewer: The area of the surface?

Brian: Yes… of the surface on the given domain….

Other students, as Luis, showed confusion:

Luis: The area of the figure, that is, the area of the function which in this case is that figure [referring to the graph in Figure 1].

Interviewer: Like the area of a surface?

Luis: Exactly

Interviewer: So if it had units would it be like square inches or square centimetres? What units would the double integral have if *x*, *y*, and *z* had units?

Luis: Cubic

Interviewer: Cubic; then, would it be area?

Luis: It would be volume...

After some discussion:

Interviewer: …Let's suppose that this other paper that I am raising here is the graph of the function [He raised a sheet of paper] What volume are we talking about?

Luis: …The volume is the one of this paper…since I have a function and I'm integrating in the values of the function then what I'm going to get are *z*, small *z*'s of what the function is, I'd be getting the volume of the figure.

Interviewer: ... So you pointed to the paper that is floating. But, does it have a volume?

Luis: ... Yes, it has a change in *x*, it has a change in *y*, and the *z* is the one from the function, so I say that it has a volume.

Other five students showed the same confusions. The above excerpt exemplifies that a student can describe the individual components of $f(0,1)\Delta x\Delta y$ but might not be able to do the Action of putting them together to interpret it as the volume of a rectangular prism, even if they can calculate the result of the product by doing the Action of substituting the given values in the expression, as conjectured in the GD. This difficulty is possibly related to the fact that these students have not constructed space as an Object, which does not allow them to imagine what their teachers mean when they talk about a surface in space and the double integral as related to the volume under a surface (Trigueros and Martínez-Planell, 2010).

As considered in the GD, these difficulties make it impossible for students to do the necessary Actions on the function restricted to a rectangle to represent the domain geometrically as a subset of space, and to interiorize the Process to imagine the relation between a restricted region on the domain and the function. The lack of all these constructions becomes an obstacle to understand other related concepts, including double integrals, as will be shown below. Students who do not show these constructions may not follow teachers' explanations; they would be confused and resort to memorization to respond to exam questions.

Only one student, Farid, gave evidence of the pre-requisite constructions described in the GD. He was successful in explaining Riemann sums and double integrals.

Riemann sums, underestimates, overestimates and double integrals

Most students had many difficulties working with problems dealing with Riemann sums and their relation to double integrals. Even after the interviewer explained to some of them that $f(0,1)\Delta x\Delta y$ was a volume and drew it, their difficulties did not enable them to make the whole construction as Brian showed in Problems 1d, 1e, and 1g:

Brian: So the Riemann sum would be the approximation of the area [sic] under this figure [referring to the graph in Figure 1], obviously it wouldn't be as precise as the value of the integral. Let's see... so geometrically 0,1, *x*, *y*, let's draw a square here like this [he is now evaluating and drawing rectangular prisms]... 0, 1.5, maybe another square closer this way, higher... 1,1 we are still at *x* 1 and even higher here… like this, change in *x* change in *y*... change in *y* being 1/2, I don't think we get from 1 to 2 with 1/2 [He seems to believe that since $\Delta y = 1/2$ the prisms will be restricted to the region $1 \le y \le 1.5$. He might think of Δy as "change in *y*" were the "change" is taken from the initial *y* value in point (0,1).], so the integral would give this area [sic] here, a figure more or less like this...

Interviewer: You said area...

Brian: [Interrupting] Volume, I mean volume, sorry... yes, volume of the integral. This would give us something more stepwise... let's see if I can draw it here like this... 1,2,3,4, like this, a series of cubes like this, stepwise, approximating, not all of this, but only this half here… [See Figure 2].

Interviewer: Do you mean the left hand part of the solid?

Brian: Yes, the left hand part of the solid would be what is approximated with this Riemann sum.

Although Brian was able to construct the meaning of volume, his construction was not right, the boxes he drew filled only the left-hand side of the rectangle. It seems that Brian could do the Actions to construct the prisms but he did not interiorize those Actions into the Process that would enable him to imagine all the constructions needed to relate Riemann sums and double integrals.

An interesting result of this experience was that even though students showed many difficulties during the interview, some of them, like Brian, showed evidence of doing some of the expected constructions during the interview. Others reflected during the interview and constructed meaning. This was the case of Victor who had considered $f(0,1)\Delta x\Delta y$ as an area. When discussing Riemann sums, and after being told that this product represents a volume he explained:

Interviewer: So you drew a little box.

Victor: Exactly a little box as we know that delta x would be 2 and delta y 1; a rectangle with width 2, eh, length 2 with 2 and height 1.

Interviewer: Then how do you compare those volumes.

Victor: Ok, now I understand, this f(0,1) delta x delta y is only the volume up to this point, I mean up to a certain height, and then, the double integral on that same area that we put on xy is, let's say, the same box but with a height that varies with the function. Now this is it!

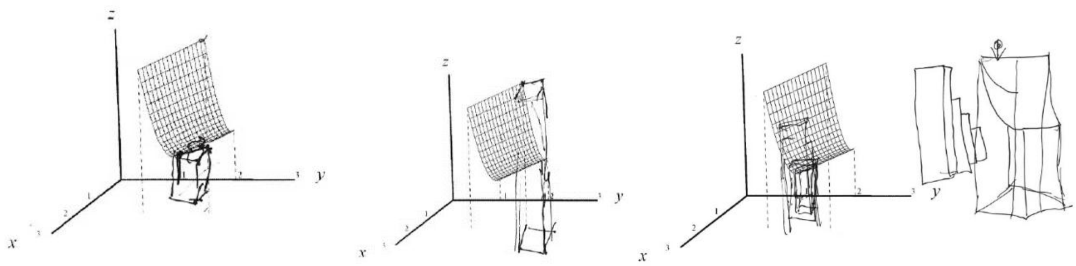

Figure 2: Brian's drawing for Problem 1d, 1e, 1g (respectively)

Interviewer: But which is larger, what is smaller, are they equal?

Victor: No, no, they are completely different, the larger is that obtained from the double integral since the height is higher.

Victor: *f*(2,2) delta *x* delta *y* is the box with dimensions over *D*…and then this is the same equation as before, but *f*(2,2) is higher so the volume there is larger.

Victor could do the Actions needed to compare volumes of prisms obtained from different values of the function. Another student, Farid, evidenced he could imagine forming one term of a Riemann sum, as discussed before. He also showed to have done the constructions necessary to imagine volumes of prisms and their role in Riemann sums. When comparing the volume of the prism in Problem 1d with the double integral:

Interviewer: And what does that represent? [Referring to the double integral.]

Farid: That represents the volume between the surface and the plane, the domain...

Interviewer: Then, how do those two numbers there compare?

After some doubts:

Farid: Represented this part [referring to the product], now, this product would be smaller, than the double integral, because this here is a, I represented it as a cube, given the value of *f* at that point, while this is the double integral of everything, of all the function *x*,*y* over *D*, so this value seems bigger

Interviewer: Which one?

Farid: The value of the double integral over *D* of *f*(*x*,*y*)*dA*

He was able to compare the volume of the prisms with the double integral. When he had to decide if there would be a prism with volume equal to the value of the double integral, a problem that was impossible for all the other students, he explained:

Farid: … The thing that comes to my mind when thinking on an inequality is the sandwich theorem…that there... must exist then a value for *x* and *y* that could be named *a* and *b* that is equal to the double integral on *dA*.

When discussing Riemann sums with a specific partition, most students could not work with the problem, even with help from the interviewer. As was shown by Brian in the previous example, some students, including Victor, imagined drawing several prisms or boxes that shared the base, and only had different heights. Those students showed they could do the Action of changing the height of a given prism but not that of partitioning the domain into small areas of the same size. Victor could describe the sum of the prisms' volume, at first he said that the Riemann sum was always an approximation to the volume under the surface, although later he reconsidered:

Victor: No, the Riemann sum is an approximation and if you take more subintervals, ah! If you take more subintervals, those were the squares, that one uses, the Riemann sum is a closer approximation and that approximation would be closer with more subintervals, and the double integral is the exact value.

Only Victor and Farid seemed to have interiorized the Action of forming a partition into a Process they coordinated with the Process of selecting heights for each subrectangle into the Process of calculating the volume corresponding to the prisms to approximate the volume under the surface.

DISCUSSION AND CONCLUSION

Results from this experience show that most of these students demonstrate a very limited understanding of two-variable functions and of those concepts associated to the construction of the double integral of a two-variable function and its geometrical interpretation. Only two students showed some understanding, although one of them relied mostly in memorized facts that he could use appropriately in most cases. This student seems to have constructed meaning for some of those facts during the interview. Students' responses show the importance of the predicted constructions included in the Genetic Decomposition. In this investigation we related observed difficulties with specific mental constructions in the GD that students seemed to lack. The importance of the pre-requisite constructions in learning this difficult topic was underscored. Its lack became an insurmountable obstacle to understand even the most basic ideas leading to the learning of the double integral.

A more encompassing understanding of function in different representation registers proved to be indispensable. Results indicate that students who could only perform Actions constructed a confusing network of concepts where the properties learnt about one-variable function are not well differentiated from those of two-variable functions. This inhibits their possibility to make those constructions involved in the understanding of 3D space, functions, and their domains. Fluency in operating within and across different representations plays an important role in the construction of two-variable functions as an Object, instead of considering them as simple correspondence rules containing one or more variables.

These results emphasize once more the importance of spending more time on helping students to construct the notion of two-variable function. But, even when two-variable function has been constructed as a Process, the notions of volume under the surface and the role of the Riemann sum in the construction of the double integral constitute fundamental constructions in the learning of double integrals.

The genetic decomposition proved useful in determining and underscoring those mental constructions that are needed to learn double integrals with meaning. It also reveals the subtleties involved in learning the double integral. After classroom use of specially designed activities, future studies may reformulate the same interview problems and also extend them to explore other ideas of the integral calculus.

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Teaching and learning continuity with technologies

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We developed a digital tool aiming at introducing the concept of – local - continuity together with its formal definition for Tunisian students at the end of secondary school. Our approach is a socioconstructivist one, mixing conceptualisation in the sense of Vergnaud together with Vygotski's concepts of mediation and ZPD. In the paper, we focus on the design of the tool and we give some flashes about students' productions with the tool and teachers' discourses in order to foster students' understanding of the continuity.

Keywords: teaching and learning of analysis and calculus, novel approaches to teaching, continuity, digital technologies

The definition of continuity of functions at a given point, together with the concept of continuity, remains a major difficulty in the teaching and learning of analysis. There is a dialectic between the definition and the concept itself which make necessary the introduction of the two aspects together.

The definition of continuity brings FUG aspects in the sense of Robert (1982). This means first that it permits to formalize (F) the concept of continuity. But it also allows to unify (U) several different images (or situations) of continuity encountered by students: in Tall and Vinner (1981), several emblematic situations of continuity are established (see below) and the definition aims at unifying all these different kinds of continuity. Moreover, the definition of continuity allows generalisations (G) to all other numerical functions, not already encountered and not necessarily with graphical representations, or more general functions inside other spaces of functions. As Robert (1982) stresses for the definition of limit of sequences, notions which bring FUG aspects must be introduced with a specific attention to mediations and especially the role of the teacher.

Our ambition is then to design a technological tool which allows on one hand students activities concerning the two aspects of continuity and, on the other hand, allows the teacher to introduce the concept of continuity with its formal definition, referring to the activities developed on the technological tool. As it was noticed in the first INDRUM conference, papers about introduction of technologies in the teaching of analysis remain very few.

We first come back to well-known concept images and concept definitions of continuity. Then, we explain our theoretical frame about conceptualisation and mathematical activities. This theoretical frame leads us to the design of the technological tool which brings most of the aspects we consider important for the conceptualisation of continuity. Due to the text constraints, the results of the paper are mostly in term of the design itself and the way the tool encompasses our theoretical frame and our hypotheses about conceptualisation (with tasks, activities and opportunities for mediations). Then, we can give some flashes about students' activities with the software and also teachers' discourses to introduce the definition of continuity, based on students' mathematical activities on the software.

CONCEPT IMAGES AND CONCEPT DEFINITIONS OF CONTINUITY

No one can speak about continuity without referring to Tall and Vinner's paper about concept images and concept definitions in mathematics, whose particular reference is about limits and continuity (Tall and Vinner, 1981). Tall considers that the concept definition is one part of the total concept image that exists in our mind. Additionally, it is understood that learners enter their acquisition process of a newly introduced concept with preexisting concept images.

Sierpinska (1992) used the notion of epistemological obstacles regarding some properties of functions and especially the concept of limit. Epistemological obstacles for continuity are very close to those observed for the concept of limit and they can be directly relied to students' concept images, as a specific origin of theses conceptions (El Bouazzaoui, 1988). One of these obstacles can be associated to what we call a *primitive concept image*: it is a geometrical and very intuitive conception of continuity, related to the aspects of the curve. With this concept image, continuity and derivability are often mixed and continuity means mainly that the curve is smooth and have no angles. Historically, this primitive conception leads Euler to introduce a definition of continuity based on algebraic representations of functions. This leads to a second epistemological obstacle: a continuous function is given by only one algebraic expression, which can be called the *algebraic concept image* of continuity. This conception has led to a new obstacle with the beginning of Fourier's analysis. Then, a clear definition is necessary. This definition comes with Cauchy and Weierstrass and it is close to our actual formal definition.

We also refer to Bkouche (1996) who identifies three points of view about continuity of functions which are more or less connected to the epistemological obstacles we have highlighted. The first one is a *cinematic point of view*. Bkouche says that the variable pulls the function with this *dynamic* concept image. The other one is an *approximation point of view*: the desired degree of approximation of the function pulls the variable. This last point of view is more *static* and leads easily to the formal definition of continuity. These two points of view are also introduced by Robert (1982) when she studies the introduction of the formal definition of limit (for sequences). A third point of view is also identified by Bkouche that is the *algebraic point of view*, which is about algebraic rules, without any idea of the meaningful of these rules.

At last, we refer to more recent papers and specifically the one of Hanke and Schafer (2017) about continuity in the last CERME congress. Their review of central papers on concept images about students' conceptions of continuity leads to a classification of the eight possible mental images that are reported in the literature: *I : Look of the graph of the function : "A graph of a continuous function must be connected"* - *II : Limits and approximation : "The left hand side and right hand side limit at each point must be equal"* - *III : Controlled wiggling : "If you wiggle a bit in x, the values will only wiggle a bit, too"* - *IV : Connection to differentiability : "Each continuous function is differentiable"* - *V : General properties of functions : "A continuous function is given by one term and not defined piecewise"*- *VI : Everyday language : "The function continues at each point and does not stop"* - *VII : Reference to a formal definition : "I have to check whether the definition of continuity applies at each point"* -*VIII : Miscellaneous*

We can recognize some of the previous categories, even if some refinements are brought. Mainly, concept images I, II, IV and VI can be close to the primitive concept image whereas VII refers to the formal definition and V seems to refer to the algebraic approach of continuity.

CONCEPTUALISATION OF CONTINUITY

We base our research work on these possible concepts image and concepts definition of continuity. However, we are more interested in conceptualisation, as the process which describes the development of students' mathematical knowledge. Conceptualisation in our sense has been mainly introduced by Vergnaud (1990) and it has been extended within an activity theoretical frame developed in the French didactic of mathematics. These developments articulate two epistemological approaches: that of mathematics didactics and that of developmental cognitive psychology as it is discussed and developed in Vandebrouck (2018).

Broadly, conceptualisation means that the developmental process occurs within students' actions over a class of mathematical situations, characteristic of the concept involved. This class of situations brings technical tasks – direct application of the concept involved - as well as tasks with adaptations of this concept. A list of such adaptations can be found in Horoks and Robert (2007): for instance mix between the concept and other knowledge, *conversions between several registers of representations* (Duval 1995), *use of different points of view*, etc. Tasks that require these adaptations of knowledge or concepts are called complex tasks. These ones encourage conceptualisation, because students become able to develop high level activities allowing availability and flexibly around the relevant concept.

A level of conceptualisation refers to such a class of situations, in a more modest sense and with explicit references to scholar curricula. In this paper, the level of conceptualisation refers to the end of scientific secondary school in Tunisia or the beginning of scientific university in France. It supposes enough activities which can permit the teacher to introduce the formal definition of continuity together with the sense of the continuity concept. The aim is not to obtain from students a high technicity about the definition itself – students are not supposed to establish or to manipulate the negation of the definition for instance. However, this level of conceptualisation supposes students to access the FUG aspects of the definition of continuity.

Of course, we also build on instrumental approach and instrumentation as a sub process of conceptualisation (Rabardel, 1995). Students' cognitive construction of knowledge (specific schemes) arise during the complex process of instrumental genesis in which they transform the artifact into an instrument that they integrate within their activities. Artigue (2002) says that it is necessary to identify the new potentials offered by instrumented work, but she also highlights the importance of identifying the constraints induced by the instrument and the instrumental distance between instrumented activities and traditional activities (in paper and pencil environment). Instrumentation theory also deals with the complexity of instrumental genesis.

We also refer to Duval's idea of visualisation as a contribution of the conceptualisation process (even if Duval and Vergnaud have not clearly discussed this point inside their frames). However, the technological tool brings new dynamic representations, which are different from static classical figures in paper and pencil environment. These new representations lead to enrich students' activities – mostly in term of recognition - bringing specific visualization processes. Duval argues that visualization is linked to visual perception, and can be produced in any register of representation. He introduces two types of visualization, namely the iconic and the non-iconic, saying that in mathematical activities, visualization does not work with iconic representations (Duval, 1999).

At last, we refer on Vygotsky (1986) who stresses the importance of mediations within a student's zone of proximal developmental (ZPD) for learning (scientific concepts). Here, we also draw on the double approach of teaching practices as a part of French activity theory coming from Robert and Rogalski (2005). The role of the teacher' mediations is specifically important in the conceptualisation process, especially because of the FUG aspects of the definition of continuity (as we have recalled above).

First of all, we refine the notion of mediation by adding a distinction between procedural and constructive mediations in the context of the dual regulation of activity. Procedural mediations are object oriented (oriented towards the resolution of the tasks), while constructive mediations are more subject oriented. We also distinguish individual (to pairs of students) and collective mediations (to the whole class).

Secondly, we use the notion of proximities (Bridoux, Grenier-Boley, Hache and Robert, 2016) which are discourses' elements that can foster students' understanding – and then conceptualisation - according to their ZPD and their own activities in progress. In this sense, our approach is close to the one of Bartolini Bussi and Mariotti (2008) with their Theory of Semiotic Mediations. However, we do not refer explicitly at this moment to this theory which supposes a focus on signs and a more complex methodology than ours. According to us, the proximities characterize the attempts of alignment that the teacher operates between students' activities (what has been done in class) and the concept at stake. We therefore study the way the teacher organizes the movements between the general knowledge and its contextualized uses: we call ascending proximities those comments which explicit the transition from a particular case to a general theorem/property; descending proximities are the other way round; horizontal proximities consist in repeating in another way the same idea or in illustrating it.

DESIGN OF THE TECHNOLOGICAL TOOL

The technological tool called "TIC-Analyse" is designed to grasp most of the aspects which have been highlighted above. First of all, it is designed to foster students' activities about continuity aspects in the two first points of view identified by Bkouche: several functions are manipulated – continuous or not – and for each of them, two windows are in correspondence. In one of the window, the cinematic-dynamical point of view is highlighted (figure 1) whereas in the second window the approximation-static point of view is highlighted (figure 2).

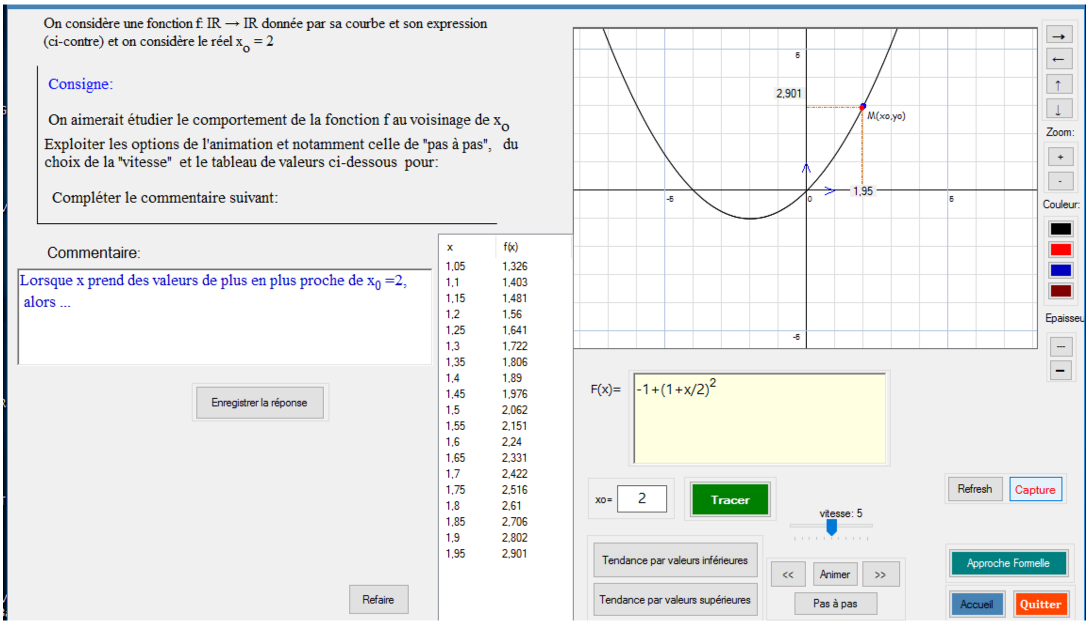

Figure 1: two windows for a function, the dynamic point of view about continuity

The correspondence between the two points of view is in coherence with Tall's idea of incorporation of the formal definition into the pre-existing students' concept images. It is also in coherence with the importance for students to deal with several points of view for the conceptualisation of continuity (adaptations).

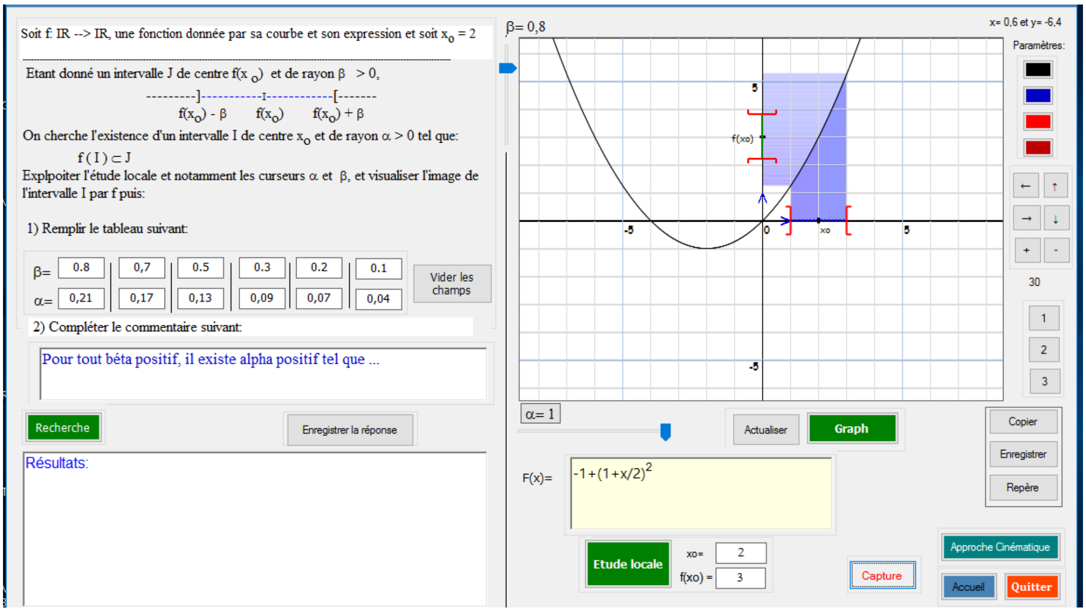

Figure 2: two windows for a function, the static points of view about continuity

In second, the functions at stake in the software are extracted from the categories of Tall and Vinner (1981). For instance, we have chosen a continuous function which is defined by two different algebraic expressions, to avoid the algebraic concept image of continuity and to avoid the amalgam between continuity and derivability. We also have two kinds of discontinuity, smooth and with angle.

There is an emphasis not only on algebraic representations of functions in order to avoid algebraic conceptions of functions. Three registers of representations of functions (numerical, graphical and algebraic) are coordinated to promote students' activities about conversions between registers (adaptations).

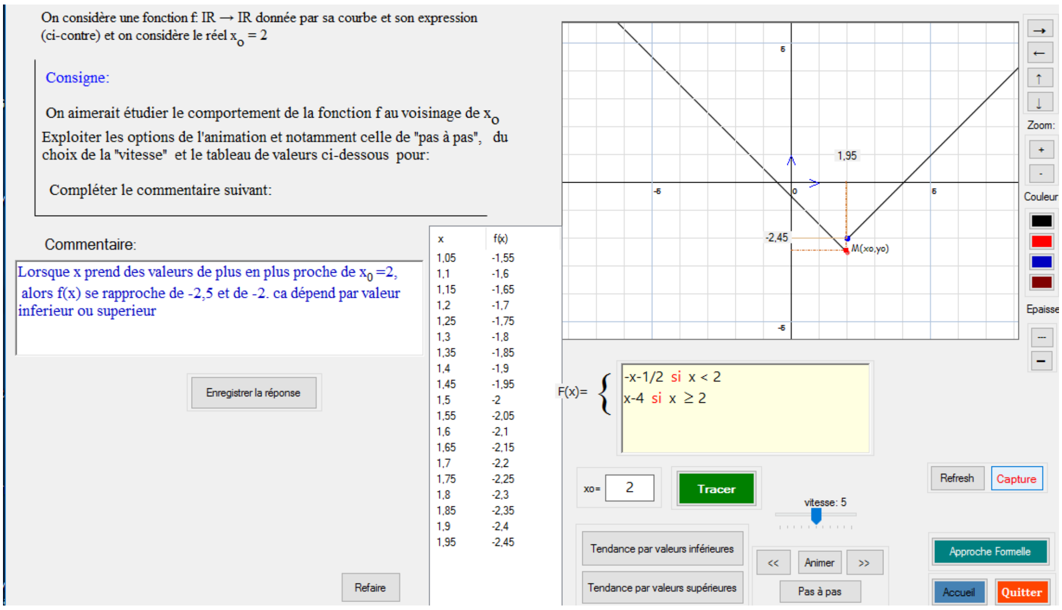

Figure 3: example of commentary given by a pair of students in the dynamic window

The design of the software is coherent with the instrumental approach mostly in the sense that the instrumental distance between the technological environment, the given tasks, and the traditional paper and pencil environment is reduced. However the software produces dynamic new representations – a moving point on the curve associated to a numerical table of values within the dynamic window; two static intervals, one being included or not in the other, for the static window – occurring non iconic visualisations which intervene in the conceptualisation process.

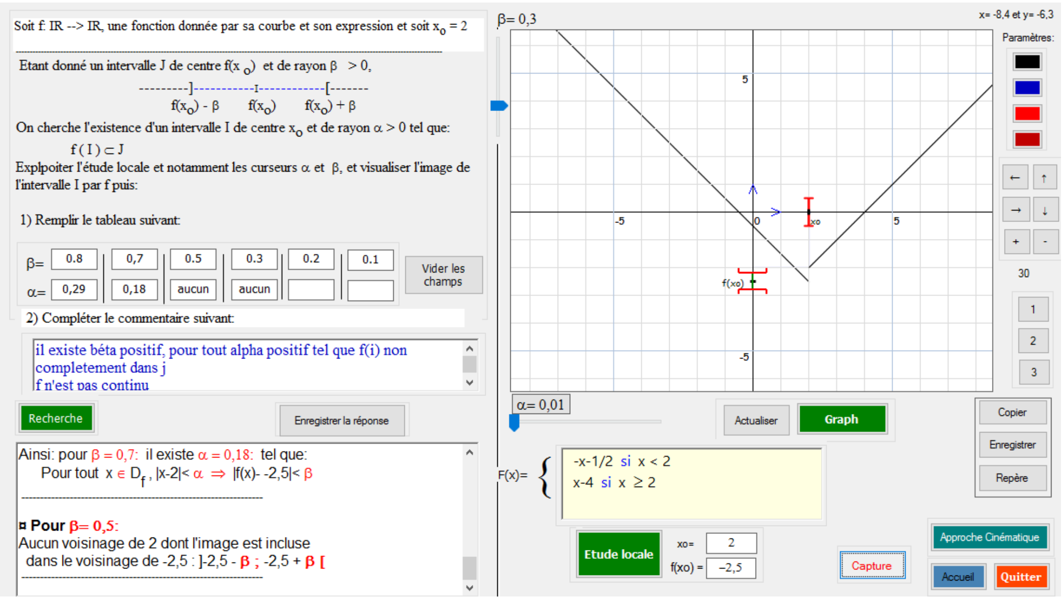

Figure 4: example of commentary given by a pair of students in the static window

The software promotes students' actions and activities about given tasks: in the dynamic window, they are supposed to command the dynamic point on the given curve – corresponding to the given algebraic expression. They can observe the numerical values of coordinates corresponding to several discrete positions of the point and they must fill a commentary with free words about continuity aspects of the function at the given point (figures 1, 3). In the static window, they must fill the given array with values of α, the β being given by the software (figures 2, 4). Then, they have to fill a commentary which begins differently according to the situation (continuity or not) and the α they have found (figures 4, 5).

As we have mentioned in our theoretical frame, students are not supposed with these tasks and activities to get the formal definition by themselves. However, students are supposed to have developed enough knowledge in their ZPD so that the teacher can introduce the definition together with the sense and FUG aspects of continuity.

STUDENTS ACTIVITIES AND TEACHER'S PROXIMITIES

The students work by pair on the tool. The session is a one hour session but four secondary schools with four teachers are involved. Students have some concept images of continuity but nothing has been thought about the formal definition. The teacher is supposed to mediate students' activities on the given tasks. Students are not supposed to be in a total autonomy during the session according to our socio constructivist approach.

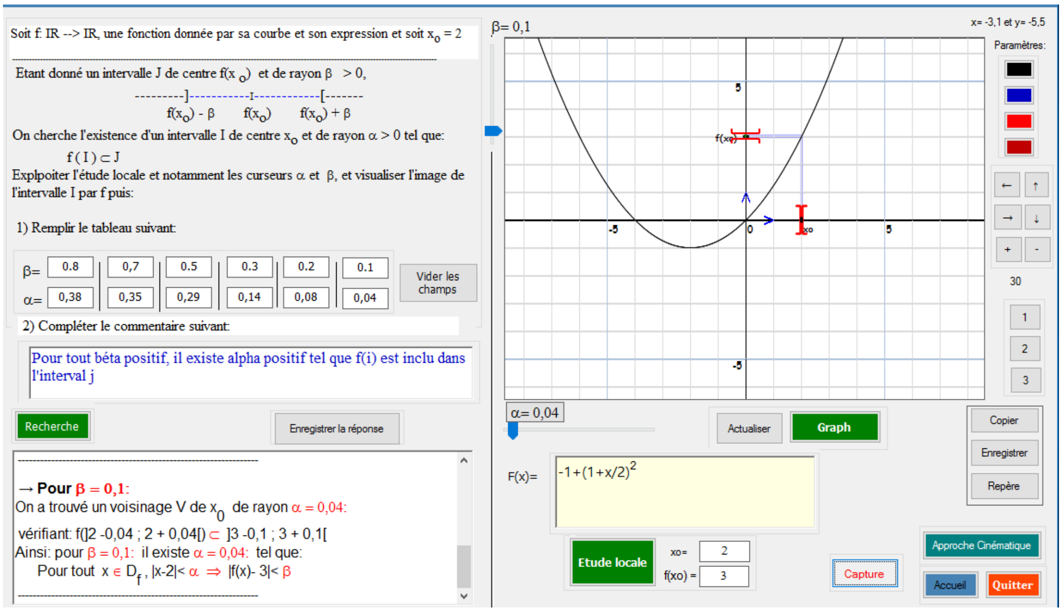

Figure 5: example of commentary given by a pair of students in the static window

We have collected video screen shots, videos of the session (for each schools) and recording of students' exchanges in some pairs. Students' activities on each tasks are identified, according to the tasks' complexity (mostly kinds of adaptations), their actions and interactions with computers and papers (written notes), the mediations they receive (procedural or constructive mediations, individual or collective, from the tool, the pairs or the teacher) and the discourses' elements seen as "potential" proximities proposed by the teacher.

It appears that the teacher mostly gives collective procedural mediations to introduce the given tasks, to assure an average progression of the students and to take care of the instrumental process. Some individuals mediations are only technical ones ("*you can click on this button*"). Some collective mediations are most constructive such as "*now, we are going to see a formal approach. We are going to see again the four activities* (ie *tasks*) *but with a new approach which we are going to call formal approach...*". The constructive mediations are not tasks oriented but they aim at helping students to organize their new knowledge and they contribute to the aimed conceptualisation according to our theoretical approach.

As examples of students' written notes (as traces of activities), we can draw on figure 3 and 4. A pair of students explains the dynamic non-continuity with their words **"***when x takes values more and more close to 2 then f(x) takes values close to -2,5 and -2. It depends whether it's lower or higher*" (figure 3) which is in coherence with the primitive concept image of continuity. The same pair of students explains the non-continuity in relation to what they can observe on the screen: "*there exists β* *positive, for all α positive* – already proposed by the tool in case of non-continuity - *such that f(i) not completely in j… f is not continuous*". We can note that the students are using "completely" to verbalize that the intersection of the two intervals is not empty. However, the inclusiveness of an interval into another one is not expected as a formalized knowledge at this level of conceptualisation. Their commentary is acceptable. Students are expressing what they have experimented several times : for several values of β (β = 0,3 in figure 4), even with α very small (α = 0,01 in figure 4), the image of the interval]2- α, 2+ α[is not included in]-2,5- β, -2,5+ β[. Concerning a case of continuity, the students are also able to write an acceptable commentary (figure 5) "*for all β positive, their exists α positive* – already proposed by the tool in case of continuity – *such that f(i) is included in j.*"

Students' activities on the given tasks are supposed to help the teacher to develop proximities with the formal definition. It is really observed that some students are able to interact spontaneously with the teacher when he wants to write the formal definition on the blackboard. This is interpreted as a sign that the teacher's discourse encounters these students' ZPD. Then the observed proximities seem to be horizontal ones: the teacher reformulates several times the students' propositions in a way which lead gradually to the awaited formal definition, for instance "*so, we are going to reformulate, for all β positive, their exists α positive, such that if x belong to a neighbour of α … we can note it* $x_0 - \alpha$, $x_0 + \alpha$*….*"

Of course, it is insufficient to ensure proof and effectiveness of our experimentation. The conceptualisation of continuity is an ongoing long process with is only initiated by our teaching process. However, we want to highlight here the important role of the teacher and more generally the importance of mediations in the conceptualisation process of such a complex concepts. We only have presented the beginning of our experimentation. It is completed by new tasks on the tool which are designed to come back on similar activities and to continue the conceptualisation process.

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TWG 2: Mathematics for engineers
- Mathematical Modelling
- Mathematics and other disciplines

The ecological relativity of modelling practices: adaptations of an study and research path to different university settings

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This paper focuses on the problem of the ecology of mathematical modelling practices at university level through the systematic variation of teaching institutions. Our aim is to deal with the variety of constraints appearing when modelling is implemented in university classrooms, and to study the way new teaching proposals can overcome them. Within the framework of the anthropological theory of the didactic, a teaching and learning proposal in terms of study and research paths in tertiary education shows new possibilities to surmount some of these constraints. The paper presents the design and successive adaptations of an SRP about an urban bike-sharing system according to the specificities of different university institutions and the reactions obtained by the students and lecturers.

Keywords: Modelling; anthropological theory of the didactic; research and study path; ecology; institutional relativity.

INTRODUCTION

The starting point of this research is delving into the problem of studying the variety of constraints appearing when mathematical modelling proposals are implemented in university classrooms, impeding their regular development, and to study the way new teaching proposals can overcome them. Several research projects have highlighted the existence of strong constraints impinging on the *large-scale dissemination of mathematics as a modelling activity* in current educational systems at all school levels (Doerr & Lesh, 2011; Kaiser & Maaβ, 2007). We use the term *ecology* to refer to the institutional conditions allowing and the constraints hindering the way a given activity is produced, transposed, taught and learned in a given educational setting.

In previous research developed in the framework of the anthropological theory of the didactic (ATD), we propose the use of a general frame to detect and place the institutional constraints hindering the possible large-scale dissemination of modelling activities based on a hierarchy of levels of didactic co-determinacy (Chevallard, 2002). In Barquero, Bosch and Gascón (2013), we use this general frame to detect constraints appearing at different levels, from the specific ones related to how mathematical contents are proposed to be taught at school, to the more general ones regarding the general organisation of school activities and the role assigned to schools in our societies. This ecological analysis shows how institutional constraints are anchored in deep-rooted practices and are difficult – for teachers and also for researchers – to notice since they appeared as "the natural way of doing". For instance, Barquero et al. (2013) characterise and empirically contrast the predominance of "applicationism" as the dominant way of interpreting, describing and conceptualizing mathematical modelling in natural sciences university degrees. Under its influence, modelling is understood as a mere application of previously constructed knowledge, as if the construction of knowledge were independent of its use. At a more general level, in many schools the prevailing pedagogy is still strongly influenced by the paradigm of "visiting works" (Chevallard, 2015), according to which school knowledge organisations are presented as interesting monuments to visit, instead of as useful tools to provide answers to problematic questions.

In this paper, we focus on going one-step to study the *ecological relativity* of modelling practices in university institutions. As it is described in Castella (2004) and Sierra (2006), each institution endures an institutional relation with knowledge, in particular, with mathematical knowledge. Consequently, each institution establishes a set of specific conditions and constraints that can favour or, on the contrary, prevent certain teaching and learning processes and knowledge constructions to be appropriately developed. It is in this aspect where we want to look more carefully. Therefore, we focus on analysing the emergence, persistence and scope of the conditions and constraints for development of modelling through a variation of university institution. In our research, we work on the use of the *study and research paths* (SRP) as epistemological and didactic model (Chevallard, 2015; Winslow et al., 2013; Barquero et al., 2018) where mathematics are conceived as a modelling tool for the study of problematic questions. We here present an SRP based on an urban bike-sharing system inaugurated in Barcelona in 2007 that has been experimented in three different university settings. The starting point of this SRP is the difficulty to get a homogeneous distribution of bicycles in a city with many sloping streets. We present the successive transformations of the SRP to three different university settings, according to the specificities of each institution, and to the reactions from students and lecturers. Some of the commonalities found show the stable constraints hindering the development of the SRP, whereas the differences detected bring new insights about the conditions to surmount them.

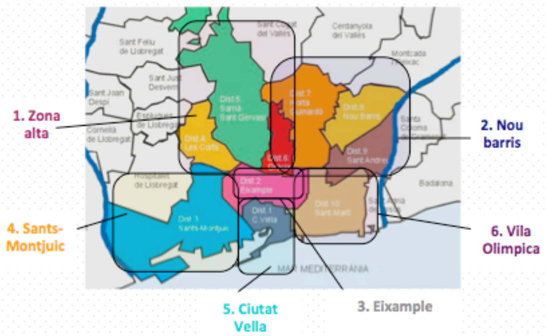
DESIGN OF AN SRP ABOUT A SHARING-BIKE SYSTEM

In the following we describe the initial design of the study and research path (SRP) about the sharing-bike system whose starting point is the generating questions (Q_0) about *how to improve the distribution of bikes in the 'Bicing' system to provide a better service to users*. When working with the *a priori* design of the SRP, there are foreseen several derived questions from Q_0 that needs from a progressive modelling process. In general terms, the modelling project was organised around the following questions that structured the two phases the *Bicing* project:

$Q_{(A)}$: How can we describe the daily flow of bikes between stations? What is the natural behaviour of the system when it is left alone (without redeployment)?

$Q_{(B)}$: How can we predict the bikes' redeployment needs? Which changes can be proposed to improve the current policy of bikes redeployment in the city?

Linked to these questions, we consider real data from *Bicing* about the distribution of bikes among the different bikes' stations. We, the researchers and the experts who collaborated with us, agreed to organise these data in certain city areas according to the similarities different stations shared on the pattern of daily bikes trips and routes followed. Finally, we decided to present the data organised in six areas (as shown in Table 1), which corresponds to the origin-destination matrix (OD matrix) containing the potential number of daily bikes' uses. Each number $\{od_{ij}\}$ means the average of the amount of bike traveling in a day from area j and arriving to area i.



	ARRIVALS	z1	z2	z3	z4	z5	z6
	DEPARTURES	Z_ALTA	N_BARRIS	EIXAMPLE	S_MONTJUIC	C_VELLA	V_OLIMPICA
z1	Z_ALTA	2009	297	2084	1088	589	646
z2	N_BARRIS	207	2356	424	29	149	701
z3	EIXAMPLE	1153	401	4332	1900	2263	1179
z4	S_MONTJUIC	886	32	1649	2594	1071	153
z5	C_VELLA	196	113	2072	895	3572	1264
z6	V_OLIMPICA	101	462	1058	504	1004	2151

Table 1: Origin-Destination matrix with daily bikes' trips

To face the first question $Q_{0(A)}$, and going beyond the descriptive analysis of the data contained in the OD matrix, models based on recurrent sequences of order $d > 1$ can be considered, which are equivalent to matrix recurrent sequences $X(n) = f(X(n-1))$ where $X(n) = (x_1(n), x_2(n), \ldots, x_6(n))$ is the vector with the bike distribution in each of the six areas at time n. Next we summarize the *a priori* design in terms of hypothesis (H), questions (Q) and answers (A) delimited by the researchers about the models that might be used in an implementation of the SRP.

One of the easier assumptions we can work with is considering that:

$H_{(A)1}$: There is no redeployment of bikes in the system and the bike flows between stations is the same every day.

$Q_{(A)1.1}$: Then, if we deploy different amounts of bikes in each station, what will be the distribution of bikes after 1, 2, 3,…, n days?

The model that can be considered under these assumptions is:

$$X(n) = M.X(n-1) \rightarrow X(n) = M^n . X(0) \text{ for } n > 0 \quad (1)$$

where M is the *transition matrix* (or *transition probability matrix*) obtained from the OD matrix, where $\{m_{ij}\}$ is the percentage of transition between two areas. That is, the potential number of daily travels with origin in j and arriving to i $\{(od_{ij})\}$ divided by total amount of departures from j ($d(j)$). When working with this first model, several questions can appear:

$Q_{(A)1.2}$: Working with the transition matrix and with different $X(0)$ at the beginning of the day, which traits from the trajectory of $X(n)$ can be underlined?

$Q_{(A)1.3}$: Does it exist any fixed point X^f to which the sequence $X(n)$ converges to? Do all $X(n)$ converge towards a fixed point X^f? Is it possible to calculate X^f in advance?

$Q_{(A)1.4}$: Which relation there exist between X^f and the *n*-power of the transition matrix?

And, it can easily appear questions about the limitations of the hypothesis assumed and models built, such as:

$Q_{(A)1.5}$: How can include other factors that are important for *Bicing*, such as: the total amount of trips made by a bike, the potential demand of bikes, the available bikes?

Introducing questions about how to improve our hypothesis and the models to be more realistic with the system we want to analyse can open many possibilities. One possible new reformulation of the hypothesis we can work with is:

$H_{(A)2}$: We assume that (1) each bike trip takes about *t* minutes, (2) the entire fleet of bikes does not move every *t* min, (3) the total number of bikes that moves in period *t* depends on: (a) the potential demand for bike trips, and (b) the amount of bikes available.

At this point, there appear more complex models where it is important to frame the time *t*, for instance, $t = 30$ minutes (which it is the average of a bike trip in *Bicing*). Then, we can define $B_i(t)$ as the number of bikes in an area at time t and $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t), B_5(t), B_6(t))$ as the vector with the bikes distribution in each area. Then, if we define the departures as $D(t) = (D_1(t), D_2(t), D_3(t), D_4(t), D_5(t), D_6(t))$ and the arrival as $A(t) = (A_1(t), A_2(t), A_3(t), A_4(t), A_5(t), A_6(t))$, $B(t)$ can be modelled by:

$$B(t+1) = B(t) - D(t) + A(t+1) \quad (2)$$

where $D(t) = \min [\text{demand\_trips}(30 \text{ min}), B(t)]$ and $A(t+1) = M \cdot D(t)$, with M the transition matrix in time periods *t*. When this second model is considered, several questions can guide the study process:

$Q_{(A)2.1}$: Using this model (2), and considering different initial distribution of bikes at the beginning of the day $B(0)$, which will be the bike distribution $B(t)$ at the end of the day? And, if the system is left alone, after 2, 3, 4, …, 30 days?

$Q_{(A)2.2}$: Which traits can we underline about the trajectory of $B(t)$ through the simulation of model (2)? Are there also some fixed points to which the sequence $B(t)$ converge?

$Q_{(A)2.3}$: Is there any relation between the fixed points X^f we reach with the ones detected with model (1)?

$Q_{(A)2.4}$: Which relationship is there between the first and second models, defined in (1) and by (2)? Which of the two models do integrate more realistic conditions about *Bicing*?

In the next section we retake this a priori design of the SRP in terms of Q_0 and the likely hypothesis and derived questions $Q_{(A).n}$ to analyse the particular implementation of the SRP about Bicing project in the different university institutions. Besides underlying the adaptations that were necessary to the SRP in each university institution, we focus on the most important conditions (common or not) that favour the development of the SRP, and consequently of the modelling practice. In most of the occasions, these conditions and constraints were phrases by the students and lecturers involved in the implementations or by the survey and interview done at the end of each implementation.

ECOLOGICAL ANALYSIS OF THE SRP IN DIFFERENT UNIVERSITY INSTITUTIONS

First SRP adaptation: The 'Bicing project' at the University of Copenhagen

The first implementation of the SRP about the bikes' distribution in the *Bicing* system took place in the University of Copenhagen (UC). Twenty-three students participated in this implementation. They were taking the course called *MathMod* (Mathematical Modelling), which was an optional course in the third year of the Mathematics degree. The course run over seven weeks, plus two extra weeks to prepare their final team project. The course had three weekly sessions of two hours each. In general terms, the first session was a lecture, the second was a practical or exercise-based session to practice the content introduce in the previous lecture and, the third one, to work in teams in the computer room to simulate by Mapple some models introduced along the course or to work on the team final project. The teaching course proposal was based on the realization of four short projects (mini-projects), linked to some practical activities. These mini-projects mostly consisted of being introduced to some pre-existing models in the lectures sessions to then asked students to put them into practice in the practical sessions. Some example of the project composing the course are: "Mini-project 1: Using the Malthusian and logistic models to predict population evolution" or "Mini-project 4: The Lotka-Volterra models".

In the academic year 2009/10, the author of the paper participated in this course as researcher and the lecturer offered the opportunity of implementing the SRP about *Bicing*. It was integrated as the fifth (and last) project of the course. The SRP implementation ran over two weeks, with six sessions of two hours. At the end of each week, students working in teams had to deliver a report with their temporary results of the *Bicing* project. It was necessary to *break with the above-mentioned organisation* of the course sessions and to set up time for the presentations by the lecturer-researcher and for students' presentation. There, students could compare their proposals and to collectively agree how to follow. During the first week, once the generating question Q_0 was presented by the lecturer-researcher, students agreed to firstly focused on $Q_{0(A)}$ from where students developed most of the path described in the previous section about model (1). In the second week, we (students and instructors) worked on how to reformulate the $H_{(A)2}$ and $Q_{(A)2}$, as most of the groups noticed that in model 1 there were considered some unrealistic assumptions. Due to time restrictions, we could not go further the second model. Finally, each team had to deliver a report one week later the ending of the project with some suggestions for Bicing about how to improve their bike replacing system, $Q_{0(B)}$. Figure 1 summarizes the path followed in this first implementation.

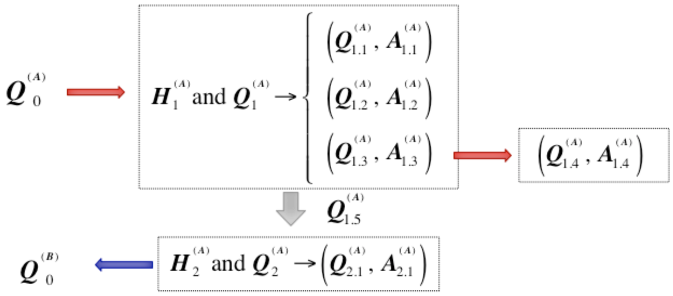


Figure 1: Summary of the path follow in the first implementation of the SRP at UC

We counted on different conditions that favour that the SRP progress fruitfully. First, as it was the fifth project of the course, and the course was explicitly focus on modelling, students and lecturers shared a *common discourse* to refer to modelling. This was an important condition for modelling to be noticed (Barquero et al. 2013). Secondly, the second mini-project was about Leslie matrices and transition matrices. It thus facilitated that students autonomously posed many new questions, such as $Q_{(A)1.3}$ and $Q_{(A)1.4}$ and, thanks to the previous work developed with Maple, students easily worked on calculating and simulating sequences and studying their convergence. On the contrary, there were also some constraints that were manifested by students mostly at the beginning of the SRP implementation. When we started with the Bicing project, students were astonished by the new responsibilities that they were asked, such as: formulating hypothesis, looking for and building models, testing models' appropriateness, formulating new questions, writing a report without any predetermined structure, etc. Although their initial confusion, consequence of a big rupture with the didactic contract established in the course, they started assuming these new responsibilities. In the previous activities of the course, students were only asked to "apply" the models they had been introduced to. So that, breaking some rules of the didactic contract and make students responsible of several new tasks in the modelling process were the main constraints we had to surmount. In fact, the course organisation shown many traits (and constraints) derived from "applicationism" (Barquero et al. 2013). For instance, it was assumed (throughout the course organisation) that the mathematical models had to be introduced in advanced and then applied to different situation, models that are rarely questioned and hardly reformulated. When the Bicing project started, many students' resistances appeared that reflected the implicit assumptions about what modelling was suppose to be and what we (as students and as lecturer) were asked to do. At the end of the course, when students were asked through a survey and with the interviews with some of them, they stated how interesting it was this last project for several reasons. Some of main reasons mentioned by the students were: the openness of the questions, the possibility to delimit the questions to face, the necessity of clearly understanding the modelling process (the hypothesis assumed, the models' construction and their validation), the possibility to compare teams' proposals and results with the rest of the groups who could have been working differently, possibilities to discuss the limitation of the models proposed and make them evolve.

Ecological relativity of the second SRP adaptation implemented at UAB

The second implementation of the SRP was the following academic year at the Universitat Autònoma de Barcelona (UAB). There was a course called "*Mathematical modelling workshop*" which started in 2009/10 with second-year students of Mathematics degree. It was the first edition of the course, which was compulsory, with a total of 45 students participating. The didactic organisation of the course was different from the previously described at the UC. The main aim of the course was to develop a project in working teams (composed of 4-5 students) that students selected from a list provided by the lecturers of the course. Running in parallel, there were planned some short activities about modelling. The first year this course was implemented, one of the modelling activities planned was the 'Bicing project'. It ran over 5 weeks, with two 2-hour sessions per week. We invested more than the double of time than in its first implementation. Similarly, students were asked that at the end of each week they had to deliver a report with a synthesis of their advances in term of: (a) questions they had focused on, (b) hypothesis assumed and mathematical models considered, (c) temporary answers and (d) new questions to follow with). At the end of the Bicing project, each working team had to deliver a final report as summary of the whole modelling work developed. In general terms, the modelling process students and instructors followed in this occasion was not so different concerning $Q_{0(A)}$, although now none of the students' working team tackled the second phase of the project with $Q_{0(B)}$, or posed any questions about the properties of the n-power of transition matrices, such as: $Q_{(A)1.4}$ or $Q_{(A)2.3}$.

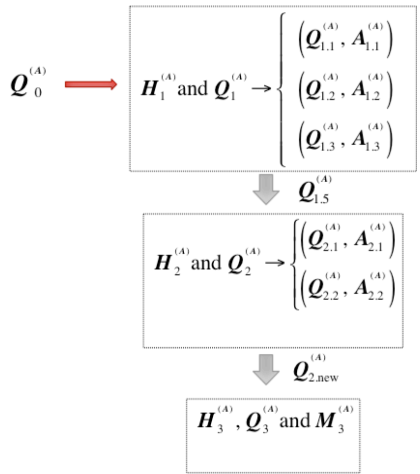


Figure 2: Summary of the path follow in the second adaptation of the SRP in UAB

One important novelty (and extension of the SRP) was that some students asked about the possibility of working with partial matrices, for instance, by considering different OD matrix to describe differently the bikes' flow in the morning and in the afternoon. Students had checked in the web how many bikes were available at different time frames and they had concluded that there were different patterns of bikes disposition depending in the daily time frame. The instructors asked to the experts we worked with about the possibility of having these new data. The external experts provided us two new matrices: one for the morning pattern, from 05:00h to 14:30h, and the other for the afternoon, from 14:30h to 00:00h. With this new data, the modelling process concerning $Q_{0(A)}$ was extended towards the construction of a third model, built upon the two previous ones (1) and (2), and taking into account these two different OD matrices. Figure 2 summarizes the path followed in this occasion and the extension it supposed for the first phase of the Bicing project.

If we focus on analysing the conditions and constraints we detected in the second implementation of the SRP, we have to mention that in this occasion it was the lecturer of the course who expressed more clearly some important constraints. He expressed, in an interview at the end of the implementation, that we had invested too much time with the project. He manifested that students needed to work more independently and there was no need of planning common discussions among all the working groups. His main request was to let students work independently and ask them to present their finding at the end of the course. Reactions that were on an opposite sense than the ones expressed by the Danish lecturer, who expressed that the activity was too open and too less guided for students. We can say that these reactions corresponded to their spontaneous teaching models that both lecturers implicitly defended. In this second implementation, it shared traits of a *modernist* teaching model (Gascón, 2001), by considering knowledge construction as an individual process, also private. That is why the lecturer preferred not planning any teaching device where to share and collectively talk about the modelling work developed, and where to question, debate and agree about the questions, tools and strategies to follow along the modelling process. As the course organisation at the UAB showed, each team was supposed to work most of the time independently in their project, and it was not until the end of the course when they explained their results. We could observe several inconveniences, linked to important constraints, which were more evident in the following courses when the lecturers planned short modelling activities as complement to the working group project of the course. First, students showed a lack of terminology and of a common discourse (shared with lecturers) to talk and write about the modelling activity developed. Second, the main outcome from the students modelling work was their final presentation of the project at the end of the course. It was delivered as a report that mostly contained the final models and models simulation, as if all the intermediate modelling work may remain in the private space of each group. Consequently, most of final reports showed a poor progression of the models considered and of the tools to contrast and validate them.

CONCLUSIONS AND DISCUSSION

It has to be highlighted that the two adaptations of the SRP presented in this paper were done under advantageous conditions. First, it was experienced with students of the Mathematics degree who were taking a course on mathematical modelling and with lecturers who are experts on modelling. Second, in both cases, the schedule and programme of the course were flexible and we had longer sessions (2-hour sessions two or three times per week) than the prevailing university settings use to offer.

Nevertheless, one could think that we may detect similar conditions and constraints in these two university setting, but it is important to see how different institutions established different relations with the knowledge at the stake, in this case, with the teaching of mathematics modelling. Then, for example, some conditions that appear in the first implementation can become strong constraint for the second one. For instance, it was the case of the necessity of sharing a common discourse to talk about and analyse modelling practices, which was an important condition underlined in the first implementation, becoming a constraints in the second one.

But, if we move away from these "optimal" university conditions, do we find similar constraints? Which of them are sensitive to be surmounted? How to overcome some of the most important constraints? To face these questions, and follow enquiring into the institutional relativity of the conditions favouring and the constraints hindering modelling practices, we proceeded with the third adaptation of the SRP. It was redesigned and later implemented with first-year university students of business and administration degree (4-year programme) in IQS School of Management of Universitat Ramon Llull in Barcelona (Spain) during the entire academic year 2013/14. In this occasion, the Bicing project was extended (called now "Cycling project") to become the central project developed along the three terms of the mathematics first-year course. The SRP was broken into three branches. The one described in this paper (in section 2) was implemented during the third term, only focusing on the first model (1). During the entire course, not only the initial structure of the SRP was extended, but also we pay special attention to which teaching devices and strategies could help to overcome some of the most common constraints for modelling and to create appropriate conditions for modelling and for the SRP. We are in the process of analysing them in depth with the aim of extending our knowledge about the ecology of the SRP and its institutional relativity.

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$C'(x) = C(x+1)-C(x)$? - Students' connections between the derivative and its economic interpretation in the context of marginal cost

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The derivative concept plays a major role in economics. Therefore, students of economics should have a proper understanding of the concept and its application in economics. One important competence for these students is to interpret the derivative in economic contexts. In books of economics the derivative is commonly interpreted as amount of change while increasing the production by one unit. However, from a mathematical point of view, this interpretation does not directly correspond to the derivative. In the study presented here, it was investigated to what extent students can make an adequate connection between the derivative as a mathematical concept and its economic interpretation mentioned above.

Keywords: derivative, students of economics, economic interpretation, concept image, marginal cost.

INTRODUCTION

The derivative plays an important role in economics. It is used to solve optimization problems, to describe and characterize economic functions, and in marginal analysis, in which the impact of small changes from the current state is examined (example: the effect of small changes in the price of a product on the demand) in order to make optimal decisions. Hence, students of economics should have an adequate understanding of the derivative concept in order to be able to use it in economics in a reflective manner. The study presented here focuses on students' of economics understanding of the derivative after their Calculus course with special emphasis on its economic interpretation, which is essential for the ability to apply the concept in economics. It is part of a larger research project about the understanding of the derivative in mathematics for students of economics (my PhD-thesis, supervisor: Rolf Biehler) at the Centre for Higher Mathematics Education in Germany (khdm).

LITERATURE REVIEW AND EMBEDDING OF THE RESEARCH

There is a lot of research about students' understanding of the derivative. Concerning the interpretation of the slope and the derivative in contexts, different difficulties are documented. Typical mistakes are the slope/height confusion or the graph-as-picture error (Beichner, 1994; Çetin, N., 2009; Carlson, M., Oehrtman, M. & Engelke, N., 2010). An interesting study, which included a task to interpret the derivative in the context of motion explicitly, was conducted by Bezuidenhout (1998). He asked students to interpret $S'(80) = 1.15$ if $S(v)$ is the stopping distance of a vehicle in metres in dependence of the velocity in km/h. Many students overgeneralized that the derivative is the acceleration or the velocity itself. Many also had problems with the units. These students did not understand the derivative as rate of change of the given function S properly.

While there is some research of students' understanding of the derivative in physical contexts (examples mentioned above), there is little research related to economic contexts. Wilhelm & Confrey (2003) showed that students cannot automatically transfer their knowledge about rate of change from a physical context to the context of money. Hence, an economic interpretation of the derivative should be explicitly taught in a Calculus course for students of economics. But even if an economic interpretation of the derivative was covered in the students' Calculus course, problems occur. Mkhatshwa & Doerr (2015) showed that many students talked about marginal cost (the derivative of a cost function) as amount of change when solving economic problems, although it was underlined in the Calculus course that the derivative is a rate. This indicates a rather superficial understanding of the connection between the derivative as a mathematical concept and its economic interpretation. A similar result is also found in Feudel (2017). Students' answers in a task to interpret $P'(73) = 0.2\,\frac{GE}{ME}$ (GE = units of money, ME = units of quantity) of a profit function P economically indicated that many students were not aware of the numerical differences and the differences in the unit between the derivative and its economic interpretation as additional profit. However, in these two studies the students were not obliged to reveal their ideas about the connection between the derivative as a mathematical concept and its economic interpretation explicitly. The study presented here directly focuses on this connection.

THEORETICAL BACKGOUND OF THE STUDY

The economic interpretation of the derivative

To be able to use the derivative in economics, students need to be able to interpret its values in economic contexts. However, understanding the interpretation of the derivative commonly used in economics is a special challenge for students because it does not directly correspond to any of the usual representations of the derivative as limit of the difference quotient, slope of the tangent line, local rate of change, or as instantaneous velocity. If $C:[0;\infty) \to [0;\infty)$ is a cost function (the variable x represents the output of a product), the derivative $C'(x)$, called marginal cost, is often interpreted as additional cost of the next unit. However, if one takes this interpretation literally it corresponds to the difference $C(x+1)-C(x)$, which differs from the derivative in its unit and in its numerical value. Since the students already have previous knowledge about the derivative from school, e.g. as slope of a function at one point, this might confuse them. Hence, the economic interpretation of the derivative should be carefully connected to the students' previous knowledge, and justified for economic contexts in the students' of economics Calculus course. A typical justification is via the approximation formula $C(x+h)-C(x) \approx C'(x)\cdot h$ for

$h \approx 0$. Since $h = 1$ can be considered as small in economics, the numerical values of $C'(x)$ and $C(x+1) - C(x)$ are close to each other, and they can be identified (a more detailed explanation can be found in Feudel (2016)).

The above mentioned perspective on marginal cost as being defined as derivative and interpreted as additional cost of the next unit coincides with what is taught in mathematics courses for students of economics (see e.g. (Sydsæter and Hammond, 2013)) and with what is presented in some books of economics like Breyer (2015). However, marginal cost can also be defined as additional cost of the next unit like in Blum (2003). In this case the derivative is viewed as method of calculation of the additional cost. Nevertheless, the problem to justify the identification of the two different mathematical objects in economic contexts also remains in this approach.

The notion of concept image to describe students' conceptual knowledge

The economic interpretation of the derivative and its connection to the pure mathematical concept, as it was explained above, should be part of students' of economics conceptual knowledge of the derivative concept. To describe students' conceptual knowledge I will refer to the notion of *concept image* by Tall & Vinner (1981), which describes the total cognitive structure associated to a concept. This includes all mental pictures, properties and associated processes. In the case of the derivative students' of economics concept image should contain its representations, the differentiation rules, its connection to the concepts of monotonicity and convexity, its use as a tool for optimization problems, and in particular an adequate economic interpretation of the derivative. Since the common economic interpretation of the derivative as amount of change while increasing the production by one unit is a different mathematical object, it should be in particular carefully connected to the rest of the students' concept image, called synthesizing in literature (Dreyfus, 2002).

Knowledge concerning the derivative covered in the students' Calculus course

In the Calculus course for students of economics in which the study took place (University of Paderborn 2015, Germany), the sessions involving the derivative began with the definition of the derivative as limit of the difference quotient. Alongside with the symbolic definition, its representations as slope of the tangent line (tangent line introduced as limit of secant lines) and as rate of change were introduced. Afterwards, the unit of the derivative in the case of a cost function C was discussed and justified via the symbolic definition of the derivative. In the second lecture the economic interpretation of the derivative in the context of marginal cost was introduced, which is essential for the study presented here. Two possible economic interpretations of the derivative were presented in the lecture:

1. Interpretation as approximation of the additional cost of the next unit

This interpretation was justified via the approximation formula $\Delta C \approx C'(x) \cdot \Delta x$, which was derived from the definition of the derivative by using the approximation aspect of the limit. The terms ΔC and $C'(x)\cdot\Delta x$ were also visualized on the board with the help of the tangent line.

2. Interpretation as additional cost of the next *marginal* unit

It was visualized that the mistake between ΔC and $C'(x)\cdot\Delta x$ becomes smaller if $\Delta x \to 0$. This results in the asymptotic equation $dC = C'(x)dx$ in which the lecturer called the "fictive infinitely small quantities" dx and dC marginal units.

Some lectures later the concepts of monotonicity and convexity and their connection to the derivative were discussed. The sessions finished with optimization problems.

All the topics covered in the lecture were also practised in small groups, in which the students had to solve problems. Relevant for the study presented here is, that these problems also included a task to interpret the value $C'(5)$ of the cost function $C(x) = 8x^2 + 10x + 700$ in an economic context (in the way presented above).

METHODOLOGY OF THE STUDY

The study aimed to find out to what extent students of economics can make an adequate connection between the derivative and its economic interpretation after their Calculus course. Hence, eight economics students who successfully completed their Calculus Course at the University of Paderborn were interviewed. Each interview lasted about 30 minutes. The interviews were structured by four tasks:

1. Consider the cost function C that is given by the following equation: $C(x) = \frac{1}{1000}x^3 - \frac{1}{4}x^2 + 21x + 500,\ x \geq 0$. The output x is given in units per quantity, the cost $C(x)$ is given in Euro. Determine the marginal cost for an output of $x = 100$ units of quantity. Determine the unit (of the marginal cost), too.
2. Is the derivative $C'(x)$ the same like the additional cost while increasing the production from x units of quantity by one unit? Justify yours answer.

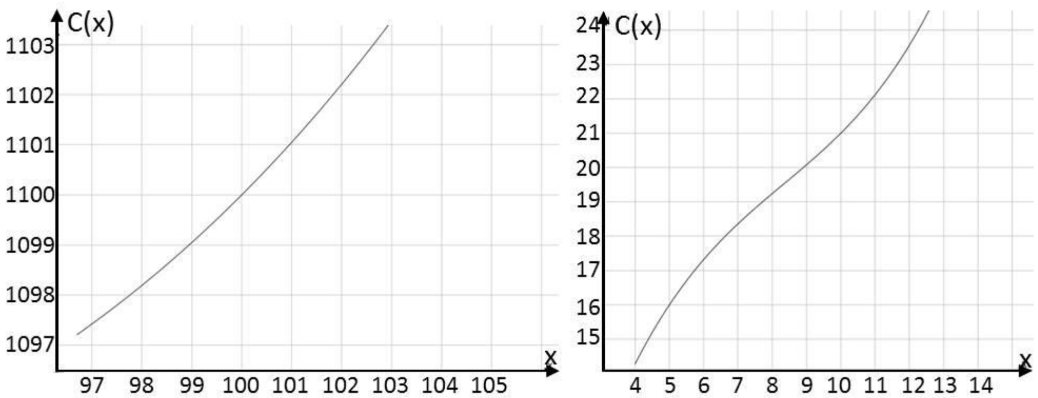

Figure 1: Graphs of the cost function from task 1 and the cost function from the task, in which the additional cost for the 11th unit had to be determined

3. Justify with the picture (left graph of figure 1) why the numerical values of $C'(x)$ and $C(x+1)-C(x)$ of the function C from task 1 are almost identical (for $x=100$).
4. Justify with the definition of the derivative why it can often be assumed in economics that the values of $C'(x)$ and $C(x+1)-C(x)$ are almost identical.

The tasks were not always presented to the students in written explicitly, but were sometimes given by the interviewer verbally during the interview process.

Task 1 was an introductory task with the aim to find out which of the two mathematical objects ($C'(x)$ or $C(x+1)-C(x)$) the students associated first with the term “marginal cost”. Task 2 was central in the interview. It aimed to find out to that extent students knew the differences between the derivative and the additional cost of the next unit. If the students claimed both objects to be equal they got the additional task to determine the additional cost of the 11th unit of a function, which was only given graphically (see right graph of figure 1). Its aim was that the students could no longer use their algorithm to determine the derivative and would use the difference $C(x+1)-C(x)$ instead. This should make the students’ concept image of marginal cost incoherent and provoke a cognitive conflict to make them rethink their ideas about the identity of the derivative and the additional cost of the next unit, and to reorganize their concept image. With the interviewer’s help the students were then led to the differences between the derivative and the additional cost of the next unit.

Task 3 had the aim to find out if the students could justify the identification of $C'(x)$ and $C(x+1)-C(x)$ within the graphical representation with the help of the tangent line (similar to the visualization presented in the lecture). Task 4 finally aimed to find out to what extent the students have internalized the justification of the identification of $C'(x)$ with $C(x+1)-C(x)$ in economics symbolically via the approximation formula $C(x+h)-C(x)\approx C'(x)h$ for $h\approx 0$ (as it was taught in their Calculus course). The results of tasks 3 and 4 are not discussed in the paper in detail due to limited space.

The interviews were recorded, transcribed, and interpreted. First, individual cases were interpreted line by line. Later the results between different individuals were compared. To ensure reliability, the author’s interpretations were discussed with colleagues of the Centre of Higher Mathematics Education in Germany (khdm).

SOME DETAILED RESULTS

Altogether eight students were interviewed. Due to limited space, two students (Holger and Lisa) with different understandings of marginal cost were chosen, whose interview parts referring to the tasks 1 and 2 are presented here in detail.

Holger’s understanding of marginal cost

Holger immediately solved the first task to determine the marginal cost at $x=100$ for the function $C(x)=\frac{1}{1000}x^3-\frac{1}{4}x^2+21x+500, x\geq 0$ by calculating $C'(100)$. This means his first association of marginal cost was the derivative and not its economic interpretation. His result was $C'(100)=1$. He did not mention a unit himself. After having been asked for the unit by the interviewer explicitly he mentioned "Euro", but could not justify it. This shows that he had an incoherent concept image of marginal cost: he associated the derivative for calculations but did not think of marginal cost as a rate (otherwise the unit would have to be Euro per unit of quantity).

In the next part of the interview Holger was explicitly confronted with the question if a definition of marginal cost as additional cost of the next unit would represent the same mathematical object like the derivative. He started thinking about it and then agreed. Therefore, the interviewer tried to provoke a cognitive conflict by giving Holger the task to determine the additional cost while increasing the production from 10 units by one unit (see right graph of figure 1). Holger first wanted to use the derivative again, but then started to think about the task again:

32	Holger:	You would have to imagine the derivative. Then you would see the additional cost. The derivative is nothing else than the slope at a point. If we take any point.
33	Interviewer:	Here is one explicitly given. We search for it at a particular point.
34	Holger:	Here is one given, 10 units of quantity. So we have $x=10$ where the total cost is 21. Now we need the cost if one more is produced. Well, but we do not need this because if we are at 11 the cost is 22 point something. So the additional cost has to be one point, yes 1.1.

Holger now used the difference $C(x+1)-C(x)$ to solve the task. To the following question of the interviewer if the value would have been the same by using $C'(x)$, Holger agreed. He then determined the slope at $x=10$ graphically and got the solution "round about one" (correct value: 1, see right graph in figure 1). After the interviewer emphasized that Holger just said "round about", Holger claimed that he cannot determine the value exactly by graphical means. Hence, the interviewer asked Holger afterwards to determine the cost difference for the cost function C from task 1 given by an equation. He now got the result 1.051 and justified the "error" compared to $C'(100)=1$ as follows:

77 Holger: One nearly gets [1], but only nearly. This is probably due to rounding.

One can see that Holger was really convinced that the derivative and the additional cost of the next unit are exactly the same, even if the calculated values differed. The interviewer now emphasized that there was no rounding involved. He then pointed to the graph of the cost function (left graph in figure1) and underlined that one can see the error in the graph, too. After the interviewer had asked Holger again to determine the value of the derivative by graphical means, now for the cost function of task 1 (left graph in figure 1), Holger found for himself a resolution of the conflict:

94 Holger: Oh, the reason is, because it is not exact. The origin of the derivative was to determine the slope at a point. To achieve this you take one point left and one point right of it, which have the same distance, and the slope between.

Afterwards you try to make this distance as small as possible, as you could think of, but we cannot reach the one point, but in our mind we want to reach it. And I assume this the very small rounding mistake, no, not rounding mistake, but this small difference is due to the fact that you do not reach the point exactly.

Two misconceptions of the derivative occurred here. Holger did not imagine the derivative as limit of slopes of secants through $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$ for $h \to 0$ but as limit of secants through $(x_0-h, f(x_0-h))$ and $(x_0+h, f(x_0+h))$ for $h \to 0$. But the important misconception that now prevented him from questioning the identity of $C'(x)$ and $C(x+1)-C(x)$ was his opinion that the "true slope" at a point x_0 was not reached by taking the limit. He imagined the derivative to be the slope of a secant through $(x_0-h, f(x_0-h))$ and $(x_0+h, f(x_0+h))$ with a very small $h>0$ (he repeated this several times later, even more explicitly than in the lines above).

To sum up, at the start of the interview, Holger identified the derivative $C'(x)$ with its economic interpretation as additional cost of the next unit. During the interview a conflict occurred due to different numerical values of these two. But instead of questioning the identity between the derivative and the additional cost of the next unit he made his concept image coherent again by attributing this error to an error between the "true slope at a point" and the derivative as result of a limiting process.

Lisa's understanding of marginal cost

Similar to Holger, Lisa also immediately solved the task to determine the marginal cost at the output $x=100$ by using the derivative. Unlike Holger she stated as unit "Euro per unit of quantity", a unit of a rate. So Lisa also associated the derivative with the notion of marginal cost first. When confronted with the definition of marginal cost as additional cost of the next unit she replied:

20 Lisa: Yes, I really thought about this last semester. In the economic subjects we really learn it this way. […] And I always had to say: If you increase x by one unit, y increases by these many units, eh? This is really the case. But I have, since I had mathematics last semester, always thought that you learn it differently in mathematics. In mathematics you say, if you increase x by one marginal unit, y increases by these many marginal units.

Unlike Holger, Lisa did not identify the derivative with the cost of the next unit. She even felt a conflict between the knowledge about marginal cost she learned in her maths course and the actual use of marginal cost in econometrics. Her remarks point out that she was of the opinion that marginal cost is not the additional cost of the next unit but of the next marginal unit. In which way she understood the term "marginal unit" was not clear yet. Therefore, the interviewer asked about this term:

21 Interviewer: Now the question, what is a marginal unit?

22 Lisa: Well, a marg/ Shall I draw it?

The interviewer gave Lisa the graph of the function of task 1 (left graph in figure 1).

32 Lisa: Well, one unit could be from 100 to 101 we said. So you increase x by one unit from 100, eh? From the actual output. Then I would be here [pointing on $C(101)$].

After the interviewer's comment to determine the accurate value by calculation she got $C(101)-C(100)=1.051$ (compared to $C'(100)=1$), and continued as follows:

48 Lisa: And a marginal unit I imagine very, very small. Here I would go a right very, very little bit to the right and then a very little, little bit upwards.

We see here that Lisa understood a marginal unit as a very small, but finite unit. This understanding is also found in books of economics (and differs from the way a marginal unit was taught in the maths course as a "fictive infinitely small quantity"). After Lisa's explanation of the "marginal unit", the interviewer tried to induce a cognitive conflict by asking Lisa if the additional cost of such a small unit should not be close to zero. She then explained the following:

52 Lisa: This has to do with the slope you have. For the derivative you calculate the slope of the tangent line. The slope of the tangent line is what you calculate, isn't it?

53 Interviewer: Right, the slope of the tangent line, yes.

54 Lisa: Yes, and this is also what I get if you increase x by a marginal unit, starting at 100. Ah, what do I get? No, if you increase x by one marginal unit, the marginal cost still increase by one. I think the slope still remains one, right?

This shows that Lisa understood the additional cost of a marginal unit as $C'(100+dx)$ with dx being a very small, but finite unit. This is in her opinion numerically the same like $C'(100)$ because the slope stays the same at $100+dx$. The interviewer again asked if additional cost and slope are the same whereat Lisa agreed.

To sum up, Lisa knew that the derivative is not the additional cost of the next unit. She remembered the interpretation of the derivative to be the additional cost of a *marginal* unit from her Calculus course. She imagined a marginal unit dx to be a very small, but finite unit (and not in the way it was taught in the course as a "fictive infinitely small quantity" in the asymptotic equation $dC=C'(x)dx$) and identified the additional cost of a marginal unit in her mind with $C'(100+dx)$, which is in her opinion the same like $C'(100)$. She did not recognize the different nature of the derivative being a rate of change and the additional cost being an amount of change.

SUMMARY AND DISCUSSION

Just like in the cases of Holger and Lisa presented in detail here, the study showed that the students had difficulties to make an adequate connection between the derivative as a mathematical concept and its economic interpretation as additional cost. The author considered a connection as adequate if the students were aware of the differences between $C'(x)$ and the additional cost of the next unit, and were able to justify their identification in economics (it did not matter if they associated marginal cost with the derivative or the additional cost of the next unit first).

At the beginning of the interview no student could make an adequate connection between the derivative and the additional cost of the next unit. The majority just declared these objects to be exactly the same (like Holger).

During the interview, all students recognized the question concerning the identity of the derivative and the additional cost of the next unit as very relevant and became aware of differences between these two with the interviewer's help. However, in the process of leading them to these differences on the graphical level via the tangent line, several problems occurred (not discussed here in detail due to limited space):

1. Misconceptions concerning the derivative concept (like Holger's misconception about the derivative being not the "true slope" at the point)
2. Incomplete concept images (example: knowledge of the geometric representation of the derivative as slope at a point but no association of the tangent line)
3. Problems in determining the slope of a linear function

Only one student could make a connection on the symbolic level via the formula $C(x+h)-C(x)\approx C'(x)\cdot h$ for $h\approx 0$ like presented in the Calculus course.

Furthermore, most participants of the study had not thought about the differences and the connection between the derivative and its economic interpretation as additional cost before the interview, although these were presented in their Calculus course.

A solution to these problems in a traditional Calculus course for students of economics, in which the concept of derivative is taught first, an economic interpretation afterwards, could be to confront the students with the two different notions of marginal cost in the tutorials of the course directly, and to provoke a cognitive conflict, just like in the interview. Afterwards, one could let them try to connect the derivative and the additional cost of the next unit by themselves or in small groups, and help them individually if misconceptions or incomplete concept images of the derivative occur. Another solution could be to start with the concept of marginal cost as additional cost first, which can be approximated by the derivative as linear approximation. However, an understanding of the mathematical concept of the derivative as slope of the tangent line is also necessary in this approach.

Concerning future research, one has to emphasize that the study presented here relied on *one* perspective on the connection between the derivative as a mathematical concept and its use in economics: additional cost as interpretation of the derivative. This perspective is important because students of economics are often confronted with it their maths course and in courses about economic theory (it can be found in respective books of economics). But as explained in the theoretic part of the paper: it is not the only one. Other perspectives may further enrich the knowledge about students' of economics understanding of the derivative and its use in economics.

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Weekly homework quizzes as formative assessment for Engineering students are a fair and effective strategy to increase learning?

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A strategy to apply online weekly homework quizzes as formative assessment for Engineering students was designed and tested in order to study if it increases student's learning. The strategy was to make optional weekly online quizzes with questions not randomly generated that students may retry over and over again until to reach the correct answer, they contribute to 10% of grade but only if students get 45% or more in usual pencil and paper assessment.

The quizzes were applied to two different mathematics courses (Single and Multivariable Calculus) of two different Engineering degrees, each one to around 100 students and during a semester. Student's adherence was very high, nearly all students refer quizzes as fair and useful to learning. Students' grades were compared with several other years.

Keywords: The role of digital and other resources in university mathematics education, Assessment practices in university mathematics education, Teaching and learning of analysis and calculus.

INTRODUCTION

Frequent online quizzes have been suggested as a strategy to enhance learning by several institutions and researchers. The National Centre for Public Policy and Higher Education in the U.S.A (Twigg, 2005) consider computer based continuous assessment and feedback to be a key strategy for quality improvement in learning. According to Gibbs (2000), student assessment is an effective way to increase understanding and online quizzes force students to spend more time working productively outside of class. Tuckman (1998) refers this as being especially valuable to procrastinators. One method that can be used to address the crisis in college mathematics, according to Thiel, Peterman, and Brown (2008), is to 'provide regular assessment of progress' and they state that 'online homework and quizzes with online grading provide students with immediate feedback, the opportunity to correct their homework mistakes, and ongoing assessment of their success in the course'. Booth (2012) considers that homework should be given out at regular times, over regular intervals, on a weekly basis; proposing that learning is work and students should develop regular work habits in order to succeed. Feedback is crucial for student success but giving adequate feedback with large class sizes is difficult and therefore automated systems are a useful solution to the large class size problem.

Quizzes are part of several successful approaches with different kinds of students, both in top universities and in other higher education institutions. Examples include:

TEAL (Dori & Belcher, 2004) at Massachusetts Institute of Technology (MIT); SCALE-UP (Beichner, et al, 2007) at North Carolina State University; Peer Teaching (Lasry, Mazur, & Watkins, 2008) at Harvard University.

Particularly, in higher education mathematics teaching, several approaches have been raised but literature is not yet in agreement about the effectiveness of quizzes to enhance learning (Siew, 2003; Varsavsky, 2004; Myers & Myers, 2007; Blanco, Estela, Ginovart & Saà, 2009; Lim, Thiel & Searles, 2012; Broughton, Robinson & Hernandez-Martinez, 2013; Shorter & Young, 2011).

CONTEXT

This research took place in two mathematics' courses to Engineering students of Instituto Superior de Engenharia de Lisboa, Portugal, each during a semester. In those semesters, weekly online quizzes on Moodle (the learning management system of the institute) were made available for a week each. The AM2 course in 2013/14 was about Multivariable Calculus, the MAE course in 2015/16 was about Single Variable Calculus. Around 100 students and 3 teachers were involved in each course.

The quizzes were called 'Mini-tests' to reinforce their relevance. The 'regular' assessment involved two face-to-face tests or the First Exam or the Second Exam. For AM2, the quizzes scored up to two values proportional to the best 12 (out of 14) grades in the quizzes and it was added if the student scored more than 9.0 values (out of 20) in 'regular' assessment. For MAE, it was slightly different: the quizzes valued 10% of the grade if the student scored more than 9.0 values (out of 20) in the 'regular' assessment and if this grade was better than the 'regular' grade. In both cases the quizzes were optional.

The aim of the quizzes was not to assess students, it was to make them study more, not to postpone, not to study first the other subjects that were naturally more pleasant for them (since they belong to their study area); to make students more aware of their level of understanding (often students only realise that they cannot solve the exercises when they go to the first test, in the middle of the semester). Students are usually optimistic about their capabilities (Wandel, 2015). It was written in Moodle and teachers repeatedly reminded students that the aim of the quizzes was to make students study more and be aware of their level of understanding; that students could copy all quizzes but, probably would not get the 9.0 values required in 'regular' assessment and therefore, it not be worthwhile.

THE QUIZZES

The quizzes were produced through the 'Moodle activity: test'. It allows the introduction of images and mathematical symbols using LaTeX (see Fig. 1). The possibility of creating questions with different instances for each student was considered, but it would take much more time to create questions and students also know how to solve a problem with a constant instead of a number, so it did not seem worthwhile.

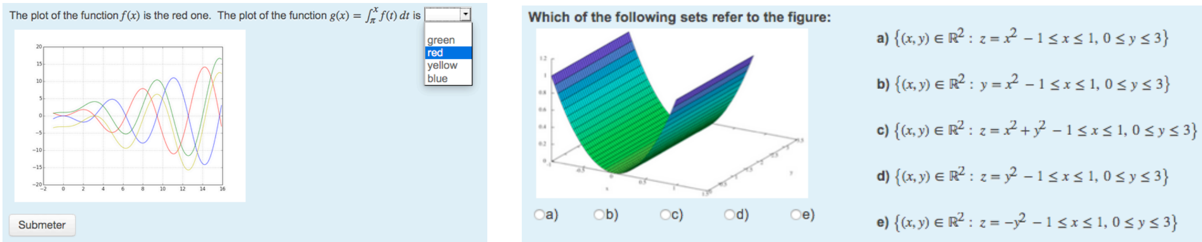

Figure 1. Multiple-choice questions including a figure and mathematical text, MAE and AM2 example. (Translated)

Whenever it was possible, we used numeric or short answers instead of multiple-choice answers since in multiple-choice answers, with a few tries, students could get the correct answer. The type of questions that we most used was 'embedded answers', because this enables a teacher to embed more than one sub-question and those sub-questions may be chosen from all the different question types: numeric, short answers, multiple-choice, true or false, etc. The 'embedded answer' question type allows the teacher to evaluate the student through their pathway and not only their final result (see Fig. 3). The feedback does not show the correct answer.

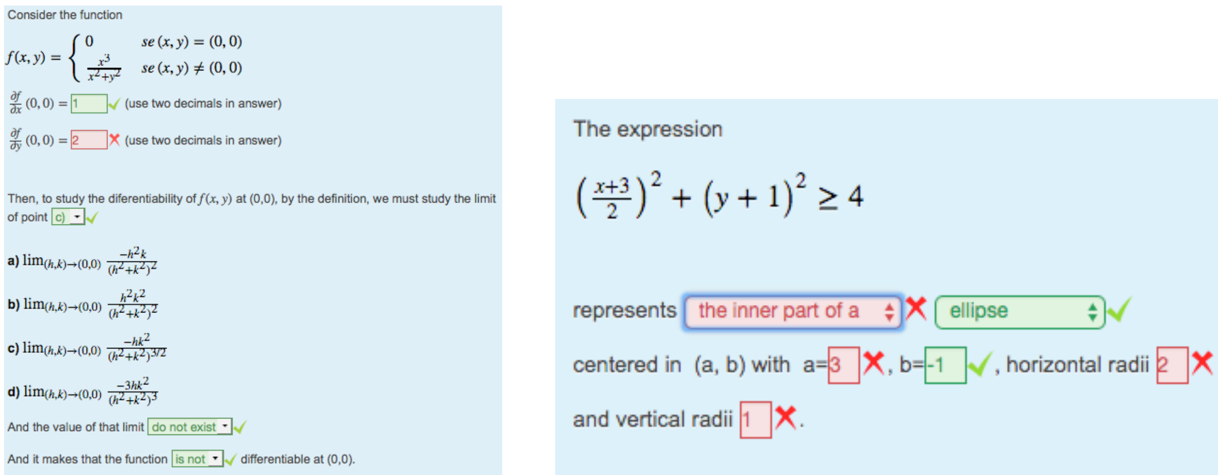

Figure 2. A question with multiple embedded questions along the path (including numerical answers), an AM2 and MAE example. (Translated)

RESEARCH DESIGN, DATA AND RESULTS

This research design is a *quasi-experience* (not an experience since not all variables could had been controlled) where two sets of quizzes were applied to two mathematics courses. The research question of this study is: are the quizzes (applied with this strategy) a fair and effective tool to increase students' learning? The strategy for application of the quizzes is that they are weekly, online, non-mandatory, count towards grades if students achieve a certain level on traditional assessment, are not randomly generated and students may resubmit without penalty. This research question was split into four sub-questions:

- RQ1: Did the students adhere to the quizzes?
- RQ2: What was students' perception of the quizzes?
- RQ3: Did students felt quizzes as unfair?
- RQ4: Did the quizzes increase students' grades?

The instruments utilized were: a students' survey about the quizzes; data from the answers to the quizzes; and course grades over several semesters. The quizzes were applied to two mathematics courses: AM2 with 104 subscribed students and MAE with 108.

The anonymous survey on Moodle was addressed to all students for each edition. The sample of students who answered the survey was reasonable. From the 104 students subscribed to AM2, all subscribed to Moodle, 65 answered the survey. From the 108 students subscribed to MAE, 94 in Moodle, 61 answered the survey. Moreover, by splitting the students by their grade at the first test (the survey was applied before the second test), the number of students answering the survey with a given grade reasonably correlates to the number of students in general who achieved that grade. Pearson correlation coefficients are $\rho = 0.6$ and $\rho = 0.5$ respectively.

Students of the institute do not have precedencies among courses and may be subscribed to a large number of courses, so it is usual that students subscribe to many courses where, in fact, they do not attempt to achieve success. We may verify this, for example, by noticing that from the 108 students subscribed to MAE only 94 were subscribed to Moodle, so the 14 remaining students did not access anything from the course: syllabus, slides, quizzes etc. Since there is no simple and fair way of identifying these students, in this research we always use the subscribed students to make measures. However, it is relevant to have in mind that it includes those 'ghost students'.

RQ1: DID THE STUDENTS ADHERE TO THE QUIZZES?

AM2 had 104 subscribed students, 79 attempted regular assessments and 76 students attempted at least one quiz. All but one of the approved students answered at least one quiz. The final quiz grade was the average of the best 10 out of 14 grades in quizzes, so it was natural that the last four quizzes had lower attendance (and for this reason we modified this rule for MAE, where the best 12 grades were chosen).

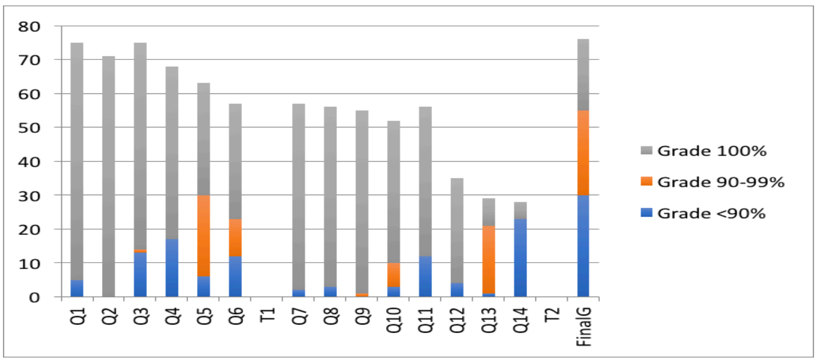

Chart 1. The number of students that answered AM2 quizzes split by grade.

MAE had 108 subscribed students, 103 completed regular assessment and 93 students attempted at least one quiz. All approved students answered at least one quiz. The final quiz grades were the average of the best 12 out of 14 grades in quizzes, so it is natural that the last two quizzes had a lower attendance (this rule changed from AM2). It is important to note that, for example, in Q5 the number of students with a total grade was lower than in the other quizzes and the number of attempts to solve the quiz was higher than in the others (326). This shows that students were, in fact, trying to reach the correct answers (this test was particularly large and complex).

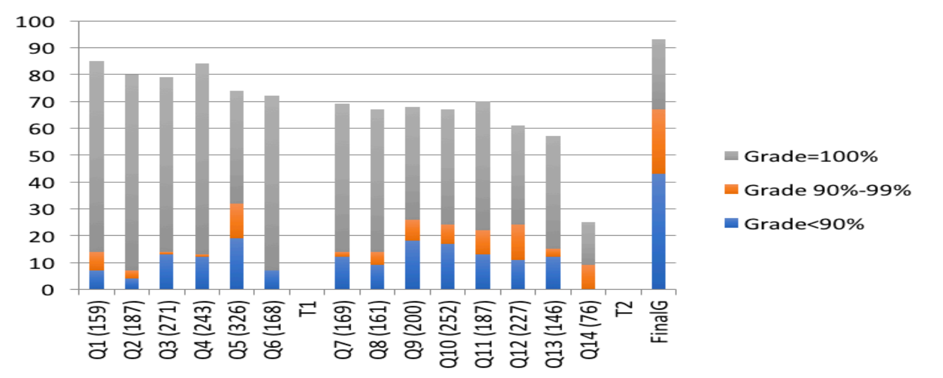

Chart 2. The number of students that answered MAE quizzes split by grade. The number of attempts to answer the quiz, registered by Moodle, is in parenthesis.

A large portion of students got a very high grade, but this was natural since students may retry without penalty and the questions were equal to all students, so it was expected that students talk to each other and reach the correct answer.

The quizzes were not mandatory and improved the grade if the student got more than 9 out of 20 values in regular assessment, so it could be expected that many students decided not to take it. However, on a regular basis, nearly half of the subscribed students answered the quizzes.

An objective result was, despite of the optional policy, that students strongly adhered to quizzes. The percentage of subscribed students that answered one quiz was 93/108=86% and 76/104=73%. All the quizzes had a high rate of attendance. Among the students that undertook ‘regular’ assessment, almost all took a quiz and a large percentage got high average grades on the quizzes.

RQ2: WHAT WAS STUDENTS’ PERCEPTION OF THE QUIZZES?

Table 1 shows that, according to the survey, none of the students thought that the quizzes were of no interest and did not care about the quizzes, while a large percentage believed that the quizzes reminded them to study, showed them the level that they were reaching and encouraged them to learn new things; some of those things they thought they understood but in fact they did not.

		AM2		MAE	
Total	65	100%	61	100%
Quizzes remind me to study the subject every week.	55	85%	50	82%
Quizzes show me there are things I thought I knew but I didn't.	48	74%	53	87%
Quizzes help me to have a better perception of the level I'm reaching.	47	72%	38	62%
I learn new things answering to quizzes.	33	51%	35	57%
Quizzes have no interest.	0	0%	0	0%
I do not care for quizzes, I just copy the results.	0	0%	1	2%
I do not care for quizzes, I not even copy the results.	1	2%	0	0%

Table 1. Students' answers to 'Select ALL the statements that you agree with' in both surveys.

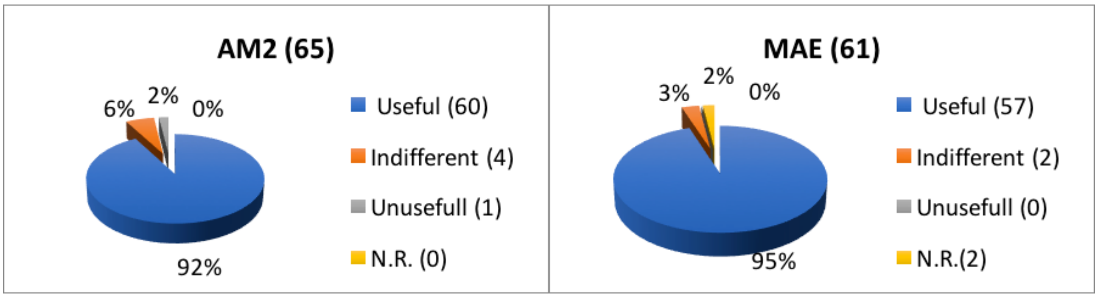

Chart 3. Percentage of students answers to 'The quizzes were…' in both surveys.

Summarising, more than 90% of students found quizzes useful (Chart 3); that they study more due to the quizzes. Students agree that quizzes remind them to study, show them that there were things that they thought that had understood but did not, encouraged them to learn new things and gave them a better perception of level that they were reaching.

RQ3: DID STUDENTS FELT QUIZZES AS UNFAIR?

In daily life as a teacher, teachers tell several times that one reason why they do not use online quizzes is because students may be cheating and it may generate unfairness. To avoid that problem, it was strongly emphasised to students that quizzes were much more relevant as formative assessments than summative assessments; students could resubmit the quiz without penalty to stimulate them to try to answer by themselves without fear of being penalised; and a clause was included that the quizzes only count towards grades if students get 9.0 values (out of 20) in regular assessments, as in Varsavsky (2004). As result, the answers in the survey to the question 'Quizzes generate unfairness?' show that very few students perceive quizzes as unfair (see Chart 5).

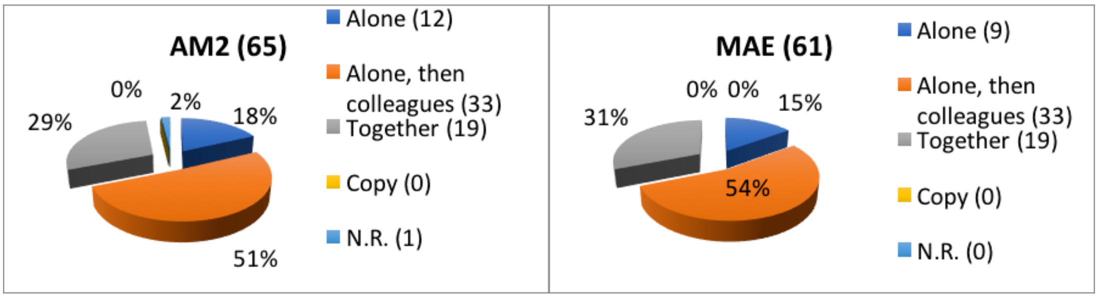

Chart 4. Percentage of students answers to 'How do you answer to quizzes?' in both surveys.

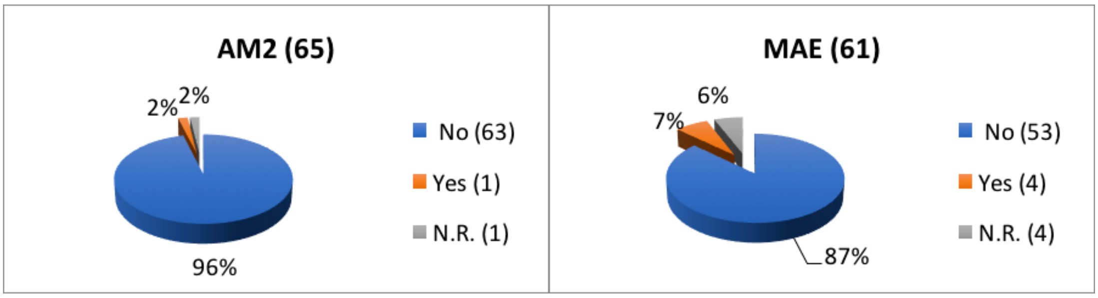

Chart 5. Percentage of students answers to 'Quizzes generate unfairness?' in both surveys.

When questioned in the survey, no student stated that they had copied the results (see Chart 4), despite it being reinforced in that question that the survey was automatically anonymous.

Therefore, with this approach, the level of unfairness of quizzes is not considered as relevant.

RQ4: DID THE QUIZZES INCREASE STUDENTS' GRADES?

Since the goal was that all students achieve a total score in all quizzes, is was expected that quiz grades would not correlate to final grades. This did occur and it was verified using the non-parametric Spearman Rho for AM2 ($\rho = 0.34$, $N = 54$, $p = 0.01$) and for MAE ($\rho = 0.28$, $N = 61$, $p = 0.03$), since data were not normal (Kolmogorov-Smirnov, $p < 0.01$).

According to Chart 6, around 70% of students that answered the survey, believe that quizzes helped them achieve a higher grade.

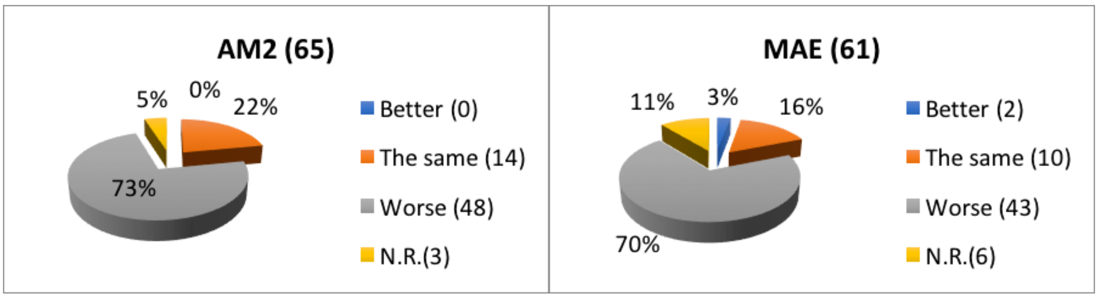

Chart 6. Percentage of students' answers to 'Without quizzes, I've scored…' in both surveys.

The data of Tables 2 and 3, relate to six responsible teachers/approaches and ten different teachers. The syllabus was essentially the same across the semesters but the approaches were naturally different. In the intervention semesters, the responsible teachers were also different. So, the quizzes were not the only different variable in that semester, thus we cannot attribute grade differences directly to the quizzes. For AM2, the pass rate nearly doubled in that semester, the average grade also increased significantly.

AM2	2010/11		2011/12		2012/13		2013/14		2014/15		2015/16	
	S1	S2	S1	S2	S1	S2	S1	**S2**	S1	S2	S1	S2
Subscribed students	101	200	128	153	90	123	80	**104**	56	66	56	108
Approv. students	27	38	31	41	20	23	12	**54**	10	19	16	33
Average appr. grade	11.7	11.8	12.3	11.7				**13.9**	12.4	11.5	11.7	11.5
Pass/Subscribed	27%	19%	24%	27%	22%	19%	15%	**52%**	18%	29%	29%	31%
Professors	A+…	A+…	A+…	A+…	A+B	A+C	D+E	**F+G +H**	D+F	D +I	J+K+ IL	J+K+ I

Table 2. Grades of AM2 students across ten semesters, the letter representing the coordinator teacher is underlined and the experimental semester is shaded.

The MAE course had, in some editions, five or six quizzes in class. It is curious to note that in the year that there were no quizzes, the pass rate was much lower. And the MAE pass grade and the average grade had the highest value in the experimental semester. However, it may have been a coincidence, we do not have enough data to reach any conclusions, it is just a positive indication.

MAE	2011/12-SI	2012/13-SI	2013/14-SI	2014/15-SI	**2015/16**-SI
Subscribed students	73	109	121	125	**108**
Pass students	17	30	58	56	**61**
Average pass grade	12.7	12.2	13.5	12.7	**13.5**
Pass/Subscribed	23%	28%	48%	45%	**56%**
Number of quizzes	0	6 in class	5 in class	5 in class	**14 online**
Professors	A	A	A	A+B	**B+A**

Table 3. Grades of MAE students across five semesters, the letter representing the coordinator teacher is underlined and the experimental semester is shaded

Summarising, as expected, quiz grades do not correlate to final grades; around 70% of respondents to the survey state that due to the quizzes they achieved a better grade.

The pass rate and the average grade increased significantly in the semesters that the quizzes were applied, which is a positive indicator but cannot be directly attributed to quizzes.

CONCLUSIONS

Two sets of 14 weekly quizzes on Moodle were available to all the engineering students on two mathematics courses (Single and Multivariable Calculus). The online quizzes were not mandatory, counted to grading if the student had more than 9 out of 20 values on traditional assessments, were not randomly generated and students could resubmit without penalty. The research question is 'Are the quizzes (applied with this strategy) a fair and effective tool to increase students' learning?'

In the answers to the survey, more than 90% of students found quizzes useful; more than 60% stated that studied more due to the quizzes; students agreed that quizzes reminded them to study; showed them that there were things that they thought they understood but did not; made them learn new things and gave them a better perception of the level that they were reaching.

The quizzes were not mandatory so students may have just ignored them. Although a large proportion of students attempted quizzes and kept answering them until the last ones.

Quiz questions were not randomly generated, so all students got the same questions and naturally, students shared the solutions with each other. To avoid unfairness, it was strongly emphasised that quizzes were important to students' formative assessment, to allow them to test themselves and get feedback on their level of understanding. Moreover, quizzes only contributed to grades if the students got more than 9 out of 20 values in 'traditional' assessments. Moreover, if a student copied many quiz results, probably would not achieve the minimum grade and it would not be worthwhile. The result was that, in the answers to the surveys, very few students stated it as being unfair. Over 70% of respondents to the surveys stated that due to the quizzes they achieved a better grade. The pass rate and the average grade increased significantly in the semesters that the quizzes were applied, which is a positive indicator, but it cannot be directly attributed to the quizzes.

This research suggests that these quizzes, with this strategy, are a fair and useful tool to increase students learning.

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The use of integrals in Mechanics of Materials textbooks for engineering students: the case of the first moment of an area

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Research has reported on the difficulties engineering students face in relating the content of their mathematics courses to what is taught in their professional courses. One way to address these difficulties is by better understanding how mathematical notions are used in professional engineering courses. This paper analyses how the notion of first moment of an area – which is defined as an integral – is used in civil engineering courses. Basing our analysis on elements from the anthropological theory of the didactic, we are currently analysing a classic Mechanics of Materials book. Our findings indicate that although first moments are introduced as an integral, the textbook's tasks do not require students to use techniques typically introduced in a traditional calculus course.

Keywords: Mathematics for engineers, teaching and learning of analysis and calculus, textbooks, anthropological theory of the didactic, first moment of an area.

INTRODUCTION

Engineering courses are usually organized into two groups: basic science courses (which are taught in the first two years, including foundational skills in mathematics and physics), and technical courses (which appear later in the programme and are more specific to each field of engineering). However, research in engineering education and mathematics education indicates that engineering students encounter many difficulties in their mathematics courses in the first years of study, which can lead to high failure rates, and in many cases, result in students dropping out of engineering programmes (Ellis, Kelton, & Rasmussen, 2014). In this sense, "poor mathematics skills are a major obstacle to completing […] engineering programs" (Fadali, Johnson, Mortensen, & McGough, 2000, p. S2D-19).

Researchers have identified some negative situations for students who pass these mathematics courses. One situation is that these students often find it difficult to relate the learned mathematical content to the content of the professional courses. For Flegg, Mallet, and Lupton (2011, p. 718) "without the explicit connection between theory and practice, the mathematical content of engineering programs may not be seen by students as relevant". Another situation is that in spite of having passed the mathematics courses (with a rather rigid structure and rare concrete applications relevant to engineering), students must apply mathematics in their engineering courses, where many new mathematical notions appear without having been encountered in the previous mathematics courses (Hochmuth, Biehler, &

Schreiber, 2014, p. 694). Faced with these problems, the mathematics and engineering education communities have been engaged in research and discussion on "how to improve engineering students' mathematics learning, and hence their service teaching" (Bingolbali, Monaghan, & Roper, 2007, p. 764).

Our current research program investigates how calculus notions are used in engineering courses, aiming at identifying possible ruptures between how notions are first introduced and used in calculus, and how they are later used in professional courses. First, we analyse how engineering textbooks present these notions, working under the principle that most tertiary instructors organise their teaching using textbooks as an important resource (e.g., Mesa & Griffiths, 2012). The manner in which mathematics notions are used in professional courses has not been the subject of much research. However, we believe this type of research could help bridge the gap between two communities. On the one hand, mathematics lecturers in engineering programs could benefit from knowing how their course content is used in professional courses; on the other hand, professional course instructors could benefit from a critical analysis of their use of mathematics, to help their students make connections between the content of mathematics and professional courses. For example, our analysis of the way integrals are used to define bending moments for beams in strength of materials textbooks for civil engineering reveals different uses of "the same" object (González-Martín & Hernandes Gomes, 2017a). Although bending moments are defined as an integral, the tasks, techniques, and justifications used in calculus courses are very different from the ones presented in professional engineering courses; this may result in students not recognising "the same" object in two different courses, and they may question the relevance of integration techniques that are not used in tasks concerning bending moments. In this paper we develop the content of González-Martín & Hernandes-Gomes (2017b) as we explore the use of integrals to introduce another engineering notion: first moment of an area. We aim to address two questions: how is the content related to integrals used in engineering to work with first moments of an area, and how does this use relate to the content in calculus courses?

Defining first moment of an area

Moments of areas are topics commonly taught in engineering courses that cover strength of materials. Due to space limitations, in this paper we focus on the first moment of an area. In civil engineering, for example, to solve bending problems one must take into account some specific geometrical characteristics of cross-sections of a bar, which is the general term for structures that include beams (Feodosyev, 1973). In this situation, the notion of first moment of an area is used to calculate the centroid of an area and the shearing stresses in transverse bending. The centroid of an area A is its geometrical barycentre and is the point C of coordinates $\bar{x}$ and $\bar{y}$ such that the following relationships hold true: $\int_A x\, dA = A\bar{x}$ and $\int_A y\, dA = A\bar{y}$.

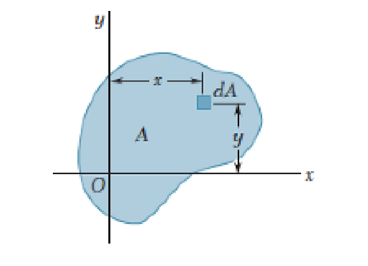

Figure 1: General area *A* with infinitesimal area *dA* in the *xy* plane (Beer et al., 2012, p. A2).

Let *A* be an area situated in the *xy* plane (Figure 1), using *x* and *y* as the coordinates of an element of area *dA*. According to Beer, Johnston, DeWolf, and Mazurek (2012, p. A2), the first moment of an area *A* with respect to the *x* axis (resp. *y* axis) is mathematically defined as the integral $Q_x = \int_A y\,dA$ (resp. $Q_y = \int_A x\,dA$). In both integrals, the index *A* in the integral sign indicates that the integral is calculated over the whole cross-sectional area. Both integrals characterize the sum of the products of each element of area *dA* and its distance to the respective axis (*x* or *y*) and are measured in cubic units (Beer et al., 2012). When an area possesses an axis of symmetry, the first moment with respect to that axis is zero, since every element of area *dA* of abscissa *x* (resp. ordinate *y*) corresponds to an element of area *dA'* of abscissa –*x* (resp. ordinate –*y*). This implies that when an area possesses an axis of symmetry, its centroid is located on that axis. For instance, in a rectangular cross-section (two axes of symmetry), its centroid *C* coincides with its geometric centre. Determining the position of the centroid is important, since several forces in a bar pass through its centroid.

To illustrate these definitions and their calculation with an example, let us consider the case of a bar with a rectangular cross-section (Figure 2). If we consider the expression above, $Q_x = \int_A y\,dA$, we can take *dA* as the area of the grey rectangle, whose dimensions are *b* and *dy*. Substituting *dA* in the integral, we have that $Q_x = \int_A y\,dA = \int_A y\,b\,dy$. Calculating this integral throughout all the vertical extension of the rectangular cross section, we obtain: $Q_x = \int_0^h y\,b\,dy$. Calculating the integral, we obtain: $Q_x = b\frac{y^2}{2}\Big|_0^h = b\frac{h^2}{2} - b\frac{0^2}{2}$, therefore $Q_x = b\frac{h^2}{2}$.

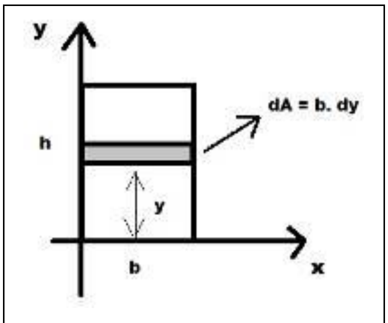

Figure 2: Determination of the first moment with respect to the *x*-axis of an area with rectangular cross-section.

THEORETICAL FRAMEWORK

As stated above, we are interested in analysing how calculus notions are used in professional engineering courses, aiming at identifying possible breaks from the content in calculus courses. For our research, we use tools from the anthropological theory of the didactic (ATD – Chevallard, 1999) because it considers human activities as institutionally situated. In this sense, knowledge about these activities and their raison d'être is also institutionally situated (Castela, 2016, p. 420). In particular, ATD offers a general epistemological model of mathematical knowledge, where mathematics is seen as a human activity through which various types of problems are studied (Barbé, Bosch, Espinoza, & Gascón, 2005, p. 236).

The key element we use in our analysis is the notion of praxeology (or, in our case, mathematical organisation or mathematical praxeology – MO hereinafter), which is formed by a quadruplet [$T/\tau/\theta/\Theta$] consisting of a type of task T to perform, a technique τ which allows the task to be completed, a discourse (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. The first two elements [T/τ] are the practical block (or know-how), whereas the knowledge block [θ/Θ] describes, explains, and justifies what is done. These two blocks are important elements of the ATD model of mathematical activity that can be used to describe mathematical knowledge. Furthermore, ATD distinguishes different types of MO: punctual, which are associated with a specific type of task; local, which integrate multiple punctual MOs that can be explained using the same technological discourse; and regional, which integrate local MOs that accept the same theoretical discourse (Barbé et al., 2005, pp. 237-238).

Praxeologies, like knowledge in general, may move from the institution where they emerge to other institutions that find them useful (Castela & Romo Vázquez, 2011). This is the case, for instance, of mathematical notions that are used to solve engineering problems. In this process, there are *transposition* effects on the concerned praxeologies (Castela & Romo Vázquez, 2011; Chevallard, 1999). We consider the work of Castela (2016), who identified that “when a fragment of social knowledge, produced within a given institution I, moves to another one I_U in order to be used, the ATD’s epistemological hypothesis states that such boundary crossing most likely results in some transformations of knowledge, called transpositive effects” (p. 420). In this boundary-crossing process, some (or all) elements of the original praxeology may evolve. Therefore, it is important to analyse the types of tasks and techniques as well as the discourses and theories employed. To that end, our research identifies specific local MOs present in professional courses; we analyse how calculus notions are used (practical block) and whether this use relates to the way the notions are usually presented in calculus courses (knowledge block).

METHODOLOGY

It is worth noting that, in order to understand how calculus notions are used in engineering courses, we have had several exchanges with an engineering teacher holding bachelor’s and master’s degrees in civil engineering, with more than 28 years of experience teaching a variety of professional engineering courses at Brazilian universities. This teacher has explained notions related to his field and has helped us identify course content in which first-year calculus notions are used.

At this teacher’s university, first moments of area are introduced during the third semester of the programme (second year), in the Strength of Materials for Civil Engineering course (students take calculus in their first two semesters). The course’s reference book is Beer et al. (2012). First moments are initially cited in chapter 4 (4.2. *Stresses and deformations in the elastic range*). We proceeded in two stages:

- First, we analysed the general structure of the content related to integrals in the calculus courses. We identified the main tasks proposed to students, grouping them according to the technological elements needed, identifying therefore the main local MOs that structure this content.
- Second, we started our analyses of the reference book for the Strength of Materials course. We identified all instances where first moments appear in the book (using key words to search in an electronic version of this book). For each occurrence of this notion, we are currently analysing the tasks presented in the book where first moments are used. For each task, we are analysing the techniques and discourses (technologies) the textbook uses. As the notion of first moment is used in different chapters of the book, where different professional notions are introduced and explained, the technological discourses are quite varied, giving place to various MOs. The next section provides specific details of our analysis.

DATA ANALYSIS AND DISCUSSION

Calculus is taught in the first year of the program over two semesters in two courses: Calculus I and Calculus II. Up until a few years ago, integrals appeared only in Calculus II, but some content was moved to Calculus I because Physics II (a course in the second semester) requires a knowledge of integrals. Integrals appear towards the end of the first course and are the main topic in the second course (the second author of this paper has taught Calculus I for 15 years and Calculus II for two years). The content covers indefinite integral (antiderivative of *f*), Riemann sum and definite integral, applications for the calculation of areas, integration by substitution, volumes (Calculus I), and integration techniques, improper integrals, and arc length (Calculus II). The main source for the calculus courses is the classic book by Stewart (2012). The content concerning integrals is basically structured using two local MOs. The first, MO_{M1}, introduces techniques for calculating indefinite integrals (immediate integration to begin with, followed by various integration techniques); however, theoretical elements justifying the different integration methods are mostly absent and those present are explained without a proof. The second MO, MO_{M2}, introduces Riemann sums to formally define integrals and interpret them as areas, and leads to the Fundamental Theorem of Calculus and the calculation of definite integrals using Barrow's rule; this leads to some applications of the integral (area, volume …). Many of the techniques used in MO_{M2} are derived from MO_{M1}.

We are currently analysing the use of first moments and centroids in the engineering textbook (Beer et al., 2012), which numbers more than 800 pages. So far, our first analyses indicate that although this content is introduced as an integral, the techniques employed do not call for integration. Our ongoing analyses of the use of first moments in the textbook are summarised in Figure 3.

Description of use	Terms used	Chapter – Sections
The term appears in a theoretical explanation. It appears with an expression using the integral sign, but no calculation is required.	First moment; First moment of an area; *Q*; centroid	4.2 (p. 245); 4.2 (p. 245); 4.4 (p. 262); 4.6 (p. 274)
		6.1 (p. 421); 6.1 (p. 421); 6.3 (p. 437)
The term appears in a theoretical explanation. It appears without an expression using the integral sign.	First moment; First moment of an area; *Q*; centroid	4.3 (p. 262)
		6.1C (p. 424); 6.1C (p. 424); 6.4 (p. 440); 6.6 (p. 454; pp. 459-460); Review (p. 467)
		8.1 (p. 559); Review (p. 591)
		9.5A (p. 651); 9.5A (p. 651); 9.5A (p. 654); 9.5A (p. 654); 9.6B (p. 666)
Concept application: It is involved in some calculations, but no calculation of integrals is required.	First moment; First moment of an area; *Q*; centroid	4.2 (p. 247; p. 248)
		6.1 (p. 422); 6.1 (p. 422); 6.3 (p. 438); 6.6 (pp. 456-457)
		8.3 (pp. 577-578)
		9.5A (p. 652); 9.5A (p. 653); 9.5A (p. 655); 9.5A (p. 656); 9.6B (p. 667); 9.6C (p. 669)
Sample problem: It is involved in some calculations, but no calculation of integrals is required	First moment; *Q*; centroid	4.3 (p. 251); 4.5 (p. 265); 4.10 (p. 326)
		6.2 (p. 429); 6.5 (pp. 443-446); 6.6 (p. 462)
		8.3 (p. 583)

Figure 3: Synthesis of uses of first moments in Beer et al. (2012).

Here, due to space limitations, we describe our analysis of two MOs present in the textbook at points where first moments come into play. It is worth noting the book advises students that they should already have completed a course in statics, that the properties of moments and centroids are explained in Appendix A, and that this material can be used to reinforce the discussion of the determination of normal and shearing stresses in beams in chapters 4, 5, and 6 (Preface, p. x).

First case: MO_{E1}

The initial use of first moments, MO_{E1}, concerns stresses and deformations in the elastic range (section 4.2 of the book). Its aim is to calculate the maximum stress that beams can resist, resulting in some recommendations about the size and shape of beams. Using some formulae, the book arrives at $\int y dA = 0$ and concludes: "This equation shows that the first moment of the cross section about its neutral axis must be zero. Thus, for a member subjected to pure bending and *as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section*" (p. 245, italics in the original). This is the first apparition of first moments in the book; however, they are not explained and the authors refer readers to Appendix A. In Appendix A, first moments and their link with the centroid are introduced in a similar manner as in this paper, using implicitly theoretical elements from MO_{M2} (namely, the interpretation of an integral). However, the book makes the connection with the centroid and deduces many integrals using geometric considerations (and the properties of the centroid), and adds "Centroids of common geometric shapes are given in a table inside the back cover" (p. A3). Therefore, this content is justified vaguely through some basic integral content (present in MO_{M2}), but mostly by using geometric considerations. The tasks in MO_{E1} calculate stresses and bending moments

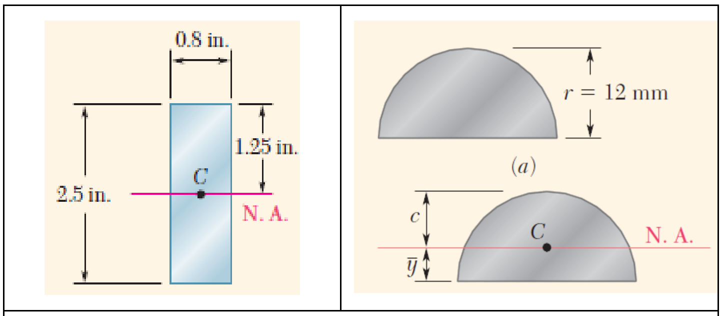

Figure 4: Left: The centroid is placed calculating the half measure of each side of the rectangle (Beer et al., 2012, p. 247). Right: The centroid is placed using geometric formulae (p. 248).

in known geometrical shapes. In the case of a rectangle (Figure 4-left), the coordinates of the centroid are deduced using geometry (and not techniques derived from MO_{M1} or MO_{M2}); the same approach is used in the case of a semicircular cross-section (Figure 4-right).

Thus, although the notions of first moment and centroid are necessary to solve tasks in MO_{E1}, the techniques employed are not based on elements derived from MO_{M1} or MO_{M2}. Students can solve the tasks present in this MO_{E1} without using any of the techniques learned in MO_{M1} or MO_{M2}, or hardly any of the technological elements present in them.

Second case: MO_{E2}

First moments and centroids are also used in chapter 6. In section 6.1A, *Shear on the horizontal face of a beam element*, MO_{E2} seeks to determine the horizontal shear per unit length (or shear flow) on a beam. Defining Δx as the length of a section of the beam, V as the shear force, ΔH as the horizontal shearing force exerted on the lower face of the element, Q as the first moment, and I as the centroidal moment of inertia, and using techniques and technological elements covered in this and previous chapters, the horizontal shear per unit length (q) is deduced as: $q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I}$. It is worth noting that the techniques used to arrive at this expression involve integrals, but they are referred to in terms of notions belonging to MO_{E2}. The above expression is used to solve tasks such as the one in Figure 5.

A beam is made of three planks, 20 by 100 mm in cross-section, and nailed together. Knowing that the spacing between nails is 25 mm and the vertical shear in the beam is V = 500N, determine the shearing force in each nail.

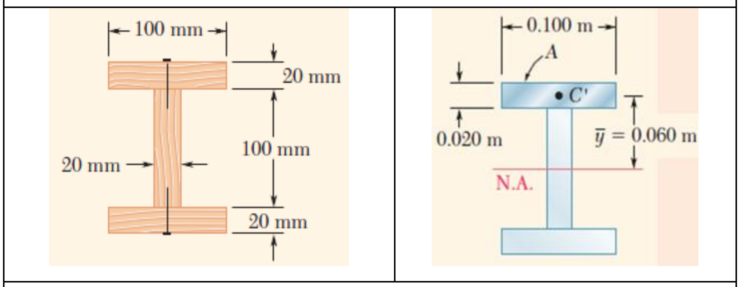

Figure 5: Task and diagrams used concerning horizontal shear (Beer et al., 2012, p. 422).

The resolution of the task is based on the determination, via different expressions, of Q and I (since V = 500N is provided) to find the horizontal force exerted on the lower face of the upper plank. For the first moment, Q, the following technique is presented: “Recalling that the first element of an area with respect to a given axis is equal to the product of the area and of the distance from its centroid to the axis, $Q = A\bar{y}$" (p. 422). The area of the cross-section of the upper plank is calculated as 0.020m × 0.1m, and the coordinate of the centroid of this horizontal plank with respect to the axis of symmetry of the cross-section is 0.05m + 0.01m (that is, half the measure of the central plank, plus half the measure of the horizontal plank). Q is thus obtained as: $Q = A\bar{y}$ = (0.020m × 0.100m)(0.060m) = $120 \times 10^{-6}m^3$. We see that, once more, the tasks to solve in this MO_{E2} involve cross-sections with geometrical shapes that make use of geometrical considerations, thus avoiding techniques belonging to MO_{M1} or MO_{M2}.

Although the technological elements of MO_{E2} refer to elements that imply the use of integrals, tasks are presented in such a way that previously deduced formulae can be used and magnitudes can be deduced using these formulae and geometrical considerations. The book later provides a table with values (Figure 6). Therefore, it is possible for students to simply memorise the formulae or use the tables to solve the given tasks without actually using any technical or technological element derived from MO_{M1} or MO_{M2}.

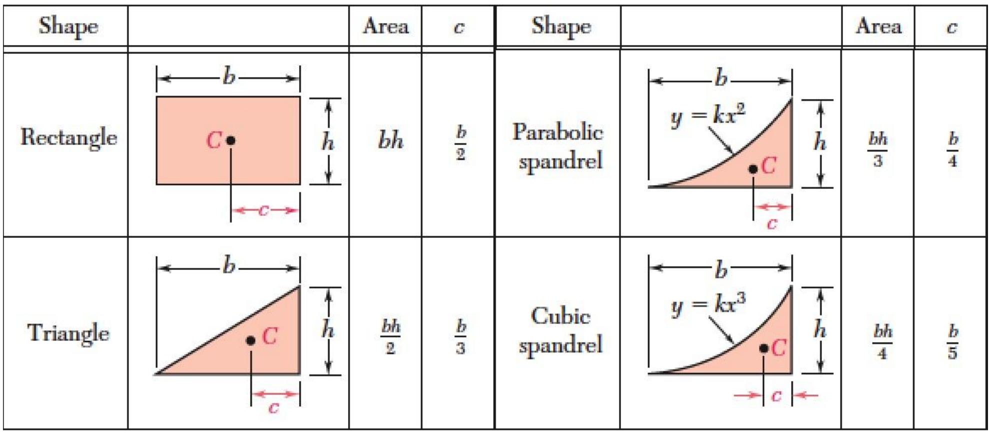

Figure 6: Areas and centroids of common shapes (Beer et al., 2012, p. 654).

FINAL CONSIDERATIONS

The data presented here, together with the data from González-Martín & Hernandes Gomes (2017a), indicate that two notions used in civil engineering (bending moment and first moment) are defined as integrals. This may often be used to justify the fact that "engineers need to learn integrals". However, our analyses show that the types of tasks and the techniques developed are not actually derived from praxeologies explored in a calculus course. In the two cases presented in this paper, both MO_{E1} and MO_{E2} have their own set of tasks and techniques, and both develop their own technological discourse, which uses the notion of integral to define their own notions and deduce properties. As Figure 3 shows, this seems to be the general situation throughout the textbook.

As Castela (2016) states, when a fragment of knowledge (in this case, the notion of integral) produced within a given institution moves to and is used by another institution, this process results in a transformation of knowledge. In the case analysed here, it is clear that all the technological discourse proper to a calculus (or even an analysis) course pertaining to the notion of integral is transformed when it is used to define first moments (and centroid) in a professional course, where explanations mostly rely on basic geometric considerations. In this case, it seems that the transpositive effects cause the notion of integral to be used very differently in both courses. The techniques presented in the Strength of Materials course make use of given formulae and geometric considerations, rendering the techniques introduced in the calculus course irrelevant for the use of first moments in MO_{E1} and MO_{E2}. This may result in students not recognising the object "integral" when they move from MO_{M1} and MO_{M2} to MO_{E1} and MO_{E2}. Students may encounter many difficulties in learning MO_{M1} and MO_{M2}, but this knowledge is not necessary to solve tasks in engineering courses, so students may question the need to learn these MOs.

It is therefore important that mathematics lecturers in engineering programs become aware of how the notions they teach are used in professional courses. Once they develop a better understanding of the techniques and tasks used in professional courses, mathematics instructors may be prompted to reflect on the mathematical praxeologies developed in their own courses and make stronger connections with the techniques used in professional courses. This could help students transition from mathematics courses to professional courses, enabling them to relate mathematical content to the content of their professional courses and better understand its relevance (Flegg et al., 2011).

We plan to continue analysing the use of integrals in professional courses in engineering. This will be the source of future papers.

ACKNOWLEDGEMENTS

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Predictors of performance in engineering mathematics

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Mathematics in university courses was identified as a main obstacle for engineering students in the beginning of their study. Since difficulties with mathematics could lead to a dropout, our research aims to analyse students' profiles referring to individual characteristics that allow identifying possible risks for students' achievement or success in the first year of study. As a first step to identify possibly risky profiles, we started to find possible predictors of students' performance. For this, we give a short overview of the research state and our derived research interest. We discuss theoretical constructs that are possibly crucial characteristics of students with respect to encountering mathematics as an obstacle. Further, we describe the method for measuring different variables of 182 engineering students. Finally, we present results referring to predictors of performance in engineering mathematics and discuss further steps of our research.

Keywords: Teaching and learning of specific topics in university mathematics, Mathematics for engineers, motivational variables, students' profiles, students' achievement.

INTRODUCTION

Besides the technical disciplines, mathematics is a crucial part of higher engineering education (SEFI, 2013). Especially in the first year, mathematics is usually taught without considering practical applications. In lectures and tutorials mathematical basics are provided for subsequent technical courses. However, engineering students "encounter epistemological/ cognitive, sociological/ cultural and didactical obstacles" (Gómez-Chacón et al., 2015, p. 2117) with mathematics struggling with the transition from school mathematics to university mathematics (Gueudet, 2008). Considering mathematical school skills, students show remarkable deficits at the beginning of their study (e.g. Knospe, 2012; Thomas et al., 2012). Empirical studies show the importance of cognitive variables, since school grades and domain-specific previous knowledge are identified as important predictors of academic achievement (e.g. Hailikari et al., 2008). Moreover, there is also some evidence that mathematics plays a crucial role when dropouts from engineering studies are regarded. Heublein (2014) stated the highest dropout rates for mathematics laden studies that were partly caused by a low motivation and partly by excessive demands in the first part of the studies. Also, an international review study mentioned mathematical competencies as part of reasons for dropping out at universities (Søgaard Larsen, 2013).

Apart from cognitive aspects, further individual characteristics are relevant in the context of learning and study success. The importance of these different aspects, e.g. socio-demographic, motivational, emotional and social aspects, are explained by different theoretical models (e.g. utilization of learning opportunities model: Schrader & Helmke, 2015; models of dropout: Heublein et al., 2010) and proved by empirical findings. The meta-analysis by Hattie (2009) summarizes the results of over 800 studies and provides an overview of factors influencing learning success in school. Moreover, there is evidence for the impact of self-efficacy beliefs (e.g. Fellenberg & Hannover, 2006), academic self-concept (e.g. Hattie, 2009) and interest (e.g. Schiefele et al., 1993a) on performance.

RESEARCH QUESTION

In view of the research state, empirical findings show insufficient mathematical skills, high dropout rates and difficulties with mathematics at the beginning of engineering study. Research concerning higher engineering education mostly deals with the improvement of mathematics teaching through developing and evaluating interventions (e.g. through integrating mathematical and technical disciplines: Rooch et al., 2013). According to the utilization of learning opportunities model, learning success does not only depend on the teaching offer but also on its utilization by students. Therefore, in this project engineering students should be explored in more detail, especially with respect to mathematics and individual characteristics. Moreover, the study of Fellenberg & Hannover (2006) gives hints that a domain-specific investigation is also empirically meaningful. Concerning the time frame, our project focuses on the first year of engineering study because the secondary-tertiary transition and dropout surveys indicate serious problems at the beginning of study. In particular, more students decide to abandon one's studies within the first two semesters (Heublein et al. 2010).

Empirical basis for the relevance of individual characteristics in learning processes exists. However, most of the studies were conducted in the context of school. Since the subject matter and learning environment changes with the transition from school to university, these results cannot be transferred immediately. In contrast, studies of higher education with a special view of mathematics are rather rare. In particular, most studies concentrate on single aspects and not on an overview of different impacting variables (e.g. Schiefele et al. 2003). Therefore, our project draws on previous findings to explore a multitude of impacts of individual characteristics in learning processes of higher education with the main aim of developing mathematics related profiles of engineering students. As a student's profile we understand the characteristic of different individual variables, e.g. performance, motivational variables like interest (Gómez-Chacón et al., 2015), or engagement (Rach & Heinze, 2013), and further the relationships among these variables. Students' profiles should allow identifying possible risks for study success in the first year of the study. The identification of students' profiles is useful because it allows the development of goal-oriented and adequate support services for students who encounter difficulties.

Since insufficient motivation and excessive demands for achievement are primary reasons for dropping out, we focus on motivational and cognitive variables as a first step to developing students' profiles. Moreover, students leaving their course of study are difficult to access, so we focus on available data like students' performance and individual characteristics with respect to the following research question:

> Which domain-specific predictors of study success or failure can be determined in the first year of engineering mathematics?

THEORETICAL CONSTRUCTS

As a main construct that impacts on students' achievement, we refer to the construct of learning motivation as an umbrella term for different motivational variables (Spinath, 2011). Firstly, we choose all of the motivational variables that are summarized by the term learning motivation in order to identify the crucial impacting factors. Furthermore, all of the constructs are connected to the subject matter, so they might play a crucial role in the transition from school to university mathematics, a situation characterised by a changing subject matter and learning environment.

A first and main part of motivation is an individual's goal orientation (Dweck, 1986). This dispositional variable involves individual's beliefs about appropriate goals as a trait referring to specific and, thus, context-related tasks (Elliot et al., 1999). A further dispositional and motivational variable is interest which is differentiated into three components: feeling-related valences, value-related valences and an intrinsic character. Interest could be understood as an individual's development of an appreciation for a specific subject like mathematics (Wild & Möller, 2009). This definition involves the necessity to regard interest context-specific. One aspect of the construct of interest, i.e. the feeling-related valences is also a part of the expectancy-value-theory of Wigfield and Eccles (1992). They derive an individual's motivation for doing a task from the individual's expectancy of the success on a specific task and the incentive value of this task. Besides the intrinsic value, which is similar to the feeling-related valences of interest, further variables, i.e. attainment value, utility value and costs are part of achievement-related values. The expectancy of a success referring to specific tasks could be understood as individuals' self-efficacy beliefs (Wigfield & Eccles, 1992) that are close to the construct of self-concept (Shavelson et al., 1982).

Learning strategies is a further umbrella term that includes variables which also could have an impact on students' achievement (Wild, 2005). Learning strategies include cognitive learning strategies like strategies for elaborating a specific issue, meta-cognitive strategies like planning or monitoring the process of learning, or strategies to use resources like a specific learning environment. Finally, although students' achievement or success is hardly to define (Heublein, 2014), it could be understood as the achievement in exams referring to a specific subject like mathematics or to proceed in a field of study like engineering despite encountered difficulties.

METHODOLOGY

Sample

Our first research step involved 182 engineering students at the University of Kassel enrolled in a calculus course in the summer semester 2017. Among the participants were 158 men and 24 women. Most of them started their second semester (67 %), though a small group of beginners is integrated (14 %). In the beginning of the semester, students of the calculus course were given a questionnaire concerning sociodemographic factors and motivational orientation towards mathematics. To achieve a high response rate, the students had two weeks to answer the questionnaire and received additional points for their permission to the final exam that could be achieved by solving weekly exercises. Students were also assured of the anonymity of their responses.

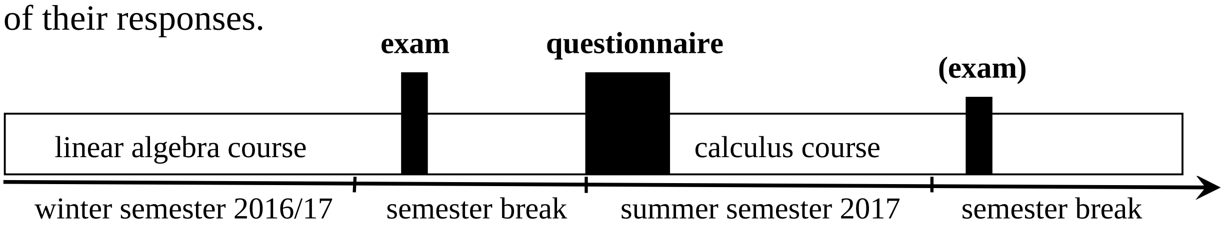

Figure 1: Time of data collection

Instruments

The so called SELLMO instrument (Spinath et al., 2012) was used to measure students' goal orientation towards mathematics. Twenty out of thirty-one items were chosen and especially referred to mathematics courses at university. Goal orientation is divided in four sub-scales with each five items concerning approaching and avoiding achievement goals, work avoidance and learning goals. One example for an item referring to learning goals is: "My aim for mathematics courses at university is to gain a deep understanding for the content."

For interest, we used a scale with nine items of Schiefele et al. (1993b) and adapted it to mathematics. One example for a negative formulated item of this scale is: "To be honest, I less care for mathematics." Referring to the expectancy-value-theory, we measured values with a scale involving six items that we developed according to Wigfield and Eccles (1992). One example for an item of this scale is: "Mathematical skills will be crucial for my future professional career." Further, we adapted each three items from the PISA study (Kunter et al., 2002) to measure students' self-concept and self-efficacy with respect to mathematics.

To measure the students' learning strategies, we focussed in the first step of our research on students' self-reports about the use of resources like lectures, tutorials and special exercises. Whereas in lectures different mathematics topics are taught, students practice in tutorials by means of additional exercises. In the special exercises they discuss their homework and resolve open questions. Finally, students' success was measured by the self-reported grade in the final exam of the linear algebra course (see Fig. 1). By contrast, further grades of final exams in the abovementioned calculus course was directly given but is not analysed yet. Finally, we collected sociodemographic variables, e.g. the type of matriculation standard, according to a questionnaire used for dropout studies (Heublein et al., 2010).

RESULTS

Results in the first step of our research firstly refer to an evaluation of the instruments concerning the quality of scales. We further proved the predicting power of different variables on the students' performance measured by the self-reported grade in the final exam of the linear algebra course that students have taken the previous semester.

Evaluation of the instruments

In the first analyses, Cronbach's alpha estimates of reliability were determined for the scales from each instrument (see Tab. 1). Measures are adequately reliable, with values ranging from .552 to .844. Most of the values are appropriate, the lowest value (.552) was found for a scale with only three items.

Construct	Number of Items	Cronbach's Alpha
Mathematics Interest	9	.844
Goal orientation:		
Approach achievement goals	5	.722
Avoidance achievement goals	5	.827
Word avoidance	5	.727
Learning goals	5	.739
Expectancy-value-theory		
Mathematics self-concept	3	.704
Mathematics self-efficacy	3	.552
Value of mathematics	6	.690

Table 1: Sample constructs and Cronbach's Alphas

Possible predictors of performance in engineering: correlations

In each of the following analyses, we defined students' achievement as the self-reported exam grade of the linear algebra course that students have taken the previous semester. We firstly assessed the relationship between students' individual characteristics referring to grades achieved in school. As seen in Table 2, the school grades and exam grades of the linear algebra course are significantly and positively correlated.

	Math school grade	Final school grade
exam grade	**.321****	**.393****

Table 2: Pearson's correlation coefficients between the exam grade and students' achievement in school (*p < 0.05; **p < 0.01)

We further proved the correlations between the exam grade and variables constituting the expectancy-value-theory. Except for interest, the correlations between the motivational variables and the students' achievement are significant. Particularly, there is a considerable relationship between the mathematics self-concept and the students' achievement.

	self-concept	self-efficacy	values	interest
exam grade	**.554****	**.363****	**.385****	.166

Table 3: Pearson's correlation coefficients between the exam grade and students' individual characteristics (expectancy-value-theory)

By contrast, the correlations between the students' achievement and the students' individual characteristics referring to the construct of goal orientation are weak and except of the working avoidance, not significant.

	AAG1	AGG2	WA	LG
exam grade	.000	.177	**.207***	.109

Table 4: Pearson's correlation coefficients between the exam grade and students' individual characteristics referring to AAG1 (Approach achievement goals), AAG2 (Avoidance achievement goals), WA (Work Avoidance), LG (Learning goals)

Using correlation analysis, we finally assessed the relationships between students' achievement and the students' engagement referring to external resources given by the attendance rate of lectures, tutorials and special exercises. However, the attendance rates seem to be independent of the students' achievement.

	Lecture	Tutorials	Special exercise
exam grade	.133	.074	.120

Table 5: Pearson's correlation coefficients between the exam grade and students' engagement

Possible predictors of performance in engineering: group comparison

To identify other possible predictors of the performance in engineering mathematics, we compared between distinct groups using t-tests. Firstly, we compared students that were enrolled in an advanced math course and students that were enrolled in usual math courses. Students in advanced math courses get more math lessons in a week and, thus, examine mathematics in a greater extent than students of usual math courses. As expected, students attending a math advanced course have significantly better grades in the exam of the linear algebra course (see Tab. 6). Further, we compared the group of students who were at a technical secondary school in which the extent of mathematics is lesser than in usual secondary schools. As expected, on average, students who had attended a technical secondary school obtained in the exam of the linear algebra course a grade of 4.0, whereas the corresponding grade for students who had not attended a technical secondary school was 3.5. Thus, students from a technical secondary school significantly perform worse in the linear algebra course than those who attended another type of school (see Tab. 6).

		Technical secondary school		Math advanced course	
		1	0	1	0
Exam grade	M	4.0	3,5	3.3	3.9
	SD	1,0	1,0	1,2	1,0
	Sig.	**.016**		**.019**	

Table 6: Exam grade of the linear algebra course depending on different subgroups (1 = attended; 0 = not attended)

DISCUSSION AND CONCLUSION

The development of engineering students' profiles referring to cognitive and motivational variables could potentially result in identifying students' risks for an undesirable low success or a dropout. For identifying possibly risky profiles, we started to find possible predictors of the students' achievement. Final school grades, maths self-beliefs, values of maths, the type of matriculation standard and the choice of advanced courses in school are meaningful predictors of performance in engineering mathematics. Hence, the results show that several variables determine the maths performance which should be considered seriously for the development of support services. However, the results reveal new questions leading to further possible research steps.

In conformity with the research state, the mathematical achievement as well as the amount of mathematics in school seem to be a predictor of the success in a final exam of the first semester. As a subsequent issue it is a crucial question if the impact of the former school time on the students' success in mathematics courses at university decreases or disappears. Moreover, it is interesting whether previous knowledge measured by a skills test has a greater effect compared to the school grades. Perhaps, domain-specific skills can be recognised as special predictors of mathematics courses in engineering. In addition, the results show that the highest correlation exists between the final school grade and students' performance. This implies further influencing variables developed in the students' school time that impact on the students' achievement at university, especially learning activities and strategies. The fact that the attendance rates seem to be independent of the students' achievement strengthens this perspective.

Concerning the motivational variables, it is interesting that not all of them have an impact on the students' achievement. The students' mathematics self-concept that is also developed in the time of learning mathematics in school shows the highest correlation to the students' achievement. Therefore, support services should not only focus on the deficits in mathematics skills but also on the assistance of students' self-beliefs. Mathematics interest shows no impact on the mathematics performance. This result could approve findings like Eilerts (2009). Since engineering students do not choose mathematics voluntarily, mathematics interest might have no predicting power in this context. In this respect, it could be also interesting to differentiate in further analyses specific groups of students, e.g. concerning gender, the school form or other variables and to investigate if different groups show different relationships between motivational variables and the students' achievement.

Regarding the method, proven scales have been used and adapted to mathematics. Only scales with a low number of items can be extended to improve the reliability. All analyses base on simple correlations. Results can be improved and deepened by using further methods like regression analyses or structural equation models, so that an investigation of indirect effects is facilitated.

To conclude, in further steps of our research, the observation of motivational variables of engineering beginners should be continued and extended to the investigation of their development. Additionally, engineering students should be surveyed in respect of learning activities and strategies that we involved in this first study only by collecting data to the use of attendance rates (external resources). A detailed investigation of students' motives for non-attendance would give more information about engineering students' learning behaviour. Thus, building upon the first results of our research, we expect to deepen the desirable insight into students' profiles in the next steps of our research.

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“Making connections” in the mathematics courses for engineers: the example of online resources for trigonometry

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Abstract: This paper concerns the teaching of mathematics for future engineers, focusing on the theme of trigonometry. We claim that the use of trigonometry in engineering courses requires different kinds of connections: connecting different domains, different concepts, frames and registers. We use here the concept of connectivity, developed in the frame of e-textbooks analysis, to analyse online courses for future engineers in France. We evidence that these courses propose some connections; but their connectivity is not developed enough to meet the requirements of engineering courses.

Keywords: Connectivity, Teaching and learning of specific topics in university mathematics, The role of digital and other resources in university mathematics education, Teaching and learning of mathematics in other fields, Mathematics for engineers.

INTRODUCTION

How can a teaching of mathematics answer the needs of engineering courses, i.e. can provide the mathematics needed to understand the course and solve the problems proposed? Which should be the features of such a teaching, and do existing courses present such features? This is the general theme of the work presented here. Previous works addressing this theme observe a gap between the mathematics taught in “mathematics courses” and the use of them to solve problems in engineering courses (e.g. Redish 2005; Biehler, Kortemeyer & Schaper 2015). Winsløw, Gueudet, Hochmut and Nardi (2018) note that several works presented at CERME conferences identify “a lack of connectedness of curricula integrating mathematics and other disciplines”. Interviewing French engineers about their mathematical needs in the workplace (Quéré 2017), and how the mathematics courses they followed as students answered or not these needs, we noticed that several of them declared that the mathematics courses “did not make enough connections”. These connections, according to the engineers, can be of different kinds: links between mathematics and the real world, between different mathematical contents, between different representations etc. We consider that this is an important issue for understanding mathematics applied to engineering and their teaching. Which kinds of connections should propose a teaching of mathematics for engineering?

Moreover, in the frame of another starting study, we are interested by the possible design of innovative curriculum resources for the teaching of mathematics for

STEMSS (Science, Technology, Engineering, Medicine and Social Sciences) courses. Before starting the design of such resources, we have investigated existing resources, trying in particular to observe whether they tried to build the kind of connections evoked by the engineers.

This investigation led us to choose a focus on trigonometry. Indeed, trigonometry appeared as extensively used in different kinds of engineering courses, and offering many possibilities of connections of all the kinds evoked above. In what follows, we firstly introduce our theoretical tools and research questions; then we present related works on the teaching of trigonometry. To exemplify the mathematical needs, we analyse the use of trigonometry in an electricity course for first year students; then we consider the trigonometry content in two online mathematics courses for engineering students.

THEORETICAL PERSPECTIVE AND RESEARCH QUESTIONS

The overarching perspective guiding our work is an institutional perspective (Chevallard, 2006). We consider that the mathematics taught are shaped by the institution where they are taught. Engineering studies at university or in engineers' schools constitute an institution, different from mathematics courses for maths majors. Within engineering studies, engineering courses also constitute an institution different from mathematics courses. Our aim here is to compare how these two institutions shape the mathematics contents, more precisely the trigonometry contents.

Our central focus is on "making connections". Students' understanding in terms of mathematical reasoning and problem-solving has been linked by several authors to "making connections" and "connectivity" (e.g. Hiebert & Carpenter 1992). Drawing on these works, we have chosen to look for:

"connections in, between, and across individuals' cognitive/learning tasks and activities, and how e-textbooks may support those (micro level); as well as for 'connected learning' between and across groups of individuals, teachers or students (macro level)." (Gueudet, Pepin, Sabra, Restrepo & Trouche to appear)

We have therefore proposed a concept of "connectivity" to analyse e-textbooks (encompassing various kinds of digital curriculum), with the intention to evaluate their potential for the building of connections for the students.

Thus, connectivity has two components: "macro-level connectivity", which considers the e-textbook as a whole; and "micro-level connectivity", where the focus is on a particular mathematical topic (trigonometry here). In this paper we only use "micro-level connectivity"; more precisely, we observe in curriculum resources available online the presence of:

- Connections between different topic areas or frames;

- Connections between different semiotic representations (e.g. text, figures, static and dynamic); […]
- Connections between different concepts […]” (Gueudet et al. to appear).

Here we want to compare the connections concerning trigonometry when it is used in engineering courses and the connections concerning trigonometry in mathematics courses for engineers, more precisely the connectivity of online courses for engineers. Our purpose is not to discuss whether trigonometry should be introduced in mathematics courses or engineering courses, but to compare how it is introduced/used in these institutions. Hence the research questions we study here can be formulated as:

- Which connections concerning trigonometry appear in non-mathematical engineering courses?
- Which connectivity, concerning trigonometry, can be observed in online resources for mathematics courses for engineers, and how does this compare with the connections in non-mathematical courses?

In terms of methods, we have searched for curriculum resources and online courses on three major websites used in France: Unisciel[1], meaning “online science university”, gathering many online courses; IUTenligne[2], a website for technicians institutes within universities; and FUN[3], meaning France Digital University, the national platform offering MOOCs. We have selected all the resources corresponding to mathematical courses for engineers or technicians on trigonometry, and have looked at the same time for resources on non-mathematical subjects using trigonometry. We have eventually chosen the theme of electrical engineering, because we have identified in it specific mathematical needs in trigonometry. Before analysing these resources, we now consider some works about trigonometry in mathematics education, and how they enlighten the connections issue.

CONTEXT AND RELATED WORKS

In France, trigonometry is firstly introduced in grade 8 or grade 9 through the definitions of cos, sin and tan as quotients of lengths in a right-angled triangle. The angles are measured in degrees. In grade 10, the unit circle and the radian are introduced, together with a new frame for cos and sin, which are now the coordinates of a point on the unit circle. In grade 11, cos and sin are studied as functions. Hence connections between frames and registers are extensively present in this teaching, and these connections can raise difficulties for the students (Berté *et al.* 2004). For

[1] http://uel.unisciel.fr

[2] http://www.iutenligne.net/

[3] https://www.fun-mooc.fr/

example, grade 9 students already use for solving geometry exercises the "$\cos^{-1}$" key on their calculator; nevertheless, this key refers to a reciprocal trigonometric function, which is only presented at university, and moreover belongs to the functional frame (Bueno-Ravel & Gueudet 2010).

The international research on the teaching and learning of trigonometry acknowledge the existence of all these different registers and representations and investigate their consequences. Kendal and Stacey (1997) compare two teachings in grade 10 in Australia, using respectively ratios and the unit circle to introduce sin, cos and tan; they conclude that the ratio method appears as a better choice. Also at the university level, trigonometry remains a difficult subject for the students (Weber 2005). Mesa and Goldstein (2017), studying the presentation of trigonometry in college textbooks, have evidenced that these textbooks propose different conceptions of angles, trigonometric and inverse trigonometric functions; depending on these conceptions, some problems can be delicate to tackle for the students. The textbooks do not try to link different conceptions, and do not highlight which one is more relevant for a given problem.

Trigonometry clearly requires many connections between frames and registers. It is moreover linked with many different mathematical subjects (geometry, functions, but also complex numbers); and is extensively used in physics. Several researchers have also studied trigonometry within physics courses. Chiu (2016) studied the impact of a new curriculum in Taiwan, where contents of physics requiring trigonometry are taught before the corresponding mathematics courses. She observes that, while curriculum designers are positive on the possible consequences of a teaching of trigonometry by the physics and then by the mathematics teachers, the students and the teachers are mostly negative about this experience. A teaching of trigonometry in mathematics courses seems necessary before using it in physics.

In his comparative study between France and Vietnam, Nguyen Thi (2013) shows that in both countries, trigonometry is present in physics courses with mathematical models for periodical phenomena, under two forms: uniform circular motion (represented by a point moving on a circle, in a graphic or algebraic register) and harmonic oscillation (represented by functions in a graphic, algebraic or vectorial register). Nevertheless only a few exercises propose modelling activities (in physics as well as in mathematics); and the two models are almost never connected.

TRIGONOMETRY IN ELECTRICAL ENGINEERING COURSES

In this section we draw on the content of several courses for future engineers or technicians, available on French websites (e.g. Piou 2014). In electrical engineering, one of the major subjects is the study of the "alternating sinusoidal regime". In circuits with such a regime, the different signals (current and voltage in particular) are of the form: $s(t) = A\sqrt{2}\sin(\omega t+\phi)$.

In a circuit where reactive loads are present (like capacitors or inductors) energy storage in the loads results in a phase difference between the voltage (u) and the current (i) waveforms. This difference is firstly introduced in the context of functions, with formulas like: $u(t)=U\sqrt{2}\sin(\omega t)$; $i(t)=I\sqrt{2}\sin(\omega t-\varphi)$, and associated with a graphical representation as two curves on the time axis (Figure 1, left part).

A connection is immediately established with a geometrical representation of these signals, through "Fresnel vectors". A signal defined by $s(t) = A\sqrt{2}\sin(\omega t+\phi)$ (where A is positive) can be represented by a vector of length A, and a direction forming an angle ϕ with the horizontal direction. Hence $u(t)$ can be represented as a horizontal vector, and $i(t)$ as a vector forming an angle ($-\varphi$) with it (figure 1, right part). Some courses also propose an interpretation in terms of complex numbers; for the sake of brevity, we do not present it here.

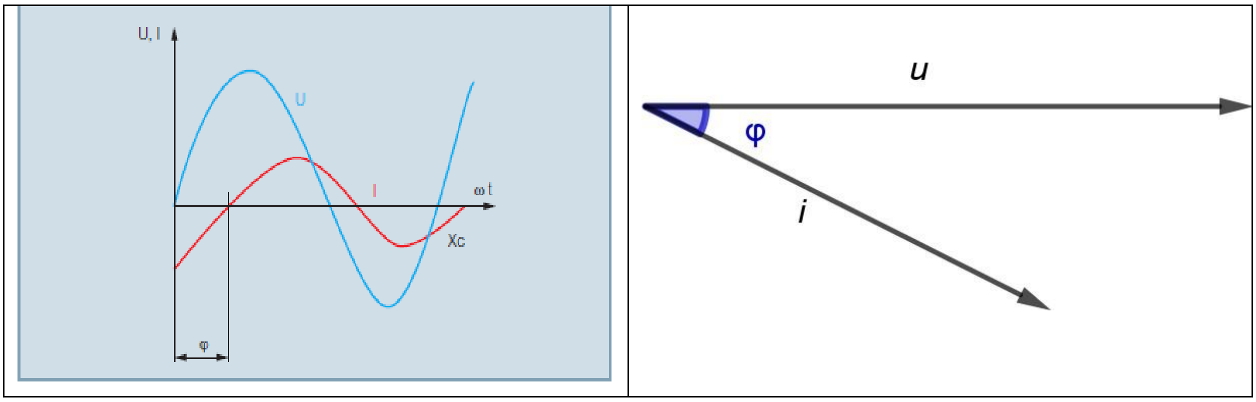

Figure 1. Signals in the alternating sinusoidal regime and phase difference. On the left: functional frame; on the right, Fresnel vectors in the vectorial frame.

We argue that the students in this case have to master connections between these two representations of signals: as two curves with a gap of φ on the horizontal axis; or as two vectors forming an angle φ. Some courses propose animated pictures or exercises to work explicitly on this connection (see figure 2 for an example extracted from a teacher's website, http://fisik.free.fr/.)

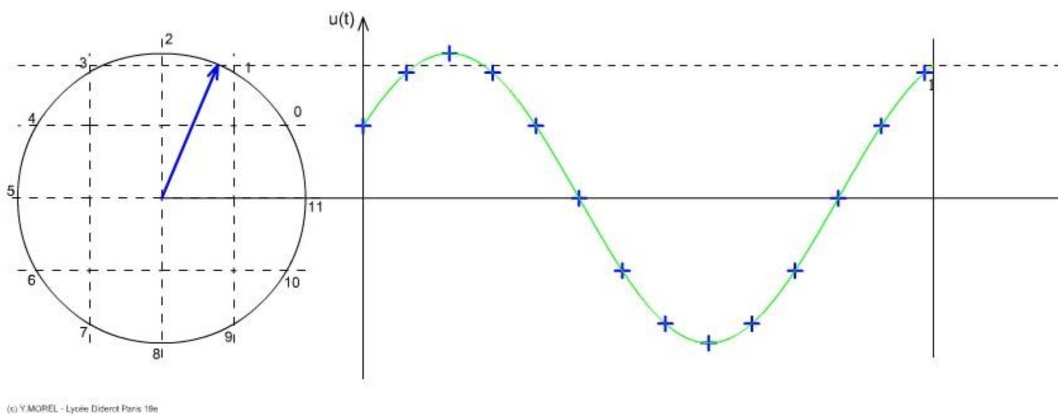

Figure 2. Connection between a rotating vector and the corresponding curve, extract of an animated picture. The blue points on the right appear when the vector rotates.

In terms of micro-level connectivity, electrical engineering courses naturally connect trigonometry with electrical engineering; they also connect different concepts: in the example we considered here, functions and vectors, and in other courses also complex numbers. Since vectors are represented as arrows, and functions represented by their graphs, these two kinds of representations are also connected in the text of the course.

TRIGONOMETRY IN MATHEMATICS ONLINE COURSES FOR ENGINEERS

A MOOC presenting the basics of mathematics for future engineers

The MOOC "Basics of mathematics", available on the FUN platform (freely available after inscription) presents its objective on its first page: "This MOOC aims at revising the basic notions of mathematics, needed to start engineering studies". The MOOC lasts 12 weeks, corresponding to 12 chapters. Chapter 2 is entitled "trigonometry". It comprises 7 course videos (from 5 to 12 minutes), interactive multiple-choice questions, and a final assessment. Most of the notions presented in chapter 2 of the MOOC are taught in France at secondary school; nevertheless the reciprocal functions (like $\sin^{-1}$) are only presented during the first year of university.

The course proposes almost no connection with engineering activities. The first video says that "Trigonometry appears in many domains, like drawing plans, navigating or mechanics"; the last video mentions the task of "triangulation", without definition. Three exercises are associated with each video. Most of these exercises are situated in the geometrical frame; some of them are in the frame of trigonometric equations. There is only one exercise in the whole chapter with a concrete context: computing the length of a cable joining the top of a pole to the ground, over a mountain with slope of 15%. The final evaluation is composed of two problems; the first one has a concrete context, measuring lengths, and the second one concerns trigonometric equations.

The definitions of cos, sin, tan are introduced in the geometrical frame in the right triangle (video 1). In video 2, a software (Maple reader) is used to establish a connection between cos, sin and tan in the unit circle and their graphs as functions (figure 3). The animated picture supports the discourse of the teacher explaining this link. This picture associates in fact the unit circle with the graphs of the functions, but also with the triangle. In fact the radius of the circle can be changed; it is not only the unit circle, but any circle; and the words "Adjacent" and "Hypotenuse" on the left of the screen link the circle and the other geometrical view as a triangle.

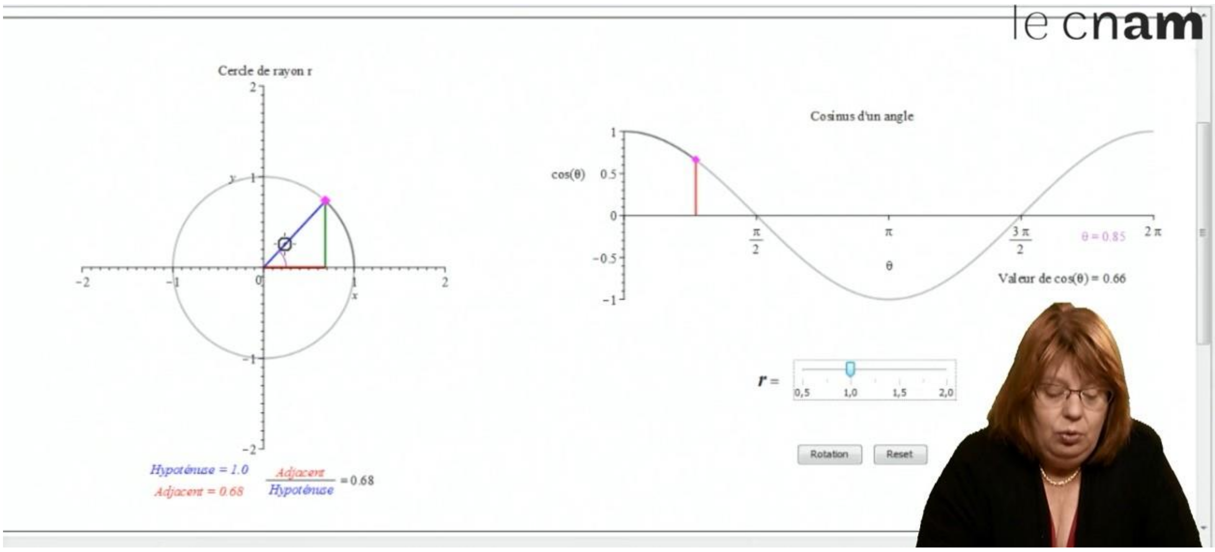

Figure 3. A dynamic representation making connections in the Mooc, case of cos.

This dynamic image is clearly used to connect the geometrical frames (triangle and unit circle) and the analytical frame; nevertheless, it might be very difficult to understand for participants who do not remember their school courses.

As a summary, we retain that micro-level connectivity in this MOOC comprises some connections between concepts and representations, including dynamic representations; but almost no connections with other domains or engineering contexts. Moreover the connections with dynamic representations can remain unclear for students because of a lack of explanations.

“Mathematical tools for physics”, an online course

“Mathematical tools for physics”[4] is an online course addressed to first year students in physics, freely accessible (without inscription). It belongs to a complete first year course, which is always organised in three sections: “learning” (course) “practice” (exercises) and “self-assessment”. It comprises 11 chapters; chapter 6 is entitled “Circular trigonometry – Hyperbolic trigonometry”. This chapter comprises 8 subsections; here we are only interested in the 4 subsections concerning circular trigonometry.

After recalling the definition of an angle, the first subsection defines sin, cos and tan in the frame of the unit circle. Nevertheless, an animated picture proposes a link with the frame of the right-angled triangle. A subsection about formulas associates a functional frame: specific values of sin, cos and tan and register of the unit circle. Then two sections are dedicated to the properties of the “direct circular functions” and of the “reciprocal circular functions”, and only mention the functional frame. This course makes no link with physics or any real-life context.

[4] http://uel.unisciel.fr/physique/outils_nancy/outils_nancy/co/outils_nancy.html

There is only one problem, divided in three questions, in the "practice" section. Interestingly, it is a problem of physics: "the Compton effect" (scattering of a photon by an electron). To solve this problem, the students must master contents of physics: the law of conservation of energy and quantity of movement (and the associated formula). The initial model is in the frame of vectors; these vectors are projected on the two axes, and the students have then to use trigonometric formula, and finally to work in the frame of functions. The difference between the wavelengths before and after the scattering is indeed of the form $\lambda(1-\cos\theta)$, where θ is the angle characterizing the direction of the photon after the scattering; the students must observe that this function is increasing over $[0, \pi]$: a larger angle corresponds to a larger change in the wavelength.

The "assessment" section comprises five parts: 2 on circular trigonometry, 2 on hyperbolic trigonometry and one entitled "composition of vibratory motions". There is here again a connection between trigonometry and physics. Nevertheless its mathematical part remains entirely in the functions' frame.

Finally, concerning the micro-level connectivity of this online course, we retain that it proposes some connections with physics in the exercise and assessment part (but no such connection in the course part). The connections between concepts and between registers are restricted to the case of a single problem.

CONCLUSION

Trigonometry is a domain of mathematics where many different concepts (angles, vectors in geometry; functions) and semiotic registers (triangles, arrows, circle in geometrical register; curves in a graphical register; equations etc.) can be connected. It is recognised as a difficult domain for students. Nevertheless, within mathematics some exercises are limited to a single register: the study of trigonometrical functions for example does not always require thinking in terms of angles. Using trigonometry in engineering courses, on the opposite, always requires such connections. The students must be able to associate an expression like $s(t) = A\sqrt{2}\sin(\omega t+\phi)$ both with a function and its graph; and a vector represented by an arrow. Engineering courses have a high degree of micro-level connectivity, for trigonometry. Our analyses of two online courses of trigonometry for engineers in France lead us to observe that they have a reduced level of micro-level connectivity: limited to connections between concepts and semiotic registers for the first one, while the second one on the opposite offers more connections between trigonometry and physics, but reduced connections between concepts and semiotic registers.

Our exploratory work leads to formulate recommendations for mathematics courses for future engineers, concerning trigonometry (and possibly other topics). Our study of engineering courses confirms that developing the ability of these students to make connections is an important aim. Connections between frames, between registers, but also connections with engineering are possible, as evidenced by Nguyen Thi (2013).

Online resources could propose such connections, and moreover could draw on the possibilities offered by dynamic representations and various kinds of software. Contributing to the development of such resources is an important aim for mathematics education research, to address the need for students to make a relevant use of the mathematics they learn at university in and for their engineering courses.

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Engineering students' engagement with resources in an online learning environment

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In this paper, we investigate how undergraduate engineering students interact with an online learning environment provided to them in a Calculus course. The constituent resources of this environment include tutorial videos, textbook and MyMathLab – an online interactive system for mathematics. A qualitative case study involving a small group of students has been conducted. We investigated which resources these students used and the manner in which they incorporated these resources in their online mathematical work.

Keywords: Students' interactions with resources, the role of digital and other resources in university mathematics education, mathematics for engineers.

INTRODUCTION

In recent years, digital resources are increasingly used for teaching and learning of mathematics (Borba et al., 2016; Pepin, Choppin, Ruthven, & Sinclair, 2017). The presence of wide range of digital resources in terms of their functionalities allows various possibilities of creating digital environments for students to learn mathematics. Each digital environment might afford unique interactive and learning opportunities; therefore, empirical research closely looking at students' engagement and the opportunities for their learning in such environments is well needed. This study deals with one digital learning environment provided to undergraduate engineering students for practicing mathematics. The aim is to explore students' interactions with the constituent resources of this environment to elucidate the learning opportunities in this environment.

Adler (2000) introduced the term *resource* to embrace several agents such as physical, human and cultural tools and aids intervening in a teacher's activity. In this paper, however, we distinguish between digital and classical resources and focus on students' work with resources. The use of digital resources is relevant in the context of engineering mathematics in the sense that engineers during their professional activities rely on technology for solving mathematical tasks (van der Wal, Bakker, & Drijvers, 2017). The framework for mathematics curricula in engineering education (Alpers et al., 2013) recommends the use of technology aimed at fostering engineering students' mathematical competencies. In the next section, we present the theoretical framework, and the subsequent section contains introduction to the constituent resources of the online learning environment.

THEORETICAL PERSPECTIVE

In order to study students' interactions with the resources, we employ the *documentational approach to didactics* (Gueudet, Pepin, & Trouche, 2012; Gueudet & Trouche, 2009) which is grounded on Rabardel's work (Rabardel, 2002) and enlarges the instrumental approach (Trouche, 2004) in mathematics education. One important distinction between the two approaches lies in the extension of the concept of *artefact,* in the former approach, to *resource* which allows considering wider set of materials intervening in the teachers' and students' activities. A resource can be conceptualised "as both noun and verb, as both object and action that we draw on in our various practices (Adler, 2000, p. 207)". Thus, the approach has the potential to take in consideration material, human and cultural resources such as language, time, mathematics teachers, etc. Moreover, a resource is never isolated but belongs to the wider set of resources (Gueudet & Trouche, 2009).

While one focus of this approach is on the teacher's work with the resources, the study of students' use of resources can provide the overview of their actual use (Gueudet & Pepin, 2016). Also, this approach has the potential to provide rich analyses if used to evaluate students' work in terms of interactions with different resource systems (Trouche & Pepin, 2014) or with a particular resource (Aldon, 2010). We will employ this approach to analyse how students interact with available resources.

In particular, we analyse students' techniques when working digitally in mathematics (Artigue, 2002). A technique is perceived as "a manner of solving a task (Artigue, 2002, p. 248)". While students work on mathematical tasks in a digital environment, they might adopt paper and pencil based techniques or instrumented techniques. The obvious and easily observable objective of each technique is to reach the goal of the activity i.e. to produce the results whereas the contribution of a technique to the learning of involved mathematical concepts might not be easily recognisable. The former corresponds to pragmatic value while the latter corresponds to epistemological value liked to each technique.

We seek to explore the kind of techniques implemented by the students in the digital environment to make sense of how students interact with this environment while working on mathematical tasks. Furthermore, realisation of the values attached to the students' instrumented techniques will also help to understand the role of digital resources in their learning (Guin, Ruthven, & Trouche, 2005). There are several resources involved in present situation, therefore, we confine to the general features of corresponding techniques in the present paper. By this, we mean to consider students' general organisation of digital work with several resources related to all contents in a Calculus course. We ask the following question: How do engineering students incorporate resources during their work in an online learning environment?

THE SETTING

This study took place in a Norwegian public university during the spring of 2017. Undergraduate students enrolled in electronics engineering program participated in this study. In their Calculus course, students were offered an online learning environment such that they could work remotely by interacting with the provided resources. These resources were made available to them electronically to work and proceed through the course. There were no mandatory lectures, and they could access the lecturer in the case they needed additional support. The final examination was also in digital format where the students were allowed the access to tools and aids.

The resource system comprised MyMathLab environment, tutorial videos coupled with the notes, and the textbook. The students' homework and the formative assessments were administered online through MyMathLab system. MyMathLab is an interactive learning system for practicing mathematics online (figure 1). While this system provides an online platform for homework and assessments, it also facilitates students in solving the tasks by providing help and feedback. Students can seek help through utilising "help me solve this" or "view an example" functions in the system. The former lets the student solve a similar task by guiding on each step whereas the latter shows a similar worked-example. The interactive nature of MyMathLab system allows considering it as a resource which can potentially influence students' activity in this course.

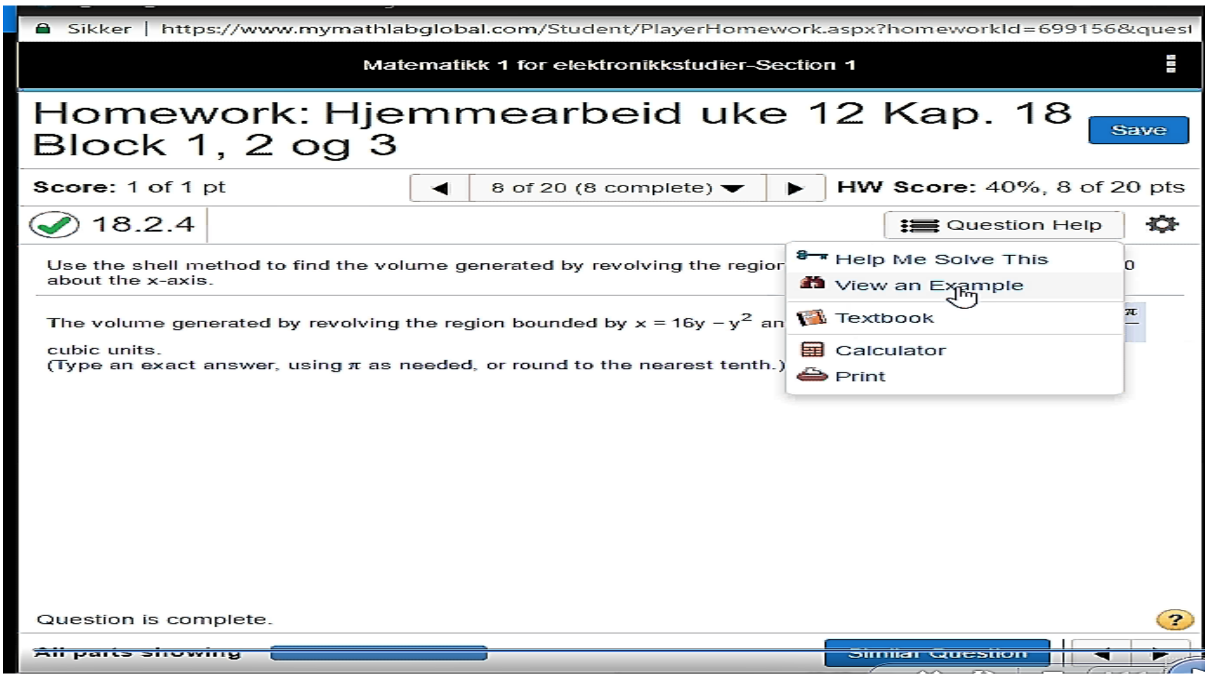

Figure 1. Interface of MyMathLab environment.

The tutorial videos are created by the lecturer, and recorded by using a document camera. Each video deals with a specific section in the book and is named accordingly. In these videos, the lecturer explained the topics in the book and worked through the relevant examples occasionally. The notes pertaining to the video tutorials were available online. The length of these videos varies depending on nature of the concerned topics. The tutorial videos replaced lectures and it was expected that students would watch the videos to learn mathematical topics. The textbook served as the central resource in the sense that MyMathLab and tutorial videos were based on contents in the book.

In this course, a compulsory task was the group project in which students were required to prepare a question bank related to integration. That question bank was needed to be programmed in the STACK environment, a computer aided assessment platform. Maxima is the programming language used in the STACK, thus they were required to learn Maxima to complete the project. The intention was to make students familiar with programming language and its use in mathematics.

RESEARCH DESIGN AND METHODS

The case study research design (Yin, 2013) has been followed in this study. A group of three students has been observed over the semester. The methods used to generate data include group observations, semi-structured interviews, individual weekly journals and field notes. Using multiple methods for data collection contributed to triangulation of data.

In order to be able to observe participants' activity, we requested them to work at campus each week for which they agreed. During these sessions, they worked on their routine work including homework and assessments. Video recordings of their group work accompanied with the screen recordings to follow the activity on their computer screens have been collected. Screen recordings of their individual work external to these group sessions have also been collected. Furthermore, weekly journals containing self-reports about their use of resources were included to get the detailed overview. The journal was provided to participants in tabular format which they filled and submitted electronically each week. In the journal, they were asked to specify the resources they used and state how the use of a particular resource helped them in their work each week. The semi-structured interviews were held occasionally to understand the emerging patterns in their use of resources. During the group work sessions, participants communicated in their native language whereas the interviews were held in English. Both the group sessions and the interviews were transcribed.

We analyse participants' weekly journals, a semi-structured interview in the middle of the semester, screen recordings, and the field notes for reporting on students' use of resources in their work. This interview is being counted on because the participants were inquired about the general manner in which they used the resources. The observations, screen recordings and the field notes are being counted on while identifying participants' techniques during their work.

ANALYSIS

Participants' weekly self-reports about use of resources

Table 1 presents the overview of participants' use of several resources as they reported in their journals. The manner in which they used them in their work and their evaluations of resources have been extracted from their journal inscriptions.

Resource used	How they incorporated resources in their work	Comments about resources (if any)
Tutorial videos	Watched to get information to complete homework	Easy to understand through videos
MatRIC videos	Skimmed through the video at amplified speed	
Own note	Used the already solved similar problems in the notes, to recall the problems (methods for solution)	
Textbook	Read through the book, found formulas to work on homework, got questions from book (during project)	
Maxima	Programmed tasks in Maxima for the project, used while doing homework, solved tasks using Maxima	Programming in Maxima is hard but when it is done, all the problems are easy to solve
WolframAlpha	Used as a shortcut to get answers, compared answers obtained from Maxima, got help with solving difficult tasks	Easier to use than Maxima, Faster than using calculator, useful when the answer is in the form of expression instead of numbers
MyMathLab	Worked on homework, learnt specific topic, solved some questions with higher difficulty	Powerful tool, easier to get help and information online
Internet		
Lecturer's notes		Tailored" for the tasks at hand, the most relevant piece of information
Youtube vidoes	Watched Maxima tutorials	
Mathway and other online calulators	Solved questions	Severely increase the probability to get the correct answer, and therefore the overall score.
STACKS	Made some questions in STACKS	

Table 1: Overview of participants' use of resources.

The three participants, Tor, Per and Jan, used MyMathLab almost every week because homework and assessments were required to be done in this system. As regards the textbook, Tor did not report the textbook in the journals rather he used the lecturer's notes. While in Per and Jan's weekly reports, they pointed out few ways in which they used the textbook on different occasions. The textbook served as a source of getting questions, checking answers to those questions, getting help with formulas, and going through examples in the book. During their project work, they consulted the book to take questions and subsequently checked the answers for those questions.

The tutorial videos were reported to be used by Jan and Per during their work. Jan occasionally watched the videos and when specifying about the kind of help, he used the word *understand* linked with this resource such as "to try to understand how to calculate…" and "to understand the calculation behind the math". Per has also mentioned the use of videos and commented, "I easily understand it when someone explains me the way of solving a problem". Tor did not mention any tutorial video provided by the lecturer, however he watched few videos on other platforms, MatRIC TV (an online resource containing videos aiming to support students in their transition from high school to university) and YouTube, once for getting introduction to *partial integration* and at another occasion to learn Maxima – the programming language.

It can be seen that participants used some other resources in their work such as online calculators, WolframAlpha, Maxima and internet (cf. Table 1). Tor named several online calculators including Mathway (https://www.mathway.com) and WolframAlpha (https://www.wolframalpha.com) to solve the tasks and to compare the answers they got in Maxima while working on the project. He mentioned that he used online calculators for saving time, however, he wrote, "I did not learn anything doing this, but it severely increases the probability to get the correct answer, and therefore the overall score". Wolfram Alpha has also been used by Per and Jan in order to verify whether the answers they got were correct. While working on the project, they picked some questions from the book and programmed in Maxima. To check the answers to those questions, they used WolframAlpha.

After completing the group project that involved learning Maxima, this programming language became an important resource for them to solve tasks in homework and assessments. Both Per and Jan began making programs for solving each task to liberate themselves from calculations. Per inscribed in a weekly journal, "(I) used Maxima to make a program to solve the problems in an easy way. This is hard to make, but when it is done, all the problems are easy to solve". Tor did not seem to use Maxima a lot, he spent some time on learning how to use Maxima for solving tasks in one week, and then spending some more time in the next week, he rather chose to focus on MyMathLab. He inscribed that, "it's (MyMathLab) a more powerful tool and it's easier to attain help and information online".

In response to a question about using videos in a semi-structured interview, Per explained his way of working on homework using the provided resources.

Per: These topics I think are quite hard to learn all by yourself. When I get a new topic, I first try to solve it myself, if I can't do that I try to look at the examples in MyMathLab… and if I don't completely understand the examples I take a look at Olav's (lecturer) video…mainly the examples' videos because then I get to see the practical kind of way to do..to solve questions.

Int: How would you rank the provided resources? Which one do you first consult with?

Per: First, I will try to do it myself because then I think I… remember and learn it the best because then I have to think and ….and if I can't do it that way…then I will try to look at example just to get a few hints. If that does not work then I watch the videos because I can't look at the notes (provided by the lecturer)…I have to get explanation of what he is doing step by step.

Tor's response was somewhat similar as he replied:

Int: Did you use any video while working on last week's homework?

Tor: No, I think MyMathLab seemed sufficient so far.

Int: Ok. So which resource did you use for getting introduction to the new topic?

Tor: I tried first MyMathLab but it went fine so I just carried on. …I check the notes and watch the videos if I get stuck..

Int: So, you turn to the videos when you get stuck.

Tor: When it is a new topic, then I just skim through his notes, but since we have integration from a couple of weeks now, I am pretty confident and go straight with it.

While Jan responded to the same question as follows.

Jan: I did not watch that many videos. I mostly use MyMathLab and just see the examples…and if I can't get it from there then I go to…to the book because it is faster… and eventually go to the videos if I do not get constructive help from there.

The participants preferred MyMathLab during their work for being the source of quick and most relevant help in comparison to the other available resources. This approach of working on the tasks saved them time and effort to search for the required piece of information from other resources such as the videos and the textbook. However, the use of MyMathLab can be considered more pragmatic as both Per and Jan mentioned that the kind of help they get from MyMathLab is in the form of examples which contributes more towards producing the results.

Another approach was to watch the videos when the help from MyMathLab was not *sufficient* as evident through participants' responses in the interview. The use of videos has not been preferred much but participants reported that they consulted the videos when they needed to *understand* something. As discussed earlier, the help and feedback in MyMathLab concern the task only as it offers the formula and solution-steps for the task. They might have needed to consult the videos to learn the concepts involved in those tasks in case when just knowing the solution steps in a question did not work. In the journal data, Jan and Per wrote that they used the videos to *understand* thus it indicates the epistemic value linked to usage of videos.

Observing participants' activity helped in finding that the use of different resources affected their manner of working on tasks i.e. techniques. We seek to categorise the participants' techniques pertaining to different resources they used, and by considering their motives behind use of each resource helped in recognising the pragmatic and epistemic value of their techniques. It is found that they increasingly used the digital tools to solve the tasks in MyMathLab environment with the progression in the course. This led to the use of more instrumented techniques instead of paper and pencil techniques promoted in the lecturer's videos and through MyMathLab. For instance, Tor mentioned in his weekly journals and it is observed in the screen recordings of his individual work that he used several calculators to work on homework as well as assessments. The participants themselves perceived this technique of using online calculators to solve the task as pragmatic.

Two of the participants used Maxima in their work as evident from journals and could be seen through the screen recordings of their work. They wanted to be pragmatic in order to make their future work easier. Making programs for each task for the first time can not be considered as merely pragmatic as Per mentioned that he found it hard. The difficulty in making programs may be linked to their knowledge of programming in order to code mathematical tasks. However, the extent to which it contributes epistemically in learning mathematics is not covered in present paper.

DISCUSSION AND CONCLUSION

In this study, we observed how a small group of three students interacted with the resources when provided with an online learning environment in their Calculus course. The environment allowed self-regulated learning and students could work remotely on their homework and assessments. To make sense of the opportunities for students' learning with resources in this environment, we explored their manner of incorporating the resources in general organisation of their digital work. Furthermore, we discussed the epistemic and pragmatic potential of participants' techniques.

In terms of resource usage and the corresponding techniques, participants opted for the resources and the techniques which were pragmatic in terms of producing results for the assigned tasks. Pragmatic techniques involved the use of online calculators, using help in the MyMathLab to produce the results for tasks. Watching videos for learning mathematical concepts seemed to be time consuming and hence not preferred much. Participants appropriated the programming language to work on the tasks with the motive to be more pragmatic and produce results easily in their work. An important factor which is likely to cause the preference for more pragmatic instrumented techniques was the online final examination where they could use the resources. As for students, it is quite important to prepare according to the examination to be able to score better.

This case study provides an example of a self-regulated learning environment created for students to work independently. Our findings suggest some general prospects which are worth paying attention when assigning online homework to students. Combination of an online homework with online examination is likely to cause students to use unexpected use of resources and techniques, for instance, online calculators and solution tools in the present case. This observation also relates to the nature of tasks posed in an online homework environment. Variety in the nature of tasks, such as open-ended tasks, may lead students to interact with resources epistemically.

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Central dialectics for mathematical modelling in the experience of a study and research path at university level

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This paper presents the a posteriori analysis of a study and research path (SRP) on comparing reality and forecasts of the number of users of certain social networks, which appears as a teaching and learning proposal for mathematical modelling. We analyse the main elements of the SRP that have been experienced with a first-year course at university in management sciences degrees in two consecutive courses, 2015/16 and 2016/17. We focus our analysis on two essential dialectics for mathematical modelling to be developed: the questions-answers and the media-milieu dialectics. In particular, we take empirical results from the two successive implementation of the SRP to outline through which mechanism these two dialectics could be prompted.

Keywords: Mathematical modelling, study and research paths, dialectics, questions-answers, media-milieu.

INTRODUCTION: THE SRP AS TEACHING PROPOSAL FOR MATHEMATICAL MODELLING

The starting point of the research is the problem of inquiring into the conditions that can help, and constraints that hinder, that mathematical modelling can be integrated and developed in the teaching and learning of mathematics into current educational systems, in particular, at university level. Researchers and practitioners agree that teaching should not be focused only on the formal transmission of knowledge, but also should provide students of the tools for enquiring into the study of real phenomena and integrate mathematics as an essential modelling tool. This change requires moving from a more traditional pedagogical paradigm of transmission of knowledge, which mostly focuses on introducing students to already built mathematical knowledge, to a paradigm of inquiry where the solving of problematic questions leads learning processes and motivates the study of new knowledge.

In the particular case of the research on modelling and their applications and on inquiry-based approaches some big steps have been made showing how, under certain suitable conditions in different educational levels and curricular frames, modelling activities may be successfully put into practice (Artigue & Blomhøj, 2013). However, although school institutions and researchers agreed that modelling should play an important role for a change towards a new pedagogical paradigm, the real situation in school and university is not satisfactory (Stillman et al. 2013) and the dissemination and long-term survival of these teaching proposals based on modelling follows as a big challenge for mathematics education (Galbraith 2007, Burkhardt 2006).

> In the case of applications and modelling a shared excitement unites many who have enthused about early experiences in the field, for example when students unleash latent power that for whatever reason had remained fettered in their previous mathematical life. However this very exhilaration can work against further progress, both individually, and particularly at a system level, by creating a sense of adequate achievement that obscures the reality that there is so much more to do.

In our research, developed in the framework of the anthropological theory of the didactic, we focus on the use of the study and research paths (SRP) as epistemological and didactic model (Chevallard, 2015; Winslow et al., 2013; Serrano et al., 2013) to face the problem of moving towards a functional teaching of mathematics and, particularly, where mathematics are conceived as a modelling tool for the study of problematic questions. According to Barquero and Bosch (2015), the starting point of an SRP should be a lively question of real interest for the community of study (students and teacher/s). The study of Q_0, called the *generating question,* evolves and opens many other *derived questions* Q_1, Q_2,…, Q_n. The continuous looking for answers to Q_0 (and to its derivative questions) is the main purpose of the study and an end in itself. As a result, the study of Q_0 and its derived questions Q_i leads to successive temporary answers A_i that can be helpful in elaborating a final response $R^\heartsuit$ to Q_0. These first characteristics can be associated to the first level of analysis of the SRP that we here consider, it consists in the dialectics establishing between the questions posed and the likely answers appearing (*questions-answers dialectic*) which also provide the basic structure of an SRP to be implemented and to be enriched after each implementation. This first layer refers to the evolution of questions to be faced and the necessary knowledge to be used. Another central dimension for an SRP is the *media-milieu dialectics*, which constitutes the second level of analysis. As described in the aforementioned investigations, the implementation of an SRP can only be carried out if the students have some pre-established responses accessible through the different means of communication and diffusion (that is, the *media*), to elaborate the consecutive provisional answers A_i. These *media* are any source of information, such as: textbooks, treatises, research articles, class notes, or the teacher acting as main media. However, the answers provided are constructions that have been elaborated to provide answers to questions that are different to the ones that may be put forward throughout the mathematical modelling process. Thus they have to be re-constructed according to the new needs. Other types of *milieus* will therefore be necessary to test the validity and appropriateness of these answers. This second level of analysis put attention to the evolution of the students' milieu.

With this aim, we present an analysis of a particular SRP about the evolution of users of certain social networks that we will analyse in term of these two central dialectics and, more concretely, focusing on two critical questions:

(1) How to enhance dialectics between posing questions and looking for answer as engine of the modelling process? How to transfer to students the responsibilities of posing questions and looking for answers? (2) What *milieu* is necessary for students to facilitate a rich development of modelling? How a richer media-milieu dialectics can be developed?

DIDACTIC ANALYSIS OF A MODELLING PROCESS: THE CASE OF AN SRP ABOUT THE EVOLUTION OF THE NUMBER OF FACEBOOK USERS

We focus on analysing the case of an SRP on *Comparing forecasts against reality in the case of Facebook users' evolution*. The first time it was experienced was during the winter term of the academic year 2015-16 with first-year students of Business Administration Degree and of Innovation Management (BAIM), all from the 'Escola Superior de Ciències Socials i de l'Empresa-Tecnocampus', Pompeu Fabra University (see Barquero, Monreal, Ruíz-Munzón & Serrano, 2017). During the academic year 2016-17 it has been implemented again in the same university degree. The SRP has run in a modelling workshop that was optional activity for students during these last two academic years. In this paper we analyse and compare both implementations by using two central dialectics: the questions-answers and the media-milieu dialectics.

The initial situation starts from real news about a research performed by Princeton in 2014, in which it was predicted that Facebook would lose the 80% of its users before 2017. Hence, the generating question Q_0 presented to students is about: *Can these forecasts be true? How can we model and fit real data about Facebook users' evolution to provide our forecast the short- and long-term evolution of the social network? How can we validate the conclusions of Princeton?* The experimentation was structured in three interconnected phases linked to the generating question Q_0, building up the a priori design of the SRP, then reflected in the design of the c-book unit. A first phase that focuses on the open research of real data about Facebook users, a second one focused on finding mathematical models (mainly based on elementary functions) that may provide a good fitting to real data, and a third one about the use of these models to forecast the behaviour of the social network in short, medium- and long-term in terms of number of users and about how to decide about best and most reliable model.

Previously, during the first term (4 ECTS of the subject) students had been getting familiar with the main properties of some groups of functions (polynomial, rational, irrational, exponential and logarithmic functions) as well as with basic topics on differential calculus and its applications to the study of the monotony and optimization of one real variable functions. Actually, before starting the first session we asked the students to answer a test on some of the mathematical tools that mainly make the workshop up, as indentifying some types of elementary functions or the concept of fitting model in certain scatter plots.

In the first experimentation 27 students, working in ‘consultant teams’ of 3-4 people, got the order from MS2 Consulting (‘Mathematical Solutions Squared’) previously described as Q_0 and they were asked to deliver a final report by the end of their work as an oral presentation as response to the MS2 request. The implementation combined face-to-face sessions in the teaching device called ‘Math modelling workshop’ (in a total of six 90-minutes weekly sessions) for the miss-in-common of the junior consultant teams’ partial reports, with work out of the classroom. For the second experimentation 12 students (18 students started the workshop, but they left it in the second session due to external matters) worked also in teams of 3-5 people. This time we opened a Moodle virtual classroom to provide the students the teaching aid of the workshop, as well as some communication and collaborative tools (forum, a different wiki for each phase, etc.) to write their progress and pose their new questions. The generating question Q_0 was presented in a small dossier, next to the initial subquestions of each of the SRP phases (Q_1, Q_2 and Q_3). The workshop run over seven face-to-face 90-minutes sessions before the final session, in which students should present their conclusions in an oral presentation in front of an external committee with representatives from MS2 Consulting.

Next we sketch in the case of the two implementations how the different dialectics were prompted by both: (a) the design of the unit (by its initial design but also by the different changes introduced according to students’ requirements: new questions and answers not envisioned, new media required, etc.) and (b) the didactic gestures and devices to manage its implementation.

Integrating the dialectics of questions-answers as engine of the SRP

The *a priori* design of the SRP was basically the same in both implementations, structured in three interconnected phases linked to Q_0, which guided the design of the workshop throughout its implementation. A first phase focuses on the open search of data about Facebook users; a second one focused on mathematical models (mainly based on elementary functions) that might provide a good fit to Facebook users data; and a third part focused on the use of these models to provide short-, medium- and long-term forecasts about the number of users of Facebook and on how to decide on the best and most reliable model. Figure 1 (and the explanation below) shows the link between different questions (Q_i) that were planned as likely to appear in the real implementation of the SRP and some expected answers (*Ai*) from the working teams. The only difference of the second design with respect to the first one was motivated for the context in which Q_0 was presented originally: the predictions made by Princeton were supposed to happen in 2017, and this year was present tense for the students of the second experimentation. Hence, we decided to make the same questions, but giving freedom to students of focusing in any other social network students were interested in.

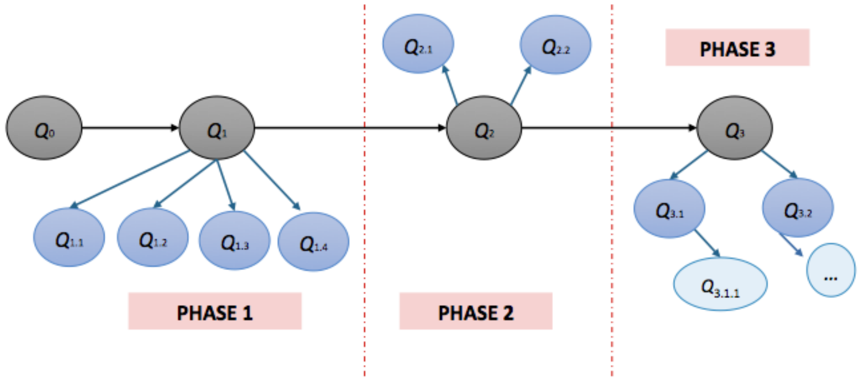

Figure 1: Tree of questions and answers of the different phases of the SRP

Q_1: Which data sets about the users of the social network are better to consider in our research? → A_1: Each group look for the data to be used and shared; the whole community agree on the terminology (year, period, units, etc.) and on the dependent and independent variables to take into account.

$Q_{1.1}$: Which time intervals may be considered? $Q_{1.2}$: How can data be well-organized? $Q_{1.3}$: How to organise and visualise data? $Q_{1.4}$: What can we say about the growth tendency of the data analysed?

Q_2: Which mathematical models provide the best fitting of data about the network users? → A_2: Each consultant group is asked to propose and justify three mathematical models fitting real data.

$Q_{2.1}$: Which models (based on elementary functions: linear, parabolic, exponential, etc.) may fit the data? $Q_{2.2}$: How can the coefficients of the model be determined?

Q_3: How can we decide about the 'best' fitting model? Can we use this model to predict the future evolution of users? → A_3: The teams need to create tools to justify why a mathematical model/s is/are the 'best' with respect to: (a) fitting data and (b) forecasting the evolution of users.

$Q_{3.1}$: How can we compare the error committed between reality and forecasts provided by models? $Q_{3.2}$: Can be the same model used for the short- and long-term forecasts?

Let us now comment the main features of the *a posteriori* analysis of the experimentations, referring here to the questions-answers dialectics level.

Regarding the first phase, we should remark the ease with which the students found real data about the evolution of the social net. The students mainly found the information by means of a graphical representation. This fact strongly determined their analysis, since they mainly focused in the graphical analysis growth tendency of the data, but not in their numerical versant, making $Q_{1.4}$ being treated before the other ones (it was considered that students would have data in table format before having graphs). With respect to the first experimentation, the fact that many groups found the same data triggered an intense debate and interchange of ideas among them, which took us to consider a brainstorming session about the previous hypothesis in the classroom, and as a consequence, the duration of the first phase was extended from 3 to 4 sessions. Due to the wealth of answers collected during the brainstorming we asked the students to deliver a first report in a poster format, synthesizing their findings, conclusions and new questions made by them. In the second experimentation the fact that students could choose a social network implied a disruption with the usual *topos* of the students in the process of study, since they were responsible on the delimitation of the field of study. They noticed about the difficulty of finding reliable data of some of their choices (Snapchat, Instagram, Twitter…), so finally only Facebook and Instagram were object of study, and not only the number of users with respect to the time, but also other variables that could have a relation. Another question that raised here was the role of the intervals of time of the data obtained, and how to work when data are not regularly spaced in time. These questions enriched the *a priori* design of the SRP. The presentation of the first phase was done on the third session, and there had not been interchange of ideas with other groups during the first phase. Furthermore, we asked students to present their plan of work: the questions that they wanted to deal with, when and how. This showed that each group had planned the next steps in many different ways and with many different variables. Nevertheless, the lack of time and our interest in the study of one real variable function made us proposed the students to use only the variables “Time” and “Users”.

Let us focus now on the second phase. In both experimentations the analysis of the different proposals made arise a non-expected aspect: the use of piecewise functions. Then the expected answer to Q_2 about the consideration of models based elementary functions (linear, quadratic, exponential, etc.) was extended. In the case of the first experimentation, since many groups worked finally with very similar data on the worldwide evolution of FB users, we took two new decisions: (a) give each team a second set of different data, corresponding to different geographical areas, in order to contrast their hypothesis and extend their study; and (b) ask for more than one fitting model for each data set. This was not necessary in the second experimentation, since each group had different data sets. Besides, in both workshops new questions and answers appeared at this stage with respect to the change of tendency of the fitting models, in accordance to a particular action or to decisions of the corresponding social network (IPO, new rival social nets, purchases of the company, new developments, etc.), which determined the moments of change of tendency. Furthermore, in the second experimentation we let the students choose a software for representing their data and the functions. This made question $Q_{1.2}$ emerge again, since they needed to adapt their data to the different software used. Just one group decided to use *Geogebra*, so they were provided the applets we used in the previous experimentations (Barquero et al. 2017). Instead, another one decided to make interpolation in order to find functions fitting their data, so they used *Symbolab* and added some questions about how to solve non-linear systems of equations. The third group worked with *Excel* for representing a scatter plot, and used linear and non-linear regression. This motivated a big change in the SRP, since question $Q_{3.1}$ emerged naturally in the exposition of their findings at the beginning of the third phase (since they have used the R-squared of their model given by the software). This gave birth to an interesting discussion on different ways of measuring the error, and the professors had to present this question as a central matter.

Concerning the third and last phase, in both experimentations we only had two face-to-face sessions of the workshop, but were not enough for a rich development of Q_3. Although this time constraint, in the first implementation of the SRP there were some applets designed and made available for students to help on the simulation of models and its contrast to real data. It helped students to delve into Q_3, but not many new questions appeared from this work. With respect to question $Q_{3.2}$ only one group dared with long term forecasts to give a date for the moment in which the users of the chosen social net would start decreasing. Both implementations finished with a final presentation of their modelling work and conclusions to an external committee.

Before finishing, we should remark that in both implementations the common discussions, presentations and brainstorming session became the main device for students to formulate and organize new questions, debate answers and contrast them.

The progressive enrichment of the milieu: the media-milieu dialectics

Since we have the first layer of analysis of the SRP in terms of the arborescence of the questions-answers, it is important to ask when, where and how questions can arise and answers can be developed. It is at this new level when there may appear the different elements taking part of the milieu, composed of varied elements: questions, temporary answers, pre-existing answers in or out school, means to validate answers, experimental data, etc., accessible through different kind of media (textbook, lectures, website resources, etc.). The relation among these elements can be analysed through the *media-milieu dialectics*. The constant dialectics between the search for data (for instance, real data about users of social networks, or about the company changes) and pre-existing answers (ways to organise data, common models to fit population evaluation, elementary functions, tools to control error, etc.) that exist in different media available for students (web resources, contents of Mathematics course, answers from lecturers from other courses...) and the creation of the appropriate means (milieu) to integrate (or refuse) them has been central in our SRP. Let us stress the importance of some of them.

In the first phase of the SRP, it was important to some groups the topics worked in another course called ‘Introduction to digital communities’ (running in parallel to the workshop) who helped on providing a general sense and functionality to Q_0 and to show how the students could look for real data and some techniques to organise them. All these elements took part of the *media* accessible to students, at the time it enriches students' *milieu* mainly composed at this stage of the data sets that each team chose to work with (shared and debated early with the whole class in the first implementation, and before the second phase in the second implementation; even strongly in this case, since the variety of data found was higher and let them make comparisons between different social networks). All these elements helped them to prepare a first report with the first temporary answer A_1 (a poster format given in the first implementation, and a face-to-face presentation in the second). Here we should remark the importance of making their plans explicit (especially in face-to-face sessions) to construct a common frame to be the source of new questions, as well as and to integrate in their milieu new concepts about modelling, and ideas of other groups that could help them. It is in the second phase in which we find more differences. In the first experimentation, the a priori design contained some *Geogebra* applets proposed to help students to explore different models based on elementary functions (Q_2). These applets provided the main *media* for students to visualize data jointly with model simulation, and also took part of their *milieu* as main tools for contrasting, comparing and deciding on the 'best' models to choose. Nevertheless, there were other tools not planned in the *a priori* design (as piecewise functions, or Gaussian functions, most of them part of their milieu, since they had been introduced in previous courses) but provided by designing new applets. In the second experimentation only one group used these applets, so their path followed was more similar to the first ones; but two groups decided to use other software mentioned above (that they could know from Statistics or other subjects), which made the main difference with the first implementation: meanwhile the first applet seemed to drive students to apply only a trial-error method, tools like interpolation or regression made students arise an earlier answer. Here again the common forum stated as a face-to-face session motivated an enrichment of the student's milieu. Regarding the third phase, there were several important questions that were not addressed properly, such as $Q_{3.1}$ about the way to measure the differences between data and forecasts, but here there is a main difference between both implementations. In the first one the students assumed and uncritically used the milieu made available through the design of an applet, a sort of *black box* to get immediate answers. Instead, in the second one students had to construct their own tool for measuring the error, and one group made it with *Excel*. Just one group could answer $Q_{3.1}$ but the answer was totally produced by them, so they could communicate it to the rest of the class, extending the appropriate milieu to other groups.

FINAL REFLECTIONS

First of all, we should mention that students are not in general motivated to validate their results after a work of research, since a lecturer will finally do it. In this workshop students were responsible to validate or justify every decision they made by the end of each phase. And this is the main reason why other questions arise and contribute to enrich the a priori design of the SRP.

In this paper we focus on the case of an SRP on comparing forecasts against reality in the case of the evolution of the number of users of certain social networks to show the use of two dialectics: the one of the questions-answers and of the media-milieu, corresponding to two of the three complementary level of didactic analysis of teaching and learning processes (Chevallard, 2008). Besides their analytic use, they suppose a productive framework to enrich teaching and learning practices, in particular, on modelling.

In what concerns to the *questions-answers dialectics*, the generating question Q_0 about the controversy of the article by Princeton was adopted by the students with a great interest from the very beginning and, up to the end of the process, was kept alive. From the two presented implementations we can underline very important conditions that were created. First, the flexibility of the lecturers and designers team that were opened to readjust the schedule according to students' team work. Furthermore, they were very attentive to integrate in their presentations all new questions and means that the students asked for. Second, students were very active on the sessions to share their proposals, making derived questions emerge naturally, some of them planned in the a priori design, some others that extended the initial proposal. Regarding the *media-milieu dialectics*, in the case of this SRP, we took several decisions along the implementation of transforming the media offered to students to help them in the modelling process and also to observe the impact new media had on students' milieu. Nevertheless, giving students the chance of using their own ICT tools, as was decided for the second experimentation, enriched the *media-milieu dialectics*, since it helped to arise other different answers that had not happened during the first experimentation. We may insist again on the role played by very important contributions, such as collaboration with other subjects, focusing some workshop sessions on discussing external answers that students brought, the creation of applets to foster students' experimental work, among other interesting aspects.

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Task design for Engineering Mathematics: process, principles and products

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We present and analyse principles and process employed at the Danish Technical University to use authentic problems from engineering (APE) in a first year mathematics course, along with some of the products (actual student assignments).

Keywords: Mathematics for Engineers, Task Design

INTRODUCTION

Mathematics is as important in most branches of engineering, as engineering is to the prosperity and development of contemporary society. Thus, it is of great importance to investigate exactly *what* mathematics is needed by (future) engineers, and how it could be effectively taught to them; such research is only emerging (see e.g. Winsløw et al., to appear, sec. 2.5). In the common case where mathematics is taught in separate "service" courses which cater to several different study programmes, these questions may be considered in entire separation: a syllabus for the mathematics course is decided based on needs in the different study programmes of engineering which include the course, and subsequently the syllabus is delivered by mathematics faculty. This amounts to a complete separation of external and internal didactic transposition, in the sense of Chevallard (1992), where the selection of mathematical contents to be taught may be based on needs and priorities from the engineering disciplines, while the actual teaching is carried out according to generic standards and methods for teaching mathematics. The aims of the overall study programme (in engineering) are only considered in the external transposition of the mathematical knowledge (see Fig. 1). We can call this model a *parallel model* for teaching mathematics to engineers, as the internal didactic transposition runs in parallel to the rest of the programme and does not interact with it (while it is certainly intended that the students' learning serves in other courses, later on).

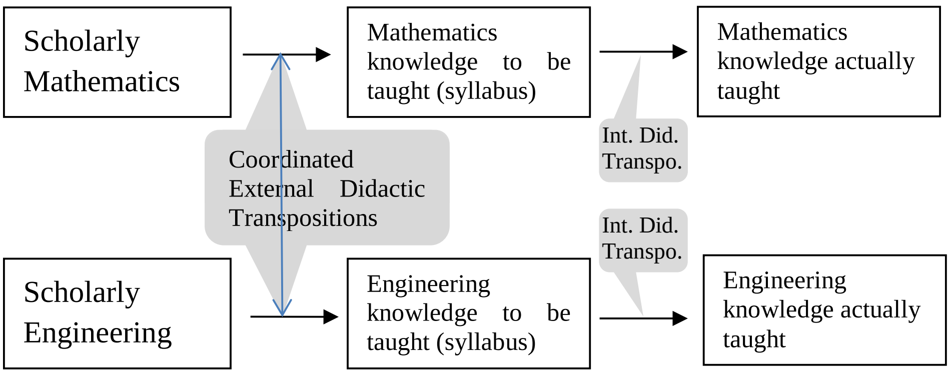

Figure 1: The parallel model for didactic transposition in engineering education

In the literature on university mathematics education, it is widely agreed that the parallel model has drawbacks:

- students may experience the mathematics teaching as unmotivated and difficult, which is reflected in relatively high failure and attrition rates for some engineering programs (e.g. Baillie & Fitzgerald, 2010).
- the knowledge they acquire in the mathematics course may not transfer readily to engineering contexts, in the sense that students are able to invest the knowledge acquired in mathematics courses when they need to do so in other courses of the programme (e.g. Britton et. al., 2005).

Motivated by these well-known problems, the model in Fig. 1 has been modified, in many universities, by various attempts to relate the internal didactic transposition of mathematics more closely to the rest of the engineering programs (e.g. Kumar & Jalkio, 1999).

One of the most common ideas to further such an interaction is that to include, in the mathematics course, more or less simple examples and student assignments where engineering problems are solved based on mathematical methods and theoretical notions (see, for instance, Härterich et al., 2012). A main challenge here is that university mathematics teachers usually have no in-depth knowledge of any engineering discipline, let alone of all the specialties which the course they teach caters to. Of course they may then ask engineering specialists for help to identify *authentic problems from Engineering which can be solved using the mathematics to be taught in their course* (we abbreviate this kind of problems as APE). In that way, "Scholarly Engineering" may exercise a more direct influence on the internal didactic transposition of mathematical knowledge (cf. Fig. 1). In this paper, we investigate some general questions related to the implementation of this (quite common) idea at the level of the internal didactic transposition:

RQ1. How could the identification and transposition of APEs be organised, given the academic and institutional separation of university mathematics teachers from their colleagues in engineering?

RQ2. What didactic variables (cf. eg. Gravesen, Grønbæk and Winsløw, 2017) are relevant to the construction of assignments based on APEs?

It is clear that answers to these questions will depend on institutional conditions and that even when such conditions are given, one will at most obtain very partial answers in the sense of reasonably validated *examples* of organisations (RQ1) and didactic variables (RQ2). As always in education, transfer of "answers" from one context to another will require some adaptation and interpretation, but this limitation may in fact be less important for the above questions, given the relatively high similarity of mathematics courses, the engineering programs they serve and the institution which deliver them. For these reasons, and given the importance of the matter already argued, it appears worthwhile to present such locally and partially validated answers. Concretely we will present and analyse the process, principles and products of APE design done at the Technical University of Denmark since 2000. Founded by H. C. Ørsted in 1829, this is one of the most prestigious Schools of Engineering in Europe, and by far the largest in Denmark.

A TASK DESIGN PROCESS

Mathematics 1 (hereafter Mat1) is the basic mathematics course for 17 different B.Sc.Eng.-programmes at the university, catering to about 1100 students per year. The course occupies 1/3 of the students' time during the first year, and covers complex numbers, basics of Linear Algebra, Ordinary Differential Equations including linear systems, and multivariate and vector Calculus up to Gauss' theorem. Besides ensuring a technical foundation for later work, the university also considers a common course on mathematics as important to the formation of an engineer identity.

Most of the course is quite traditional, however with intense use of the computer algebra system *Maple*. Exercises with easy applications to engineering occur. However, during the last four weeks of the course, the students work on a "project". This is an assignment containing about 20-30 more or less challenging tasks, related to a mathematical model related to an APE. The model is usually given in the assignment, and while some new mathematics may be introduced, the starting point is Mat1. Each project assignment is presented in a text of varying extent (ranging from 4 to 29 pages, averaging 11); it is those texts which we aim to analyse in this paper. The students do the projects in groups, hand in a report of about 20-50 pages, and defend their work during an oral exam, which accounts for 25% of their grade.

The groups can choose their assignment from a list of 4-5 projects, in part depending on the study programme, with titles like those shown shown in Fig. 4. As the titles suggest, the project problems come from many different areas of Engineering. Every year, new projects are added and some are dropped; and the details of retained projects are updated based on teachers' experiences. The elaboration of new projects is a particularly delicate undertaking. When the first projects were done from 2000-2006, a systematic effort was deployed to engage researchers from the university – both applied mathematicians and researchers from Engineering at large - to propose project topics; they were then, mostly, drafted or adjusted by the course responsible. Some are still used in revised form.

It is the task of the course responsible to organise the production and revision of projects. The initiative can come from teachers at the course or other mathematicians, who identify a more or less classical APE which can form material for a project; then, the motivation is often that some specific parts of Mat1 can be worked on in new ways. But the initiative also frequently come from colleagues from other departments. In some cases, their motivation is personal fascination with mathematics in a more or less current APE, and possibly ongoing collaboration with mathematics colleagues in this relation (reflecting an interaction between Scholarly Mathematics and Engineering, which could be added to Fig. 1). In other cases, they propose an attractive current APE for a Mat1 project, in order to attract students to their specialty later on – these colleagues then, sometimes participate for free as supervisors on the students' project work.

Summing up what the process involved in Mat1 could contribute to RQ1, at least two venues can be identified in relation to Fig. 1:

- Scholarly (applied) Mathematics and other basic sciences such as chemistry and physics, where the main source of motivation is mathematical contents related to Mat1; but work done here can still involve or lead to APE. Project proposals from this source are typically mathematically "rich" but are not often related to current research.
- Scholarly Engineering, often with current APE's; the elaboration of a project typically necessitates considerable adaption to fit Mat1, and is often tailored to the interest of students from a small range of study programmes.

Finally, the genesis of a project may involve a mixture of both sources, when the APE is identified by scholars with a deep involvement in both areas.

DIDACTIC VARIABLES AND PRINCIPLES

To present and analyse the project assignments which have appeared over the last 15 years, we have defined 10 *didactic variables* (DV) which are relevant to classify them according to the aims which have, explicitly or implicitly, been pursued (Fig. 2). Each variable has, in principle, a non-numerical range, but can be determined with relatively high objectivity for each assignment, based on the text. The detailed presentation of any project in terms of the variables will, naturally, be difficult to compare with others when given in this form. So when considering all projects it appears useful to assign indicatory numerical values to the DVs on a scale from 0 to 2: for instance, to assess the breadth of Mat1 contents which a given project requires the students to work with, 0 indicates that only one topic (such as systems of linear equations) is involved, 1 that a few topics from both Calculus and Linear Algebra are involved, and 2 that the project combines many topics. Naturally, this "grading" is not absolute but relative to other projects (cf. also Fig.2). In the next section, we outline a concrete assignment and explain, at the same time, how the numerical values of the other DV's are set. The variables were initially formulated in by the authors (based on the first authors' many years of involvement in the design) and subsequently validated and adapted during the actual analysis of assignments. The variables thus constitute a concrete answer to RQ2, which is of course a partial answer based on experiences from context we described. In the rest of this paper, we provide more explanation on how the variables can be used to analyse concrete projects and, potentially, to direct and systematize the design of student assignments.

For each DV, Fig. 2 also includes a brief description of the more or less explicit aims which have been pursued in the construction of projects over the past 17 years. The brevity required in the Table format does not allow for much nuance. We note that what is ideal use of *Maple* is not subject to complete consensus among the teachers of the course, or in relation to the rest of the institution. On the one hand, some course teachers consider that students should use *Maple* whenever it is useful; while others, including colleagues from other departments, often insist on the value of students' mastery of basic manual computation (as reflected in the aims for DV3 given in Fig. 2). The variables DV4-10 all describe aspects of the relationship between the internal transposition represented by the assignment, and Scholarly Engineering (cf. Fig. 1). Their values are thus of specific importance to go beyond the parallel transposition.

Didactic variable (DV):	**Aim of designers:**
DV1 What breadth of content areas from Mat1 are needed to solve the assignment? What depth of use?	As many as possible, preferably involving new combinations. Depth beyond "standard tasks" required.
DV2 What new mathematical contents are introduced?	Contents in continuation of Mat1, not excessive for students to cope with
DV3 How must/can *Maple* be used?	*Maple* should mostly be used to:
- DV3a How *essential* is the Maple use? - DV3b What types of Maple functions (numerical, symbolic, graphical…) are relevant? - DV3c Are the relevant use known or new to students? - DV3d Is there *black box* use of *Maple* ? - DV3e What parts of the *Maple* use are prescribed?	avoid tedious computations, and for tasks which the students could not handle otherwise
DV4. What is the "theme" and source of the problem the project attacks	Origin in APE, if possible source in paper or ongoing research in engineering
DV5. Breadth of engineering problem – are more disciplines involved?	Ideally more than one branch of engineering involved
DV6. How is the mathematical model established and worked with?	Ok if model is given in the assignment, but the students should work with its details and structure
DV7. How realistic is the model?	As much as possible for the students
DV8. How are data used?	Data from the source, used as there
DV9. Should the students look up information outside assignment?	This is not a main aim, except students should use Mat1 course material
DV10. How complete answers does the model give to the main problem?	Clear and definite answers/points, to give students a satisfying experience

Figure 2. Didactic variables for the analysis of project assignments.

PRODUCTS

A total of 37 projects have been proposed during the past 10 years. Not all projects are used every year, and all are revised before use, in the light of past experience, new needs in the course, and in a few cases, updates to the APE and its solution from

Scholarly Engineering. We first give a relatively detailed presentation and analysis (based on the DVs) of one project; at the same time, we describe how each of the 10 DVs is assigned a value as described above, for a rough analysis of the projects. Then we present an overview and rough analysis of the whole inventory of projects.

In-depth presentation of one project

We now take a closer look at one of the projects, entitled: *Heat flow in a house – simulation and dimensioning*. The assignment is relatively long, 18 pages, including about 5 pages of data. The first paragraph outlines the underlying APE:

> The building sector accounts for about 40% of the total energy consumption in Denmark. It is a common assumption that there is a large unrealized potential for reducing the consumption (…) in a financially sound way. This requires knowledge of the physical processes which affect the energy consumption of buildings, the financial aspects of the construction and maintenance of buildings, as well as the mathematical methods used to compute these.

It turns out that the energy flow in the building is modelled as an analogy of currents in electric circuits (cf. Fig. 3). The project is based on a genuine APE, and bibliography of the assignment includes a reference to Nielsen (2005) which is the essential source (DV4 = 2), along with a "pricelist" from the construction industry, and the last part of the assignment draws on a simple model of investments and interest. Relative to other projects, this assignment involves a relatively broad area of Engineering fields, and DV5 is set to 2. The introduction acknowledges that the model proposed in assignment is "a bit simplified", but in fact it still gives similar results; we assign DV7 to 2, in spite of some problems (see end of this section).

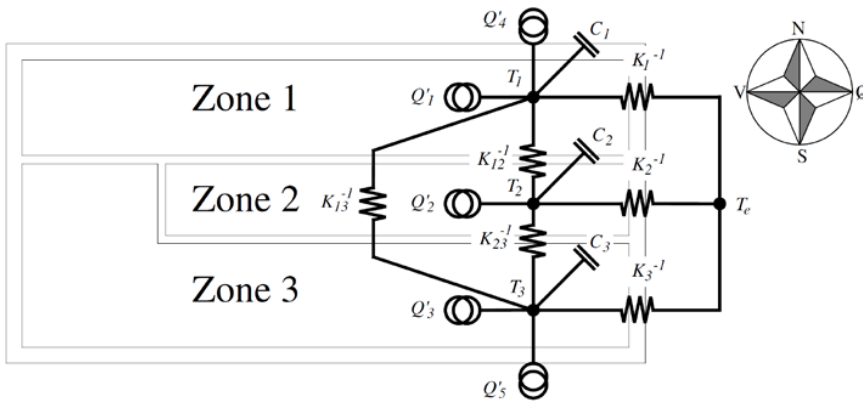

Figure 3. A figure from the project assignment "Heat flow in house".

The central model, illustrated in Fig. 3, concerns a house with three rooms, called "climate zones". Here Q'_k are the internal and external heat sources (heaters and sunlight), while C_k are the heat capacities of the rooms and K_* are the heat transmission coefficient of the walls of the house, reflecting that these walls involve a variety of layers. Before introducing the final model house, the students' work with the simpler case of a one-room house, and an external temperature T_e which is a given sine function. Based on further assumptions, this leads to the model for the internal temperature T_i, as a function of time t:

(*) $$C\frac{dT_i}{dt} = K(T_e - T_i) + P(T_{set} - T_i)$$

where T_{set} is the desired internal temperature (constant), and P is the performance of the internal heat source. While (*) is just a first order ODE, it still gives rise to interesting Engineering tasks: the investigation of stationary solutions, the performance needed to ensure an average temperature of 19.8^0C, and the thickness required to respect given limits on the oscillation of T_i. The full model consists of a system of three differential equations which are similar to (*) but with an added complexity due to the heat contribution from sunlight which, moreover, is investigated with two different models. The students must also take into account a model of the walls involving layers of materials to be computed using authentic data. Finally, to take into account the cost of construction, the students are given a simple mathematical model for the total economy involving investment, interest, and operation costs; the mathematics is very simple but still gives rise to interesting questions regarding how to optimize, for instance, insulation thickness. Throughout, the students use real data (DV8=2) but these are all given, so DV9=0. Throughout the models are given to the students, and while students are given full and extensive explanations, they are not really asked to do more than apply them; thus DV6 = 1.

The project draws on a broad range of Mat1-topics: harmonic oscillations and complex exponential function, single and coupled differential equations, solved using advanced matrix algebra, involving both eigenvalue problems and quadratic forms. Thus DV1=2, while DV2=0 as almost no new mathematics is introduced (the exception being the argument required to justify the stationary solution to the system of differential equations, which involves an extended eigenvalue problem).

While most of the tasks can in principle be solved manually, the visualizations of temperature variations corresponding to different parameter values decisively require a tool like *Maple*. The assignment moreover invites to numeric experimentations, possibly based on graphical representations, and standard use for tedious operations like inversion of matrices, make the overall potential of *Maple*-use relatively average for projects; we thus assign DV3=1, even if the realized use by some students in some cases goes beyond a mere use of techniques known from the rest of the course.

In real practice, the project also suffers from some flaws. Some of the questions lead to less interesting results (like tedious computations leading to a requirement of a four-doubling of the wall thickness in order to reduce an already negligible oscillation of 0.12^0C for T_i). More serious is the breakdown of the model when taking into account the contribution of sunshine at low temperatures such as 8^0C, where the stationary answer cannot be found. It can be argued that such problems often arise with simplified models, but it still leaves a negative impression on many students, which might be avoided by future revisions. Altogether, we consider DV10=1.

Inventory of project assignments

Fig. 4 lists the inventory of projects used in the last 10 years, analysed using the DVs.

	DV (cf. Fig. 2)									
Project name (shortened in a few cases)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Oscillations in Axle-bearing Systems	2	0	1	2	1	1	2	2	1	2
Micro/Nano Cantilever Based Mass Sensor	1	2	1	2	1	1	2	1	1	1
Enzymatic Hydrolysis of Cellulose	1	1	1	0	1	2	1	1	0	1
Modelling 2D Halbach permanent Magnets	1	2	2	2	1	2	2	1	0	2
Factorization of Integers	0	2	2	0	1	0	1	0	0	1
Heat flow in house – simulation, dimensioning	2	0	2	2	2	1	2	2	0	1
Quantum Mechanics in a Nutshell	2	2	2	2	2	1	2	2	0	1
Red Blood Cells – Optimization in Nature	1	2	2	1	1	2	1	0	0	2
Utilization of the Waste Product Whey	1	0	1	2	1	2	1	1	0	2
Forced Pendulum	1	2	2	1	1	2	1	0	0	2
Stability in Chilled Tank Reactor	1	2	1	1	1	1	1	1	0	2
Optimization of Work Cycles	2	1	2	2	1	1	1	0	0	2
GPS and Geometry	1	2	2	2	2	1	1	1	2	1
Oscillations in Grid Constructions	2	2	1	2	1	2	1	0	1	1
Groundwater Flow in the Forest Vestskoven	1	2	2	2	1	1	2	2	0	1
Internet Hit lists	1	2	2	2	1	1	2	2	1	2
Short Circuit in Electric Networks	1	2	2	1	1	1	1	0	0	2
Simulation of Stretch Reflex	1	1	1	2	1	1	1	0	0	1
Parking Orbits of Satellites	2	1	0	1	1	2	1	0	1	2
Solar Energy Absorption in Curved Glass houses	2	0	2	1	1	1	1	0	0	1
Flow in Chemical Reactors	2	0	1	1	1	2	1	0	1	2
Finite elements in One Dimension	1	2	2	2	1	2	1	0	1	1
Geodesic Curves	1	2	2	0	0	2	2	0	0	2
The Brains Glycose Metabolism	1	2	2	2	1	1	2	1	1	1
Resistors and Markov Chains	1	2	2	1	1	1	1	0	0	2
Dosage of Anaesthesia	2	1	1	2	1	2	1	1	0	2
Anthrax – Attack, Escape and Rescue	1	2	2	2	1	2	1	1	2	2

Decomposition of PCE	1	1	1	2	1	1	1	2	1	0
Modelling Concrete Moulding	1	2	2	2	2	1	2	2	2	1
Soap Membranes	1	2	2	1	1	1	2	0	0	1
Distribution of Electrons in Semiconductors	1	2	2	1	1	1	2	0	0	1
Methane Concentration Profiles in Soil	1	2	1	1	1	1	1	1	0	2
Train Running in the Alps	2	1	2	1	2	2	1	0	1	1
Proteins' 3 Dimensional Structure	0	2	2	2	1	1	1	2	1	1
Reaction Kinetics	1	2	2	1	1	1	2	1	0	2
Error Correcting Codes	0	2	2	1	1	1	1	0	0	2
Phononic bandgaps	2	0	2	2	2	2	1	0	0	2

Figure 4: Inventory of current projects with values of the didactic variables

A number of interesting tendencies can be identified in the above table, including apparent dependencies of some variables, potentials which appear relatively unexplored (like DV9), etc.; some of these are still not fully analysed. We stress that a simple sum of the values of didactic variables, for a specific project, cannot be construed as a measure of the "didactic quality" of the assignment. One reason is that the variables are not of equal importance (in particular, DV1, 4 and 10 are essential). But more importantly, one cannot always construe the number two as being objectively "the best possible value" of the DV; the aims listed in Figure 2 are open to debate and the viewpoint of teachers and designers may differ. A good example is DV4, where we have given "2" for projects with a clear APE, "1" for projects with an authentic problem from basic science (e.g. Chemistry) and "0" for projects which are not based on an APE but on a (*prima facie*) purely mathematical problem, such as the project "Geodesic curves". One can argue that a project of the "0" type can also be of high quality as a project for engineers, in view of the importance in several branches of the mathematical problem (in the example, DV2=2 and indeed, geodesic curves have multiple applications in many branches of engineering, see e.g. Patrikalaksis and Maekawa, 2010, 265-291). A similar uncertainty must also be pointed out for other variables such as DV3, where the further graduation suggested in Fig.2 could be useful to provide a more nuanced picture than in the analysis in Fig. 4, where "2" merely means that *Maple* is indispensable for large parts of the project.

SUMMARY AND OUTLOOK

We have presented the principles, process and products of a relatively longstanding effort to integrate elements of scholarly Engineering (APE's) in the internal didactic transposition of basic mathematics in a course catering to a wide range of Engineering programmes, going well beyond isolated "applications" of a Calculus or Linear Algebra. We have emphasised the multiple dimensions which such an effort needs to consider, in order to maintain the link with the mathematical knowledge to be taught within the module in question, and to establish non-trivial links with Scholarly Engineering (cf. Figure 1). Certainly, the concrete inventory of variables can be developed and adapted further, and we believe it can eventually become a valuable explicit basis for the discussion of aims (right column in Fig. 3) of projects in our and other similar contexts. More importantly, considering such explicit variables could be an important tool for systematizing the design process, both as a check list for constructing new projects and (in combination with the analyses behind Figure 4) to identify potentials for enriching existing projects. We expect that the variables will also become useful guidelines for investigating the effects of the project work in this course as a means to facilitate the transition to later courses where mathematics is so fully integrated into the Engineering knowledge to be taught that the latter is in practice as inseparable from mathematics as music is from sound.

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Mathematical modelling and activation – a study on a large class, a project-based task and students' *flow*

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We studied how engineering students in a large class (n=346) can be activated by a project-based task, in which they have to model mathematically the motion of an object. The students had to throw an object, use (1) their smart phones for filming, and (2) tracker software for capturing the motion. Through a poster, they had to report their video analysis. We framed activation through the concept of flow, which is a state of being fully absorbed by an activity. We administered a web-based questionnaire (response rate 69%). The results show that such a project-based task is feasible with >300 students and activated them: three out of five experienced flow. Also, we validated the theory that for experiencing flow, a task must be perceived as challenging and that one's skills should match that challenge.

Keywords: flow, large class, mathematical modelling, mathematics for engineers, novel approaches to teaching, project-based tasks.

INTRODUCTION

Harris et al. (2015) studied engineering students' values regarding mathematics finding that not many first-year engineering students have a positive stance towards mathematics. The students see mathematics as a hurdle in their studies, and they are disappointed by the mathematical demands in the first year of their studies. Some even indicate that they wouldn't have chosen the engineering direction if they had known about the mathematics demands before. Nevertheless, mathematics needs to be part of engineering studies, because alumni from engineering studies, such as engineers, managers, researchers, etc., need mathematical modelling competencies to describe, analyse, and predict phenomena to solve problems at the workplace (Alpers et al., 2013). This means, that in particular mathematical modelling needs to be included in engineering studies. It can be integral part of the mathematics curriculum, but the learning of mathematical modelling can also take place in other disciplines, such as physics, where mathematical models are used to describe and analyse physical phenomena. The study described in this paper centres on a mathematical modelling task situated within kinematics (the physics of movement).

In university first-year studies, engineering students often attend large-scale lectures and have tutorial sessions to practice examination-like exercises. However, research has demonstrated the advantages of activating, inquiry-based tasks over these traditional instruction methods (De Jong, Linn, & Zacharia, 2013; Freeman, et al. 2014). This means that we need research in engineering education into what mathematical modelling tasks can be activating, how these can be organised, how students experience these tasks, what task characteristics create challenges, etc. Moreover, studies with large groups are scarce; the review by Freeman et al. (2014) shows that most studies on students' activation are carried out with small or medium size classes (up to 110 students). With more than 300 first-year students, we can add to the research on how engineering students in large classes can be activated.

Sullivan et al. (2011) describe *challenging tasks* as requiring students to: plan their approach, especially sequencing more than one step; process multiple pieces of information, with an expectation that they make connections between those pieces, and see concepts in new ways; choose their own strategies, goals, and level of accessing the task; spend time on the task and record their reasoning; explain their strategies and justify their thinking to the teacher and other students. We used a task format that fits this description: a *project-based task,* which is a task that cannot be completed within limited time, which has a clear, but not straight-forward goal, there are various approaches to tackle it, and results must be presented through a product, such as a written report or an oral presentation (Blomhøj & Kjeldsen, 2006).

THE TRACKER PROJECT TASK

Domínguez et al. (2015) did research with a group of 20 engineering students and asked them: *a child is throwing a candy to another; make a mathematical model of this movement*. This modelling task is an open-ended task with characteristics of 'a challenging' task (Sullivan et al., 2011): students need to sequence more than one step; process multiple pieces of information and connect the throwing and the model; choose their own strategies, goals, and level of accessing the task; spend time on the task. We adapted the task in the following way: (1) students could choose whatever movement of whatever object: throwing a ball, jumping their skate board, etc.; (2) students were asked to use their *smart phones* for filming, as nearly all students nowadays have smart phones with high quality cameras; (3) students were asked to download *tracker* software (http://physlets.org/tracker/), which captures motion in videos based on contrasts and yields a table of time and position coordinates (measurements). We made a tutorial video on the use of Tracker. The measurements were to be mathematically modelled (i.e. create a formula that approximates the movement). The required, final product was a poster, in which students presented their reasoning – another characteristic of a 'challenging task' (Sullivan et al., 2011). The poster had to contain the video analysis, including a discussion of the accuracy of their mathematical model in comparison to the measurements. The task had to be done in groups of two or three. Collaboration was convenient, because one student alone cannot throw and film simultaneously. In our communication with the students we indicated the task as *the Tracker Project*.

It was our first time to implement a project-based mathematical modelling task with such a large group. Unlike earlier studies (e.g. Domínguez et al., 2015) we did neither have a group of 20 students, nor uniform equipment, nor sufficient staff. We couldn't learn from earlier experiences, as – to our knowledge – there are no reports of similar studies carried out with more than 300 students. The few studies on the activation of students in large classes centre on using clickers in lectures (Freeman et al., 2014). Thus, we didn't immediately want to focus on students' learning, but instead, first study the feasibility of such a task with such a large group, with the variation of cameras, and with students who have little experience with open-ended tasks. We felt that we – as lecturers – should first take the opportunity to learn how it worked in practice, whether students liked the task and how they engaged with it.

In this paper, we report on our research into the extent to which students' were activated by the modelling task. By activation, we mean – for the time being – that the task grasped them and that they liked working on it. Thus, our study is on students' attitudes, which is an aspect of their *affect*. Based on Harris et al. (2015), we expected the engineering students to have preconceived beliefs about mathematics, and we wanted to avoid that our research would be contaminated by their biases. Therefore, we undertook our research by limiting the use of the word mathematics in our communication with students. Abundant use of the term mathematics could trigger memories and bias of traditional mathematics education, which could interfere with their evaluation of the Tracker Project Task.

THEORETICAL FRAME

Recent research in the field of mathematics education and affect conceptualize the latter in terms of complex, dynamic systems and participatory environments (Pepin & Roesken-Winter, 2015). However, while distinguishing between aspects of affect (values, emotions, beliefs, attitudes, etc.), these researchers don't differentiate between aspects of mathematics education. Yet, mathematics education contains many aspects, such as instruction formats, teacher attitudes, tasks, etc. These become invisible when researchers address mathematics holistically and ask students to mark their (dis-)agreement to statements such as: 'mathematics is my favourite subject'. (Dis-)agreement to such a statement gives little room for nuances and contexts. A student partly agreeing with this item might rather have said: "mathematics with this particular teacher is my favourite subject, but last year it was the opposite" or "mathematics could be favourite, if it had relevance for my future".

We wanted to study students' affect through an activity that differed from standard activities within traditional mathematics education. Thus, we sought an activity-based conceptualisation of affect. An activity-based perspective in mathematics education aligns with a socio-cultural perspective. One of its promoters, Lerman (2000), describes mathematics as a socio-cultural practice embedded within a community. Within a school institution, mathematics is a practice embedded in a community of a teacher and a group of students, its rules, language, etc. The activities consist, among others, of explanations by the teacher, and work on tasks by students. This practice differs markedly from mathematics as a practice within a research community, whereby the actors organize mathematical patterns, solve creatively a non-routine problem by using mathematics, and actors may reach different answers. Describing mathematics socio-culturally as a practice embedded within a community entails focusing on the activities undertaken by the actors, which are mediated by language, tools, etc. Using an activity-based conceptualisation of mathematics enabled us to relate affect to distinct activities and not to mathematics holistically, whereby we could distinguish mathematical activities as having different contexts. In our study the activity was guided by the Tracker Project Task and students had to use mathematics within a kinematics context.

We sought an activity-based conceptualisation of students' affect with respect to them being activated. Activation is an aspect of attitude, just like boredom or anxiety (Pepin & Roesken-Winter, 2015). For this conceptualisation, we turned to a concept, which describes "a state in which people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it" (Csíkszentmihályi, 1990, p.4). Nakamura and Csikszentmihalyi (2009) describe how they observed rock climbers, gamers, painters and researchers during their challenge, and how these people got absorbed in their activities, felt happiness, forgot about time and basic needs (eating, resting), and were intrinsically motivated (motivated by the activity itself, not by an external incentive). They coined this state: *flow*.

Flow is an activity-based concept: without activity, there cannot be an experience of *flow*. *Flow* is an experience of an individual, yet, the activity is culturally embedded (e.g. gamers play a game created by others, painters expose their work). In fact, social activities can intensify *flow* through group cohesion (group *flow*). We will use students' self-reported experience of *flow* as an operationalisation of their activation through the Tracker Project task. *Flow* has also been studied in mathematics education (a.o. Armstrong, 2008; Drakes, 2012; Liljedahl, 2016), observing that many students in traditional mathematics classes is to not experience *flow* at all.

Figure 1 (left) illustrates how *flow* depends on the perceived challenge of a task and perceived skills of a person engaging in the task (Nakamura & Csikszentmihalyi, 2009). If the activity is too challenging for the skills, then the task may cause anxiety. If the activity is too easy for the skills, then the task may cause boredom. When challenge and skills match, a person engaging in a task may experience *flow*. In later work, Csíkszentmihályi and colleagues adapted the diagram, adding more affective states, and stating that *flow* can be only experienced when a participant perceives the task as more than averagely challenging, and that he/she thinks to have the skills that match this challenge, see Figure 1 (right). The older diagram still appears in recent studies (e.g. Liljedahl, 2016). Therefore, we opted to use our study to empirically validate the old versus the new theory and see whether flow occurs only when the actor perceives a more than average challenge. Our research was guided by two questions. The first was empirical: To what extent did the Tracker Project Task make students experience *flow*? The second was about the choice of flow diagram: Can one of the Csíkszentmihályi diagrams of *flow* be confirmed by plotting skills, challenge, and *flow* into one diagram?

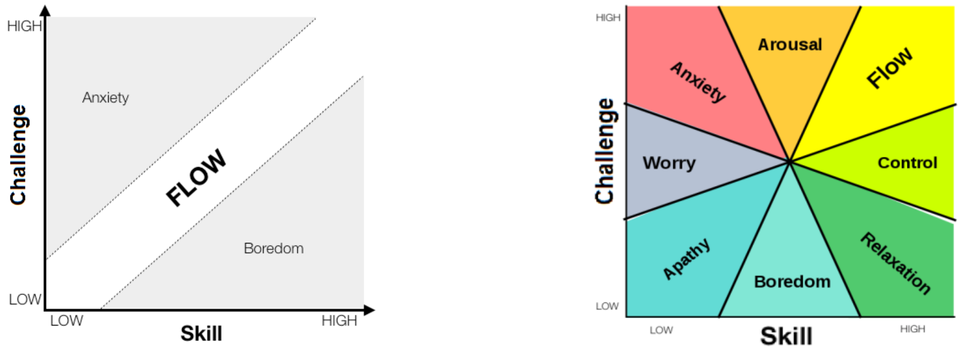

Figure 1: *Flow* and other affective states related to task challenge and a person's skills (adapted from Nakamura & Csikszentmihalyi, 2009)

METHODS

In the Spring of 2017 we offered the Tracker Project task to all first-year students in engineering at our university (Mechatronics, Electrical Eng., Data Eng., Renewable Energy, ICT). There were 346 students for whom the task was mandatory.

The research design for studying students' activation in terms of *flow* was a survey. We collected data through a digital questionnaire. Participation in the survey was voluntary, but encouraged with prizes of NOK 500 (approx $50) for three randomly drawn participants. After removing seven participants (four had constantly chosen a 3 as answer, three were 2nd-year students for whom the task wasn't mandatory), we had n=239 students. This response rate of 69% is very high (Bryman, 2015).

Based on instruments from earlier research (Armstrong, 2008; Egbert, 2004), we developed 15 items in alignment with the task. Each item consisted of a statement, asking students for their (dis-)agreement on a 5-point Likert scale, from 1 (strongly disagree) to 5 (strongly agree), see the Appendix. Five items were designed to measure students' perception of *flow*. For this, they could indicate, for example, whether they forgot about the time, and whether they even would do the task if it wasn't obligatory. By having several items related to *flow*, a participant's score is indicator of the extent to which he/she had experienced *flow*. Five other items were designed to measure students' self-perceived *skills* (e.g. 'the Tracker technology was easy to use' or (inverted) 'It was complicated to find the right formula of the model'). And a further five items were designed to measure students' perception of the task's *challenge* (e.g. 'during this task I started thinking about other movements (what if..?)' and (inverted) 'this task was more for secondary schools').

We make a difference between *flow* as a concept (written in italics), and the scale of Flow (with a capital letter). The concept of *flow* is a psychological state of a person, and therefore it cannot be measured. However, we assume that it can be approximated by a score on the scale of Flow. A student's score on this scale results from his/her answers to the five questions in our questionnaire. The score on the Flow scale is calculated by adding the scores on the five questions. As the score on one question ranges from 1–5, the score on the Flow scale ranges from 5–25. Likewise for respectively, *challenge* and the Challenge scale, and *skills* and the Skills scale. To increase reliability, within each scale one or two questions were inversely posed, and the scoring was inverted, too. As measure of reliability (internal consistency), we calculated Cronbach's Alpha: the Skill scale yielded 0.55, the Challenge scale yielded 0.73, and the Flow scale yielded 0.63. A scale is considered unreliable if Cronbach's Alpha is less than 0.5 (Bryman, 2015). Thus, the three scales can be considered as being reliable.

RESULTS

We observed students everywhere on campus, flying paper helicopters, riding skateboards, or throwing apples, cats or balls. We received more than 100 posters in our Virtual Learning System. As explained before, in this study we didn't want to focus on students' performance (the precision of their measurements, their understanding of modelling, the depth of their analysis, etc.). Instead, we focused on the feasibility of an activating tasks for massive students groups, which would show in their activation in terms of *flow* as measured through the questionnaire. Second, we aimed at seeing whether the measurement reproduced one of the two flow diagrams. The Appendix shows mean scores on all items.

The mean score on four items in the Flow scale is higher than 3.5, being well on the positive side. This indicates that a majority of the students experienced a state of *flow* to quite an extent, in particular with respect to losing track of time, and not being easily distracted. Only item 14 was answered below the middle range. This item focuses on doing the task even if at some costs (Csíkszentmihályi, 1990), which translates in our study to: one out of four would even do the task voluntarily.

Scale	**mean** (std dev)
Flow (5 items)	17.0 (3.1)
Challenge (5 items)	15.5 (3.3)
Skills (5 items)	18.8 (2.7)

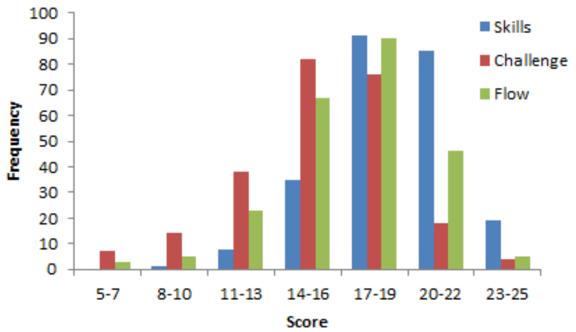

Figure 2: Scores to Skills, Challenge, and Flow scales (n=239)

When adding the students' scores on the five questions, we obtain their score on the scale Flow. See Figure 2 (right) for a bar graph. This graph shows the frequencies of scores (number of students with certain scores). The green bars of the Flow scale show a skewed distribution. On this Flow scale, 31 students (13%) scored 13 points or lower, 67 students (28%) scored in the middle range of 14–16 points, and 141 students (59%) scored 17 points or higher. When we take 17 points as a threshold, then three out of five students experienced *flow* to quite an extent. The table in Figure 2 presents mean scores on the scales for Skills, Challenge and Flow (minimal score = 5, middle score range = 14–16, maximal score = 25). The scores on Skills are highest: generally, students perceived themselves as highly skilled; the low standard deviation indicates a high agreement among students. The scores on Challenge are around the middle range; these scores are most "normal" (making a Gauss curve).

To validate the Csíkszentmihályi diagrams (Figure 1), we created a scatter diagram. Each student was represented by a dot defined by his/her Skills score on the x-axis and his/her Challenge score on the y-axis, see Figure 2. The resulting diagram shows a scattered distribution, which means that there is no correlation at all between the scales Challenge and Skills (r = 0.097). In this diagram, we added the third scale, the one for Flow, by colouring the dots depending on the student's Flow scores. These scores range from red to orange (13 or lower), via yellow (middle range, 14–16) to green (17 or higher). Roughly, one can discern overlapping red, yellow and green areas. The red area is more visible at the bottom showing the students who experienced little *flow* (13% of the students). These students indicated that the task posed little challenge, independently of their perceived skills. The yellow area runs from bottom right to the centre showing the students who experienced medium *flow* (28% of the students). These students either indicated low challenge and high skills, or medium challenge and medium skills. The green area is the largest with the majority of students (59%). It is in the top-right, fading towards the centre, showing the students who experienced *flow* to quite an extent. These students indicated that they perceived the task as challenging, and they perceived themselves skilled.

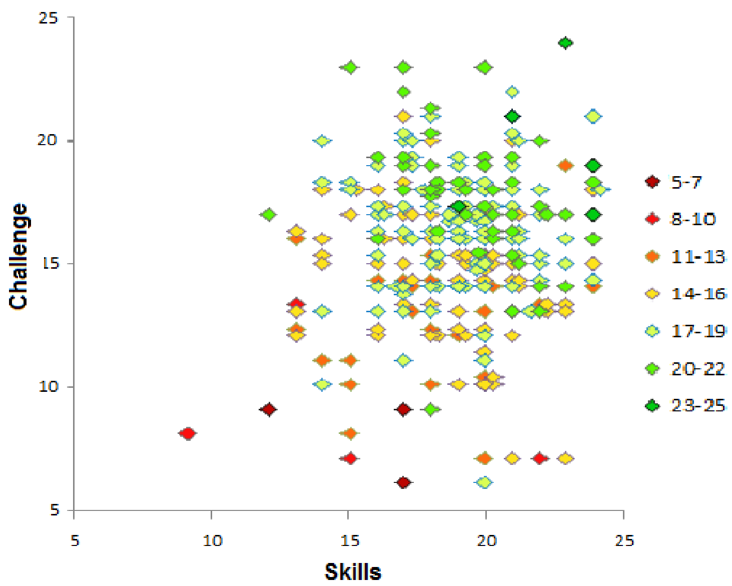

Figure 3: Flow score indicated by color, as depending on Skills and Challenge

This colour distribution of Flow does not confirm the earlier Csíkszentmihályi diagram (Fig 1, left), as the green dots do not centre on the diagonal. Instead, the green dots are more to be found in the area where the later Csíkszentmihályi diagram (Fig 1, right) situates *flow*: the task must be perceived as quite challenging, and this challenge must match one's skills. A majority of the students in our study indicate that they perceived the Tracker Project Task as such.

In light of the different regions in the later Csíkszentmihályi diagram, we also see many students who may fit into the affective states of 'control', 'arousal' and 'relaxation'. Only few students may fit the more negative affective states of 'apathy', 'boredom', 'anxiety' and 'worry'.

CONCLUSION, DISCUSSION, RECOMMENDATIONS

We studied whether a project-based task was feasible with a class of more than 300 students, that is: whether the task activated individual students. The first research question asked: to what extent does the Tracker Project Task make students experience *flow*? The results from the survey showed that a majority of the students (59%) experienced *flow* to quite an extent, forgot about time and wanted more of such activities. This result was confirmed by anecdotal evidence of their boasting stories in the tutorials of them throwing objects, and the high response rate to the survey. This means that the Tracker Project Task activated a majority of the students and that they had positive attitudes towards it. Thus, an activating mathematics task can be feasible with a large class of engineering students, even if they are known to have a negative stance towards mathematics (Harris et al., 2015).

The Tracker Project Task was designed to be challenging with characteristics such as: expecting students to process multiple pieces of information, that they make connections between those pieces, choose their own strategies, and explain their strategies to others (Blomhøj & Kjeldsen, 2006; Sullivan et al., 2001). We observed other characteristics in the Tracker Project Task that activated the students. First, the task had a clear goal, which was understandable to all students. We observed this through the few questions that we got from the students on how to carry out the task. Thus, the task was easily accessible, also known as *having a low floor*. Second, the better students were able to challenge themselves further, allowing for *a high ceiling.* Third, the use of readily-available technology (cameras in smart phones, tracker software) may have captivated the engineering students, who are known to be technology minded. Fourth, the task was a mathematical modelling task embedded in engineering practices, whereby mathematics served non-mathematical purposes; this showed students the relevance of mathematics to their studies, and contrasted with bare mathematics tasks that alienate and demotivate students.

Our second research question pertained to the theory of *flow* and how it can be conceptualized in a diagram (Nakamura & Csíkszentmihályi, 2009). Our data reject the earlier theory that *flow* depends on the alignment of skills and challenge. Instead, our data support the later theory that *flow* occurs when the participants perceive the task more than average challenging, and that their skills should match this challenge.

Furthermore, we take from our study that the concept of *flow* proved useful for activity-based research on affect in mathematics education.

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APPENDIX

Mean scores on all items (1=lowest, 3=middle, 5=highest).

Flow questions	mean (std dev)
(Inv) This Tracker task took too much of my time	3.67 (0.88)
Time was flying when we worked in this task.	3.40 (0.92)
(Inv) I was easily distracted when we worked on this task.	3.55 (0.91)
I would do this task even if it wasn't obligatory.	2.60 (1.13)
I would like to have more of such practical tasks.	3.70 (1.02)

Skills questions	mean (std dev)
The Tracker technology was easy to use.	3.89 (0.88)
(Inv) It was complicated to find the right formula of the model.	3.38 (0.92)
The aims of the task were clear to me.	3.96 (0.91)
During this task I had full control over what we did.	3.77 (1.13)
Filming the movement of an object was easy.	3.76 (1.02)

Challenge questions	mean (std dev)
This "Modelling med Tracker" task made me curious.	3.61 (0.75)
Making a poster made me feel like a "real scientist".	2.52 (1.03)
(Inv) This task is more suitable for secondary schools.	2.58 (0.95)
This task helped me to better understand the theory.	3.39 (0.88)
During this task I started thinking about modelling other movements (what if..?).	3.31 (1.12)

Using schematic representation of resource systems to examine how first year engineering students use resources in their studies of mathematics

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Very little research has been done on how students use resources when studying mathematics. My project aims to examine this both quantitatively and qualitatively. The qualitative data collection includes hierarchical focus interviews, with schematic representation of resource systems as a supplement. The intent is to reduce the degree of co-producing answers and imposing terminology on the students.

Keywords: Students' practices at university level, the role of digital and other resources in mathematics education, mathematics for engineers, documentational approach, hierarchical focus interviews, schematic representation of resource systems.

INTRODUCTION

The focus of my PhD-project is engineering students' use of resources for learning mathematics. Data collection takes place at three Norwegian universities. I focus on which resources they use, to what extent and in what situations, as well as their rationale for how they use resources.

THEORETICAL FRAMEWORK

I use the documentational approach (Gueudet & Trouche, 2009) to examine those questions. Relevant to the approach is the term 'document' which is the joint entity of a set of resources and schemes for utilizing them in certain situations to achieve certain goals. Also relevant is the term 'resource system' for the set of all resources an individual is using, structurally organized. For instance, what resources to use in what situation can be part of the structure. The framework is designed to examine teachers' practices and professional development, but can be adapted to examine students' use of resources as well. One of the focuses of the framework is how students' documents develop (called 'documentational genesis'). I look at students' practices during their first year of university, when I expected a lot of development to occur.

METHODOLOGY

I use both qualitative and quantitative means for data collection. I do so because I want to study resource with some depth, while also getting an indication of the variety of uses. Here, I will focus on qualitative data collection. I used hierarchical focus interviews (Tomlinson, 1989). These contain strategies to reduce the degree to which the interviewer co-produces the answers. Prior to the first interview (of three), I asked each student to draw a ‘mind map’ about their use of resources in mathematics. I used their mind maps to create schematic representations of their resource systems (SRRS), inspired by Pepin, Xu, Trouche and Wang (2016). From a pilot interview in spring 2017, I theorized that the construction of a mind map helped students structure their thoughts about using resources prior to the interview.

RESULTS

Qualitative data collection spanned the fall semester of 2017, with nine students from three different universities. All students created a mind map during the first interview. In the other 18 interviews, they made changes to their mind maps a total of eight times. The students seemed comfortable talking about their use of resources after constructing a mind map. Their descriptions of their mind maps also yielded interesting insight into how they perceived their use of resources.

The students structured their mind map in several ways. They all contained resources, but five students also had categories in their mind maps, three had situations, five had what purpose they used certain resources for and one had features they appreciated about certain resources. The resources that the most students put in their mind maps were the textbook (nine), fellow students (seven), lectures (six), exercises (six), lecturer (five), pencil and paper (five), Wolfram Alpha (four), calculator (three) and google (three).

CONCLUSION

It is difficult to discern whether differences in students’ mind map structure only represent stylistic choices, or meaningful differences in how they perceive their use of resources. While a student’s mind map can say much about their documents, analysing the mind map in a vacuum is insufficient. However, when mind map construction is combined with an interview, the two forms of data may shed some light on one another. The interviews may also benefit from the construction of the mind map, as it gives students some time to consider their use of resources prior to the interview.

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Mathematics Teaching for Economics Students, But How?

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The research presented in this poster addresses the poor performance of many economics students in their first mathematics course at university level. Two different universities are involved in the research, trying to answer the question of "how to structure teaching in mathematics to economics students at university level to strengthen their mathematical competences?"

Keywords: Teaching and learning of mathematics in other fields, curricular and institutional issues concerning the teaching of mathematics at university level.

How to structure teaching to improve economics students' mathematical competences? Research papers within economics are often heavily mathematical. Mathematics is an indispensable tool in studying economics and in the economist's working day. The Norwegian Association of Higher Education Institutions (2011) therefore stipulates that mathematics should be a useful tool for students in the learning of other subject areas within economics.

Research has proved that mathematical knowledge in algebra and arithmetic is a good indicator for performance in introductory economics courses (e.g. Ballard & Johnson, 2004). However, many universities are struggling to find satisfactory formats for teaching mathematics as a service discipline, particularly in their economics undergraduate degree courses.

At the University of Agder there is a first year, first semester course in mathematics for economics students. The proportion of students failing this course, has for several years been about 40%. This is an alarming issue and I have experienced similar problems at Åbo Akademi University in Finland, the university where I studied for my master's degree. Currently these two universities have taken opposite directions regarding the teaching of mathematics to economics students.

Both universities face the issue of students' diverse mathematical backgrounds. At the Åbo Akademi University the course in mathematics for economics students has been moved to the first semester, to more naturally be a continuation of school mathematics. At the University of Agder, a diagnostic test, compulsory for all economics students, is being implemented. The test is followed by a preparatory mathematics module; although not compulsory it is recommended that students with weak mathematical background take it, prior to the main course in mathematics. The preparatory module will consist of online, self-study material with a clear structure, implemented in the Canvas digital learning environment with 3-4 workshops.

The overall goal of the research is to improve the teaching of mathematics for economics students and to optimize their learning. This poster is about the first stage in the research process and considers productive opportunities for research on the diagnostic test and the online preparatory mathematics module. Students studying and participating in activities online produce a huge amount of data which gives the opportunity to use Learning Analytics (LA) as a research method to better understand the mathematical needs of economics students and their use of mathematical resources. Peña-Ayala (2017, p. 6) writes: “LA in the context of higher education is an appropriate tool for reflecting the learning behaviour of students and provide suitable assistance from teachers or tutors.” I thus want to find out what factors in the preparatory mathematics module contribute or do not contribute to the students' progress in learning mathematics. Because of the preparatory module being optional there is also the possibility to find out amongst students recommended to make use of the preparatory mathematics module, did those who took part in the module perform better in the main mathematics course examination than those who did not take part?

At this stage, there are no concrete results to report. Data will be collected and analysed in autumn 2018. The theoretical framework to be used is under consideration. The theoretical framework will underpin the formulation of research questions. Clow (2013) argues for Learning Analytics being atheoretical. There are limited studies linking learning theory to learning analytics, but for example Macfadyen and Dawson (2010) acknowledge the importance of effective student centred learning and mention, as an example of the social aspect for learning, the possibility of learner-to-learner communication within digital learning environments, and thus they look into the socio-constructivist paradigm.

At the time of the conference I hope to be more knowledgeable about what theoretical framework could guide the proposed research. It will be a valuable opportunity to discuss the theoretical framework with more experienced scholars in the field during the conference.

As Peña-Ayala (2017, p. 68) writes: “LA aims at developing models, methods, and tools that can be widely used, whose deliverables are reliable and valid at a scale beyond a course or cohort to provide benefits for learners and educators without distracting or misleading them” I hope the proposed research will provide new knowledge about teaching of mathematics to economics students.

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Non-standard Problems in an Ordinary Differential Equations Course

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We report first results from a teaching intervention in an ordinary differential equations (ODEs) course for engineering students. Our aim is to challenge traditional approaches to teaching of Existence and Uniqueness Theorems (EUTs) through the design of problems that students cannot solve by applying well-rehearsed techniques or familiar methods. We analyse how the use of non-standard problems contributes to the development of students' conceptual understanding of EUTs and ODEs.

Keywords: existence and uniqueness theorems, design research, non-standard problems, commognitive theory, mathematical discourse.

INTRODUCTION AND BACKGROUND TO THE STUDY

Although ODEs are an important topic in the engineering curriculum, students experience difficulties with mastering ODEs and with the very concept of a differential equation (Arslan, 2010). In our study, the lecturer, a mathematician, devised a set of non-standard problems (see below, Problem 1 of 6) to challenge students' conceptual understanding of the EUTs. These problems formed an assessed piece of coursework.

1. (a) Verify that

$$y(x) = \frac{2}{x} - \frac{C_1}{x^2}$$

is the general solution of a differential equation

$$x^2 y' + 2xy = 2.$$

(b) Show that both initial conditions $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solution. Does this fact violate the Existence and Uniqueness Theorem (EUT)? Explain your answer.

Figure 1. One of the problems in the study

We analysed how solutions changed and developed as students worked on the problems. Students' discussions in small groups were audio-recorded, transcribed, and then analysed using constructs from commognitive theory (Sfard, 2008). We are currently in the initial phase of the data analysis aimed at answering the following question: How do non-standard problems contribute to the development of students' mathematical discourse and further their conceptual understanding of fundamental notions and results in an ODE course?

RESULTS

For Problem 1a (P1a), students could use one of two solution methods: M1 (substitution) and M2 (integration). Working on the problems, several students changed their approach. In the final script, only one student produced a correct and complete solution (M2) while 14 (of 19) students used M1 verifying that a given function is a solution (which is sufficient for the particular solution), but failed to explain why this solution is the general one (hence incomplete M1). We conducted similar analyses for Problem 1b (P1b).

We present one extract (for P1b) as an example of our analyses of students' group discussions using commognitive constructs - narrative, routine, ritual, substantiation.

S12. The first idea was just to try to solve for C and I got the same constant, so that's OK. And I checked for asymptotes and I got one on x=0, so I noted that the equation is split to get two curves, at least, according to calculator we got it split about zero.

S11. So it's undefined at zero.

S12. Undefined at zero, so we get two different curves and both solutions work. We do not have a continuous curve which happens to intersect at these two points […]

S14. It's not continuous for $x = 0$?

S12. No. So if we take an interval from -3 to 1, it's discontinuous in this interval, so it's not a curve that happens to just hit these two points, it is two individual curves that have the same solution. So it's correct in just a tiny area.

S11. That was my argument as well. As the theorem states, there is a continuous interval but here it is split into two which contain two different t_0's.

S13. The theorem says, that there is a unique solution for every interval where the function is continuous. Since there are two intervals and there are two solutions, it does not conflict with the theorem. […]

Note that S12 is using two different visual realizations of solutions, first the algebraic representation and then the graph plotted by calculator. He shows that the realizations are not equivalent, they do not produce the same result. We see how the student demonstrates the ability to solve the problem by developing the realization tree and employing the mathematical object of "continuous solution" (discursive object). S11 is not so sure at the beginning, he is guided by S12 (considered "more experienced" interlocutor) and adopts the narrative offered by S12. S13 concludes by reformulating the expression "does not violate the theorem" as "does not conflict with the theorem".

We see how students worked to substantiate the narrative. This routine can be characterized as the exploration. Students gradually improved their abilities in developing and endorsing the EUTs narratives while working on all six tasks during the group discussions.

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Students' understanding of mathematical models in three Norwegian biology courses

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Keywords: mathematical models, mathematical understanding, biology students

INTRODUCTION

Development in Biology over the last three decades have greatly increased the need for using mathematics in this field. This has influenced biology students which need to develop mathematical knowledge to be able to work with contemporary biology models, including frameworks that are applicable in analyzing the overwhelming flow of biological data (i.e. Labov, Reid & Yamamoto, 2010). Although a quantitative approach is often used in university biology courses, yet they remain largely qualitative and descriptive (Nelson, et. al., 2009). In Norwegian context, where this study will be conducted, there is an increased awareness towards mathematics from the Departments of Biology. However, there is a lack of information on students' understanding and usage of mathematics within this context. This poster addresses this issue with a particular focus on the use of mathematical models (MM) as dynamic tools that allows us to observe various aspects of students' understanding.

UNDERSTANDING OF MATHEMATICAL MODELS

Mathematical understanding has been the scope of many researchers. Sfard (1994) distinguish between *operational* and *structural* way of understanding. While the operational understanding includes highly manipulative skills and use them as principal means in their quest after meaning, the structuralist is more capable of direct-grasp understanding. She defines as reification the transition from an operational to a structural way of thinking, and states that this transition is a basic phenomenon in the formation of a mathematical object.

According to Niss (2012), a *mathematical model* can be defined as mapping (translation), f, from a mathematical domain, D, to a mathematical realm, M. In this context, D and M, represent not only sets of objects but also collections of relationships, phenomena or questions, while f operates on objects and the relationships, phenomena or questions. It is important to point out the distinction between mathematical model and mathematical modeling. While mathematical model is a tool for facilitating quantification, analyses, predictions and gaining insight for a real-world situation, mathematical modeling is the process of the creation of such tool (GAIMME, 2016). In this study, I focus only on the use of mathematical models as a tool that allows one to observe different shades of students' mathematical understanding, and not on the process of mathematical modeling. Observing biology students while engaging in biology problems that use mathematical models as their representation, I aim to describe their way of thinking and reasoning – as operational or structural thinking.

METHODOLOGY

This study is a descriptive case-study in a naturalistic paradigm. For the pilot phase of this study, have been selected three different biology courses from a Norwegian university (two courses of bachelor level and one in master level). These courses have been selected considering their use of mathematical models. In these courses, mathematics is implicitly presented using mathematical models (i.e., population dynamic models). All students taking these courses have taken previously at least one mathematical course (mostly, Calculus course or Statistic course).

For collection of the data, I plan to video-record sessions in these courses when students are engaged in group-work. I will use semi-structured interviews with some of the students after some sessions, and personal written notes.

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TWG 3: Number, Algebra, Logic

Contemporary Research Practices in Discrete Mathematics - A way to enrich the understanding of Discrete Mathematics at University Level

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This paper is part of a thesis about discrete mathematics and its teaching in higher education. The literature on the didactics of discrete mathematics questions this branch at different levels: its integration in teaching, the particularity of its affective dimension, and its epistemological specificities especially in the fields of proof and modeling. We seek to epistemologically define this field and to characterize its corresponding mathematical activity by studying the processes of knowledge construction, the types of problems, the specificity of concepts and proofs, and also the existing links between discrete mathematics and other disciplines. This epistemological study has a didactic purpose of defining and analyzing the teaching of discrete mathematics in higher education.

Keywords: teaching and learning of number theory and discrete mathematics, teaching and learning of logic and proof, higher education, functional definition, epistemology.

INTRODUCTION AND CONTEXT

Research on discrete mathematics has rapidly developed in its methodologies, in the way it is viewed by mathematicians, and in its range of applications. Discrete mathematics has been described by *TSG-17 Teaching and Learning of Discrete Mathematics* at the ICME-13 (2016) as a comparatively young branch of mathematics with no agreed-on definition but having old roots and emblematic problems. Moreover, it is a robust field with applications to a variety of real world situations, and of on growing importance to contemporary society (Hart & Sandefur, in press). Over the past several decades, discrete mathematics has proved to be an important part of the recommended program for students of computer science (Maurer, 1997; DeBellis & Rosenstein, 2004; Grenier & Payan, 1998; Borwein, 2009; Epp, 2016; Rosenstein, 2016). Epp (2016) points out the strong necessity for engaging students in abstract thinking for the course of discrete mathematics and its applications in computer science. Discrete mathematics also seems to be a very important tool for research in biology and chemistry. On the other hand, discrete mathematics has been influenced by a variety of mathematical results, methods, and representations (group theory, number theory, geometry, algebraic combinatorics, graph theory, and cryptography). Their integration and combination in a profound theory is essential for research in discrete mathematics (Heinze, Anderson, & Reiss, 2004). A recent publication that looks into the future of mathematics, *The Mathematical Sciences in 2025* (Committee on the Mathematical Sciences in 2025,

2013) identifies two new drivers of mathematics: computation and big data. For both, it describes how discrete mathematics plays an important role like discrete mathematics algorithms for mathematical processing, dynamical systems in ecology, networks in industry and the humanities, and discrete optimization (p.77).

The growing importance of discrete mathematics leads us to define this field for an educational purpose. We seek to develop a "functional definition" [1] of discrete mathematics, in order to use it to analyze and design didactical situations. Specifically, we are concerned with the university level. We first present the state of art in the teaching and the learning of discrete mathematics mainly at secondary level pointing out some of its epistemological aspects. We then state our research questions and describe the methodology aimed at developing a "functional definition" of discrete mathematics. Our research is inscribed in a "contemporary epistemology" [2] that draws on interviews with mathematicians. Our working hypothesis is that such interviews can update and enrich our functional definition. Finally, we discuss some preliminary results of the interviews with mathematicians and close with some concluding remarks.

TEACHING AND LEARNING OF DISCRETE MATHEMATICS-SUMMARY OF A STATE OF ART

This young field of mathematics with numerous interconnections has no agreed-on definition shared by mathematicians (Maurer, 1997; Hart & Martin, 2016) and has blossomed in several directions. There exist different attempts to define discrete mathematics, by mathematicians (like in the United States) and by mathematics educators (like in France). These attempts depend on the epistemological posture of the authors and on the intended function of the definition (e.g. to enable mathematicians to define a field, mathematics educators to characterize a domain, and teachers to present a topic of mathematics…etc.). For example, in an attempt to define discrete mathematics, a mathematician proposed two standard approaches toward this definition (Maurer, 1997): by specifying properties or by lists of topics. The defining lists are too many (courses aiming for computer science majors, algorithm-oriented course, finite mathematics course for social science and business majors, high school course) (Maurer, 1997). Mathematics educators have also proposed a definition of discrete mathematics such as the following:

> "The main idea is that discrete mathematics is the study of mathematical structures that are "discrete" in contrast with "continuous" ones. Discrete structures are configurations that can be characterized with a finite or countable set of relations" (Ouvrier-Buffet, 2014, p. 181).

Moreover, discrete mathematics acquires particular objects and methods (Grenier & Payan, 1998). However, these attempts to define discrete mathematics are not all-inclusive as they overlook many characteristics of the concepts and proofs involved in this field. In our opinion, what is also important for the didactics of mathematics is to uncover the specificities of this field of mathematics in comparison to others.

Recent research in discrete mathematics, computer science, and mathematics education has led to a serious discussion of the principles of proof, the teaching and learning of proof, the validity of computer-based proofs or of visual proofs etc. The distinction between the terms reasoning, proving, augmenting, demonstrating, and the complex relationship between argumentation and demonstration (argumentation considered as an epistemological obstacle for the learning of proof), calls for a debate and an in-depth analysis (Balacheff, 1987; 1999; Reid & Knipping, 2010). In their book, *Proofs in Mathematics Educatio*n, Reid and Knipping included several examples in their discussions, and there might be a reason that a large number of these examples come from discrete mathematics (Reid & Knipping, 2010). In the special issue of ZDM (2004) and in the ICME 13 monograph (2016), discrete mathematics continues to be promoted as the essential mathematics in a 21st century school curriculum. Its power lies in the opportunity it provides for supporting reasoning, problem solving, modeling, and systematic thinking in the school curriculum. Besides, recursion and recursive thinking seem to be powerful modeling and problem solving strategies throughout mathematics in general and in the teaching and learning of discrete mathematics in particular. The latter has been highlighted in the studies of part III of ICME 13 monograph entitled *recursion and recursive thinking*. They describe the integration of recursive thinking with iterative as well as algebraic thinking, and they present the benefits of this integration as means to deepen the students understanding of each of the geometry of transformations and covariation of variables.

Some epistemological aspects of discrete mathematics pointed out in didactics

Researchers in didactics of discrete mathematics have proposed several characteristics, of epistemological nature, of discrete mathematics. These characteristics are the result of their research aiming at investigating the place and role of discrete mathematics in education, analyzing the teaching and learning situations, integrating new content into the curricula, studying the place and role of proof in the curricula, and examining the mathematical expression (symbolic and visual) and the use of language. Accordingly, several aspects have revealed such as: problems in discrete mathematics encourage the development of heuristic and affective processes (Goldin, 2016), there exists a specific relationship between discrete mathematics and proof-existence of different situations that provide different views on proof (Grenier & Payan, 1998), there exist different models in discrete mathematics which necessitates the work on modeling (Grenier & Payan, 1998), discrete objects and situations are easily accessible (Grenier & Payan, 1998; Maurer, 1997; DeBellis & Rosenstein, 2004), there exist different definitions of different natures for discrete objects (Grenier & Payan, 1998; Maurer, 1997; Ouvrier-Buffet, 2011; 2006; Balacheff, 1987), and the fact that examples from discrete mathematics enhance the semantic development of mathematical concepts and proving skills (Alcock, 2009). Discrete mathematics provide the opportunity to develop students reasoning ability, communication skills, problem solving ability, and modeling skills, as well as mathematical habits of the mind that are specifically cultivated by studying discrete mathematics such as algorithmic problem solving, combinatorial reasoning, and recursive thinking. In short, as Hart & Martin (2016) say, discrete mathematics is *empirically powerful* as a tool to enhance modeling and solving fundamental contemporary problems, and it is *pedagogically powerful* in that it can be used in the curriculum to simultaneously address content, process, and affect goals of mathematics education.

RESEARCH AIM AND RESEARCH QUESTIONS

The importance of discrete mathematics in both research and in education has been highly marked and extensively studied in the literature. However, the inclusion of discrete mathematics in school curricula faces challenges worldwide. There are countries like Hungary and Germany in which discrete mathematics has been taught since a long time and as early as primary years of school. In France, the recent introduction of graph theory for grade 12 classes of specialty "ES" (economy and social) represents an official entry of discrete mathematics into the classrooms, yet this integration is still far from that of other European countries. In the United States, since 2000 discrete mathematics had been integrated into the curricula such as "combinatorics, iteration, and recursion, and vertex-edge graphs…" as mathematical topics at school level (K-12) (NCTM, 2000, p. 31). Yet, the new Common Core State Standards for mathematics that were developed in 2009 and adopted soon afterwards by most of the states in the United States excluded discrete mathematics (Rosenstein, 2016). Rosenstein explains in his paper that the reasons for this exclusion are: (1) the shift in focus from college-readiness to calculus-readiness, (2) the desire to expand the STEM pipeline by ensuring that students take more calculus at secondary level, and (3) the concerns for international assessments. He calls out the international mathematical education community to have an active role in introducing discrete mathematics into the curricula of their countries' schools by developing their own curriculum material to promote a broader curriculum. However, although discrete mathematics is taught in a shy manner in some countries, this does not mark the existence of didactics of discrete mathematics, as a well-structured branch of mathematics in the same way there exists the didactics of algebra, calculus or geometry. Discrete mathematics exists at the frontiers with other fields like computer science. Hence, the teaching of discrete mathematics constitutes a challenge (a complex choice of topics with a high demand for instruction). We believe that proof processes of discrete mathematics are abundant, diverse, and particular, and we aim at exploring this aspect and its connection with other mathematical domains.

The literature led to the following research questions: how can we define "functionally" discrete mathematics (that is how can we describe its epistemological aspects, the links between discrete mathematics and other domains, and what are the most recurrent types of problems that arise), and how can we describe the teaching of discrete mathematics at university level. Their treatment, based on the "contemporary epistemology", will contribute to the delimitation of the field of discrete mathematics, hence an objective of our study. In particular, this treatment will update and enrich the conceptions [3] of mathematics educators about discrete mathematics and lead to the development of teaching and learning situations.

Towards a Functional Framework

Therefore, our research aims at further investigating the above questions and exploring the reality of the teaching and learning of discrete mathematics. Our objective is to develop a "functional framework" for discrete mathematics in order to conduct didactical studies of discrete mathematics. In this way, a "functional definition" of discrete mathematics will have two main functions: (1) to delimit the mathematical domain of discrete mathematics (epistemological level) and (2) to open new horizons for the integration of this field in teaching (didactic level).

The epistemological aspects of this framework are a very important asset and often not taken into consideration explicitly by university teachers. Indeed, Artigue (2016) claims the existence of a disconnection between the mathematician's experience as researchers and their experience as teachers. This might be caused by the absence of the epistemological dimension in their work as educators. The importance of developing these epistemological aspects is linked to the following characteristics as stated by Radford (2016), quoting Artigue (2016): (1) epistemology allows the reflection on the manner in which objects of knowledge appear in the school practice, (2) epistemology offers means through which we understand the formation of knowledge (historical production and social production), and (3) epistemology allows the reflection on the notion of *epistemological obstacle*. Accordingly, this first function of our "functional framework" concerns the delimitation of the field of discrete mathematics, by its contents, its types of problems, and to highlight the specificity of the work on proof in relation to other mathematical domains. The place and role of modeling in discrete mathematics will also be investigated. As discrete mathematics interacts with other mathematical fields, we will also need to characterize the links between discrete mathematics and arithmetic, number theory, algebra among others. Moreover, since the epistemological definition of discrete mathematics is linked to that of computer science (via the problems of counting and combinatorics among others), we will be specifying the links and interactions between these two scientific domains, explicitly relying on the "contemporary epistemology", i.e. the current problems and interactions between discrete mathematics and computer science. Finally, we will integrate into our definition a strong didactical perspective by studying the place and the role of discrete mathematics in the articulation between secondary and university education (particularly between university education and teacher training). We are also interested in investigating the process of evaluation conducted at the university level of the concepts and procedures proper to discrete mathematics. Ultimately, our purpose is to be able to make coherent epistemological propositions for the teaching of discrete mathematics at a given level.

RESEARCH METHODOLOGY

Our research methodology to address our first research question, which is “how to characterize discrete mathematics at the epistemological level?” is based on a contemporary epistemology relying on the experiences of researchers in the field of discrete mathematics who are also instructors of discrete mathematics at the university level. Our approach is inspired by several previous work relying on interviews with mathematicians and mathematics educators such as Nardi (2008). We will also base our work on the notion of praxeology of Chevallard, particularly sequences of praxeologies, for the elaboration of our framework in order to describe, analyze and structure specific contents at the heart of the teaching and learning process. The work of Hausberger (2017) on *structuralist praxeology* in Abstract Algebra could be an inspiring example. He uses a historical and epistemological study of structuralist thinking and practices combined with a study of few textbooks to develop his notion (Hausberger, 2017). In our study, we will be considering the choice of particular emblematical textbooks of discrete mathematics at university level along with the interviews to study the teaching practices.

Our study is an exploratory one in which we will conduct interviews with the researchers aiming at reinterpreting the literature findings, investigating the coherence between the literature and teacher practices, and identifying other epistemological aspects. We have conducted interviews with instructors of discrete mathematics at each of the Lebanese University and the Mathematical Society in France. In accordance with the literature findings, we have developed a questionnaire that included open-ended questions concerning the definition of discrete mathematics, types of problems, particularly proofs, in discrete mathematics, and the utility of discrete mathematics at university level (teaching and learning). We have noted important aspects of discrete mathematics, which will enrich our “functional definition”, and they will be presented as soon as we complete the rest of the interviews. At the methodological level, Table 1 represents our first approach to analysis. However, to better frame the conceptions of the researchers, we will be developing in parallel other analyses methods. This will be done using two complementary approaches: the first based on the praxeologies of Chevallard and the second relying on a theoretical model regarding “conceptions” (Balacheff, 2013).

Axis	Criteria for analysis
Conception on the definition of discrete mathematics (in teaching and in research)	Identify different points of view for researchers (since the definition is not agreed-on)
Topics from discrete mathematics	Identify and categorize topics
Conception on proofs in discrete mathematics (in teaching and in research)	Identify types of problems, types of reasoning, characteristics of concepts, place and role of modeling

Links between discrete mathematics and other disciplines (in teaching and research)	Categorize links; are they being worked in class?
Learning of discrete mathematics	Identify objectives, learning outcomes, learning difficulties, student behavior

Table 1-Criteria for analyzing the interviews with the researchers in discrete mathematics

To test our questionnaire, we have conducted two pilot interviews with two graph theorists, one in Lebanon and the other in France. The interviews were recorded and transcribed. We have selected some instances from the pilot study, and they will be presented in this paper in the following section.

PRELIMINARY RESULTS

Researchers' conceptions about the definition of discrete mathematics

In order to analyze the conception of the interviewees about the definition of discrete mathematics, we tried to elicit some epistemological aspects of discrete mathematics. The pilot interviews showed that for the two interviewees Michel (researcher and instructor of graph theory in Lebanon) and Bertrand (researcher and instructor on graph theory in France) discrete mathematics is difficult to define, and sometimes it is easier to define what is not discrete. Both interviewees used the term "separable" to describe discrete objects:

Bertrand: […] so basically one could say that discrete mathematics concerns objects that can be separated […] (our translation)

Michel: […] the elements can be manipulated separately […] (our translation)

Interesting examples illustrating this important aspect of the definition discrete mathematics ("separable") will be presented for discussion during the conference. However, the interviewees had different opinions regarding the teaching strategies and the origin of student difficulties. Michel focuses on the teaching of concepts whereas Bertrand puts more emphasis on the methods and strategies (through games and experimentations).

Michel: […] in the courses, I try to convey the basic ideas like in graph theory: definition of graphs, adjacency matrices, standard objects such as […] (our translation)

Bertrand: […] in fact, it is to train for reasoning skills ... and by the extrapolation to critical thinking [...] that is by working on the problems I have proposed like […] (our translation)

It is this discrepancy between teachers' perceptions of discrete mathematics and their corresponding teaching practices at university level that we intend to further explore in our study.

Researchers' conceptions on proofs in discrete mathematics

In order to characterize proofs in discrete mathematics in general, we made an attempt at identifying the types of proofs used in discrete mathematics. We find that the proofs by contradiction, by induction, and by recurrence are the most used. The exhaustive proofs are also used frequently, and according to the interviewees, this is due to the fact that oftentimes problems require very complex strategies, which compels the students to perform case-by-case analysis. Apparently, this exploratory phase of problems is a remarkable requisite of topics in discrete mathematics more than in other branches of mathematics.

Moreover, heuristic processes show in the students' development of methods, to find approximate solutions instead of exact solutions to problems. According to Bertrand, it is widely used in the experimentations for proof and in the mathematical investigation processes. For Bertrand, in discrete mathematics, heuristics consist of taking particular cases (like combinatorial optimization problems), extirpating to arrive at a clear solution (questions of tiling and stacking), and modeling illustrations especially in difficult problems.

We have also noticed that the "proof" activity in mathematics has a different status than the "demonstration" activity. It is affirmed by the interviewees that there is a difficulty for students in writing proofs:

Michel: […] they feel at ease, they understand everything that is explained but they feel unable to reproduce […] (our translation)

Bertrand: […] we think it's clear and that we are convinced; however when asked to write, to formalize, we do stupid things …] (our translation)

Therefore, we notice that the place of proof in discrete mathematics is not well defined and needs more investigation especially when it comes to its characteristics and the distinctions between the terms proof, demonstration writing, argumentation, etc.

CONCLUDING REMARKS

Currently, we limit our work to researchers of discrete mathematics particularly graph theory. For the rest of our work, we plan to complete the analyses of interviews with the researchers to further develop the state of the art (to further explore proof, modeling and their particularities in relation to discrete mathematics). At the conference, we will present some more refined results of these interviews along with to the questionnaire used. This mapping along with the review of literature will allow us to better develop the criteria that would ultimately lead to a functional definition of discrete mathematics. We are also interested in exploring the teaching practices of researchers in order to make informed suggestion on the training of instructors at the university. An extension to this work might possibly be in interviewing researchers in contiguous disciplines like computer science, algebra, or number theory.

[1] We aim at constructing a representation of discrete mathematics that presents the concepts, types of problems, proof processes and strategies, reasoning skills and other particularities of the field of discrete mathematics.

[2] The adjective “contemporary” indicates that our research focuses on the researchers’ practices in *statu nascendi*. We have conducted interviews with mathematicians to this end.

[3] In this paper, we use the word “conception” in the common sense, not yet in any specific theoretical sense.

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Analysing regressive reasoning at university level

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ABSTRACT:

This paper focuses on the epistemic and cognitive characterization of regressive reasoning in resolving strategic games. It explores the use of the Finer Logic of Inquiry Model as a tool for the analysis of the regressive reasoning. It reports the results of a study carried out on 32 undergraduate students who are studying a Mathematics Degree in a university of Spain.

Key words: *Teaching and learning of specific topics at university level mathematics, Teaching and learning of logic, reasoning and proof, Regressive reasoning, Logic of Inquiry, Strategy games.*

1. INTRODUCTION

The method of analysis has proved to be extremely stimulating in various fields, and has played a crucial role in the emergence of the modern world-view. The combination of the two branches of analysis and synthesis has been applied to several fields of artificial intelligence, theoretical computer science, and in programming methodology (Peckhaus, 2000; Grosholz, Breger, 2000). For many engineering students and mathematics undergraduate students, learning the method of analysis in tertiary education mathematics is a critical issue. They have the challenge of incorporating it in different disciplines related to the design and production of products and services, such as, Project Management, Systems Engineering and Design Science. They have no theoretical and methodical basis (Koskela and Kagioglou, 2006). A conscious integration of regressive reasoning in mathematics university learning raises the need for articulation between epistemological and cognitive aspects. Regressive reasoning is not completely logically determined, but has elements of contingency, creativity and intuition. The purpose of this text is to highlight the potential of Finer Logic of Inquiry Model (Arzarello 2014) as a tool for the didactical analysis of the regressive reasoning. This model has been used at secondary level education, not being used at tertiary level so far.

Here we will report the results of a study carried out on 32 undergraduate students studying a Mathematics Degree at a Spanish University, using strategy games in order to promote the regressive reasoning. The choice of strategy games is justified by antecedents to this study in which they have been shown to be a key tool for teaching problem solving and regressive reasoning (Gómez-Chacón, 1992).

The present research is primarily exploratory for two reasons: 1) Regressive reasoning has been scantly analysed in mathematics and educational psychology; 2) the use of the Finer Logic of Inquiry Model methodology to analyse data from mathematical thought at tertiary education is a new development. The theoretical background and empirical studies related to regressive reasoning needs to be developed.

2. REGRESSIVE REASONING

In mathematics, progressive reasoning alone is not exhaustive to fulfil the tasks of solving problems. Great mathematicians like Pappus, Descartes, Leibniz, in their discussions about analysis and synthesis, emphasize this fact (Peckhaus, 2000). Regressive reasoning is known by different denominations: regressive analysis, backward solution, method of analysis, etc. This process includes different ways of proceeding in problem solving: backward strategy, strategy of assuming the problem solved, Reductio ad Absurdum, beginning at the end of the problem, etc.

Pappus was the mathematician who has contributed substantially to the clarification and exemplification of the method. In the seventh book of his Collection he deals with the topic of Heuristics (methods to solve the problems).Where he exemplifies the method of analysis as the method of synthesis, therefore making the development of this reasoning clearer. Pappus defines the method of analysis as follows: “In analysis, we start from what is required, we take it for granted; and we draw correspondence (ακολουθον) from it and correspondence from the correspondence, till we reach a point that we can use as a starting point in synthesis. That is to say, in analysis we assume what is sought as already found (what we have to prove as true).” (elaboration by Polya, 1965 and by Hintikka and Remes, 1974). Subsequently he points out: “This procedure we call analysis, or solution backward, or regressive reasoning.” (Hintikka and Remes, 1974) And on the Method of Synthesis: “In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. This procedure we call synthesis, or constructive solution, or progressive reasoning.”(Hintikka and Remes, 1974)

In summary, the following was considered backward reasoning: the practice that involves the making of a number of arguments from the bottom of the problem and proceeds through logical correspondences which allow to obtain something known or to be reached through other paths. The analytical method consists of a procedure that starts with the formulation of the problem and ends with the determination of the conditions for its solution.

3. FINER LOGIC OF INQUIRY MODEL (FLIM)

Trying to overcome the static approach of habitual logical mathematical reasoning, Hintikka (1996, 1999) developed what he calls *Logic of Inquiry*. The idea, already elaborated by ancient Greek philosophers, is building knowledge through a questioning process, implicit or explicit. The knowledge is the result of research generated by a specific question. The philosopher introduces it as the “logic of question and answer”.

In his approach he considers Game Theory and game semantics to support formal epistemic logic. Hintikka overcomes the limitations and excessive abstractions of Tarski's Definitions of Truth (Sher, 1999), which leave the process used to reach the truth unexplained. He introduces a top-down definition of truth (Hintikka, 1995) unlike the classical and tarskian bottom-up view, highlighting the regressive way of proceeding in problem solving from an epistemological point of view. Hintikka (1995) retakes the idea of Wittgenstein's language games and some aspects of Game Theory, elaborating on a theory where the centre is “a path towards the formulation of a truth that, instead of proceeding recursively from atomic to complex formulas, reverses the approach and proceeds from the more complex ones to their simplest constituents”. In this research, the study of games will try to explain this interlacing between game theory and strategic rules that allows the student to win.

The FLIM elaborated on by Arzarello (2014) sought to propose a concretion of Hintikka`s proposal to be used in the Didactics of Mathematics. More specifically, he explained the elements needed to analyse the interactions between strategic and deductive components of students’ resolution protocols. This model allows for the structuring of the resolution in two components: Inquiry Component (IC) and Deductive Component (DC).

In the Inquiry Component the subject alternates a series of questions, answers and explorations, according to Hintikka's *Logic of Inquiry*. Its purpose is to meet the aim of the problem, solving conjectures that gradually rise from results of two explorations:

- *Exploration:* in order to analyse and understand the situation in which the subject is involved
- *Control:* in order to verify the ideas or conjectures that came out during the development of the activity.

In the above Component, the cognitive dimension of reasoning is necessary. From a cognitive point of view, the progressive-regressive reasoning movement has been highlighted by studies such as those of Saada-Robert (1989). The psychological model for solving mathematical problems focuses on the distinction between two phases of the resolution: investigate why things are like this (backwards, until reach a plausible hypothesis -abduction- or a known fact) and verify this investigation (forward, codified by the classical logic). Based on Saada-Robert's model, Arzarello (2014) and Soldano (2017) characterized this cognitive dimension through the sequence of actions in three different modalities: ascending, neutral and descending.

Ascending modality (A) refers to the path towards the formation of ideas and conjectures after a phase of exploration. Descending modality (D) characterizes the transition from a conjecture to an investigation. The purpose of descending modality is to find an equivalence between the object of thought (the conjecture, the idea) and the object of work (the problem and its resolution). Neutral modality (N) marks the change between the ascendant and the descendent; it is the moment in which a conjecture is formulated. Observable actions in the subjects are: formulations (of questions, of resolutions plan, of conjectures), affirmations, explorations and controls.

In the Deductive Component the subject is not directly involved in the investigation and verification of conjectures and uses a language with a logical nature to formally formulate the truth. Three specific modalities are added: detached modality, logical control and deductive modality (Arzarello, 2014; Soldano, 2017). Detached modality is the moment in which a conjecture, which has not arisen immediately after an exploration, is formulated. Logical control is the time when an exploration-control is done without using instruments. It is characterised by the use of formal language. Deductive modality characterises control phases where instruments are involved. Deductive Steps and Logical Chains are added to the Inquiry Component actions.

Inquiry and Deductive components are not often well differentiated during problem resolution where the subject passes from one component to another, even more than once. We can say that the typical components structure is nested in this way: (IC ~ (DC ~ (IC ...))) with "~" that expresses the passage from one component to the other.

Observable actions		**Modalities**
General	*Specific*	
Verbal Handwritten Gestures Others (gaze, …) Silent	Question Affirmation Conjecture Exploration Control Plan formulation Deductive step Logical chain	Ascendant Neutral Descendant Detached Logical Control Deductive

Table 1

Table 1 summarizes some observable actions and their modalities according to the definitions given and that will be considered in the analysis.

4. AIM AND METHODOLOGY

Aim

The aim of this paper is to show an evaluation tool for examining how regressive reasoning develops in university students. In particular, how the FLIM can be a valid tool to analyse the interplay between cognitive and epistemic in the regressive reasoning.

Participants and instrument

Data were collected in 2014 from 32 (19 women and 13 men, aged between 21 and 23) Caucasian undergraduates working toward a BSc. in mathematics. All of the participants were in their last year of academic studies. They were following advanced courses in several areas of geometry, algebra, probability and analysis. With regard to solving problems, the students had been introduced to the problem solving heuristics. They had not received any special training about backtracking heuristics.

The work dynamic started with individuals being given paper and pencil with which they need to resolve two games, each lasting one and a half hours. Figure 1 shows the problem which we will analyse in the results section. Strategy games allow for the natural development of regressive reasoning. These games are disconnected from the mathematical content which forces the student to use their mathematical knowledge acquired in their university degree.

The Triangular Solitaire (Gómez-Chacón, 1992) is a game for a single person that requires a board with 15 boxes as the figure shows.

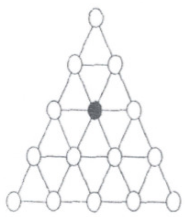	*These are the rules:* *1. Place the pegs in all boxes, except in the one marked in black.* *2. The player can move as many pegs as they like as long as they are able to jump over an adjacent peg and onto an empty space (along the line). At the same time, he "eats" the peg that was jumped over and that peg gets taken out of the game. All pegs move in the same way. Pegs can move around the table in any direction.* *Objective: The player wins when there is only one peg on the table.*

Figure 1

Students were given the game and asked to describe their approaches to solving the problem on protocols including: thought processes in the resolution, explanations of the difficulties they might face, and strategies they would use in order to solve with paper and pencil. A qualitative analysis was chosen to examine the resolution protocols of the students through the “Finer Logic of Inquiry Model” (Arzarello 2014). A general analysis of 32 students took place before a case study was carried out. In this paper we describe an individual student case in order to show a deep understanding of the tendencies of the behaviour related to the sequences of actions and movement between modalities of reasoning. The protocols analysis, at a macroscopic level of this case, provides the identification of reasoning difficulties and way of using backward reasoning that determined success or failure in the resolution. It’s worth noting that Student M (see section 5) is a key informant of the group because he belongs to 60% of students that use the backward strategy and incorporates graphical representations to achieve the transition between modalities.

5. REGRESSIVE REASONING USE (CASE STUDY)

Regressive reasoning use varies among the group of students. Let us examine a case study. A student (Student-M) has combined regressive reasoning with different strategies and auxiliary constructions: drawings, graphical representations. Student-M indicates difficulties in creating the solution because of the actions which are needed for discovering the solution and because of the recognition of representational equivalences. The visualisation and representations which are used help during the resolution process; Student-M performs continuous control over its own resolution process. She is able to slightly modify the strategy or even change it completely to reach the solution. For analysis purposes, Student-M's protocol has been divided into the following phases: familiarisation, exploring and carrying out the strategy, results verification. According to the Finer Logic of Inquiry Model, this student's protocol is mainly characterised by the inquiry component. This begins with the first part of the protocol, corresponding to the *familiarization phase*. The entire protocol has been translated highlighting the parts where student M uses backward reasoning (in *Italics).*

Student M protocol

1 To accomplish the exercise, I'm going to number the holes on the board in order to leave a trace of the movements I'm doing. At the beginning, all the holes are filled except number 5.

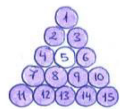

Figure 3

2 I observe that you can only start with two movements 14-9-5 or 12-8-5.

3 Since this is an equilateral triangle, I think it does not matter what the starting movement is because they should lead to "symmetrical" solutions.

4 I'll start to do it roughly.

5 The steps I'll take are: 14-9-5; 7-8-9; 12-13-14; 2-4-7; 11-7-4; 10-9-8; 3-6-10.

6 *At this point, I note that the only way to eliminate 1 would be to move 8-5-3*.

7 *Here I notice that [with these movements] the game cannot be solved because the 4 cannot be eliminated and the remaining pegs cannot eliminate each other.*

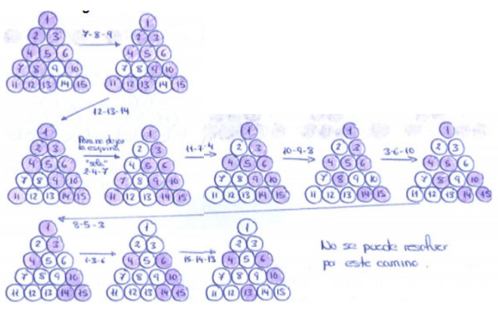

Figure 4

8 I realise that I can try to go backwards, that is, starting with just one peg in one position and undo the jumps trying to fill the table with the exception of a hole.

9 *Looking at the board, I think that maybe the fact that the last piece stays on the board (the peg from which I start to move backwards), in a position that you can come up with many jumps, facilitates the strategy.* These places are positions 4, 6 and 13 because you can get to them with 4 jumps.

10 To fill up the game table I will have to do 13 moves, because there are 15 holes, an initial peg and an empty final hole.

11 Let's start only with peg 13.

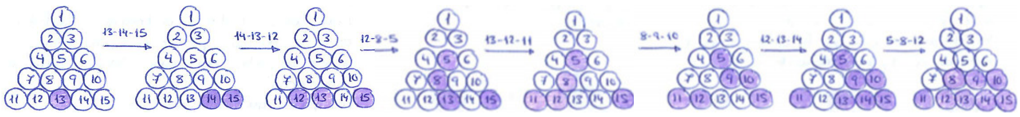

Figure 5

12 Here I already notice that *I do not reach the solution because I will never fill the top corner due to the absence of a peg in the 3rd row*; I should do 11-7-4 leaving corner 11 without a peg [so that the top corner will be filled].

13 Let's start with the reason for the various steps:

13-14-15: *I want to start filling the corners as soon as possible because these holes are the hardest to fill up* (the peg is in hole 15 and I will not move it anymore).

14-13-12: Random movement.

12-8-5: *I want to leave hole 12 free to get to the next step at corner 11.*

8-9-10: I want to leave hole 8 free to retrieve peg 12 (to fill 13 and 14) in the next step, so I can complete it later [the row].

12-13-14: I want to complete the row below.

5-8-12: I want to complete the row below.

14 I think trying to fill the centre was not a good strategy...

15 … so now I'm going to try to fill the outside of the triangle, that is, [I'll try to] undo the jumps to the corners and sides. (Playing normally would involve jumping to the centre avoiding corners and sides if possible.).

16 I also get stuck [on the fact] that by eating pegs or undoing the jumps, the movements that are made are triangular. So I will try to fill the smaller triangles contained in the big triangle.

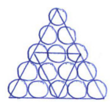

Figure 6

17 First, I will fill the lower right triangle.

18 *Now I'm going to fill the upper triangle; to do so (Since i do not want to remove the peg I placed in position 1), I have to get some pegs in the 4th row that, undoing the jump fills the 2nd and 3rd row. I undo the jump with the 9.*

19 *Now you have to fill the lower left triangle.*

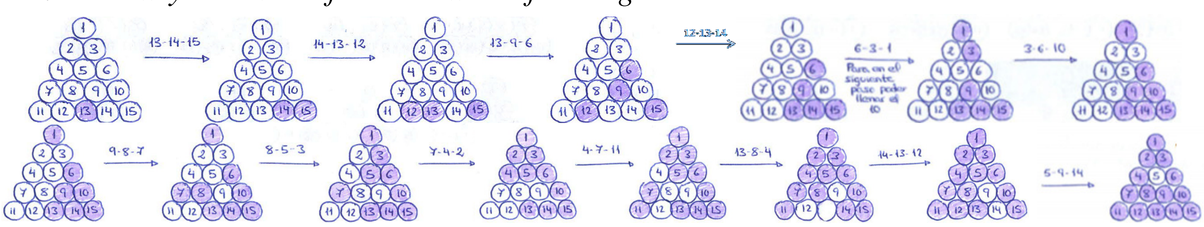

Figure 7

20 Now I just have to write the jumps in the correct order

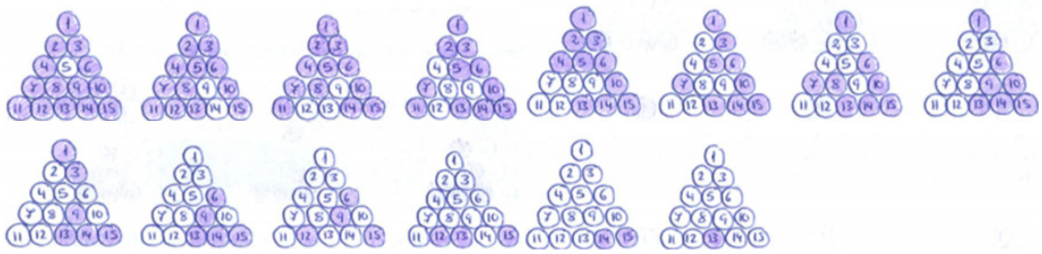

Figure 8

The following table shows actions and cognitive modalities associated with each protocol line and figure; a check (✓) indicates the lines where the regressive reasoning is used. The last column of the table shows different strategies involved.

Familiarization phase				
Protocol parts	***Action***	***Modality***	***R.R***	***Strategy***
Lines 1-4 y Fig. 3	Exploration	Descendant		
Line 5	Affirmation	Neutral		
L. 6-7 and Fig. 4	Exploration	Ascendant	✓	
Explore and carry out the strategy				
Line 8	Plan	Neutral		Backward
Line 9	Exploration	Ascendant	✓	Begin from the end
Line 10	Affirmation	Descendant		
Line 11 y Fig. 5	Exploration	Descendant		
Line 12	Affirmation	Ascendant	✓	
Line 13	Exploration	Ascend/Descen	✓	
Line 14	Affirmation	Ascendant		
L. 15-16 and Fig. 6	Plan	Neutral		Auxiliary construction
Lines 17 y Fig. 7	Exploration	Descendant		
Line 18	Exploration	Ascendant	✓	
Line 19 y Fig. 7	Exploration	Ascendant	✓	
Results verification				
Line 20 y Fig. 8	Control	Detached		

Table 2

This cognitive analysis shows that the first two resolution phases are characterised by a continuous alternation of explorations and plan formulations together with an alternation of descending and ascending modalities. The second resolution phase involves the continuous use of the going backward strategy. Subdivision of the board into rows and then into triangles is fundamental to reach the solution. Student-M modifies the strategy slightly by adding new elements in the resolution (board subdivision into rows and triangles) typical of problem solving using regressive reasoning. Crucial points of backward reasoning are reached in the ascending modality (see ✓ in Table 2) where ideations occur. A routine that can be established regarding the use of modalities is A~N~D~(A~N~D~(A~…)). The neutral modality marks the transition between A and D and it is characterised by the incorporation of auxiliary constructions as generating tools of new knowledge (epistemic transaction).

In the third phase of the resolution, by writing and graphically representing the steps taken to reach the solution, Student-M (in detached modality) checks the result obtained by going backwards.

6. CONCLUSION

Analysis with the FLIM model allows to model student's cognitive movement in a logical concatenated way. The strategic aspects are more dominant in the ascending and descending modality, while the epistemic ones are prevailing in the neutral modality. Our study confirms results obtained by Soldano (2017) (with upper secondary school students in geometry): the ascending modality characterises the backward way of thinking, while descending is the cognitive modality that characterises the progressive way of reasoning. However, most likely, abductive reasoning has been used in the formulation of conjectures in ascending modality, but we cannot be sure of it by only analysing the protocol, we need to complete this information by interview. This is an open question for further research.

At a phenomenological level, this method allows us to analyse the development of strategic aspects within the cognitive modality movement to reach the solution. But it mainly focuses on cognitive modalities while it doesn't distinguish between the strategic principles that are used. Through this tool it's possible to emphasise that regressive reasoning involves auxiliary intuition elements that are necessary to achieve the solution; these aspects are developed by looking at the consequence and looking for the premises. A larger sample size with two different tasks, find the winning strategy and mathematically solving the game, would allow us to advance in the development of the tools for evaluating regressive reasoning.

7. ACKNOWLEDGMENT

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A Study of Students' Reasoning About "There exists no …"

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In this paper, we report findings from two studies of students' engagement in metatheoretical tasks drawn from a model of the reasoning requirements of a proof by contradiction. The studies aimed to explore students' engagement in the tasks, the extent to which they were successful, and the similarities and/or differences between students' and mathematicians' approaches. Findings indicate students tend towards syntactic, logical theory approaches while mathematicians gravitate towards semantic, mathematical theory approaches. Drawing on interview data, it is shown that students may use symbols to avoid employing fragile content knowledge, yet encounter further difficulties by viewing quantifiers as appended symbols.

Keywords: metatheoretical reasoning, proof by contradiction.

METATHEORETICAL DIFFICULTIES: AN OVERVIEW

Mariotti, Bartolini Bussi, Boero, Ferri & Garuti (1997) argue, "what characterises a mathematical theorem is the system of statement, proof and theory" (p. 183). By this they mean a statement and its proof are situated within a theory from which one draws not only axioms, definitions, and theorems (i.e., a mathematical theory) but also rules of inference (i.e., a logical theory). The fact that proofs are contingent on both a mathematical and a logical theory is best illustrated by the fact that there are statements that are valid in some mathematical theories (e.g., Euclidean geometry) that are not valid in others (e.g., Hyperbolic geometry). Thus, their validity depends on the mathematical theory referenced. Moreover, if one does not employ standard logic, further shifts in the status of theorems may occur; a point illustrated by Brouwer's rejection of his Fixed-Point Theorem, after adopting intuitionistic logic.

Building on this model of mathematical theorems and the theory of Cognitive Unity, Antonini and Mariotti (2008) demonstrated that students' difficulties with indirect proofs may occur within the mathematical theory or the logical theory. To demonstrate the latter, a compelling example is given where a university student, Fabio, describes difficulties accepting a proof by contraposition because of the movement back and forth between the statement-to-prove and the contrapositive.

> Fabio: Yes, there are two gaps, an initial gap and a final gap. Neither does the initial gap is comfortable: why do I have to start from something that is not? [...] However, the final gap is the worst, [...] it is a logical gap, an act of faith that I must do, a sacrifice (Antonini and Mariotti, 2008, p. 407, *sic*).

Indeed, Fabio speaks both of his acceptance of the proof of the contrapositive and his difficulties accepting the contrapositive proof as a proof of the statement-to-prove. Antonini and Mariotti refer to these difficulties as *metatheoretical*, for they are at the level of the logical theory that is applied to the mathematical theory. Their work is of interest, for it raises many questions: To what extent are novices successful when engaging in metatheoretical tasks? What approaches do they employ? Do their approaches differ from mathematicians'? The purpose of this paper is to take a preliminary step towards answering these questions. Specifically, we report on two studies of participants' responses to metatheoretical tasks drawn from a model of the reasoning requirements of a proof by contradiction, which is described below.

PROOF BY CONTRADICTION AND ITS REASONING REQUIREMENTS

In this section, our aim is to model the reasoning requirements of a proof by contradiction of a universally quantified conditional statement. To aid our discussion, we consider a specific example: *Theorem 5. For all positive integers n, if n mod(3) ≡ 2 then n is not a perfect square*. To prove the theorem by contradiction, one must correctly negate the universally quantified conditional statement and take the resulting statement as one's primary assumption. Such actions require one accept (at least at an intuitive level) that for a conditional statement to be true universally, it must not be the case that there is some element in the universe of discourse for which the premise is true (has a truth-value of *true*) and the conclusion is false (has a truth-value of *false*). In the case of our example, we assume "*There exists a positive integer n, such that n mod(3) ≡ 2 and n is a perfect square*." As shown by Wu Yu, Lee, & Lin (2003), this task is far from trivia for students 17-20 years of age. Moreover, as Antonini and Mariotti (2008) note, the validity of the work is determined by theorems that reside within the logical theory (i.e., the metatheory).

Having assumed the negation of the statement-to-prove, one must now explore the consequences of this assumption and identify a contradiction. Three aspects of this work are important. First, to carry out this work one must move back to the mathematical theory, for it is here that the contradiction will reside. Second, one's goal is open-ended, for one does not know in advance where one will find the contradiction. In fact, there may be many. Third, one must know one's commitments with regard to the mathematical theory. Otherwise, one will not have the means to recognize a contradiction. This point was made by Sierpinska (2007) who argued, "sensitivity to contradictions in mathematics requires theoretical thinking … (thinking) concerned with internal coherence of conceptual systems" (p. 1-54). Once the contradiction is identified, one's work is not done. One must make sense of it.

In our example, we claimed that an integer existed but having produced a contradiction we now know that such a number cannot exist. Hence, one must conclude,[i] *there exists no integer n, such that n mod(3) ≡ 2 and n is a perfect square*. And it is at this point that one is faced with the very requirement that Antonini and

Mariotti's (2008) Fabio rejected; namely, seeing the proof of this statement as a proof of the statement-to-prove. In other words, having shown S*: *There exists no integer n, such that n mod(3) ≡ 2 and n is a perfect square*, one must recognize (from a logical standpoint) that one has proven S: *For all positive integers n, if n mod(3) ≡ 2 then n is not a perfect square*; that is, we must recognize S* implies S since S* is a non-identical but logically equivalent form of S. As Antonini and Mariotti (2008) note, this work relies on theorems in the logical theory rather than the mathematical theory; that is, it is metatheoretical. Thus, a proof by contradiction imposes two unique metatheoretical requirements. First, at the beginning, when one must produce the negation of a statement. Second, at the conclusion, when one must recognize that S* implies S. And, it is the latter requirement that is the focus of the reported studies.

AN OVERVIEW OF THE STUDIES

The reported studies examined students' engagement and extent of success in the metatheoretical reasoning requirements that arise at the conclusion of a proof by contradiction. All studies were conducted at a minority-serving university, where the majority of students qualify for need-based financial assistance and are first-generation university students. Study 1 explored the extent to which *novices* (*i.e.*, students without prior logical training or who have limited training) are successful evaluating claims of the form S*implies S. Study 2 explored students' and mathematicians' approaches to and success with metatheoretical tasks. The aim of the combined studies was to explore the reasoning practices that may inhibit or support students' metatheoretical work and consequently, play a role in the extent to which students reach or fail to achieve cognitive unity in relation to indirect proofs.

Study 1 Methods and Findings

To explore novices' success with metatheoretical reasoning tasks prior to instruction, 46 university students were surveyed. The surveys were administered on the first day of a "Basic Set Theory and Logic" course that served as the universities' first logic course and their "Introduction to Proof" course. Prior to the course, students would have been enrolled in computation-focused courses on calculus and differential equations. Included on the survey were two tasks that asked students to compare a pair of statements and determine, "Can you prove *Statement A* by proving *Statement B*?" (Figure 1.) Task 1 involved a universally quantified statement and an incorrect alternative. Task 2 involved the same Statement A and a correct alternative.

Task 1. *Statement A.* For every integer *n*, if *n* is a perfect square, then *n* has an even number of factors. *Statement B.* There exists no integer *n* such that *n* has an even number of factors and *n* is not a perfect square. ☐ Yes, you can prove Statement A by proving Statement B. ☐ No, you cannot prove Statement A by proving Statement B.
Task 2. *Statement A.* For every integer *n*, if *n* is a perfect square, then *n* has an even number of factors. *Statement B.* There exists no integer *n* such that *n* is a perfect square and *n* has an odd number of factors. ☐ Yes, you can prove Statement A by proving Statement B.

☐ No, you cannot prove Statement A by proving Statement B.

Figure 1. Study 1's Task 1 and Task 2

Survey results indicated that of the 46 students surveyed, 50% were successful at Task 1, 47.8% were successful at Task 2, and 24% were successful at both tasks.

Study 1 Discussion

The findings of Study 1 demonstrate that most of the students did not enter the *Basic Set Theory and Logic* course reasoning in ways aligned with the metatheoretical requirements of indirect proofs, as the rates were at or below guessing and less than a quarter successfully answered both questions. While the findings are not startling, they provide a warrant for further research. Indeed, prior to Study 1 there were no studies of novices' responses to such tasks prior to instruction. Thus, the findings warrant the following questions: Do novices' difficulties persist after instruction? Do students' and mathematicians' approaches differ?

Study 2 Methods

Study 2 aimed to explore university students' and mathematicians' extent of success and approaches to the metatheoretical task in Figure 2. Participants were 21 students drawn from the same student population as Study 1 and 6 mathematicians. However, the Study 2 students had completed the *Basic Set Theory and Logic* course. As the course focused on set theory and logic in the service of proof writing, the instruction on set theory and logic was limited to basic properties, terms, and definitions, as well as symbolizing practices, and then on specific proof techniques and/or strategies. All participants took part in video-recorded interviews during which the task was presented on a large piece of paper. The participants were given as much time as requested and then asked to explain their answer to the stated question.

Question: Can you prove Theorem 5 by proving Statement A?
Theorem 5. For all positive integers n, if n mod(3) ≡ 2 then n is not a perfect square.
Statement A. There exists no positive integer n such that n mod(3) ≡ 2 and n is a perfect square.

Figure 2. Study 2, Interview Task

To identify approaches the analysis focused on which theory (mathematical or logical) the participant worked in and how they engaged in that theory. Responses were considered *mathematical theory approaches* (MTA) if the participant was observed: (1) explicitly exploring mathematical statements, definitions and/or terms; and/or (2) constructing a proof of either statement. Responses were considered *logical theory approaches* (LTA) if the participant was observed: (1) posing explicit questions of equivalence; (2) constructing truth-tables and/or working with symbolic logic; and/or (3) citing logical theorems or practices. In addition to the approach, participants' responses were analysed for the form of engagement. Specifically, coding noted participants' use of syntactic and/or semantic reasoning, with *semantic* referring to reasoning that employs meanings and multiple representational systems and *syntactic* referring to rule-based reasoning within a representational system.[ii]

Study 2 Findings

In Table 2, we report the percentage of correct responses. The reader will notice that among the 21 students five types of responses were observed: *yes*, *yes-no-yes*, *no-yes-no*, *no*, and *don't know*. *Yes* refers to students who, after a period of exploration, decided without hesitation that one can prove Theorem 5 by proving Statement A. *Yes-no-yes* refers to students who repeatedly switched answers, expressed hesitation and doubt, but ultimately choose "yes." *No-yes-no* were similar to yes-no-yes but were students who repeatedly switched answers and ultimately choose *no*. *No* refers to students who reached, with evident certainty, the decision you cannot prove Theorem 5 by proving Statement A. *Uncertain* refers to students who, after deliberation, responded to the prompt by remarking they "didn't know."

Prompt: *Can you prove Theorem 5 by proving Statement A?*		
Student Responses	**N**	**%**
Yes	6	28.6%
Yes-no-yes	5	23.8%
No-yes-no	4	19.0 %
No	5	23.8%
Uncertain (Don't Know)	1	4.8%
Mathematician Responses		
Yes	6	100%

Table 2: Student and Faculty Response by Category

As seen in Table 2, less than one-third of the students (28.6%) who had completed the *Basic Set Theory and Logic* course stated with certainty, *yes* one could prove Theorem 5 by proving Statement A. And, nearly as many (23.8%) reached this conclusion with significant hesitation (*Yes-no-yes*). Furthermore, 42.8% argued either with certainty (*No*) or with hesitation (*No-yes-no*) that you *cannot* prove Theorem 5 by proving Statement A. These findings indicate instruction had little impact on the students' success with the metatheoretical requirements of a proof by contradiction. In contrast, (without surprise) all of the mathematicians replied *yes*.

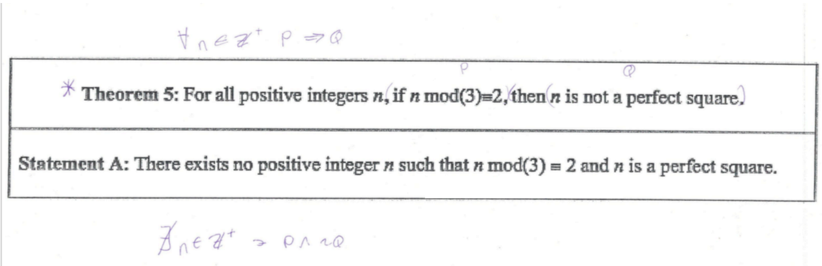
Since findings that indicate the prevalence of difficulties are of little use without information on the nature of students' engagement, we turn to the analysis of participants' approaches. This analysis focused on the question of which theories the participant engaged with and their form of engagement (see Table 3).

	Student Response By Type and Form				
(*n* = 21)	**Mathematical Theory Approach (MTA)**		**Logical Theory Approach (LTA)**		**NE***
Response Type	**Semantic**	**Syntactic**	**Semantic**	**Syntactic**	
Yes		1		3	1
Yes-no-yes			3	3	
No-yes-no				4	
No	1		1	3	
Don't know			1		

Table 3: Student and Faculty Response by Type and Form *(NE is no evidence)

Looking at Table 3, the reader will notice that the majority of the students (18 of 21; 85.7%) engaged in a logical theory approach (LTA). For many this work occurred symbolically, with 15 of the 18 (LTA) students replacing the open sentences with the symbols (e.g., P, Q, ~P or ~Q or P(*n*), Q(*n*), etc.) and the phrases *for all* and *there exists no* with ∀ and ∄, respectively.[iii] Indeed, except for one LTA-semantic (Don't know) and two LTA-syntactic (No-yes-no), the students worked symbolically. When asked about the use of symbols many students noted their discomfort with the content, "*mod is really rough in my memory right now*", and that "*it's easier to work with symbols*." Thus, the symbolic approaches enabled the students to avoid content for which they lacked confidence in their mathematical understandings.

For 3 of the LTA students their symbolic approach led to a quick and definitive *yes*, as shown Figure 3. The reader will notice the student initially focuses on the relationship between the quantifiers and then on how translating from ∄ to ∀ requires one to act on the open sentences by negating a sentence of the form (P∧ ~Q).

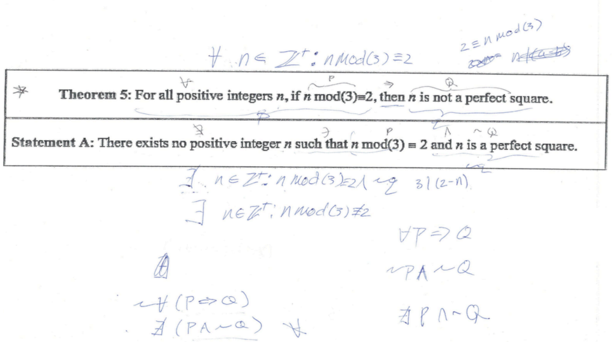


Student A: Yes, you can prove Theorem 5 by proving Statement A … because when you say there exists no that implies ... well that's a for all statement and then you have to negate the umm … the umm … (writes P∧ ~Q).

Figure 3: Student A's (Correct) Syntactic-LTA Response

In contrast to those who readily replied *yes*, nearly half of the LTA students experienced a significant amount of hesitancy and doubt (Yes-no-yes; No-yes-no). Many of these students articulated difficulties with the phrase "*there exists no*" while, at the same time expressing certainty regarding the logical relationship between *for all* and *there exists* (i.e., they asserted the negation of one quantifier produced the other). Among these students, it was not uncommon for them to argue that *there exists no* means *nothing* and that, "nothing is the opposite of everything," a point which left many confused having already noted *for all* and *there exist* were "opposites" in logic. For nearly a third of the students (6 total) recognizing *there exists no* as the opposite of *for all* and the open sentence "*n* mod(3) ≡ 2 and *n* is a perfect square" as the negation of "if *n* mod(3) ≡ 2 then *n* is not a perfect square" led to the conclusion Statement A is the negation of Theorem 5, as illustrated in Figure 4.

Two aspects of this approach are important to note. First, the student compares the quantifiers (*for all* and *none*) and then compares the open sentences. Hence, the quantifiers are not seen as variable-binding operators that act on open sentences but rather as appended symbols. Such reasoning enables the student to translate $\nexists$ into its "opposite" $\forall$ independently of translating $P\wedge \sim Q$ into its "opposite" $P\Rightarrow Q$. Second, such reasoning relies on the student incorrectly viewing $\nexists$ $(P\wedge \sim Q)$ as being of the form $(\sim\exists)$ $(P\wedge\sim Q)$ rather than as of the form $\sim[(\exists)$ $(P\wedge\sim Q)]$.



Student B: they're opposites […] this (Statement A) is the negation of Theorem 5 …it's saying for all of them, it's saying none of them [...] Yeah, (writes $\forall(P\Rightarrow Q)$) and (writes ~ symbol before $\forall(P\Rightarrow Q)$)) is (writes $\nexists$ $(P\wedge \sim Q)$).

Figure 4: Student B's (Incorrect) Syntactic-LTA Response

In addition to the LTA responses, two MTA responses were observed. In the MTA-semantic response, the student spent his time considering numbers that satisfy n(mod 3)≡2 and trying to understand the structure of a number that would disprove Theorem 5 or Statement A. Eventually, this student decided Statement A was false and, therefore, could not be used to prove Theorem 5. In the MTA-syntactic response, the student immediately remarked, "it's by contradiction." The student then proceeded to determine if Statement A provided the needed claims for such a proof:

Student C: by contradiction […] he's claiming that there is no positive integer n, … such that (points to Statement A's open sentences) […] so, he's saying there is no positive integer n here so you can use that argument (points to open sentences again) and … so, yeah, you can put those together and prove it.

As seen in Table 4, the mathematicians' responses were quite different, with all but one engaging in an MTA. Though not shown, it is important to note that in three of the five MTA-semantic responses, the mathematician spent the majority of the time proving (or considering how they would prove) Theorem 5.

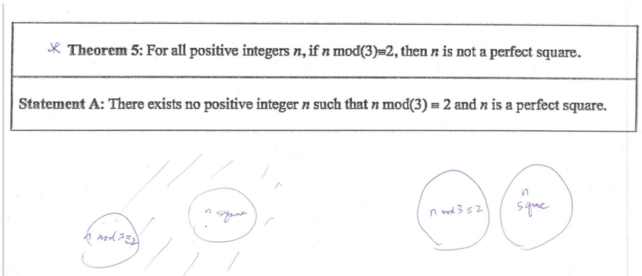
	Faculty Response Category			
(n = 6)	**Mathematical Theory Approach (MTA)**		**Logical Theory Approach (LTA)**	
Response Type	**Semantic**	**Syntactic**	**Semantic**	**Syntactic**
Yes	5		1	

Table 5: Faculty Response by Type and Form

This work lead all three to realized they would use a proof by contradiction and in so doing, prove Statement A to prove Theorem 5. In the other two MTA-semantic responses, the mathematicians repeatedly rephrased the statements, while explicitly noting the everyday meanings of the words, until they had convinced themselves that the statements were "essentially the same.' This work was often well-situated in the mathematical theory, as seen in the transcript below where the mathematician speaks of "turning around" Statement A and "running through" sets of numbers.

Mathematician α: I tend to take statements like that [Statement A] and try to rephrase them, so … for me, I would say, what does that actually say? It says that, umm, whenever, umm, a positive integer n is congruent to, umm, is congruent to 2 mod 3 then n cannot be a perfect square … like I … I try to turn it around … I'm sitting here almost hesitant about whether or not I've even done it correctly. But let me think … so, umm, let's see, so there exists no positive integer n such that these two things are true … so that's … what is that the same thing as saying, it's saying that, umm, if you ran over the positive integers n which were congruent to 2 modulo 3 you are never going to hit a perfect square but then that's what this is saying (point to Theorem 5), umm, if I think of for all positive integers n and this part is true, that n is congruent to two modulo three, then I am never going to hit a perfect square. So, … umm, actually, I think, umm, I would almost rephrase these things as being equivalent but I am feeling a little bit hesitant about that.

Here, it is important to note that in addition to Mathematician α, three other mathematicians expressed hesitancy with regard to their own reasoning; e.g., Dr. β remarked "*just doubting myself for some reason*." In each case, the mathematician was asked "Do you have some doubts about your answer?" and all responded "No." Thus, the participants appeared to be applying inferences with a high degree of (perhaps intuitive) certainty, while also doubting their own judgements of those inferences. Finally, two other observations are of note. First, like the students, one mathematician translated the statements into symbols. However, they immediately pushed the paper away saying, "I am not going to do that." Second, in the case of the LTA-semantic approach, the mathematician translated both statements into Venn diagrams (Figure 5) and then, by comparing the diagrams, reasoned through the task.



Dr. β: I'm going to draw some sets. … Statement A says to me the sets of *n* mod(3) … congruent to 2 and perfect squares … (long pause)… are disjoint. Right. There exists no positive integer … so this says for all positive integers if *n* mod(3) is 2 expressed then … this will be expressed as a containment and ….it's not a perfect square … perfect square is on the outside …….. and, umm, let's see, if-then means that ….(long pause) that

that set is inside that set or is it the other way … that implies that …(long pause). Yeah, it looks like it (laughs quietly).

Figure 4: A Mathematician's (Correct) Semantic-LTA Response

Study 2 Discussion

Study 2 aimed to explore students' and mathematicians' success with and approaches to a metatheoretical task. The data demonstrate that post-instruction, novices continued to struggle with the metatheoretical requirements of proof by contradiction and gravitated towards syntactic-LTA approaches, while the mathematicians tended towards semantic-MTA approaches. Furthermore, students' remarks indicated their use of syntactic-LTA approaches enabled them to avoid perceived content knowledge weaknesses, whereas the mathematicians drew heavily on this knowledge to produce proofs and explore concepts. The study also revealed a tendency among students who struggled with the tasks; namely, a tendency to view quantifiers as appended symbols rather than as variable-binding operators that *act on* open-sentences. Though far from providing definitive evidence, the study contributes to the literature by highlighting the logical complexities novices may encounter when producing or comprehending proofs by contradiction, given the approaches they gravitate towards.

CONCLUDING REMARKS

One question raised by the studies is, why didn't the students' reasoning progress, even after completing the *Basic Set Theory and Logic* course? Certainly, the lack of progress may be due to poor instruction, an insufficient curricular treatment, or the cognitive demand of the tasks. Turning to the curricular materials used, Chartrand, Polimeni, and Zhang's (2008) *Mathematical Proofs: A Transition to Advanced Mathematics*, one finds little in the ways of support for the metatheoretical tasks studied. This text includes an introductory chapter on logic with two subsections on quantifiers. In these subsections, the quantifiers *for all* and *there exists* are defined and discussed with regard to the variations of these phrases used in mathematics (e.g., for some; at least one, etc.). Neither are quantifiers discussed as variable-binding operators nor is the phrase "there exists no" or the symbol $\nexists$ mentioned. The same is true in a latter chapter focused on proof by contradiction, where emphasis is placed on moving from "for all" to "there exists" when proving by contradiction without mention of what one must do once one determines something "does not exist." Thus, the lack of progress may be tied to an insufficient curricular treatment of the topic.

Turning to the mathematicians' responses an alternative rationale for students' persistent difficulties becomes evident. As discussed, most of the mathematicians in Study 2 expressed a lack of confidence in their own reasoning, while none wished to change their answer due to an intuition (i.e., "gut feeling"). Hence, it seems reasonable to conclude that the task was cognitively demanding. Consequently, even with instruction, we might expect low success rates among undergraduates, who are at the early stages of the education and lack the content knowledge experts employed.

Lastly, in an interesting study of effective proof comprehension strategies, Weber (2015) found mathematicians preferred students, "rephrase theorems in their own words" and that students not use the strategy, "rewrite the theorem in first-order logic." These views reflect the practices of the mathematicians in Study 2, for none used the latter strategy, while nearly all used the former. However, their rephrasing of the statements relied on their extensive content knowledge; namely, as a tool for inferring meanings. Hence, the findings raise questions regarding whether or not the mathematicians would use these approaches if they were working with unfamiliar (or difficult) content. Indeed, it seems that we must be careful inferring instructional recommendations from the mathematicians' practices. Many appeared to generate inferences automatically – a practice that seemed to inhibit them from rationalizing their judgements; as illustrated by the mathematicians' repeated expressions of hesitancy.

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[i] An alternative conclusion is the mathematical theory is inconsistent. This progression, therefore, assumes a stance where inconsistencies are rejected. This issue is beyond the scope of the paper.

[ii] The proposed definitions of syntactic and semantic are heavily influence by those of Weber and Alcock (2009) but are not identical for they are not tied to the representational system of proof.

[iii] In American mathematics texts ~ symbolizes "not." The symbol ¬ may be more familiar to some.

Students' problems in the identification of subspaces in Linear Algebra

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The goal of the study presented in this paper is the investigation of students' problems with exercises concerning central topics of linear algebra courses at university level. We present the results of our analysis of students' work on an exercise about subspaces of $\mathbb{R}^2$. *We evaluated the written solutions of the task as well as transcripts based on videos taken of student groups working on the problem. We identified and classified descriptions of vector spaces and subspaces that varied widely and demonstrated highly different skills in working with geometric or formal algebraic objects. We analyzed how far students could progress in a complex reasoning process, and identified those steps in the reasoning process on which students needed support to continue.*

Keywords: Linear algebra, vector space, subspace, proof, tutorial groups.

PURPOSE AND BACKGROUND

Problems in teaching and learning of linear algebra have a long history in many countries (Dorier & Sierpinska, 2001). Frequently, the abstract character and the formalism of mathematics that students have not been exposed to in school before is named as a central obstacle (a variety of studies are outlined and evaluated in Dorier, Robert, Robinet & Rogalsiu, 2000). Since vector spaces are a central part and moreover of special importance for almost all disciplines related to mathematics at university, special attention has been paid to them (Dorier, 2000, Stewart, 2017). Generally, students do not develop a clear concept of vectors at school level (Mai, Feudel & Biehler, 2017), and the more abstract approach to this subject taught at universities is described as being "out of reach" by some students (Stewart, 2017). Wawro, Sweeney and Rabin (2011) investigated concept images of subspaces in interviews with students and identified recurring concept images, distinguishing between a subspace as a *part of a whole*, a *geometric object*, and an *algebraic object*. The introduction of first concepts in tutorial meetings in linear algebra, with a special focus on the behavior and influence by the tutor, has been studied by Grenier-Boley (2014).

CONTEXT AND DESIGN OF THE STUDY

In this study, we investigated the problems of students shortly after their first encounter with vector spaces and subspaces at university level. The participants of our study were students with major mathematics or computer science, enrolled for bachelor of science or bachelor of education (for secondary school, "Gymnasium"), most of them in their first semester. In our study, we collected data from students working on tasks in groups during their tutorial group meetings (1.5 hours), where the tutors were advised to answer questions but to only intervene when the students had substantial problems to continue. The students worked on exercises about the content of a recent lecture under the supervision of a tutor. In this context, we assigned special tasks that we developed ourselves together with the lecturer and his assistant, but we did not influence the course design otherwise. We will report only on one of them in this paper. The course can be considered to be typical for a beginners' lecture in linear algebra, which normally is rather abstract, and was given by an experienced lecturer. During their tutor meetings, the students worked on our exercises on separate sheets that we collected, scanned for later analysis and gave back to the students in the next meeting without any grading or corrections. We gathered between 78 and 130 written works on each exercise. Moreover, we also took video recordings of groups of 2–4 volunteering students working on these exercises. They worked on the exercise under the supervision of a student tutor who was part of the research team and familiar with our a priori analysis of the task. The experienced tutor was advised to help the students if they struggle with the exercise in the same way as she would do in an ordinary tutor group meeting. We were interested in identifying important didactic variables. The results obtained by analyzing the first implementation of the exercise about vector spaces are currently being used for designing a second implementation in the course Linear Algebra I.

The task for students in our study and preliminary research questions

In this paper, we concentrate on an exercise about subspaces and vector spaces (see figure 1) that was part of the exercise sheet during week 7 of the course, immediately after the notion of subspaces had been introduced. Students are taught analytical geometry and linear algebra at school level, where vectors are introduced as tuples (or classes of arrows), but they do not as a rule have a clear concept of a vector (cf. Mai, Feudel & Biehler, 2017). Students know equations of planes and lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, without considering them as subspaces, because this notion is not taught at school level.

The following exercise was designed in order to provide two different kinds of learning potentials (as described in Gravesen, Grønbæk and Winsløw, 2016):

1. Linkage potential: In part a) to e), our intension was to motivate the students to activate their school knowledge concerning the description of geometric objects using equations; we hoped that they would recognize the sets as descriptions of lines, points, parabolas etc., and connect this knowledge with the new concepts of vector spaces and subspaces.
2. Research potential: Part f) of the exercise was created in order to engage the students in a research-like activity. Even if achieving a complete solution seemed unrealistic for most of them, we were interested in how the students would approach this open question. They had to formulate a hypothesis and use abstraction to identify and construct subspaces. The exercise can be seen

as a “mini research project” that differs in type from standard exercises.

Exercise Which of the subsets of $\mathbb{R}^2$ given in part a) to e) are vector spaces with respect to the addition and scalar multiplication defined on $\mathbb{R}^2$?

a) $M_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 = 0\}$

b) $M_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 = 1\}$

c) $M_3 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -2, x_2 = -1\}$

d) $M_4 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2^2 = 0\}$

e) $M_5 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq 0\}$

f) Try to find all subspaces of $\mathbb{R}^2$. Make it clear to yourself that you indeed found *all* subspaces. A formal proof is not necessary.

Figure 1: Exercise on subspaces (translated from German)

The parts a) to e) can be solved by a formal check whether the properties of subspaces are satisfied by the provided sets. As this was learned in the previous lecture, this is a standard task. Geometric ideas are not necessary, but we hoped that students may do geometric interpretations of the sets to develop a geometric meaning of subspaces and non-subspaces of $\mathbb{R}^2$. Task f) is different, because this is the first time that this type of question is asked. Students may use the results from a) to e), that have provided examples and counterexamples of subspaces, to find the zero space, all lines through the origin, and the whole $\mathbb{R}^2$ as subspaces and give reasons why they are subspaces on some level. The challenging question is whether or why these are *all* subspaces of $\mathbb{R}^2$. Research questions concerning f) are: How many students identify the zero space and the entire $\mathbb{R}^2$ as subspaces? Are all lines through the origin identified as subspaces? Which arguments do students provide for considering a line through the origin as a subspace of $\mathbb{R}^2$? How do they reason, when exploring, whether there are more subspaces in $\mathbb{R}^2$ or whether they have already found all?

We were also interested in the sources of knowledge students used, such as their results on a) to e), parts of the lecture, or geometric interpretations related to their school knowledge, and concerning the videographed tutorial sessions, which kind of support by the tutor they can use in their reasoning process.

METHODOLOGY AND DATA COLLECTION

For the analysis of the written work of the students, we followed the method of Biehler, Kortemeyer and Schaper (2015), by comparing each solution with the so-called *student expert solution* (SES), which is a sample solution based on the idealized actual knowledge of the students at this point of the lecture. Moreover, the student expert solution contains additional meta-information about the solution, for example, several alternative opportunities for solutions and explicitly written-out learning objectives. In relation to Brousseau’s theory of didactic situations, this method can be seen as a special approach to the development of an *a priori analysis*. We evaluated the written work in a two-step procedure: In a first step, we categorized the solutions by correctness and collected peculiarities and mistakes. Based on this and the *SES*, we developed a detailed coding system for deeper analysis. The recorded videos have been transcribed in order to allow a detailed qualitative analysis.

A PRIORI ANALYSIS OF THE TASK

In the lecture, the definition of vector spaces was given in a typical traditional, abstract way. The zero space and the vector spaces K (trivial vector space over K) and $\mathbb{C}$ (the latter together with component-wise addition and multiplication) had been presented by the lecturer as first examples. Apart from this introduction, the students had only seen the following (relatively abstract) non-trivial examples for vector spaces in the lecture: (VS1): K^{n}, the "standard vector space", where K is any field, with n ∈ ℕ, including the definition of addition and the scalar multiplication (component-wise), (VS2): $K^{\mathbb{N}}$, the vector space of sequences over the field K, with the component-wise operations. Subsequently, subspaces of vector spaces had been defined to be subsets of vectors spaces that are vector spaces themselves with respect to the same operations. Following this, they had learned that a sufficient criterion for proving that a nonempty subset W of a vector space V over the field K is a subspace is to prove that ly $\forall\, v, w \in W \Longrightarrow v + w \in W$, and secondly $\forall\, v \in W, \forall\, a \in K \Longrightarrow av \in W$. As examples of subspaces, the trivial subspaces $\{0\}$ and V were nominated without proof. Moreover, for both vector spaces (VS1) and (VS2), there was an abstract example for a subspace given, and we state the first one of them here since it will be of use for our later analysis:

(S1) The set $L := \{(x_1, \ldots, x_n) \in K^n \mid \forall\, 1 \leq i \leq m\colon \sum_{j=1}^{n} a_{ij} x_j = 0\} \subseteq K^n\}$ is the solution space of a homogeneous linear equation system $\sum_{j=1}^{n} a_{ij} x_j = 0$. It was shown that this set is indeed closed with respect to addition and scalar multiplication and is a subspace. Note that this example can be applied to $\mathbb{R}^2$, if we choose $m = 1$. The subspaces in L are the lines through the origin expressed by linear equations. This interpretation could be done by students on the basis of school knowledge. The lecturer did not provide this specialization himself.

For our later analysis, the following distinction is central. All provided examples have in common that sets are characterized by equations (subspaces defined by *relations)*. In contrast, 1-dimensional subspaces could also be defined by *explicit construction:* for instance for any $x \in V$: $L_x \coloneqq \{v \in V \mid v = \lambda x, \lambda \in K\}$. The latter way of defining subspaces was not yet a topic of the lecture, which will turn out to be an obstacle for some students. Moreover, the students had not seen any geometric interpretation or visualizations of vector spaces or subspaces, in particular no (concrete) examples of subspaces in $\mathbb{R}^n$. In the following, we will give an overview about possible approaches and steps to part f).

Step 1: Find some subspaces. With the knowledge from the lecture, the trivial subspaces (the zero space and $\mathbb{R}^2$) can be named. To find nontrivial subspaces, one can identify again the set M_1 of the previous part a) of the exercise as a subspace. Starting with this set, one could generalize from numbers (like 2 and 1, as chosen in part a)) to a general form with coefficients, and give the set $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$, with $a, b \in \mathbb{R}$ not both being zero. Instead, if one abstracts from the mathematical language used in the exercise before, these sets could also be expressed constructively as $L_x \coloneqq \{v \in V | v = \lambda x, \lambda \in K\}$. Supported by Dorier, Robert, Robinet and Rogalsiu (2000), we expected difficulties to translate the relational representation into the constructive representation and vice versa. Alternatively, with the knowledge from the lecture, one could apply the example (S1) given in the lecture to the space $\mathbb{R}^2$, and describe the subspaces in terms of the solutions of homogeneous linear equation systems. This reasoning can be done just algebraically. It could also happen that students use geometrical terminology concerning lines through the origin.

Step 2: Verification of the subspace properties. In order to reason why the subsets given in step 1 are subspaces, one could either refer to the solution of part a) of the exercise or (for the trivial subspaces and in case of the use of the solution spaces of homogeneous linear equation systems) to the lecture. In case of a geometric description ("lines through the origin"), either geometric or algebraic arguments have to be provided to verify the subspace properties.

Step 3: Why are these all subspaces? The final challenge is to reason if and why all subspaces of $\mathbb{R}^2$ have been found. This can be done algebraically, but we did not expect our students to complete this reasoning process in the given time, since it requires a development of several successive algebraic arguments. Based on their school knowledge, the students could recognize the descriptions of geometric objects by equations in part a) to e) and abstract from the previous results, leading to the conclusion that lines through the origin are subspaces, but no other lines, single points or other collections of points. At this point, a successful reasoning based on school knowledge could be done constructively, based on geometric arguments. Trying to construct "bigger" subspaces than just the lines through the origin, a student could build the union of two different lines and check whether this set is a subspace. Alternatively, he or she could try to find the minimal subspace that includes one line g_0 through the origin and an additional point x_0 not lying on this line. He or she could come to the conclusion that this has to be the whole $\mathbb{R}^2$. A formal argumentation here is that every point can be represented as a linear combination of a point $x_1 \neq (0,0)$ from the line g_0 and x_0, but even if the student does not come to this conclusion at this point, he or she could have the idea to consider the line through the new point x_0 together with the original line, and therefore check this new set for the subspace conditions. He or she could check the closure of addition or come to the idea that further points have to be added to the union in order to get a subspace. Since this type of reasoning seemed to us more likely to be achieved with the previous (including school) knowledge of the students, the tutors in the normal tutor group meetings as well as the tutor in the video study were advised to guide the students along this reasoning process if they struggled in approaching the problem. Based on this sample solution process description, we tried to answer the following research questions in our analysis:

1. How far in this three-step process would the students come when they work on this exercise? Would they even be aware of the need to do step 2 and 3?
2. Would they favor one of the described approaches to the problem (geometric, algebraic), and would they use the constructive or the relational way to describe the 1-dimensional subspaces? Would they approach step 3 in a constructive way, building up subspaces starting with just one point, as described above, or would they find other ways (purely algebraic?)?
3. Finally: Would they recognize that parts of exercise f) could be solved by an application of the example (S1) given in the lecture?

Since we posed the question in part f) in a relatively weak phrasing, we could not expect the majority of students to give a fully structured, formal reasoning in this exercise, in particular for the steps 2 and 3. But we were interested if the exercise itself would stimulate the students to give reasons for their answers and, in particular, how they would argue in this case.

RESULTS

To find answers to our questions, we analyzed the written works as well as the video recordings of the students working on part f).

Work on part f): Written exercises

From the written works of 116 students on this exercise, just 48 handed in solutions for part f). This is most likely due to the fact that the time was very limited, so many students just did not come to part f). We analyzed their work with respect to the three steps of the solution as described in the a priori analysis.

Trivial subspaces	$\mathbb{R}^2$	33
	Zero Space	32
1-dimensional subspaces	Solution with any description of the 1-dimensional subspaces (some students used more than one description)	33
	- Relational description: $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$	24
	- Constructive description: $L_x := \{v \in V \mid v = \lambda x, \lambda \in K\}$	4
	- Geometric descriptions: “line containing zero”, “line through origin”	12

Table 1: Frequency of the nominations and descriptions of the subspaces

Step 1: Which subspaces do they find? How do they describe them? Do they use previous parts of the exercise or name the set considered in part a)?

The results in table 1 were collected by counting how often the three types of subspaces were mentioned in the solutions. Hereby, each notion of a subspace counted, as long as it was clear enough to denote the required set. How did the students describe the 1-dimensional subspaces? We distinguished between "geometric" descriptions, using expressions like "line containing zero", "line through the origin", relational descriptions using a set like $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$ or something mathematically equivalent (see a priori analysis for a definition of this category) or constructive descriptions like $L_x \coloneqq \{v \in V | v = \lambda x, \lambda \in K\}$. Some students used more than one description in their solution. Apart from this, it was interesting to see that only 8 students did mention any part (mostly a)) of the previous exercise in part f). It is not clear if those who could not give any (nontrivial) subspace actually never recognized that the set M_1 from part a) is a subspace (since the word "subspace" was not used in part a)), or if they just forgot about it before they started with part f). Moreover, it is interesting that the trivial subspaces, which we expected the easiest to find, were not nominated more often than the 1-dimensional subspaces. We were also surprised to see that only 2 of the students did refer to example (S1) (see a priori analysis) from the lecture, concerning the solution spaces of homogenous systems of linear equations.

	No reasoning	Incorrect reasoning/ unclear approach	Partial reasoning	Complete reasoning
Step 2	34	5	7	2
Step 3	35	6	6	1

Table 2: Frequency of reasoning in part f)

Step 2: Do they show that the given sets are subspaces? How do they argue?

Most students did not give reasons (see table 2 for results), but within those who did, we distinguished between approaches that did not go in the right direction (for instance students just answered by listing all properties of a subspace without proving them or claimed that it was "clear" that the spaces are subspaces), students who did give a correct approach or a partial proof (they mentioned that closure must be proved, but did not, or just checked the addition or the scalar multiplication, or just checked an example etc.) and complete solutions with full reasoning (using example (S1) from the lecture in both cases).

Step 3: How do they reason that they found all subspaces of $\mathbb{R}^2$?

Within the 13 solutions that had some kind of reasoning (see table 2 for results), we distinguished again between unclear or vague approaches to reason the completeness of the given list of subspaces (for instance the statement "there are no other possible, because one cannot multiply vectors"), promising but incomplete approaches (some students gave reasons why lines not going through the origin cannot be subspaces, but did not consider other subsets, or just discussed the closure of one of the operations) and complete reasoning (just one case, again applying example (S1) from the lecture).

Work on part f): Video recordings

We give a summary about three groups of students that we recorded during their solution process on part f) under the perspective of our research questions. Due to limitations of space, we cannot document the method used to analyze the transcripts and the students' interaction in more detail.

Group 1: The first group tried to find subspaces by systematically going through the list of properties, and found the zero space to fulfill them. Then, they remembered the set proven to be a vector space in part a) and generalized it to a set of the form $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$ after a discussion whether the coefficients are arbitrarily exchangeable without harming the subspace conditions. They discussed the closure of the vector space operations in this set, but referring to the proof they had given in part a), they convinced themselves quickly that there was nothing else to prove. After this, they also identified the full space $\mathbb{R}^2$ since there is no claim for a subspace to be a proper subset. At this point, they were asked by the tutor if and why they found all subspaces now. They had the idea to consider the set $\mathbb{Z}^2$ and, referring to their knowledge about groups, discussed the closure of operations on this set before they could finally rule it out to be a subspace by the fact that the scalar multiplication with elements from $\mathbb{R}$ is not closed on $\mathbb{Z}$. The tutor then asked them to consider the set $M_{a,b}$ geometrically. They start to consider the tuples of coefficients (a, b) in the plane instead of the equation $ax_1 + bx_2 = 0$. With another hint from the tutor, they found out that the set $M_{a,b}$ whose elements are described by the equation $ax_1 + bx_2 = 0$ denotes lines in the plane, and discussed the closure of the operations for these lines. The students did not develop an idea themselves to give arguments why they had found all subspaces. However, the students were able to follow the geometric constructive reasoning of the tutor (see a priori analysis).

Group 2: The second group came up with the idea to apply example (S1) from the lecture. After some discussion and a bit help from the tutor, they found that the subspaces defined there are the solutions of one or more linear equations, each having two coefficients. The central difficulty for them was to see that the number of coefficients is fixed to 2, but there could be an arbitrary number m of equations in a system of linear equations that is still defined in $\mathbb{R}^2$. It was a real discovery later that $m = 1$ provided descriptions as are provided in $M_{a,b}$. Up to this point, they did not consider the trivial subspaces at all. They struggled a bit to write down the concrete subspaces they could find this way in terms of algebraic expressions, but managed it with some help from the tutor. Asked whether they found all subspaces, they did not develop the idea to consider the spaces as lines in the plane on their own, but after the tutor came up with this idea, they were able to work with this concept after a short phase of orientation in which they convinced themselves that the geometric objects stand for the same subspaces they worked with before. Just at this point, they identified the trivial subspaces too. Step 3 (to reason that all subspaces were found) was only solved with tutorial support (similar to group 1).

Group 3: The third group used the previous parts a) to e) in their reasoning and started with the subspace found in a), but immediately identified this set to describe a "line through the origin", which gave birth to a generalization to all lines through the origin. They continued to orally communicate in geometric terms, but decided to write down the set using the relational algebraic expression $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$. They named the trivial subspaces without further comments. The proof given in part a) sufficed for them for a reasoning of step 2, and they started to discuss step 3 quickly. They used references to part b) to e) to rule out other types of possible subspaces. The group thought they had finished at this point. It was the tutor who pointed out that step 3 was not yet satisfactorily answered. Different from the other groups, they took up the tutors input to construct other subspaces geometrically and in order to find out that such subspaces have to be equal to $\mathbb{R}^2$. With some minor help from the tutor, they finished this step quickly, needing much less time for the full task than all other groups.

CONCLUSION AND DISCUSSION

As a result, we can state that students at this point in their studies were able to find and describe (using varying descriptions) subspaces of $\mathbb{R}^2$, but the question to find all subspaces was a serious obstacle for the students. Moreover, the step to translate the algebraic description of the subspaces into a geometric view, where reasoning could be done with less formality, was a further obstacle for them, since they seemed not to connect or apply their geometric knowledge from school to the new problem.

It seems like a geometric approach to this kind of problems is not a natural, automatic behavior of students at this point of their education. This result resonates with the observations from Wawro et al. (2011), who stated that intuitive geometric notions can be the preferred approach of first year students to the concepts of subspaces, but also cause problems if they their geometric intuitions are inconsistent with the formal definition. It is worth pointing out that our students did not, in opposition to the results of Wawro et al., automatically identify (often mistakenly, if there was no respect to a necessary embedding) the $\mathbb{R}^1$ as a subspace of the $\mathbb{R}^2$. A possible explanation for this result is the fact that our results were obtained shortly after the introduction of subspaces in the lecture, where Wawro et al. interviewed their students when they already have had more time to develop a concept image of subspaces, including some misconceptions.

Most students did not connect the different parts of the exercise, appearing in different "languages" (like the sets in the parts a) to e) and the open question in part f)) to solve the problem in f). With some help, especially with the request to consider the sets geometrically, they were able at least to understand reasoning on this basis, and some students actually could even give proofs or approaches to proofs on their own. We came to the result that the students needed more guidance and preparation to solve this problem, and in particular support that helps them to deal with each step and even sub-step of the solution of the problem in part f). In our subsequent study in winter term 2017/2018, we are investigating if explicit indications in a) to e) to consider the sets geometrically and a rephrasing of part f), splitting it up into more explicitly described steps, have a decisive influence on the students' ability to solve task f).

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A TDS analytical framework to study students' mathematical activity

An example: linear transformations at University

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Our research focuses on the teaching of linear transformations in "Classes Préparatoires aux Grandes Écoles". The theory of didactical situations, jointly with Peirce's semiotics, constitute the main theoretical framework of our works and allow us to analyse student's reasoning in situations of oral evaluation. We illustrate the use and utility of this framework with the study of student's mathematical activity when they are faced to situations involving complex concepts such as linear transformations in polynomial spaces.

Keywords: linear algebra, semiotics, theory of didactical situations, transition, tertiary level

INTRODUCTION

Our work deals with a double object: linear transformations as a structuring concept of the teaching of linear algebra and a particular institution at the undergraduate level, the Classes Préparatoires aux Grandes Écoles. Grounded on didactical motivations, our epistemological analysis allows us to exemplify the crucial role of linear transformations for the emergence of linear algebra concepts. This epistemological part of our research, mainly based on the works of Dorier and Moore, leads us to the use of Peirce's semiotic and to enrich the analytical model of Bloch and Gibel (2011) which is rooted in the Theory of Didactical Situations (TDS). The purpose of this paper is twofold:

- give some responses to the following question: which reasoning forms are actually produced by a student during the different stages of a situation of oral evaluation?
- show the utility of our framework to analyse the signs and arguments produced and thus take part to the development and enrichment of this model.

Within the French didactic tradition, we remind briefly the theoretical tools used in the elaboration of the framework. Then, we expose our model, using the terminology of Bloch and Gibel. Equipped with these tools, we use the model to analyse the arguments produced by students. But, at first, we succinctly introduce the institution of Classes Préparatoires aux Grandes Écoles by highlighting the differences with the University, especially regarding the transition phenomenon with secondary level.

I. THE CLASSES PRÉPARATOIRES AUX GRANDES ÉCOLES: A FRENCH POST-SECONDARY LEVEL INSTITUTION

The Classes Préparatoires aux Grandes Écoles, which we can translate as Higher School Preparatory Classes, are part of the French tertiary education system for over two centuries. They consist in two really intensive years which act as a preparatory course to train undergraduate students for their further enrolment in one of the French graduate schools called G*randes Écoles*, such as École Polytechnique, École Normale Supérieure, École des Hautes Études Commerciales, also known as HEC School of Management … The enrolment in one of these Grandes Écoles depends on the rating obtained in national competitive and demanding examinations.

We summarize the main differences between the Classes Préparatoires and the University in the following table. Thus, we highlight some facts that Winslow (2007) showed to think about the study of the transition phenomena from the secondary to the post-secondary level.

Classes Préparatoires	University
Full-time teacher	Part-time teacher, part-time researcher
One teacher by class One class per teacher	Several teachers for one class Several classes for one teacher
Non adoption of semesters	Adoption of semesters
Common national curriculum	Local curriculum
Non degree course	Degree course
In High School	At University
Class councils Report cards	No class council No report card
Selection of students	No selection of students

Table 1: Comparison of CPGE and University

As Winslow (2007), Castela (2011) and more recently Farah (2015) wrote, these differences have deep didactic implications, relative to the theoretical knowledge and praxeologies, to the problem solving approaches, to the evaluations and to the personal homework just to name a few. In our experimental work, we studied some arguments produced by second year students from one of these Classes Préparatoires during an oral evaluation on linear algebra.

As noted by Bloch and Gibel (2016) for calculus, in order to recognize and analyse the reasoning forms actually produced by a student,

> it is necessary to classify the objects, signs and reasoning processes they have to cope with during resolution of calculus problems. (Bloch & Gibel, 2016, p. 44)

II. A MODEL TO ANALYSE STUDENTS' REASONING

We then need a tool to modelize the students' productions, more precisely to identify, while in a mathematical situation, which is the knowledge they rely on during their activity; we want to identify the structure and functions of students' reasoning processes in this situation. Briefly, this model should allow us to seize the complexity of the reasoning processes a student has to cope with during the resolution of mathematical problems. By reasoning process, we mean valid or erroneous ones according to the work of Gibel (2015) about reasoning.

This model takes its origins in the Theory of Didactical Situations (Gonzalez-Martin, Bloch, Durand-Guerrier, and Maschietto 2014) and in the semiotics of Peirce. We briefly recall the main theoretical elements of TDS and semiotic used in the elaboration of the model.

The TDS theory

TDS relies on a two basic premises concerning the mathematical activity and the learning of mathematical knowledge. For TDS, the mathematical activity consists of distinct stages: a situation of action, followed by a situation of formulation and then a situation of validation phase. To take the learning activity into account, TDS adds two more stages: the phase of devolution and the phase of institutionalization. TDS defines three fields to construct and analyse such situations. The theoretical field which the domain of elaboration of fundamental situations relative to a knowledge. The a priori experimental field which envisages a situation at a specific level of teaching, taking into account the didactic repertory as defined by Gibel (Gibel, 2004). The a priori analysis of the situation, which checks if the conditions of devolvement are fullfilled, takes place at this second level. The third field is the field of eventuality where the situation is actually implemented. In short, TDS is a didactical framework which tries to implement situations with adidactical parts and offers tools to analyse the teaching and learning activities. These adidactical stages allow students to face a heuristic phase of research and then, through a confrontation to the elements of an adequate milieu, to test, validate or invalidate their conjectures. The notion of milieu appears to have a central role. TDS organizes situations with up to seven logically successive phases, but in our work we will mainly work with three of them: a heuristic one, grounded in a problem, and a formulation and validation one, composing the adidactical moments of the situation, and then the institutionalization by the teacher or with his/her help, which is the didactical moment. The dynamic of the nesting of the situations with the paired levels of milieu illustrates the dynamic of the learning processes involved. The following chart (Bloch 2006, Bloch and Gibel 2016) sums up the levels of milieu paired to the different phases of situations corresponding to the experimental situation

M1 Didactical milieu	E1: reflexive subject	P1: P. planner	S1: situation of project
M0 Learning milieu : institutionalization	E0: generic student	P0: professor teaching	S0: Didactical situation

M-1 Reference milieu: Formulation and validation	E-1: The subject as learner	P-1: Professor Regulator	S-1:Learning situation
M-2 Heuristic milieu: action, research	E-2: The subject as an actor	P-2: P devolves and observes	S-2: Situation of reference
M-3 Material milieu	E-3: epistemological subject		S-3: Objective situation

Table 2: Structuration of the didactical milieu

The levels M-2 and M-1 are the ones that will allow us to identify, describe and analyze the elements (signs, processes …) associated to the emergence of an argumentation within the proof process. More precisely, as Bloch and Gibel (2016) write

> The place where we hope to see the expected reasoning processes appear and develop is located at the articulation between the heuristic milieu and the reference milieu. (*ibidem*, p.46)

Thus, the TDS theoretical framework allows us to consider not only the reasoning processes produced, but their functions within the situation and the levels of argumentation they rely on: it already gives us a glimpse of the multidimensionality of our model.

The semiotic tools

To take into account the semantic dimension, *i.e.* the meaning of the signs and arguments produced, with certain accuracy, we need semiotic tools. The signs a student produce during an adidactical situation, whether formal or informal, are the only observable phenomena that can sustain our semiotic analysis: roughly speaking, the signs produced (syntactic aspect) are in relation to an object (semantic aspect) creating a instantiated sign (pragmatic aspect) relative to the milieu, the didactical repertoire and the repertoire of representations as defined by Gibel (2015). This triadic relation, linking the sign, the object and the student's instantiated sign, led us to use Peirce's semiotics. In this paper, we only analyse the relation between a sign and its object, its content. But in his thesis, one the authors (Lalaude-Labayle 2016) conducted a semiotic study, relying on the full Peirce's triadic classification of signs as 'put in algebra' by Marty (1990). Applying the universal categories to the relation between a sign and its object refers to the notions of icon, index and symbol, describing the abstraction level of this relation. An icon is a sign which stands for an object because of its physical resemblance: a drawing, a triangular matrix represented with triangles … An index is a sign physically connected to its object: the columns of the pivots in a reduced echelon form of a matrix *M* is an index that points to the basis of *Im(f)*, where *M* is canonically associated to *f* ... A symbol is a sign that refers to its object by virtue of a law: *ker* is a symbol whose object is the concept of kernel of a linear transformation.

The logical inferences

In our work we understand the term 'reasoning' in it broadest sense. More precisely, by reasoning we mean a sequence of representations, following some intern and potentially explainable rules that lead to reach some explicit goal. Postulating, as it is the case within the TDS theory, some rationality of the students, we need to define these rules or inferences. Peirce saw mathematics as the science of drawing necessary conclusions, studying what is and what is not logically possible. But, since one does not think about logical propositions but about and with signs, Peirce broadened the notion of inference. He then distinguishes three kinds of rational inferences: abduction, induction and deduction. Deduction, or necessary reasoning, deduces a proposition *B* from a proposition *A,* where *B* is a formal logical consequence of *A*. Induction goes from the particular to the general; it allows inferring *B* from *A*. Abduction allows inferring *A* as a probable explanation of *B*. So, deduction proves that something must be, induction shows how something effectively operates and abduction suggests that something could be.

The framework to analyse students' productions

To analyse students' processes of reasoning, Bloch and Gibel (2011) develop a multidimensional model. They focus their didactical analysis on three main axes. The first axis is related to the level of milieu and so to the phase of the situation in which the student produces his/her reasoning (*cf.* Table 2). The second axis of the model is linked to the notions of didactical repertoire, of organizational system and of a repertoire of representations. It studies the functions of the reasoning produced and is in close relation to the first axis. As Bloch and Gibel (2016) state it, they aim at

> linking these two axes, showing how the reasoning functions are linked specifically to the levels of milieu and how these functions also *manifest* these levels of milieu. (*ibidem*, p.47)

Semiotic analysis of observable signs constitutes the third axis of the model. Marc Lalaude-Labayle enriched the model by adding a fourth axis about the forms of inference applied. This logical axis links the second and third axis by setting out and clarifying the organization of the reasoning signs and their functions. This fourth axis helps to 'make visible' the organization within the system of representations and its actualization. We sum up this model in the following table

	Milieu M-2 Heuristic level	**Milieu M-1** Formulation, validation	**Milieu M0** Institutionalization
Nature and functions of reasoning	R1.1 SEM - Decision of a working frame (DOO) - Decision of transformation (semiotic register)	R1.2 SYNT/SEM - Generic calculations - Formulation of underpinned conjectures (right or wrong)	R1.3 SYNT - organization of the signs - formalization and certification of validations

	- Decision of calculation - Heuristic tools; errors - Exhibition of an example /a counter ex. - pattern research/identification	- Decision on a mathematical object	- Formalization of proofs within the mathematics involved theory
Level of use of symbols	R2.1 SEM Icons or indices depending on the context (schemas, intuitions…)	R2.2 SYNT/SEM Local or more generic arguments: indices, calculations	R2.3 SYNT Formal and specific arguments related to the chosen frame
Actualisation of the repertoire	R3.1 SYNT/SEM - Ancient knowledge - Enrichment at the heuristic level(patterns, praxeologies ...)	R3.2 SYNT/SEM Enrichment at the argumental level: - statements - organizational system	R3.3 SYNT - Formalized proofs - Signs within the relevant theory - theoretical elements
Forms of reasoning	R4.1 - deductive - inductive - abductive	R4.2 - deductive - inductive	R4.3 - deductive

Table 3 *– A model to analyse situations*

SEM signifies that the formulations are made on a semantic mode whereas SYNT is for syntactic mode. This model emphasizes the fact that the mathematical activity, with its reasoning processes, appears in the heuristic and reference milieu (*cf.* Table 2). These two milieus, and the articulation between them, will thus be of particular interest for our work, even if the situation is an ordinary one.

Let us insist on the fact that the use of this model relies on a precise *a priori* mathematical analysis of the situation and of its components, *e.g.* the problem to be solved. Within the TDS this step appears to be necessary to clarify the didactical *a priori* analysis.

III. A PRACTICAL USE OF THE MODEL IN A LINEAR ALGEBRA ORAL EXAMINATION

We analyse the productions of a second year student of Classes Préparatoires in the context of an oral examination. The students of the Classes Préparatoires, by group of three, pass such an oral exam once every two weeks. It lasts one hour during which the teacher asks to the three students to solve different mathematical problems. They work simultaneously and individually on a large blackboard on the problem they just discovered, in front of the teacher. The teacher helps the students by giving advices or clues. Taking into account the writings noted on the chalkboard, she/he can ask for some explanation or clarification. This can give rise to an oral or written answer and possibly to some discussion to deepen and enrich the repertoire of the representations of the student. The oral exam we analyse deals with linear algebra, and more specifically linear transformations and matrices.

The student is confronted to the following problem:

Let n be a integer, greater than or equal to 2 and let φ defined on $\mathbf{R}_n[X]$ *by: for all polynomial P from* $\mathbf{R}_n[X]$, *φ(P)=P(X+1)-P(X).*

1. *Show that φ* is an endomorphism of $\mathbf{R}_n[X]$.
2. Is φ injective? Surjective? Bijective?

In France, $\mathbf{R}_n[X]$ symbolises the vector space of real polynomials of degree at most n.

As briefly explained earlier, our model provides a framework for investigating mathematical and didactic activities in terms of milieu, focusing on the reasoning processes, signs and their dynamics and on the conditions that enable their development during the situation, be it ordinary or not. As is done within TDS, our didactic analysis is divided in several stages: a detailed and strucured *a priori* mathematical and didactic analysis, enriched with a specific *a priori* analysis of the reasoning involved; follow then an *a posteriori* analysis organized in our model.

***A priori* analysis**

From the mathematical point of view, showing that *φ* is an endomorphism of $\mathbf{R}_n[X]$ can be approached in different ways, engaging several frames: indeed the stability of $\mathbf{R}_n[X]$ under φ can be done in a purely algebraic frame using the degree and the composition rule, in a functional frame using the decomposition $P=\sum a_i X^i$ or in an algebraic frame using the linearity and showing that forall integer i between 0 and *n*, $\varphi(X^i)\in\mathbf{R}_n[X]$. To study the injectivity and surjectivity of φ, the student can again choose between several frames and several registers of semiotic representation (Duval, 2017): she/he can use an algebraic frame with an example $\varphi(1)=\vec{0}$ and then applying the theorem linking rank and kernel dimension; but she/he can also try to find precisely *Ker φ,* that is find a basis. To do this, she/he can use the functional decomposition of polynomials, she/he can solve a linear system or she/he can determine the matrix of φ in the canonical basis of $\mathbf{R}_n[X]$.

From a didactic point of view, the situation studied here is said to contain an adidactic dimension. Most of the actions, of the frames and of registers of semiotic representation are devolved to the student. So, a first difficulty that occurs for the student can be the control she/he has on the objects involved: a circular application of the definition of linearity of φ to prove its linearity, the complexity of formulas to write down *φ(P)* for a general *P*, the non operability of φ with wrong calculations of *φ(1), φ(X),* $\varphi(X^2)$ for example. For the injectivity of φ, the student can “forget” the structure of the space $\mathbf{R}_n[X]$ he is working on and try to check whether *φ(P)=φ(Q)* implies that *P=Q.* All the reasoning processes and objects involved are part of the didactic repertoire of the class the student belongs to.

A posteriori analysis: analysis of a student's production

In the following, we extract some translated excerpts of the student's answer, the whole solution can be found in Lalaude-Labayle (2016).

To solve the first part of the problem, which is an ordinary situation in undergraduate level, the student is mainly confronted to the milieu of reference, articulating objects and processes involved in his repertoire of representations. Here the heuristic milieu is not really requested. He starts by showing the stability $\mathbf{R}_n[X]$ under φ and that writes changing *X* with *X+1* doesn't change the degree of *P.* He then proves the linearity in an algebraic frame. Doing so, he makes a formalization of proofs within the required theory and thus reaches the level R1.3. The semiotic analysis shows that he uses generic arguments (R2.2) and more formal one (R2.3). These arguments and signs don't give any hint to how φ operates on $\mathbf{R}_n[X]$. Its argumentation validates its use of the didactic repertoire, and reveals some implicit assumptions: φ(P) is a polynomial is here implicit, as is its use linking degree and composition of polynomials. He uses mainly hypothetical-deductive inferences. But, as an introduction to his argumentation, the student asks himself whether deg(φ(P)≤n: he formalizes here the start of an abductive reasoning.

To study the injectivity of φ, the student applies a transformation of register of semiotic representations to formalize the link with *ker* φ, starts within an algebraic frame then uses the decomposition of *P* to study *ker φ,* but without success: he cannot make φ operate on *P* and says not to have any clue to study the kernel. During this phase, the student tried to use some deductive reasoning involving objects from the reference milieu: the lack of articulation between the heuristic milieu and the reference milieu confirm the difficulties encountered in R1.3 with an aimless organisation of the signs. The semiotic analysis underlines the lack of quantifiers which leads to a incomplete apprehension of the objects (R2.2) and reinforces the feeling of lack of goal in the reasoning. The teacher asks then the student to consider the tools he has got in his repertoire to "calculate" objects in linear algebra. With no answer, he asks the student to consider the matrix of φ in the canonical basis. Doing so, he tries to force the student to face the heuristic milieu and tries to maintain an adidactic dimension in the situation. The calculations produced confirm the fact that the student doesn't know how to compute φ: he obtains the identity matrix. This matrix doesn't appear to be an index to control the reasoning (articulation R1.2 and R1.3). With a new oral intervention of the teacher, the student writes the right matrix. Some misinterpretations are following: $\mathbf{R}_n$ instead of $\mathbf{R}_n[X]$, surjectivity is meant instead of injectivity (R3.1), φ(1)=0 is used as an symbol for ker φ=vect(1) instead of a simple index of it (R2.1). The student uses the theorem of the rank (R3.2), relying then on a deductive form of reasoning. But some of his deductions rely on the preceding explicit calculations (articulation R4.2 R4.3).

CONCLUSION

As stated in the introduction, the purpose of this paper is twofold: try to determine the reasoning produced by a student in a specific mathematical situation and show the utility of our framework to analyse the signs and arguments produced in this situation.

Regarding the first question, our analysis allows us to say that this particular student has difficulties to reach fully the institutionalization milieu (level R3): the reasoning and the articulation of the objects involved do not ease his control over his arguments and eventually lead him to aimless computations. Moreover he seems to get stuck in the reference milieu (level R2) and to hypothetical-deductive inferences. The student does not rely on reasoning made in a heuristic milieu (level R1) that would be appropriate to linear transformations and polynomials. The problem we analyze in this work contains an adidactic dimension but fails in asking the student to make effectively operate φ on $\mathbf{R}_n[X]$. In semiotic terminology, we can postulate that the at least incomplete pragmatic dimension in the reasoning leads to some confusion and lack of pertinent association between the syntactic and semantic dimensions (Bloch and Gibel, 2016).

Regarding the second question, our work seems to confirm that the model used within TDS constitutes an efficient framework, as stated in Bloch and Gibel (2011, 2016). It helps specifying the reasoning and signs on which it relies both for the *a priori* and *a posteriori* analysis and highlights the obstacles contained within the mathematical notions. Our future work should be working more specifically on the abductive reasoning and on situations encouraging students to adopt a heuristic approach.

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Leveraging Specific Contexts and Outcomes to Generalize in Combinatorial Settings

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Generalization is a fundamental aspect of mathematics, and it is a practice with which undergraduate students should engage and gain fluency. It is important for students in combinatorial settings to be able to generalize, but combinatorics lends itself to engagement with specific examples, concrete outcomes, and particular contexts. In this paper, we seek to inform the nature of generalization in combinatorial settings by demonstrating ways in which students leverage specific, concrete settings to engage in generalizing activity in combinatorics. We provide two data examples that highlight ways in which concrete and specific ideas can be leveraged to help students develop generalizations in combinatorial settings.

Keywords: Combinatorics, Generalization, Examples, Discrete Mathematics.

INTRODUCTION AND MOTIVATION

Generalization is a foundational mathematical activity, a mathematical practice that both researchers and policy-makers value (Amit & Neria, 2008; Ellis, 2007). At the undergraduate level, given the nature of abstract, advanced mathematics, it is important to learn how to facilitate generalizing activity for students. We have recently conducted a study designed to investigate undergraduate students' generalizing activity, and we explored the students' generalizing activity in the context of combinatorial problems. In this study, we aimed to examine ways in which to foster students' engagement in generalizing activity. In combinatorics, however, it is often important and even necessary to focus on specific contexts or to consider particular, concrete outcomes. Indeed, in our prior work (e.g., Lockwood 2013, 2014) and in this current study, we have found that concrete, specific instantiations of problems, outcomes, and examples are particularly important for students' combinatorial thinking and activity. We believe that in the domain of combinatorics in particular, such specific instantiations are especially important for developing combinatorial thinking. Given that we want our mathematics students to be able to reason generally in combinatorial settings, we examine the interplay between the natural need for specific contexts and outcomes in combinatorics and the desire to have students engage in meaningful generalization. In this paper, we seek to answer the following research question: *In what ways can specific examples, concrete outcomes, and particular contexts be leveraged to foster generalizing activity in a combinatorial setting?*

LITERATURE REVIEW AND THEORETICAL PERSPECTIVE

A Piagetian perspective on generalization and generalizing activity. As a broad theoretical perspective, we adhere to a constructivist view of learning, asserting that students construct their knowledge of a given situation based on their mathematical experiences. We fundamentally view generalization as being related to Piaget's notions of reflective abstraction, and we emphasize the importance of having students engage with and reflect upon their prior activity as they engage in generalization. Many researchers have studied generalization in a variety of contexts involving both school-aged children (Amit & Neria, 2008; Ellis, 2007; Rivera, 2010) and undergraduate students in a variety of areas (e.g., Dubinsky, 1991; Harel & Tall, 1991). This report contributes to the growing body of literature by examining the nature of generalization in an undergraduate combinatorial setting.

We follow Ellis (2011) and take generalization to mean engaging in "at least one of three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases" (p. 311). To describe students' activity as they generalize, we adopt Ellis' (2007) taxonomy of generalizing activity, Ellis describes three main categories of generalizing actions: relating, searching and extending. In this paper, we focus especially on *relating,* which occurs when "a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects" (p.235). In this paper, the term "generalization" need not involve a formal, final statement of a general rule or property, but rather it may refer to the results of a students' generalizing activity, even if that activity is incomplete or not normatively correct.

Combinatorial thinking and activity.

Combinatorial enumeration problems, or "counting problems," are easy to state, but they can be surprisingly challenging for students to solve. This is due in large part to the fact that counting problems are not reliably solved using prescribed, fool proof algorithms (e.g., Kapur, 1970). Solving counting problems thus provides opportunities for students at all levels to engage in rich mathematical thinking. There is ample evidence that students struggle with solving counting problems (e.g., Batanero, Navarro-Pelayo, & Godino, 1997). Although researchers have taken strides in identifying productive strategies and ways of thinking that might help address such difficulties, there remains much to learn about how we might effectively help students to count successfully.

In this paper, we examine the role of generalizing in students' counting, and we explore how to frame generalizing activity in terms of Lockwood's (2013) model of students' combinatorial thinking. Lockwood (2013) suggested that there are three key components to students' combinatorial thinking (Figure 1) and that solidifying the relationships between these components is an important aspect of successful counting. *Formulas/expressions* refer to numbers and/or variables that represent the answer to a counting problem. *Counting processes* refer to the enumeration process in which a counter engages as they solve a problem. *Sets of outcomes* are the collection of (encoded) objects that are being counted.

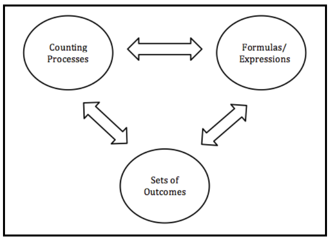

Figure 1: Lockwood's (2013) model of students' combinatorial thinking

To exemplify the model, we discuss the Horse Race Problem, which is a problem that we discuss in the Results Section. The problem states: *There are 10 horses in a race. In how many different ways can the horses finish in first, second, and third place?* Note that one counting process to solve this problem is to consider options for which horse could be first, second, or third place. We can argue that there are 10 options for which horse is first, and for any choice of which horse is first there are 9 options for which horse is second, and then for any of those there are 8 choices for which horse is third. This counting process yields an expression of 10*9*8, which is 720. This process would organize the set of outcomes lexicographically, grouped according to which horse finished first, then second, then third.

Lockwood has emphasized the importance of sets of outcomes in a number of studies. In particular, she has advocated for a set-oriented perspective toward counting (Lockwood, 2014), in which the act of counting is viewed as inherently involving structuring and enumerating the set of outcomes. In addition, she has made a case for the value of listing outcomes, demonstrating that listing outcomes was positively correlated with solving problems correctly for novice undergraduate students (Lockwood & Gibson, 2016). In this way, Lockwood has emphasized the importance of considering concrete outcomes as students solve counting problems. The point of the set-oriented perspective is that students should think carefully about what they should be solving in a given problem.

On one hand, then, this prior research suggests that it is useful for students to consider the concrete, specific outcomes that they should count. Further, when a student solves a counting problem such outcomes are necessarily tied to that problem and context. We want students to be able to think about what constitutes an outcome in a particular combinatorial situation. On the other hand, though, we want to foster generalization for students and to encourage them to engaging in generalizing activity, even in combinatorial situations. We want students to be able to develop and apply general formulas, or to be able to make general arguments about their counting processes. In this paper, we describe specific ways in which students use concrete settings to leverage general thinking and activity in the domain of combinatorics.

METHODS

We report on data from a study designed to study students' generalizing activity in the context of combinatorics. We report on two data sources. First, we report on a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) with four undergraduate calculus students, and we focus on one particular student Carson (student names in this paper are pseudonyms). The students were chosen based on a selection interview; they had not taken a discrete mathematics course in the university and were novice counters who could explain their thinking. The students were interviewed together as a group of four during nine 90-minute sessions. The interviews were audio and video recorded. During this time, the students worked both individually and together on combinatorial activities, and the interviewer often asked probing questions or asked the students to explain their work. These tasks included solving basic counting problems, coming up with general formulas for counting problems, and solving problems related to combinatorial proof.

Second, we report on an individual interview with a calculus student, Tyler, who had similarly not taken a discrete mathematics course in the university. The interview was individual and 60 minutes long. We gave Tyler tasks involving determining the number of 3, 4, 5, and eventually n-length passwords using As and Bs, and then passwords consisting of the characters As, Bs, and the number 1. We had him write tables in which he recorded the number of passwords with a certain number of As, and ultimately the tasks could yield the binomial theorem (which we do not discuss in this paper). These tasks were broadly designed to target students' generalizing activity in combinatorial tasks specifically, and we sought both to learn about students' combinatorial reasoning and about their combinatorial generalization.

The design experiment sessions and the interviews were transcribed, and we created enhanced transcripts in which we inserted relevant images and descriptions of activity into the transcripts. For the purposes of this paper, we identified two cases of Carson and Tyler as students who leveraged particular problems and situations in order to engage in generalizing activity. We focused on these students' data and identified relevant episodes that shed light on this phenomenon. We reviewed the transcripts and the videos and discussed these cases with the research team.

RESULTS AND DATA EXAMPLES

In our results, we seek to demonstrate instances in which students leveraged and use specific, concrete examples in combinatorics in order to engage in generalizing activity. These are meant to shed light on the interplay between particular situations and contexts that are important in combinatorial settings and the broader practice of generalization. We argue that specific examples, concrete outcomes, and particular contexts remain a fundamental aspect of combinatorial reasoning that can help to facilitate generalization. We offer two specific examples of how this phenomenon occurs, shedding light on the nature of generalization in combinatorial settings.

Students leverage activity on particular problems to generalize counting formulas and principles.

In this case, a student in the design experiment, Carson, repeatedly referred back to his prior work on a particular problem that stood out to him as being important. We view this as an example in which work on a particular problem can be leveraged to help students engage in generalizing activity. During initial problem solving in the first session, Carson had solved the Horse Race Problem, described previously. We will demonstrate that as he proceeded to solve other tasks and solve other problems, he repeatedly referred back to his prior activity on this problem, related it to other situations, and used it to generalize a counting formula. Carson solved the problem in a different way than we had described above, arriving at a correct expression of 10!/7!. Note, this is equivalent to the expression 10*9*8, but, as he explained below, Carson used a different counting process. He had a particular way of reasoning about this solution, leveraging the notion of division and equivalence to explain why the division by 7! makes sense combinatorially.

Carson: …So, there's 10 factorial total outcomes, and then we know for any given first 3 there's gonna be 7 factorial, because that's saying we know the first 3 horses have finished – how can the last 7 horses finish, so that's gonna be 7 factorial. But all we care about is how many given first 3s there are. So, if we divide the total number of outcomes by the number of potential of outcomes for the last 7 horses that will give us the potential number of outcomes for the first 3. If that makes sense?

Carson argued that for any particular arrangement of all ten horses, since all that matters is how the first three horses finish, we can divide by the number of identical arrangements of the last seven horses. This is a valuable way to think about these problems, and understanding and articulating this counting process seemed to be an important moment for Carson. As we proceeded to consider more problems, Carson repeatedly returned to this Horse Race Problem. We will demonstrate ways in which Carson has engaged in the generalizing activity of *relating* (Ellis, 2007) by using this problem as he approached additional tasks. In this way, this problem served as a generic example (Mason & Pimm, 1984), a way in which he could make general arguments and connect his reasoning to other problems. We now describe several of the ways that Carson leveraged this particular problem.

First, we see that Carson engaged in relating by connecting back to the Horse Race Problem as he solved other problems. For example, in solving a problem of arranging 4 of 7 books in a row on a shelf, Carson arrived at the correct answer of 7!/3!. The interviewer asked him how he was thinking about the problem, and his response below shows the connection he made to the Horse Race Problem.

Carson: Yes, kind of similar to the horse problem. You can say they're all in a race, you wanna see how many ways the first 4 books could finish in the race.

We later had the four students categorize problems they had solved, and from that exercise we asked them to generalize formulas. One of these formulas was the number of ways of arranging some number of objects from a larger set of objects. Carson had indicated that he saw several problems as being the same, and in the excerpt below, he explained why he viewed the problems as being the same. Again, he referred to the podium and the division that he had articulated on the Horse Race Problem as being a distinguishing feature of all of these problems.

Carson: So, essentially all of them are asking for a ranking of a given set of objects and asking how many arrangements there are for a given number of places, right? So, the cats are racing to get the collars you could say or the restaurants are racing to get the top five rankings in the town or the horses are racing in a race. Then each of the rankings or the collars are a ranking in the race. Yeah, then you can just divide by the duplicates for leftover ones, the ones that didn't make the podium finish or whatever amount of finishes there are or whatever podium they're asking for.

His reference to the horses and to the podium suggest to us that this continued to be a salient aspect of his reasoning. We interpret that Carson was engaging in the generalizing activity of relating (Ellis, 2007), and, in terms of Lockwood's (2013) model, he related the counting process of arranging all of the objects and then dividing by the ways to arrange the leftover objects. He also seemed to emphasize the nature of the set of outcomes (arrangements). He recognized that counting process as similar among the problems he grouped together, and he related each of those other problems to the ranking and podium language he used in the Horse Race Problem.

Further, we also asked the students to come up with a general formula for the problems they had grouped together. They did this for several problems, but we highlight the formula for the permutation problems. In trying to articulate the kind of problem they were dealing with, again Carson referred to his activity on the Horse Race Problem.

Carson: Right. I mean thinking about the method for solving this, it's the factorial from above, right? So, we have ten horses in a race. How many ways can the horses finish, but then how many of those have a unique podium, right? So, how many times are the first, second, and third place different?

The students then had a conversation about what the formula would be. They came up with the formula $a!/(a-b)!$ for arranging b objects from a set of a distinct objects, which another student, Josh, articulated:

Josh: No, I think that it would be something like if you have a objects, you would have a factorial – that's the total number of things that you can select – over a minus b factorial, where b is the number of slots that you have.

After they agreed upon this formula, Carson explained how he was thinking about this general formula they had come up with. The excerpt below shows that he referred back to the imagery of the podium, using that context to make a general argument.

Carson: … A is your total number of arrangements for the entire thing and then you want to divide by the number of ways that the places you're not selecting can be arranged, right? So, if you're selecting first, second, and third, then you have fourth through tenth and those can be arranged in ten minus three factorial ways, right? So, we can just divide by that number of arrangements [begins motioning slots with hands] for the back end to get just one for the front end because that's what we're asking for is how many ways can that podium be arranged?

We contend that in relating back to the Horse Race Problem, Carson was relating back to different components of Lockwood's (2013) model, including formulas, sets of outcomes, and counting process. This exchange suggests that Carson had a well-developed understanding of the specific problem in terms of the components of the model, and he related different aspects of the problem in different situations. From our Piagetian perspective we view Carson and the students as constructing a formula that makes sense to them, and Carson reflected upon his prior activity in order to develop a statement of a more general formula.

The Horse Race Problem came up in additional settings for Carson, including during reasoning about combinatorial proof in a later session. Ultimately, Carson acknowledged how important this problem was for him. In the final interview, when we were reflecting on the entire design experiment, Carson shared that he continued to think about subsequent problems in terms of the Horse Race Problem. We interpret that his language below means that he felt that he conceptually understood the ideas in the Horse Race Problem, perhaps in a deep way that he felt confident about.

Carson: For whatever reason, the horse race problem is the one that's in my head forever. And it must have just been where it clicked in the interview because that's kind of what I refer to. If somebody says how many ways can a horse finish in the podium, how many ways can the podium be organized, things like that. And that's kind of where I keep going back to. And I don't know why that is.

This case serves as an example of a student leveraging activity on a particular problem for a number of other activities, particularly generalizing activities of relating (Ellis, 2007). There have other examples of students drawing on prototypical problem types in combinatorics. Maher and colleagues have talked about students referring to Pizza Problems or Block Towers problem, demonstrating how students think about and use powerful particular problems in other (e.g., Maher, Powell, & Uptegrove, 2011). We build on such work by explicitly drawing attention to the generalizing activity that a specific problem fostered for students, highlighting the affordances that can stem from a student deeply understanding and justifying his or her activity on a particular counting problem.

Students compare and contrast specific examples to identify general structure.

We briefly describe an additional example in which a calculus student Tyler was relating two different situations while working on the Passwords Activity. In the interview Tyler was counting two kinds of passwords – those involving either upper case As and/or Bs, and those involving As, Bs, and the number 1 (with repetition allowed). Tyler had initially engaged in systematic listing activity to count the number of possible 4-character AB passwords. In particular, he created the list of 4-character AB passwords with exactly two As (Figure 2a), and the table of passwords according to number of As (Figure 2b).

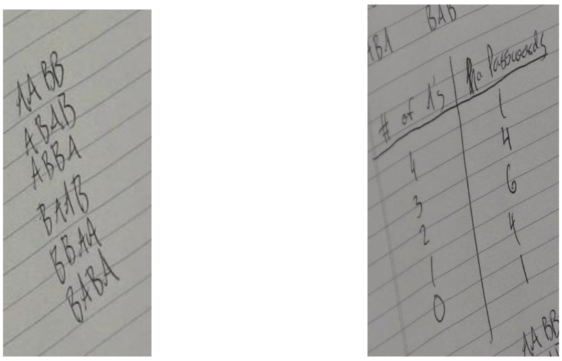

Figure 2a, b: Tyler's arrangements of 4-character AB passwords with exactly 2 As and his complete 4-character AB table

Later in the interview, Tyler was in a situation of counting 4-character passwords consisting of uppercase As, Bs, and 1s. We had asked him to create a table based on the number of 1s in the password. To complete this table, one can first consider placing the 1(s) and then filling the remaining positions with either A or B. Notably, placing As and Bs then reduced to the prior problem Tyler had solved, namely counting 4-character AB passwords. Tyler realized that there were the same number of arrangements of two types of characters, and he made a general statement about this case, recognizing that he will always have six ways of arranging two kinds of characters. Tyler was able to speak generally about counting arrangements of two kinds of characters. That is, he recognized that the tables gave him totals for the number of ways of arranging two characters, not just that they had to be As and Bs or 1s and *x*s. In the excerpt below he had been working on an extension problem, and he speaks about two different "things" that are changing, suggesting he had extrapolated a notion of arranging two characters independently of what the characters are specifically.

Tyler: Um well these are all the number of combinations I can do, um, with 2 different, 2 things that are changing, and this number of letters.

Here, we conjecture that reasoning about the particular situations and engaging with the actual outcomes allowed Tyler to make an important connection between arranging As and Bs and 1s and *x*s. The similar nature of the activity when listing in both cases allowed him to draw attention to the similar counting process in which he was engaging and the fact that the outcomes he was generating were fundamentally similar – arrangements of two kinds of characters. Ultimately this allowed him to make and use a useful generalization, and he understood the values in the rows in the AB tables as representing the number of arrangements of two kinds of characters.

We infer that Tyler engaged in relating (Ellis, 2007), and that comparing both situations allowed him to draw out some general commonalities between the two specific settings. Here, we argue that reasoning carefully about the concrete examples and actually engaging in concrete listing activity may have helped to solidify a broader combinatorial process.

CONCLUSION AND DISCUSSION

Prior research (e.g., Lockwood, 2013; 2014; Lockwood & Gibson, 2016) has emphasized the importance of having students focus on sets of outcomes as they solve counting problems. Often this focus on outcomes necessarily means that students reason about very specific contexts and concrete objects, and we view this as a fundamental aspect of counting and combinatorial activity. However, we also acknowledge that part of mathematical engagement involves looking beyond particular situations and contexts, and we have tried to demonstrate certain ways in which the particular contexts and concrete outcomes can be leveraged to facilitate meaningful generalizing activity for students.

Specifically, we offer two qualitatively different examples in which students leveraged the structure of specific combinatorial contexts to establish more general relationships. In our first example, Carson used his activity on and solution of the Horse Race Problem as a template for a specific combinatorial process, which he then used in similar contexts. Carson's generalizing activity was manifest through using this template as a means to relate combinatorial processes that he viewed as similar in some way and to connect the structure of the Horse Race problem to other cases. This specific generalizing activity demonstrates a powerful manifestation of relating, wherein Carson leveraged the structure of a known specific example as a solution for novel and abstract counting processes.

In our second example, Tyler used a connection between two situations to generalize a concrete arrangement structure. Tyler had initially recognized that his tables partitioning the 3-length passwords according to the number of As represented instead a partition of ways to arrange pairs of objects. Tyler then leveraged the specific process with which he was familiar by arranging 1s and *x*s, thus implementing the same specific counting process in a particular generalized setting.

Both of these examples involve the activity of using concrete situations to form a general relationship. These cases help to inform the nature of generalization in combinatorial contexts, offering examples of specific ways in which concrete outcomes and situations can be leveraged for use in more general settings.

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Discrete mathematics at university level
Interfacing mathematics, computer science and arithmetic

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Discrete mathematics is a recent field linked with Computer Science. We discuss its place in university mathematics curricula and in the particular case of France, where it has difficulties to find its place. We make explicit the didactical challenges posed by discrete mathematics at university level, and present DEMIPS network and its plans to tackle them. Through two detailed examples we discuss the reasons for teaching Discrete Mathematics at university level, and illustrate our conclusions.

Keywords: teaching and learning of number theory and discrete mathematics, teaching and learning of mathematics in other fields, proof, algorithms.

INTRODUCTION

This paper points out the current need for the construction of resources and debates regarding discrete mathematics at university level. We wish to emphasize the features of the French context, both from an educator's and researcher's point of view, at the intersection of didactics, mathematics and computer science. Indeed, teaching and learning discrete mathematics involves mathematical skills and heuristics (e.g. different kinds of proofs and reasoning, several ways of modelling etc.)[1] and also develops objects, concepts, methods and tools that are necessary for computer science. This link with computer science brings new types of questions to mathematics (for instance, regarding algorithmic complexity). Then, our aims are to design original situations for schools and at university level, and to construct appropriate introductory situations for computer science and maths majors.

We propose an overview of discrete mathematics in mathematics education and make a focus on the interface between discrete mathematics and computer science. Then, after presenting our research group in France, we analyse two kinds of situations.

DISCRETE MATHEMATICS IN AND FOR MATHEMATICS EDUCATION

How to define discrete mathematics?

Several mathematical topics are often gathered under the blanket term *discrete mathematics*. A first step in contributing to a thorough didactical study of discrete mathematics is to provide a satisfactory definition, or at least delimitation, of what it refers to. Several definitions exist, which either attempt to provide a general common trait to the covered topics, such as "the mathematics of discrete sets", or resort to an enumeration of objects, concepts or techniques most often associated with discrete mathematics. Most of these definitions include or are followed by a discussion on some typical difficulties, such as the distinction between finite, discrete and continuous mathematics (e.g. Maurer, 1997). To clarify the distinction between finite and discrete mathematics, the MAA (1992) places finite mathematics in the pre-calculus category and discrete mathematics in the same category as calculus.

We advocate that an interesting way to define discrete mathematics both for research and didactical perspectives (for the design of courses and of didactical engineering at university level) is to emphasize the features of the modes of reasoning that are common (or specific) to the various topics usually recognized as discrete mathematics, and the discrete nature of the structures they involve. Moreover, a classification of problems is required in order to structure a didactical analysis of the field of discrete mathematics. Furthermore, as the development of discrete mathematics has been strongly directed by the needs for computer science, the links with computer science must be explicitly explored.

In 1974, Knuth, a pioneer in computer science and its teaching made a similar analysis (Knuth, 1974, p. 329) :

> "The most surprising thing to me, in my own experiences with applications of mathematics to computer science, has been the fact so much of the mathematics has been of a particular discrete type [...]. Such mathematics was almost entirely absent from my own training, although I had a reasonably good undergraduate and graduate education in mathematics. [...] I have naturally been wondering whether or not the traditional curriculum – the calculus courses, etc. – should be revised to include more of these discrete mathematical manipulations, or whether computer science is exceptional in its frequent application of them."

We consider that these questions are still topical, even at university level, and deserve a careful didactical analysis.

Where is discrete mathematics? What questions are relevant at university level?

It is often stated that discrete mathematics can be a tool for improving reasoning and problem-solving skills (see for instance Rosenstein, Franzblau & Roberts (1997) who advocated an introduction of discrete mathematics in curricula, asked didactical questions, and made propositions that were taken into account for *Principles and Standards for School Mathematics* NCTM, 2000 for instance). Moreover, discrete mathematics is an active modern branch of contemporary mathematics with a wide range of applications in society, which is a very legitimate reason to teach it at school, high school and college. In fact, discrete mathematics courses are relevant to a wide variety of majors at university level, including mathematics, number theory, computer science, and engineering: from an epistemological point of view, discrete mathematics has an interdisciplinary nature and can provide a mathematical foundation (with specific ways of reasoning and proving, and mathematical concepts) for computer science and engineering courses. By 1989, an MAA report (Ralston, 1989) from an *ad-hoc* committee consisting of mathematicians and computer scientists recommended that "discrete mathematics should be part of the first two years of the standard mathematics curriculum at all colleges and universities”. This report also emphasizes the notions of proof, recursion, induction, modelling, and algorithmic thinking, as well as the benefits of using discrete mathematics in the secondary level to improve problem-solving skills with the transition to university level in mind (Ralston, 1989). Moreover, Epp (2016) points out the strong necessity of abstract thinking for the course and its applications in computer science. She underscores that it is done in the frame of the current curricular recommendations, prepared by The Joint Task Force on Computing Curricula (2013) of the ACM and the IEEE Computer Society, which gives discrete mathematics as one of the two largest components in the “core body of knowledge” recommended for all computer science students. Besides, discrete mathematics is in close relationship with other mathematical areas: other fields of mathematics use its methods and results, and, are useful for solving some discrete mathematics problems.

What is currently the place of discrete mathematics and its links with other scientific fields at university level? In several countries (Hungary, USA, Germany for instance), its significance in university programs is well-established and acknowledged. That is not always the case in France where the status of discrete mathematics in the first three university years is unclear, at least in mathematical curricula. However, discrete mathematics appears sporadically in few parts of mathematics curricula as probability theory (in particular combinatorics for discrete probability theory) or arithmetic. It sometimes appears in courses dedicated to the learning of proving, mathematical reasoning and problem solving, but we question whether its specificity is emphasized. One is more likely to find courses where discrete mathematics is taught for itself in computer science or mathematics and computer science curricula, where there exists a kind of common basis shared between teachers and including classical contents of discrete mathematics as can be seen abroad. These reports and recommendations coming from academic societies and the above remarks underscore two key questions for mathematics education at university level, and more specifically in France:

- What are the place and role of discrete mathematics at university level? How to design curricula and didactical engineering for the first university years ?
- What links are there between discrete mathematics and other areas (mainly of mathematics and computer science) and how are they (or should they be) practised / worked in the first university years?

These questions are particularly crucial for countries where discrete mathematics does not have a well-established status is the first university years.

What do we know from a didactical point of view?

In mathematics education, various research regarding the teaching and learning of discrete mathematics exist, focusing mainly on the primary and the secondary levels (ZDM (2004), Hart & Sandefur (in press) propose overviews). This research meets general approval, and points out epistemological features of discrete mathematics such as: discrete problems bring out different ways of proving (Grenier & Payan, 1998); discrete structures enable work on the construction of mathematical models, optimization, operational research and experimental mathematics (e.g. Grenier & Payan, 1998; Maurer, 1997); discrete concepts are accessible and problems are easy to understand (Grenier & Payan, 1998; De Bellis & Rosenstein, 2004); discrete concepts have different kinds of definitions and representations (Ouvrier-Buffet, 2006, 2011); some discrete problems are real world problems developing and using techniques from mathematics and computer science (Schuster, 2004), etc. Discrete mathematics problems are also a frame for developing and teaching algorithms; conversely, the study of algorithms requires a lot of discrete mathematics, and studying algorithms and programming can be a good way to justify the introduction of discrete mathematics contents (e.g. Modeste, 2012 & 2016). In all this research, discrete mathematics seems to be a powerful source of problems for teaching and learning mathematical proofs and processes and engaging students in developing new ways of thinking (such as recursive thinking), heuristics and problem-solving skills from primary school to university. Besides, some researchers point out that its teaching provides opportunities to bypass some of the sources of commonly-occurring negative affect in students (e.g. Goldin, 2016).

It appears that the features of discrete mathematics clearly represent challenges for university mathematics, in particular in France.

THE “DEMIPS” NETWORK – A WAY TO FEDERATE DISCRETE MATHEMATICS EDUCATION

Presentation of the DEMIPS network

In the French framework of mathematics education, there is a need to federate (isolated) research in university mathematics. Following the INDRUM momentum, the national network DEMIPS[2] supports the development of new research programs. DEMIPS’s research involves around 40 researchers in mathematics, mathematics education, physics education, computer science, and epistemology and history of mathematics and sciences, and is concerned with five main topics: three topics dealing with mathematical contents (analysis; linear algebra and abstract algebra; arithmetic, discrete mathematics and algorithmics) at the secondary – post secondary transition and at university level (the links with physics and computer science are questioned); a transversal topic (logic, language, reasoning, proofs - from both a mathematics and computer science point of view); and a specific topic dealing with the practices of teachers and teachers-researchers at university level (in mathematics, computer science and physics).

We (the authors of this paper) organize federative research in the fields of arithmetic, discrete mathematics and algorithms. The members of our group are mathematics educators (didacticians) with specific skills in teaching and learning at university level, mathematicians, and researchers in computer science. We choose to study the parts of mathematics which lie at the intersection of “classical” mathematics and theoretical computer science (for instance discrete mathematics, arithmetic, and algorithms), which interact and complement each other. As theoretical background we will follow Brousseau's theory of didactical situations (Brousseau, 1998) for its notion of didactical engineering, and the notion of scheme (Vergnaud, 1990) in order to structure our analysis of mathematical concepts. We organize our questions around key axes regarding the French university level:

- What are the epistemological features of concepts and reasoning in arithmetic, discrete mathematics and algorithms? How do they interact? (And then, how can these interactions be used to enrich the way these concepts are taught?)
- What kind of situations can one design in these mathematical areas for the university level and for pre-service and in-service teacher training? What for?
- What kind of curricula are there for this kind of mathematics at university level? What can be said about the design of these curricula?

Our research questions try to break down the barriers between scientific disciplines involving discrete mathematics. They also underline typical situations and questions common to mathematics and computer science, and try to put to use didactical analysis techniques to cast a new light on the way these questions are, or could be, tackled at university level. We develop below two examples to illustrate our work, and elaborateon the place and role of discrete mathematics at university level.

SITUATIONS AND IMPLEMENTATIONS AT UNIVERSITY LEVEL - EXAMPLES FROM DEMIPS' WORKSHOPS

We develop here two examples to illustrate the potentialities of discrete mathematics to engage students in learning modelling, proving, and mathematical reasoning and also to underscore the validity and the interest of keeping in mind the algorithmic point of view and the connections with computer science. These examples emphasize new perspectives for the teaching and learning of mathematics. The first example explore the links between mathematics and computer science in a problem-solving context and the second deals with a classical "divide and conquer"-type algorithm.

Example 1 – Discrete lines

The mathematical object concerned here is the discrete straight line (colouring squares, or "pixels", on a regular rectangular grid, in order to give the best possible visual impression of a straight line). The (real) straight line can act as a referent. Discrete straight lines are accessible through their representations (e.g. perceptive and analytical aspects of geometry) and their definitions and properties are non-institutionalized (a concept is institutionalized if it has a place in a classically taught content). Computer programmers are familiar with this concept. Professional researchers in discrete geometry (both mathematicians and computer scientists) use several definitions, but the proof of the equivalence of these definitions remains worth considering. The complexity of the underlying axiomatization of discrete geometrical concepts is actually an open and interesting problem.

Ouvrier-Buffet (2006) has analysed the evolution potential of zero-definitions (in Lakatos' sense, zero-definitions act as working definitions) of the concept "discrete straight line" in a defining situation implemented with freshmen. She underscores several approaches dealing with this concept, namely "real straight line" (What is the "nearest" pixel to a real line? What kind of modelling should be used?), "regularity" (What are the properties of the sequence of stages (called *chaincode string*)?), and "axiomatization" (What about the existence of the intersection of two discrete straight lines? Is a discrete straight line unique?). Each point of view brings about several zero-definitions. To engage into an axiomatic perspective carries great difficulty. This approach deals with both the perceptive aspect of a straight line and the axiomatic perspective. We are here confronted with two markedly different defining styles: a local one and a global-theoretical one, the latter mobilizing some implicit skills and knowledge in students (e.g. building a theory and choosing among competing definitions). The main results of this experiment underscore the ability of students to engage in a defining activity with a "neutral" but complex concept. Students do not assume an axiomatic perspective but mobilize reasoning involving approximate methods close to those used for real straight lines (and then arithmetic tools) and also the characterization of the sequence of stages of pixels (how can we modify a sequence to obtain a better regularity?) that involves recursive arguments.

From a didactical point of view, this research requires the development of a new theoretical background in order to model the defining process. From a mathematical point of view, the discrete geometrical objects, and more specifically the discrete straight lines can be approached in several ways: differential discrete analysis, the Bresenham algorithm, algorithms involving combinatorial analysis, several discretizations using algorithms which generate and study errors (Greene & Yao, Freeman & Pham, Rosenfeld), and the introduction by Reveillès of the arithmetical definition of a discrete straight line (1991). For instance, the approach to the discretization of a real straight line by checking linearity conditions is directly related to number theoretical issues in the approximation of real numbers by rational numbers. These linearity conditions can be checked incrementally, leading to a decomposition of arbitrary strings into straight substrings (Wu, 1982). The ongoing mathematical problems in discrete geometry are intimately related to questions in other fields of mathematics and computer science. The construction and the manipulation of algorithms are important for this purpose.

Example 2 – Exponentiation by squaring

A classic algorithmic problem is that of computing for some natural n the n-th power an of real number a. A naïve solution is, starting with value 1, to multiply n times this value by a. The final value one obtains is indeed the expected result, which is not very difficult to establish. The fact that this algorithm terminates is also trivially true since it contains a single bounded repetition. Finally, the complexity of this computation is clearly in Θ(n) (i.e. asymptotically bounded above and below by n), counting for instance the number of multiplications performed, and assuming that multiplication by a is an elementary operation.

This algorithm is not very efficient, considering that its running time is actually *exponential* in the representation size of n (which is of the order of log(n)). A more efficient technique relies on the observation that $a^n=(a^2)^{n/2}$ if n is even, otherwise $a^n=a.(a^2)^{(n-1)/2}$. Written as a recursive Python function, this algorithm reads as follows[3]:

```
def power(a, n):
  if n == 0:
     return 1
  elif n % 2 == 0:
     return power(a * a, n // 2)
  else:
     return a * power(a * a, n // 2)
```

We will now study a few common questions asked about algorithms, which will allow us to illustrate examples of mathematical techniques relevant to the analysis of, and discussion about, algorithms. In the following, typewriter face (as in n) will be used for formal parameters, and italic (as in n) to denote actual values.

Termination. A first question when it comes to analysing an algorithm is to determine whether or not it terminates, i.e. whether its execution on any instance of the problem (i.e. any pair (a, n)) yields a result after a finite number of execution steps or *elementary operations*. A standard technique used to prove this kind of result in non-trivial cases is the following. Assume here that there exist a_0 and n_0 such that power(a_0, n_0) performs infinitely many recursive calls. Call n_i the value of parameter n on the i-th recursive call. The sequence of naturals $(n_i)_{i\geq 0}$ is strictly decreasing, because whenever $n_i>0$, n_{i+1} is the quotient of n_i by 2, rounded down. This contradicts the fact that infinitely many calls are made, which means that the value of n eventually has to reach 0 and the function must terminate for all values of a and n.

Correctness. It remains to prove that the result is indeed correct for any instance of the problem. This is often done using some form of induction due to the intrinsically discrete and recursive or iterative nature of algorithms. In our case, we will establish that the value returned by a call to power(a, n) is indeed a^n, via a simple recurrence on the *call depth*, which is the maximal number, say k, of generated nested calls. The base case (k=0) is obvious: since there is no recursive call it must mean that n=0 and the returned is indeed $1 = a^n$. In the inductive case, assume the property holds for call depth k and consider a call of maximal depth k+1. Necessarily n must be greater than 0. If n is even, n//2 evaluates to n/2, a single nested call power(a*a, n//2) is performed and the obtained value is returned directly. This call itself has call depth exactly k, therefore by induction hypothesis its return value is $(a^2)^{n/2}=a^n$. Similarly if n is odd, n//2 evaluates to (n-1)/2, the value returned by power(a*a, n//2) is, by induction hypothesis, $(a^2)^{(n-1)/2}=a^{n-1}$, and the value returned by the main call is a*power(a*a, n//2), which evaluates to a^n. Therefore by the recurrence principle, the function returns the correct value whatever the initial value of its parameters.

Complexity. In the study of termination, we observed that in a call power(a, n), the value of n for the next call (if there is one) is divided by two (rounded down). One may observe the successive values of n more easily when it is written down in binary. Indeed, the operation of dividing a number by two and rounding down corresponds, in binary representation, to erasing its rightmost digit. The algorithm stops when n is 0, and performs one recursive call otherwise, modifying its value as we just saw. The number of nested calls for some initial value of n is therefore equal to the length, say k, of its binary representation, in other words its number of digits. Moreover, when n is even, exactly one multiplication is performed in the current call, two when it is odd. Therefore, denoting by m the number of digits equal to 1 in the binary representation of n, the total number of multiplications performed by the power(a, n) is exactly k+m, which is asymptotically bounded above by $\log_2(n)$.

Summary. We chose this example to illustrate, on a simple problem, the type of questions which can be asked about algorithms and the methods which are likely to be used to answer them. Note that in this simple case, all three properties could have been proven simultaneously using a complete recurrence on n. For our purpose, we chose a more basic and detailed approach. It would have been interesting to show how these proofs could be rephrased in the context of an iterative function. This example also tries to advocate the necessity for students in mathematics, computer science and related topics to have at least a basic understanding of various flavours of recursion and induction (including basic properties of orderings), to be able to present rigorous proof arguments (at least informally), and to possess minimal fluency in arithmetic, in order to be able to envision algorithms as objects worth studying in their own right. It is moreover often the case that the study of algorithms provides insight on related mathematical objects (here, the relationship between the value of a number and the length of its binary representation). Finally, this example illustrates a typical preoccupation of algorithmics, which is to provide more efficient, sometimes even optimal, algorithmic solutions to problems.

DISCUSSION AND CONCLUSIONS

Discrete mathematics is now considered as an entire field of mathematics, with many links to computer science. While it has entered university curricula in many countries, its status and contour are not always clear, and there are countries (such as France) where it has difficulties finding a legitimate place. Through the two examples we have developed (the discrete line and exponentiation by squaring), we have illustrated that it is legitimate to question the place that discrete mathematics occupies in university mathematics, for different reasons:

- it allows to develop situations for mathematical reasoning, mathematical heuristics, and problem solving (by its nature, but also by contrast with traditional continuous mathematics),

- many objects and techniques of discrete mathematics are required knowledge for computer science curricula; these contents must be identified and analysed from a didactical point of view, to design appropriate activities and situations,
- discrete mathematics involves specific questions and types of problems (such as complexity questions, combinatorial problems, etc.) that must be studied in order to understand their place in university curricula.

The DEMIPS network, through the topic group *arithmetic, discrete mathematics and algorithmics*, aims at addressing these questions. We pointed out the necessity to develop a didactical research on the topic of discrete mathematics at university level and its articulation with other fields of mathematics and other disciplines. This didactical research must rely on an institutional analysis of the situation in universities, and most importantly on a thorough epistemological study of discrete mathematics and its specific branches. It also requires to select and develop appropriate theoretical frameworks. Such work, started in the DEMIPS topic group, requires a plurality of viewpoints and interactions between (discrete) mathematicians, computer scientists, and didacticians of mathematics.

NOTES

1. Problems that can be identified as belonging to discrete mathematics can be found in many books aiming at developing "mathematical thinking", such as (Mason, Burton & Kaye, 1985).

2. Didactique et Epistémologie des Mathématiques, et liens Informatique et Physique dans le Supérieur: Didactics and Epistemology of Mathematics, and links with Computer Science and Physics in University Mathematics - with the support of CNRS.

3. Here a is assumed to range over floats, and n over positive integers. Note that in Python 3, n//2 computes the quotient of n by 2, whose value is n/2 if n is even and (n-1)/2 otherwise.

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Tasks for enriching the understanding of the concept of linear span

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The concept of linear span is one of the first abstract notions that students encounter in a course on Linear Algebra. Using the theoretical construct of concept image and concept definition (Tall & Vinner, 1981) along with observations about teaching and learning Linear Algebra, we present two tasks designed to enrich students' concept image regarding linear span. These tasks could be included in a problem workshop of an introductory university course on Linear Algebra. Each task is carefully created and/or selected so as to foster the ground for potential conflict factors to arise and be confronted. A preliminary evaluation shows that the tasks are well received by students and succeed in addressing certain conflicting factors.

Keywords: Teaching and learning of linear and abstract algebra; Teachers' and students' practices at university level; Linear span; Task-design.

INTRODUCTION

Linear Algebra is a subject with many applications in Mathematics and other sciences, but its teaching and learning proves to be demanding both for lecturers and students. The difficulties encountered are partly attributed to the way the subject is usually taught, as well as to students' lack of familiarity with proofs and limited knowledge of Logic and Set Theory. (Dorier et al., 2000; Hillel, 2000). Sierpinska (2000) attributes students' difficulties in Linear Algebra to their practical rather than theoretical way of thinking.

The concept of linear span seems to be quite difficult for students. Carlson (1993) states that difficulties in the notions of subspace, linear span and linear dependence / independence, if they are not addressed in time, create barriers for students. The analysis of Stewart and Thomas (2009) showed that students who were taught these concepts through formal definitions faced significant difficulties in understanding the concept of span compared to a group who were taught with emphasis on embodiment (Tall, 2004) and geometry. Moreover, they report that students have experienced several difficulties in linking the concept of span to the concept of a base. Finally, Wawro et al. (2012) propose teaching the concept through the solution of systems of linear equations and present a teaching approach through a series of realistic mathematical activities.

The main purpose of this paper is to investigate students' understanding of the concept of linear span and to use tasks to help resolve conflict factors in the students' concept image (Tall & Vinner, 1981). Based on a study of first year Mathematics undergraduates in a Greek university, we identify the misconception many students have that in a linearly dependent set each vector is in the span of the others. We use a set of design principles based on Sierpinska's (2000) remarks about theoretical thinking and Harel's (2000) principles of teaching and learning Linear Algebra, to create a set of tasks, and we present results of a preliminary evaluation of the tasks which indicate their potential to address the above misconception.

The work presented in this paper is part of the first writer's Master's thesis.

THE SETTING

The course "Geometry and Linear Algebra" is a first year mandatory course for students following the degrees in Mathematics or in Applied Mathematics at a Greek University. The course is typically taught through 4 hours of lectures and a two-hour problem workshop per week. Problem workshops are an important part in the teaching of the mandatory courses in the department. In the workshops the students are encouraged to work in groups of 5 or 6 students, on selected problems on the topics taught that week with guidance from the lecturer and a number of postgraduate or senior undergraduate students. The role of the latter is to discuss with students about the problems and the key mathematical ideas that may come up in the process. Promoting mathematical discussion among the students is a promindent element of the workshops of this course. During the semester of the study, the second writer was the lecturer of the course and the first one of the postgraduate students involved in the workshops.

During the first part of the course, students experiment with the idea of linear span in Euclidean 2- and 3-space, as an intuitive introduction to the concept. Later on, students are given a slightly modified version of the formal definition, limited to the spaces $\mathbf{R}^n$. The notion of linear span is usually described as the "subspace generated by the set S of vectors in $\mathbf{R}^n$". In relation to the general goals of the course, students are expected to familiarize with the concept of linear span in subspaces of $\mathbf{R}^n$, to be able to identify its geometrical representation in the case of $\mathbf{R}^2$ and $\mathbf{R}^3$ and to determine if a vector is in the span of a fixed set of vectors. We note the most important aspects of the concept. Firstly, linear span is a subspace, hence it is *closed* under the operations of a vector space. Secondly, *every* element in this subspace is a linear combination of some of the vectors in S. The final aspect is also very important but sometimes overlooked. In contrast to the concept of basis, there is *no limitation* in the choice of the set of generators S.

A starting point for this work was a study of the written answers given by students in response to a question in the final examination for the "Geometry and Linear Algebra" course, asking them to determine whether a vector belongs to the subspace spanned by two other vectors. The findings suggested that some students may have the misconception that in a linearly dependent set of vectors, every vector can be expressed as a linear combination of the others (see Papadaki, 2017). This misconception was found to affect students' understanding of linear span and to be a potential conflict factor (Tall & Vinner, 1981). We believe that examining the notion of linear span through tasks may offer the opportunity to confront such difficulties in a meaningful way.

THEORETICAL FRAMEWORK & DESIGN PRINCIPLES

Tall & Vinner's (1981) cognitive model of concept image and concept definition is used in the development of the task and to account for students' responses. According to them *concept image* is "the total cognitive structure that is associated with the concept" (p. 152). For each individual a concept image includes all the mental pictures (graphs, symbols, formulas etc) generated about the concept, associated properties and processes. The concept image is unique for each student and is changing over time when the student meets new stimuli. The term *evoked concept image* (Tall & Vinner, 1981) is used to describe the part of a concept image which is evoked at a specific time. Different parts of the concept image which contain conflicting aspects are called *potential conflict factors* (Tall & Vinner, 1981) and they are not evident to the individual until a stimulus causes the conflicting images to be evoked simultaneously and create confusion, in which case they are referred to as *conflict factors*.

The term *concept definition* is referring to "the form of words used to specify that concept" (Tall & Vinner, 1981: 152). The concept definition might be a reflection of an evoked concept image associated with the definition or a rote memorization of a given definition with little or no meaning to the student. We adopt Tall & Vinner's (1981) differentiation between the *personal concept definition,* constructed by the individual, and the *formal definition* of a concept, the definition accepted by the mathematical community as a whole. The personal concept definition might contain aspects not included in the formal definition and/or ignore others. Finally, the (personal) concept definition creates its own concept image, which is part of the concept image as a whole, called *concept definition image*. Tall & Vinner (1981) argue that potential conflict factors can be an obstacle in understanding the formal theory, especially the ones that are in contrast with the formal concept definition. Warwo et al. (2011) investigated students' concept images of subspace and the links students create with the formal definition of a linear subspace.

Bingolbali & Monaghan (2008) demonstrated how the construct of concept image – concept definition can be used in socio-cultural research. They argued that although concept image is unique to the individual there are aspects that are shared among students. They link these aspects to teaching and shared experiences in the department they are studying.

In this paper we adopt the original concept image – concept definition framework (Tall & Vinner, 1981) along with its more recent developments (Bingolbali & Monaghan, 2008) to design tasks that can enrich the understanding of linear span of undergraduate Mathematics students when used in situations which encourage interaction among students and tutors. We believe that this framework can be easily understood and used by mathematicians. Nardi (2006) presents evidence from discussions with mathematicians which support this idea. Therefore, we find this framework useful as a means to communicate our design and findings both to Mathematics lecturers and researchers in Mathematics Education.

In designing the tasks, we take into account Sierpinska's (2000) remarks about theoretical thinking. To be more specific, the task should have characteristics that correspond to theoretical thinking, such as opportunities for conscious reflection, connections between related concepts or different representations and attention to contradictory thoughts. Harel (2000) emphasizes the need for curricula tailored to students' needs which aid the understanding of abstract concepts in Linear Algebra. He proposes three principles that we take into account in designing the tasks. That is, the tasks should include familiar concepts that allow connection with prior knowledge and language (concreteness principle), they should justify the need of linear span (necessity principle) and allow generalization of the key ideas (generalizability principle).

We identify the following principles based on the theoretical framework, the concept of linear span as thought in the course "Geometry and Linear Algebra" as well as the needs of our students.

1. <u>Include key aspects of linear span:</u> Closure under the operations of a vector space; Every vector is a linear combination of the set of generators; No limitation in the choice of the set of generators
2. <u>Tackle potential conflict factors:</u> The difference between linear combination and linear dependence; Modes of representation (Hillel, 2000)
3. <u>Promote theoretical thinking (Sierpinska, 2000):</u> Reflection; Connections between different representations; Attention to contradictory thoughts
4. <u>The three principles of teaching and learning Linear Algebra (Harel, 2000):</u> Concreteness principle; Necessity principle; Generalizability principle
5. <u>Promote discussion:</u> among the students; between the students and the tutor

METHODOLOGY

The aim of this work is to investigate the conflict factor identified earlier through tasks that are designed to foster the ground for this conflict to emerge and to be discussed with the students. We present data collected during a preliminary evaluation of the tasks through semi-structured interviews with seven students who had attended the course "Geometry and Linear Algebra" the previous semester. The analysis of this preliminary evaluation is expected to answer the following questions: Can the tasks tackle this potential conflict factor? What are the roots of this conflict factor? Does the discussion around the task help students resolve their misconceptions? Do students find the tasks interesting and/or useful?

The following table summarizes the information about the seven participants.

	Mathematics			Applied Mathematics	
	1st Year	*2nd Year*	*3rd Year*	*1st Year*	*2nd Year*
Male	0	1	0	1	0
Female	3	0	1	0	1

Prior to the interviews each student was given a folder including the task and other necessary information. The students had one week to attempt and review the tasks before the interviews. All interviews were videotaped. To ensure confidentiality each student was assigned and referred to with an alias.

ANALYSIS

The first task is based on an exercise from the book "Linear Algebra: Concepts and Methods" by Antony and Harvey (2012). Its structure was slightly altered to fit that of the course notes (Kourouniotis, 2014). It aims to create connections with prior knowledge, known processes and language under the new context and introduce to students basic ideas linked with the concept through algebraic and geometric representations of the notion. The task is divided into three interconnected sub-tasks as a scaffolding strategy to support students.

Task 1: Consider the vectors:

$$v_1 = (-1, 0, 1),\ v_2 = (1, 2, 3),\ w_1 = (-1, 2, 5),\ w_2 = (1, 2, 5)$$

i) Show that w_1 can be expressed as a linear combination of v_1 and v_2, but w_2 cannot be expressed as a linear combination of v_1 and v_2.

ii) Explain what subspace of $\mathbf{R}^3$ is spanned by v_1, v_2 and w_1. Explain what subspace of $\mathbf{R}^3$ is spanned by v_1, v_2 and w_2. What do you observe?

iii) Show that the vectors v_1, v_2, w_1 and w_2 span $\mathbf{R}^3$, that is for every $u = (x, y, z)$ there are a, b, c, d such that:

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Show also that every vector $u \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

The first, introductory, sub-task aims to support students' theoretical thinking in the following sub-task by limiting its focus on calculations. This task was completed by all the participants without difficulty prior to the interview. The second sub-task is expected to enrich students' image of linear span by making connections between the algebraic and geometrical representations of the concept in $\mathbf{R}^3$. It may also motivate students to seek a deeper connection between Analytic Geometry and Linear Algebra. This sub-task was completed by 5 students. Finally, the third sub-task aims to create a link between the relation of the given vectors and the number of ways arbitrary vectors can be expressed as a linear combination of the elements in the set.

Sub-task (iii) proved to be the most difficult for the participants, being completed by only 2 students before the interviews.

In more detail, the students who did not complete (ii) appeared to have trouble with methodology. The students are expected to know from the first part of the course what the geometric representation of a 1-, 2- or 3-dimensional subspace of $\mathbf{R}^3$ is, therefore one has to connect this idea with the notion of linear span and check if the given vectors are linearly dependent. In both cases the students did not make this connection beforehand but the problem was quickly resolved through discussion. Apart from that, six out of the seven students found the question *"what do you observe?"* useful. This question was added to the task as an encouragement for reflection on the effect that different choices of vectors have on the outcome and to promote discussion. In particular, three of the students indicated that they might not have given a second thought to their result if it wasn't for this question. One of the students found the question stressful, although she had successfully answered it. Her reaction is significant to us at this point. Clute (1984) found that students with higher anxiety levels can benefit more from instrumental approaches. Open questions, such as the above, are not frequent in Greek secondary education. It is therefore reasonable to assume that some students would have difficulty (and in some cases anxiety) answering this question in a problem workshop.

While discussing sub-task (ii) an unexpected observation was made by two of the students. These students interestingly replied that the span of the vectors v_1, v_2 and w_1 is the vector space $\mathbf{R}^2$. This conflict factor is called by Wawro et al. (2011: p. 13) the "nested subspaces". Based on their evidence they hypothesized that this confusion has roots in students identifying any 2-dimensional subspace of $\mathbf{R}^n$ with $\mathbf{R}^2$ and suggested that lecturers must be aware of this as a potential conflict factor. Their hypothesis was confirmed in these cases too.

In trying to answer sub-task (iii) the biggest pitfall was following the same reasoning used in subtask (ii). This approach will not help answering the second part which requires from students to solve a system of linear equations. Despite the instructions included in the Task, four out of the five students who didn't complete (iii), tried to use the same approach as in (ii). Additionally, three of them faced a difficulty making use of the proposition "*for every $u = (x, y, z)$ there are a, b, c, d such that $u = av_1 + bv_2 + cw_1 + dw_2$*" and did not manage to recognize the random vector $u = (x, y, z)$ as a parameter of the problem. Instead they identified it as another variable. In each case the task was completed with the help of the interviewer but we find that subtask (iii) required more guidance from the part of the interviewer compared to subtask (ii). The fifth student managed to solve the required linear system but she could not make a connection between the infinite number of solutions and the fact that the four vectors are more than enough to describe any vector in $\mathbf{R}^3$.

The second task was created to address potential conflict factors in relation to the notions of linear combination and linear dependence in the context of linear span.

The idea for this task was based on our goal to promote theoretical thinking and discussion. The conflict is given to the student as a statement - challenge and the goal is to find an example to support the given proposition. It is expected that students will first use a trial and error approach by reaching for appropriate vectors in their example space (Mason & Watson, 2008). This approach will probably fail if students are not able to identify what are the key relations between v_1, v_2 and w in the proposition. If one's concept image includes conflicting ideas about the status of vectors in a set of generators, it might be difficult to find an example without careful prompting and discussion. Because of the nature of the problem, we believe that students would want to cross-examine their findings or get some guidance.

> ***Task 2:*** Let v_1, v_2 and w be linearly dependent vectors in $\mathbf{R}^3$. It is possible for w not to be in the space spanned by v_1 and v_2 although v_1, v_2 and w are linearly dependent. Give an example. Why do you think this can happen?

Moving on to the interviews, only one student had found an example of three vectors fulfilling the requirements of the task before the interview. In four of the seven cases clear signs of conflicting images emerged. This reinforces our preliminary hypothesis that students struggle with identifying the difference between the notions of linear combination and linear dependence. Furthermore, it might be an indication that Task 2 can help potential conflict factors to emerge and be resolved in a controlled environment. The following quotations capture these observations.

> Minos: So, what I thought was that I can have two vectors... which will be linearly independent that will span a plane in $\mathbf{R}^3$. I can of course... I am sure that I can find another third vector that will not belong in the plane but the relationship to be true... these three vectors to be linearly dependent.

Minos' evoked concept image of the linear span is geometric. He thinks of the span of the two vectors as a plane and he tries to find an example by checking vectors that are not on that plane. Of course, if the two vectors are linearly independent, adding a third vector that does not belong in their span will result in a linearly independent set. It seems that either this fact is not part of his concept image or his evoked concept image does not include this information because of the phrasing of the task.

In the following two quotations, the conflict can be directly connected to our preliminary findings in Papadaki (2017). The students seem to struggle with the idea of three vectors being linearly dependent and at the same time one of them not being able to be expressed as a linear combination of the others.

> Interviewer: Well, so for w not to belong in the span of the two other vectors it could not be written as a linear combination of them…

> Pasiphae: Yes… yes… well… But then how can they be linearly dependent? They are all together linearly dependent…

The student thinks of the two notions as equivalent. She later justifies her thinking by stating that if they are linearly dependent she can solve the algebraic equation $av_1 + bv_2 + cw = 0$ for any of the three vectors. Similarly, Ariadne describes her own experience with the task. It is worth mentioning that later in the interview Ariadne successfully refers to the (personal) definitions for both concepts.

> Ariadne: To begin with, to me it seemed absurd at first... because… what does it tell me? It tells me that they are linearly dependent, so if I solve for w, I will find a linear combination, so based on the theory it belongs to the subspace spanned by v_1 and v_2.

In Ariadne's case, it can be assumed that although her concept definition for linear dependence includes the information that the coefficients a, b and c are not all zero, in her evoked concept image this statement is replaced by none of them being zero.

The quotations depict two possible roots of students' difficulties with the task. That is, thinking of the linear span of two vectors as necessarily a 2-dimensional subspace or thinking of the algebraic representations of linear dependence and linear combination as equivalent.

Task 2 was thoroughly discussed with the students using different approaches based on the line of thinking of the students, but also influenced by the interviewer. The ideas portrayed in this task were discussed using an algebraic approach with four of the students and geometrically with two of them. In each interview the final example was found by the students using an informed trial and error approach. All six students reported that the discussion was very useful and Task 2 is important for understanding the concept. Three of them also said that this was the task that made them the biggest impression and four of the students suggested that it would be better if this task was presented to them in a problem workshop after a sequence of related more instrumental tasks.

Concluding, four of the students reported that they understand a notion better through examples and tasks. The way that students' concept image is formed through model examples and experience, is of course well known. What is important is the fact that the students are aware of this happening. This last observation is an indication why it is crucial to pay attention to the examples and tasks used in any course. There are students who are consciously depending on them and expect to understand the "mysterious" concepts that the lecturer is talking about through them.

RESULTS & DISCUSSION

The analysis of the interviews gave us very important information about how tasks can be improved and used in a problem workshop for an introductory course on Linear Algebra. Although all students indicated that they found the tasks useful they gave us opportunities to reflect upon their design and experiment with different tactics which can be used by tutors in an attempt to make the most out of these tasks.

Beginning with the first task, students appeared to have particular difficulty in subtask (iii). One reason might be that (iii) requires a shift in thinking and cannot be fully answered by using the same approach as in subtask (ii). In an attempt to resolve this issue we are also considering a slightly different version of this part of the task that forces students to begin with the shifted approach as follows:

Show that for every u = (x, y, z) there exist a, b, c, d such that:

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Conclude that v_1, v_2, w_1 and w_2 span $\mathbf{R}^3$. Moreover, show that every vector $u \in \mathbf{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

Another observation we made while discussing Task 1 with the students was that of “nested subspaces”. This is another conflict factor we didn’t take into account at first and realized it only during the interviews with the students. Our observation is in line with the hypothesis of Warwo et al. (2011).

Task 2 was fruitful both in terms of meaningful discussion and reflection. Students found Task 2 important for understanding the concept of span. We also observed manifestations of cognitive conflict which indicates that the task can be used as a means to resolve potential conflict factors. Different approaches can be used to discuss these conflicts with students (algebraically, geometrically or by trial and error). A useful tactic might be to discuss the conflicting factors using more than one representation of vectors with the same group of students.

In addition, the indications about the need of examples and tasks made by the students were of great importance. This fact depicts the necessity of well thought examples and tasks in order to help students create a coherent concept image.

This paper presents an approach on how lecturers can design tasks inspired by their observations on students’ misconceptions and taking advantage of the research in Mathematics Education. The framework could be used as guidelines for tutors that are interested in developing tasks for a Linear Algebra course based on their students needs and related research. Finally, the tasks need to be tested in a problem workshop and be compared to other tasks aiming to familiarize first year Mathematics undergraduates with the concept of linear span.

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Delineating Aspects of Understanding Eigentheory through Assessment Development

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In this report, we share insights we have gained from developing an assessment for documenting students' understanding of eigentheory. We explain the literature and theory that influenced the assessment's development and share question examples. We frame our results in terms of three eigentheory settings ($A\boldsymbol{x} = \lambda\boldsymbol{x}$, $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ and eigenspaces) and four interpretations (numeric, algebraic, geometric, and verbal). Results from our analysis include students' reasoning being influenced by setting, insights into students' struggle with understanding eigenspaces, and the importance of making connections between and across various interpretations.

Keywords: Teaching and learning of linear and abstract algebra, teaching and learning of specific topics in university mathematics.

INTRODUCTION

Linear algebra is particularly useful to science, technology, engineering and mathematics (STEM) fields and has received increased attention by undergraduate mathematics education researchers in the past few decades (Dorier, 2000; Artigue, Batanero, & Kent, 2007; Rasmussen & Wawro, 2017). A specifically useful group of concepts in linear algebra is eigentheory, or the study of eigenvectors, eigenvalues, eigenspaces, and other related concepts. Eigentheory is important for many applications in STEM, such as studying Markov chains and modeling quantum mechanical systems; however, research specifically focused on the teaching and learning of eigentheory is a fairly recent endeavor and is far from exhausted.

As part of our ongoing research program analyzing students' understanding of eigentheory (e.g., Watson, Wawro, Zandieh, & Kerrigan, 2017; Wawro, Watson, & Christensen, 2017), we created an assessment instrument focused on the multifaceted and interconnected nature of eigentheory. The purpose of this paper is to describe insights have we gained about students' conceptual understanding of eigentheory as a result of developing, using, and refining this assessment instrument.

THEORY AND LITERATURE REVIEW

We ground our work in the Emergent Perspective (Cobb & Yackel, 1996), which assumes that mathematical development is a process of active individual construction and mathematical enculturation. In this paper we focus on the former by analyzing mathematical conceptions that individual students bring to bear in their mathematical work (Rasmussen, Wawro, & Zandieh, 2015). The literature on the teaching and learning of eigentheory points to several aspects important to students' conceptual understanding. Here we summarize that literature by highlighting what we found to be important aspects for building a working model for understanding eigentheory.

Literature on student understanding of eigentheory

Thomas and Stewart (2011) found that students struggle to coordinate the two different mathematical processes (*matrix* multiplication versus *scalar* multiplication) captured in the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense of equality as "yielding the same result," an interpretation that is nontrivial or even novel to students (Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). Furthermore, students have to keep track of multiple mathematical entities (matrices, vectors, and scalars) when working on eigentheory problems, all of which can be symbolized similarly. For instance, the zero in $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ refers to the zero vector, whereas the zero in $det(A - \lambda I) = 0$ is the number zero. This complexity of coordinating mathematical entities and their symbolization is something students have to grapple with when studying eigentheory.

Thomas and Stewart (2011) also posit that this struggle to coordinate may prevent them from making the needed symbolic progression from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, which is central to determining the eigenvalues and eigenvectors of A. In their genetic decomposition of eigentheory concepts, Salgado and Trigueros (2015) posit that students need to interpret the procedure of finding eigenvectors and eigenvalues of A as determining the solution set of the homogeneous system of equations created by the matrix equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. Harel (2000) posits that the interpretation of "solution" in this setting, the set of all vectors $\boldsymbol{x}$ that make the equation true, entails a new level of complexity beyond solving equations such as $cx = d$, where c, x, and d are real numbers. When considering the notion of eigenspace in particular, Salgado and Trigueros (2015) found that students struggle to coordinate the number of eigenvectors corresponding to a given eigenvalue with the dimension of the space spanned by the eigenvectors. Thus, understanding eigentheory not only involves coordinating mathematical processes and entities but also equations and solution sets.

In addition, students have to make sense of instructors' frequent movement between geometric, algebraic, and abstract modes of description, but this may be challenging (Hillel, 2000). In fact, Thomas and Stewart (2011) found that students in their study primarily thought of eigenvectors and eigenvalues symbolically and were confident in matrix-oriented algebraic procedures, but the majority had no geometric or embodied views. In contrast, other researchers have shown how exploration through dynamic geometry software (Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Nyman, Lapp, St John, & Berry, 2010), geometric interpretations of a linear transformation (Zandieh, Wawro, & Rasmussen, 2017), or real-world contexts (Salgado & Trigueros, 2015) can help students develop conceptual understanding of eigentheory. We similarly agree on the importance of understanding eigentheory concepts in multiple ways and navigating between various interpretations, and we incorporate this complexity in our model of student understanding of eigentheory.

Working Model of Understanding Eigentheory

Regarding what it may mean to have a conceptual understanding of eigentheory, our current working model is a network of connections within and across three main settings of how eigentheory is framed. The three sets of relationships that are pertinent are: (1) relationships indicated by the eigen-equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$; (2) relationships indicated by the homogeneous form of the eigen-equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$; and (3) relationships indicated by a linear combination of eigenvectors. Within the first two settings, what is most frequently the focus of inquiry is one particular eigenvector $\boldsymbol{x}$ for either form of the eigen-equation. However, when considering the collection of all $\boldsymbol{x}$ that satisfy either eigen-equation, one arrives at the *eigenspace* of A associated with λ. The relationships between vectors in the same eigenspace are the focus of the third setting. For instance, if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are eigenvectors of A with eigenvalue λ, then all vectors that are a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ (i.e., $span\{\boldsymbol{x}_1, \boldsymbol{x}_2\} = k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2$ for scalars k_1 and k_2) are also eigenvectors of A associated with λ. Furthermore, reasoning about relationships in this third setting almost certainly involves reasoning about either the first or second setting as well. Each of these settings includes entities and relationships between those entities that may be realized in various ways. We organize this variability in our model according to four main interpretations: graphical, numeric, symbolic, and verbal.

THE EIGENTHEORY MCE ASSESSMENT

The development of the Multiple Choice Extended (MCE) assessment instrument for eigentheory grows from our prior work in student understanding of span and linear independence in which we developed the MCE-style question format (Zandieh et al. 2015); questions begin with a multiple-choice element and then prompt students to justify their answer by selecting all statements that could support their choice (see Figure 1). This format was inspired by existing conceptually-oriented assessment instruments in undergraduate mathematics and physics (e.g., Carlson, Oehrtman, & Engelke, 2010; Hestenes, Wells, & Swackhamer, 1992; Wilcox & Pollock, 2013).

Development of the Eigentheory MCE involved multiple steps. First, we compiled a database of questions about eigenvectors, eigenvalues, and related concepts from research on student understanding of eigenvectors and eigenvalues (e.g., Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Thomas & Stewart, 2011), online resources for clicker and classroom voting on linear algebra (Cline & Zullo, 2016), and previous linear algebra homework assignments, exams, and interview protocols used by research team members (e.g., Henderson et al., 2010). Second, we used research results regarding students' understanding of eigentheory from the literature, as well as our own teaching experience and theoretical thinking, to determine which questions seemed to address important aspects of what it may mean to have a conceptual understanding of eigentheory. Third, the most promising questions were edited into the MCE format, which involved moulding the problem into a multiple-choice question and developing six corresponding justification choices that required students to reason within and between various eigentheory settings and interpretations. Fourth, through multiple rounds of administering the assessment to students, analysing the data, and subsequent refinement, we arrived at the current Eigentheory MCE. It contains six questions, each with six justification choices; five questions are in Figure 1 (the sixth is omitted because of space limitations).

1. The matrix $A = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$ has $\lambda = 6$ as one of its eigenvalues. Which of the following vectors is an eigenvector of A with corresponding eigenvalue $\lambda = 6$?

(a) $\boldsymbol{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ ←**CORRECT**

(b) $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Because … (select ALL that could justify your choice)

(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement.

(ii) This vector $\boldsymbol{x}$ is the only vector in $\mathbb{R}^2$ for which $A\boldsymbol{x} = 6\boldsymbol{x}$.

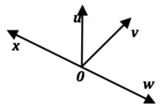
(iii) This vector $\boldsymbol{x}$ makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement

(iv) Subtracting 6 from the diagonal of A yields this vector $\boldsymbol{x}$ as a column vector of the resulting matrix.

(v) The vector $A\boldsymbol{x}$ is 6 times the magnitude and in the same direction as this vector $\boldsymbol{x}$.

(vi) The matrix A also has $\lambda = -3$ as an eigenvalue.

2. Suppose the vector $\boldsymbol{x} \in \mathbb{R}^2$, in the two-dimensional sketch below, is an eigenvector of a 2x2 matrix M with real-valued eigenvalues. Which of the vectors $\boldsymbol{u}, \boldsymbol{v}$ or $\boldsymbol{w}$ illustrated in the figure could be the result of the product $M\boldsymbol{x}$?



(a) $\boldsymbol{u}$

(b) $\boldsymbol{v}$

(c) $\boldsymbol{w}$ ←**CORRECT**

(d) Not enough information is given to know a possible result of the product $M\boldsymbol{x}$

Because … (select ALL that could justify your choice)

(i) This vector is on the same line as $\boldsymbol{x}$.

(ii) This vector and $\boldsymbol{x}$ form a linearly independent set.

(iii) This vector is in $span\{\boldsymbol{x}\}$.

(iv) This vector is a scalar multiple of $\boldsymbol{x}$.

(v) The matrix M needs to be known to determine what $M\boldsymbol{x}$ could be.

(vi) There exists a scalar c such that $M\boldsymbol{x} = c\boldsymbol{x}$.

3. Suppose A is a $n\times n$ matrix, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. Is $\boldsymbol{v}$ an eigenvector of A?

(a) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 2 ←**CORRECT**

(b) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 5.

(c) No, $\boldsymbol{v}$ is not an eigenvector of A.

Because … (select ALL that could justify your choice)

(i) $\boldsymbol{v}$ is a linear combination of eigenvectors that have the same eigenvalue.

(ii) The set $\{\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{z}\}$ is linearly dependent.

(iii) A linear combination of eigenvectors does not result in another eigenvector

(iv) $A\boldsymbol{v} = A(5\boldsymbol{y} + 5\boldsymbol{z}) = 5A\boldsymbol{y} + 5A\boldsymbol{z} = 5 \cdot 2\boldsymbol{y} + 5 \cdot 2\boldsymbol{z} = 2(5\boldsymbol{y} + 5\boldsymbol{z}) = 2\boldsymbol{v}$.

(v) $\boldsymbol{v}$ is an element of the eigenspace created by the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.

(vi) $A\boldsymbol{v} = A(5\boldsymbol{y} + 5\boldsymbol{z}) = 5A\boldsymbol{y} + 5A\boldsymbol{z} = 5 \cdot 2\boldsymbol{y} + 5 \cdot 2\boldsymbol{z} = 5(2\boldsymbol{y} + 2\boldsymbol{z}) = 5\boldsymbol{v}$.

4. An eigenvalue of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is:

(a) $\lambda = 2$ ←**CORRECT**

(b) $\lambda = 3$

Because … (select ALL that could justify your choice)

(i) The equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ when using this eigenvalue.

(ii) This eigenvalue makes $(A - \lambda I)$ noninvertible.

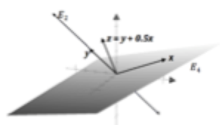
(iii) When acted on by matrix A, there is a line of vectors in $\mathbb{R}^2$ that are stretched by the amount of this eigenvalue.

(iv) There exists a vector $\boldsymbol{x}$ such that the product of the matrix A and $\boldsymbol{x}$ is the same as the product of this eigenvalue and $\boldsymbol{x}$.

(v) This eigenvalue makes $A\boldsymbol{x} = \lambda\boldsymbol{x}$ a true statement for $\boldsymbol{x} = \boldsymbol{0}$.

(vi) This eigenvalue is a solution to $\det(A - \lambda) = 0$.

5. Suppose a 3x3 matrix B has two real eigenvalues: for eigenvalue 2 its eigenspace E_2 is one-dimensional, and for eigenvalue 4 its eigenspace E_4 is two-dimensional. Also suppose that vector $\boldsymbol{x} \in \mathbb{R}^3$ lies on the plane created by the eigenspace E_4 and $\boldsymbol{y} \in \mathbb{R}^3$ lies on the line created by the eigenspace E_2, as illustrated in the graph.



If $\boldsymbol{z} = y + \boldsymbol{0.5}x$, which of the following is true?

(a) The vector $\boldsymbol{z}$ is an eigenvector of B with an eigenvalue of _____ [fill in the blank]

(b) The vector $\boldsymbol{z}$ is not an eigenvector of B. ←**CORRECT**

Because … (select ALL that could justify your choice)

(i) $B\boldsymbol{z} = B(\boldsymbol{y} + 0.5\boldsymbol{x}) = B\boldsymbol{y} + 0.5B\boldsymbol{x} = 2\boldsymbol{y} + 0.5(4\boldsymbol{x}) = 2(\boldsymbol{y} + \boldsymbol{x}) = 2\boldsymbol{z}$

(ii) A linear combination of eigenvectors is always an eigenvector.

(iii) $B\boldsymbol{z}$ is not a scalar multiple of $\boldsymbol{z}$

(iv) $\boldsymbol{z}$ does not lie in the plane E_4 or on the line E_2.

(v) $\boldsymbol{z}$ is in $Span\{\boldsymbol{x}, \boldsymbol{y}\}$

(vi) The eigenvalue of $\boldsymbol{z}$ is a weighted average of the eigenvalues of $\boldsymbol{x}$ and $\boldsymbol{y}$

Figure 1: Questions 1-5 from the Eigentheory MCE

The MCE questions were created to elicit student thinking about eigentheory within and across the settings and interpretations within our working model of understanding eigentheory. For example, the stem of Question 1 is a numeric interpretation; its given justifications for students to choose as true and relevant, we see that justification (i) is a symbolic interpretation in the $A\boldsymbol{x} = \lambda\boldsymbol{x}$ setting, (iii) is a symbolic interpretation in the $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ setting, and (v) is a geometric interpretation in the $A\boldsymbol{x} = \lambda\boldsymbol{x}$ setting (Figure 1). As students choose justifications that support their answer to the main question, they are prompted to reason about eigenvectors and eigenvalues within and across multiple settings and interpretations.

METHODS

The Eigentheory MCE was given to two introductory linear algebra classes taught by the same instructor at a large, research-intensive public university in the United States at the end of Spring Semester 2016. The course utilized the Inquiry-Oriented Linear Algebra (http://iola.math.vt.edu) curricular materials and Lay (2012) as its textbook. One class (of 29 students) received the MCE with given closed-ended justifications (see Figure 1), and the other class (of 28 students) received an open-ended version where students had to write their own justifications for their multiple-choice answer; we refer to these as Class C and Class O, respectively. Students had 20-25 minutes to work on the assessment. All student work referred to in this paper is labelled “B#.”

Analysis of the closed-ended MCE consisted of entering the data into spreadsheets and looking for trends such as: (a) common sets of justifications that students selected or did not select; (b) how selecting certain justifications may have influenced students’ multiple choice selection; and (c) instances in which we would have expected students to select what we viewed as related justifications, but they did not. We used Grounded Theory (Glaser & Strauss, 1967) to characterize the concepts students brought to bear in their justifications in the open-ended MCE, coding independently and discussing our results as a team to find emerging themes. Finally, we compared students’ responses across questions and across classes to discover further insight into student understanding of eigentheory.

RESULTS

We include four insights into student understanding of eigentheory discovered from our MCE data analysis. These selected results are organized by settings (sections 1-2) and interpretations (section 3) from our working model of understanding eigentheory.

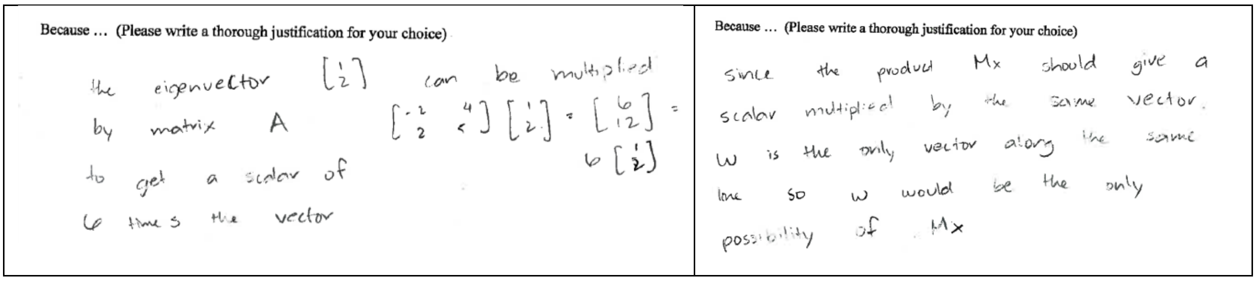


Figure 2: B65’s reasoning within the $\boldsymbol{Ax = \lambda x}$ setting.

Reasoning about relationships within $\boldsymbol{Ax = \lambda x}$ or $\boldsymbol{(A - \lambda I)x = 0}$

We found that as students respond to an MCE question, they seem to situate it within a particular setting, perhaps the setting they are most familiar or comfortable with, regardless of the setting in which the question was initially written. Furthermore, a student’s chosen setting can lead to different ways of reasoning about a problem. We present two illustrations of this from Class O: B65’s justifications for Q1 and Q2, and B66’s justifications for Q1 and Q3. First, in Figure 2, B65 seemed to situate both problems within the $Ax = \lambda x$ setting. On Q1, B65 explained that multiplying the vector $\begin{bmatrix}1\\2\end{bmatrix}$ by the matrix A resulted in six times that vector. On Q2, B65 explained that

Mx needed to be a scalar multiple, and thus the only possible vector would be one along the same line as the vector x, namely the vector w. In both cases B65 emphasized that for an eigenvector, multiplying by the matrix yields a multiple of the original vector, thus working within the $Ax = \lambda x$ setting.

Second, in Figure 3, B66 seemed to situate both problems in the $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ setting. On Q1, we infer this student first found the matrix $(A - \lambda I)$, multiplied each vector from the MCE question by it, and chose the vector that was mapped to the zero vector. Then, on Q3, although what s/he actually writes is idiosyncratic, we can infer s/he was still reasoning with the homogeneous equation, imagining the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ being mapped to zero by the matrix $(A - \lambda I)$, and thus the vector $\boldsymbol{v}$ would also map to zero. In both Q1 and Q3, B66 emphasized an action on the eigenvectors to produce the zero vector, seemingly invoking the $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ setting.

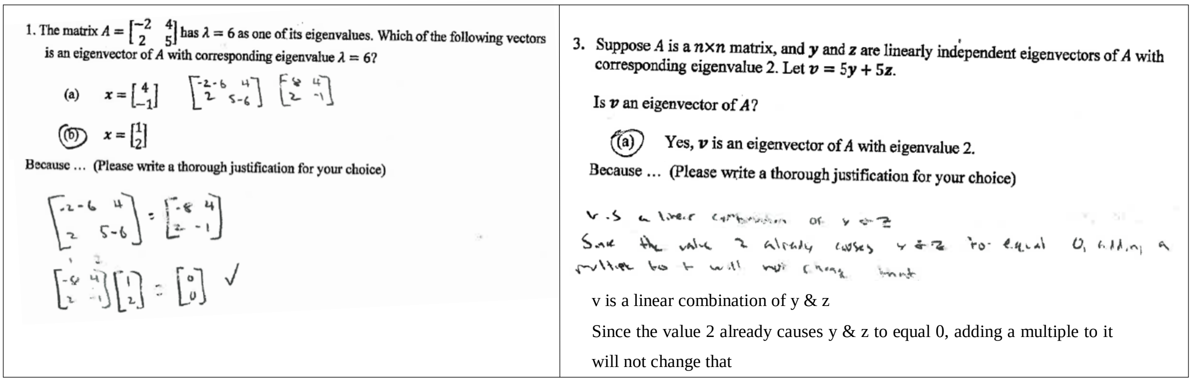


Figure 3: B66's reasoning with the $(A - \lambda I)x = 0$ setting.

We note that the stems of Q1, Q2, and Q3 are not written so as to elicit student reasoning within a particular setting. This allows for use of the open-ended assessment to measure a student's preferred setting or for the closed-ended assessment to measure whether students can interpret the problem in either setting

Reasoning about Eigenspaces

The previous section provides examples of the relationships involved in the first two settings: $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. In this section we attend to the eigenspace setting, which focuses on the relationships involved with scalar multiples or linear combinations of eigenvectors. An eigenspace, like any vector space, is closed under addition and scalar multiplication; thus, a linear combination of vectors in an eigenspace is also an eigenvector with the same eigenvalue as the other vectors in that eigenspace. When asked about eigenspaces, students may draw on these facts and/or may work within one of the previous two settings to derive these principles.

For Q3, only six (of 28) students in Class O circled the correct answer (a) that $\boldsymbol{v}$ is an eigenvector with eigenvalue 2, five chose (b) an eigenvalue of 5, and 16 chose (c) that $\boldsymbol{v}$ was not an eigenvector. Approximately half the students in each group used the phrase " is a linear combination of" as part of their justification (3 for (a), 3 for (b), 7 for (c)). Sample justifications using "is a linear combination of" are given in Figure 4.

Justification given with choosing (a)	Justification given with choosing (c)	Justification given with choosing (b)	Justification given with choosing (b)
$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ which have the same eigenvalue.	No, because $\boldsymbol{v}$ is a linear combination of the two vectors.	$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. Both $5\boldsymbol{y}$ and $5\boldsymbol{z}$ are scalar multiples of their previous form so the resultant vector will be an eigenvector as well	Since it is a linear combination of the other eigenvectors, it would also be an eigenvector.

NOTE: Typed versions are used here to improve readability of students' handwritten justifications

Figure 4. Example of four students' open-ended justifications for Question 3

Very few students in Class O gave justifications that brought in the relationship between eigenvectors and eigenvalues described in the first two settings. One such student was B66, described in the above section. In Class C, however, 13 (of 29) students chose justification (iv) (symbolic $A\boldsymbol{x} = \lambda\boldsymbol{x}$), 11 of which correctly selected answer (a). Because a much higher percentage of Class C circled (a) than in Class O, it is possible that this justification served as a hint that helped some students choose the correct answer. On the other hand, this MCE option allowed us to test whether students recognized the relevance of this set of relationships for the given question.

Although some students who answered (c) used the phrasing "linear combination," their arguments focused more on the linear independence of the vectors. The answers students gave for (c) include: "Eigenvectors must be linearly independent from each other so if $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ then it cannot be an eigenvector," [B58], and "Because they all correspond to the same eigenvalue they all must have unique eigenvectors and $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ and therefore not unique and not an eigenvector of A" [B79]. These answers focus on eigenvectors as necessarily being linearly independent or unique. This focus may come from students remembering that eigenvectors of distinct eigenvalues are linearly independent or that textbook solutions often give an eigenspace basis as the final answer, which may explain students thinking there are only finitely many eigenvectors for an eigenspace.

In Q5, eigenspaces were represented geometrically, and students who completed it were rather successful in selecting the correct multiple-choice answer (14/21 in Class C and 21/26 in Class O). However, many students still focused on finite numbers of vectors. On Q5, reasons given by some students to support the correct choice (b) similarly focused on finite numbers of eigenvectors: "Matrix B already has 3 eigenvectors so there's no room for a 4^{th}" [B59], and "$\boldsymbol{z}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{x}$, and there are already 3 eigenvectors for 3 dimensions, so $\boldsymbol{z}$ cannot be an eigenvector of B" [B66]. We conjecture these students may have been conflating the total number of possible eigenvectors (infinite) for a 3x3 matrix with the number of linearly independent vectors needed to create the bases for the 1- and 2-dimensional eigenspaces. Alternatively, B58's justification for Q5 focuses on dimension: "In a 3x3 matrix there can only be 3 dimensions to the eigenspace. E_2 and E_4 together span the entire space of $\mathbb{R}^3$ so there cannot be another eigenvector of B besides E_2 and E_4." We conjecture grasping the difference between finiteness of dimensions and infiniteness of eigenvectors may be particularly vital for understanding eigenspaces.

Reasoning Across Interpretations

We conclude our results by discussing various aspects of students reasoning across interpretations and the ways in which the MCE afforded that. In particular, we focus on symbolic versus geometric interpretations of eigentheory. On Q1, as noted in the previous section, a majority of students in Class O wrote at least one equation (symbolic interpretation), but *none* wrote anything geometric in their justifications. This could be an indication that students might favour algebraic reasoning over geometric reasoning when justifying their answers to eigentheory questions, even though the classes used the IOLA curriculum which specifically introduces eigenvectors and eigenvalues geometrically. On the other hand, it could be that the numeric interpretation that Q1 was written in did not elicit geometric interpretations from students in their open-ended justifications, or that students see symbolic justifications as more acceptable to the teacher or the broader math community than geometric ones. In a more direct way of assessing students' ability to see connections to the geometric interpretation, the closed-ended MCE gives students the geometric justification choice (v) on Q1, and roughly half (14/29) of the students in Class C selected it. Furthermore, over 80% of the total students from both classes answered the multiple choice stem of Q1 and Q2 correctly (48/57 and 51/57 respectively), demonstrating some ability to reason both numerically and geometrically about eigenvectors and eigenvalues. Because the wording of the MCE questions and justifications makes use of the four different interpretations from our working model, we are better able to assess students' understanding of the symbolic, numeric, geometric, and verbal interpretations in eigentheory, both within and cross settings.

DISCUSSION

Research on student thinking often relies on students' written work on mathematics problems as evidence of how students make sense of or reason about particular content. Our research here is no exception, with student work on the MCE revealing a variety of ways that students understand aspects of eigentheory. However, the MCE's closed-ended justifications extend a written question's ability to examine connections between settings and interpretations that students might not have initially considered or felt the need to include in their justifications. For instance on Q1 in Class O, four students wrote some form of $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ as part of their justification, ten wrote some form of $A\boldsymbol{x} = \lambda\boldsymbol{x}$, four wrote some form of both equations, and ten students did not write either equation. In contrast, on Q1 in Class C, 23 students selected both justifications (i) (symbolic $A\boldsymbol{x} = \lambda\boldsymbol{x}$) and (iii) (symbolic $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$), and only one student selected neither. Hence, when students were forced to consider the two eigentheory settings (i) and (iii), the large majority was able to see both as true and relevant. As other researchers have pointed out the importance of understanding both equations in eigentheory, it is significant that the MCE may give new insight in students' understanding of connections between these two settings.

We do see some potential limitations of the MCE. First, it is more time consuming to take than a simple multiple-choice test, and this affects the number of questions that can be asked. The MCE can also be cognitively taxing because students must consider each justification to determine its truth and relevance. Third, scoring MCE results can be complicated. We hope that further refinement and use of the MCE, as well as developing possible scoring systems, will continue to broaden and deepen the mathematical community's understanding of how students reason about eigentheory.

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TWG 4: Students' practices

The complexity of knowledge construction in a classroom setting

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We study a class of mathematics education MA students in an introductory course on Chaos and Fractals, as they grapple with the Sierpinksi triangle, and in particular with the apparent paradox that its area equals 0, while its perimeter is infinitely long. For this purpose, we network an approach for investigating the construction of knowledge in small groups with one for examining how ideas and ways of reasoning function-as-if-shared in a classroom. Our results show complexities: (i) small group work and whole class discussions mutually influence each other; (ii) ideas may function-as-if-shared in the whole class even if the majority of students have not previously constructed them in their groups; (iii) knowledge constructed in the small groups may or may not later function-as-if-shared in the whole class.

Keywords: Teachers' and students' practices at university level, teaching and learning of analysis and calculus, knowledge construction in classrooms, paradoxes

INTRODUCTION

The research presented here deals with the construction of knowledge in a student centred, inquiry-based classroom, where small group work (SGW) alternates with whole class discussions (WCDs). Construction of knowledge is usually investigated by observing small groups (1 to 4 students) of students. The reason for this is that in larger groups, the density of information for each student is low and does not allow the researcher to interpret their utterances or actions. However, intentional learning more often than not takes place in classrooms with many more than 4 students. We therefore use different approaches for analysing the SGW and the WCDs. The aim of our research is to link the two analyses by following ideas from their emergence in SGW or WCD, via their flow between SGW and WCD settings, until they possibly function-as-if-shared in the class, even though they may not have been constructed by all students. We thus aim at tracing and describing the complexity of knowledge construction across several classroom settings.

THEORETICAL BACKGROUND

The perspective we adopt for analysing the construction of knowledge during SGW is Abstraction in Context (AiC), a theoretical framework for analysing processes of constructing abstract mathematical knowledge (Dreyfus, Hershkowitz, & Schwarz, 2015). AiC methodology begins with an a priori task analysis identifying the new (to the learner) knowledge elements required or useful when solving the task. It then uses a model of three types of epistemic actions – actions pertaining to the knowing of the learners – to analyze their learning processes. The model suggests *constructing* as the central epistemic action of mathematical abstraction. Constructing consists of assembling, interweaving and integrating previous constructs to make a new construct emerge. It refers to the first time the new construct is expressed or used by the learner. Hence, while the term constructing refers to the process, the term construct refers to the outcome of the action.

The perspective we adopt for analysing WCD episodes is documenting collective activity (DCA). Collective activity of a class refers to the ways of reasoning that function-as-if-shared (FAIS) as students work together to solve problems, explain their thinking, represent their ideas, and so on (Rasmussen & Stephan, 2008). These FAIS ways of reasoning can be used to describe the mathematical activity of a group and may or may not be appropriate descriptions of the characteristics of each individual student in the group. The empirical evidence that a way of reasoning is FAIS is obtained by using Toulmin's (1958) model of argumentation, the core of which consists of Data, Claim, and Warrant. Typically, the data consist of facts or procedures that lead to the claim that is made. To further improve the strength of the argument, speakers often provide more clarification, which serves as a warrant for connecting the data to the claim. Backings provide further support for the core of the argument. For examples, see the data analysis below, e.g., in *WCD 9*. The following three criteria are used to determine when a way of reasoning becomes normative: 1) When the backing and/or warrants for particular claim are initially present but then drop off; 2) when certain parts of an argument shift position within subsequent arguments (e.g., a claim shifts to data); or 3) when a particular idea is repeatedly used as either data or warrant for different claims across multiple arguments.

In earlier studies (Tabach et al., 2014; Hershkowitz et al., 2014), we have shown how DCA and AiC combine to provide an in-depth analysis of knowledge shifts in the classroom and of the knowledge agents that initiate these shifts. In Tabach et al. (2017), we articulate why and how the two approaches are theoretically compatible. In this paper, we analyse a lesson where students dealt with an apparent paradox because of its potential to bring to the fore the complex nature of knowledge constructing processes across social settings in a classroom. Specifically, the paradox is an infinite perimeter that delimits a shape with no area, a phenomenon occurring in fractals. While paradoxes are abundant in the study of infinity, we found only two studies relating to similar ones: Sacristán (2001) examined how the coordination of visual and numerical representations supported a single student's resolution of this apparent paradox. Wijeratne & Zazkis (2015) found that their students were hindered by contextual considerations when attempting to resolve a similar paradox of a solid of revolution with finite volume but infinite surface area. Neither of these studies focused on the construction of knowledge in a classroom community.

METHODOLOGY

The setting for the research was a course on Chaos and Fractals at a US university, which formed part of the mathematics requirement toward a master's degree in mathematics education. Participants were 11 students with an undergraduate degree in mathematics, the teacher, and an instructor/observer who occasionally intervened. The teacher and instructor were both part of the research team. Classes took place during one semester twice a week for 75-minutes each; typical class periods alternated between SGW and WCD. During SGW, students worked in four stable groups; they were invited to use huddle boards - one table sized white board per group - in order to promote group communication and to facilitate subsequent whole class presentation of their work. The teacher and instructor went from group to group, trying to understand student thinking and attempting to focus students' activity on what they saw as the main issues; they did this mainly by asking questions but did not otherwise intervene in the SGW. The four stable groups will be numbered 1 (Carmen, Jan and Joy); 2 (Kevin, Elise and Mia); 3 (Soo, Kay and Shani); and 4 (Curtis and Sam). All names are pseudonyms. Groups 1 and 2 were video-recorded during SGW; the class was video-recorded during WCDs. In WCDs, groups had the opportunity to use the huddle boards to share their thinking; there were also teacher led discussions and short lectures whose aim it was to facilitate reflection on issues having been discussed by some groups.

On Day 9 (out of 24), class work was based on an activity about the Sierpinski Triangle (ST). As shown in Figure 1, the ST may be produced by a recursive procedure: Draw an equilateral triangle; connect the midpoints of its sides; remove the middle triangle to get three equilateral triangles (of side ½ of the original one); repeat these steps (including the repetition) for each of the three smaller triangles. The ST is obtained by means of the (infinite) recursion.

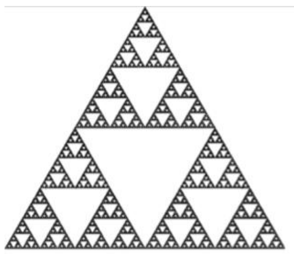

Figure 1. The Sierpinski Triangle (shown after 6 iterations)

The activity was based on a three-part worksheet. In Part A, students were asked to carry out the recursive procedure six times, blackening the removed triangles on the huddle boards. In Part B, they were asked to imagine continuing the recursion forever and to discuss the figure they would obtain, in particular its area and its perimeter. In Part C, they were asked about properties of the resulting figure, especially comparing it to its parts.

The teacher had planned for the students to come up with enough properties in Part C to enable a definition of self-similarity, and as a by-product of learning about self-similarity, to realize that self-similar objects may have finite (or even zero) area and infinite perimeter. However, as will be seen below, the class had its own emergent goals, which led us to investigate student reasoning about area (A) and perimeter (P).

Our task analysis yielded 4 knowledge elements for each of A and P: nature, process, limit, and infinity, denoted A_n, A_p, A_l, A_∞, and P_n, P_p, P_l, P_∞. *Nature* refers to the nature and properties of the region at step *k* whose area/perimeter is being considered. *Process* refers to the process of removing the triangles and computing the relevant area/perimeter. *Limit* refers to the realization that the process is never-ending and the areas form a sequence converging to zero whereas the perimeters form a diverging sequence. *Infinity* refers to the awareness that the actual figure has zero area and an infinitely long perimeter. The difference between limit and infinity may be construed as the difference between potential and actual infinity. These knowledge elements have been formulated in precise language and operational criteria have been fixed to decide whether a student or group of students has constructed each knowledge element. As an example, P_∞ is defined as follows: Eventually, there actually is a figure whose perimeter is longer than any finite curve. Operationally, we will say that a student has constructed P_∞ if the student explicitly claims that the eventual shape or figure or region has an infinitely long perimeter.

The data used in this paper consisted of transcripts from Day 9, images of the groups' huddle boards, and researcher notes taken in the classroom. SGW was analysed using AiC. We present only constructing actions here. In most cases, these will be attributed to the groups rather than individual students; exceptions will be noted. The WCDs were analysed using DCA; for each argument (numbered as A1, A2, etc.), Claim, Data and Warrant were identified, so that the criteria for ideas that FAIS could be applied.

SMALL GROUP WORK AND WHOLE CLASS DISCUSSIONS

The class on Day 9 started by watching and discussing an excerpt from a video about fractals with real world examples including a cauliflower, mountains, a magnetic pendulum and the coast of Britain; this took about 25 minutes and included WCD 1, SGW 2, and WCD 3. Then students were then asked to start working in groups on the worksheet; they spent about 26 minutes on drawing according to Part A and discussing the meaning of repeating the repetition (SGW 4, WCD 5). They used terms such as "infinite loop" and "zooming in". The focus of this paper is their work on Part B of the activity during the remaining 24 minutes, split into three episodes of SGW (6, 8, 10) and three WCDs (7, 9, 11). The class did not reach Part C on Day 9.

SGW 6 (area as process) - The teacher invited the students to develop a conjecture about area and perimeter. After a brief discussion about the perimeter showing confusion (Joy: "in one sense it's infinity, because you keep adding a little bit more. But it should approach a number, right?"), Group 1 focused on area, and attempted to compute the area after one repetition. A reminder by the teacher to produce a conjecture led to a seed of the idea of recursion (Carmen: "That's one-fourth of it, so each term maybe three-fourths of it") thus starting the construction of A_p.

Group 2 quickly came up with a formula (Elise: "So it's three fourths to the *n* of our A_1?") and spent the reminder of the time discussing what *n* means and how to denote things (e.g., A_0 for the initial area). We interpret this as having constructed A_p.

WCD 7 (computations) - The teacher asked whether the groups had come up with a conjecture, and the students reacted by presenting computational results. Kevin presented their result as the sequence $(3/4)^n$ and Joy added that they had just started in this direction after computing the area of the first triangle. Student arguments included only claims and hence were not analysed per the DCA approach.

SGW 8 (perimeter as process) - Group 2 focused on computing the perimeter (Kevin: "we have an additional *a*, we have three halves more *a*", and later Mia: "So, it's like, it's going by a scale of three over two, to the *n*"). The group also made attempts at seeing what happens in the long run (Mia: "the perimeter is just keeps getting bigger, and bigger and bigger"; Elise: "Or is there, like, a limit? That it stops?"). While this points in the direction of P_l, our interpretation is that they have not constructed P_l yet: In spite of them having identified what an expert might see as a diverging geometric sequence, they question whether it converges or not. We also note that some students may be thinking additively rather than multiplicatively.

Group 1 quickly constructed A_p (Carmen: "And then three-fourths of our three-fourths"; Jan: "It's alright, we got enough… to do the whole") and somewhat hesitatingly, A_l (Carmen: "Maybe zero?"; Joy: "No no no, because this is like three-fourths time three-fourths is nine-sixteenths, and after that would be… what? Twenty-seven over sixty-four?"; Carmen: "Is it approach… zero? I think it does"; Joy: "Okay, so you are right, it approaches zero"). There is evidence that they also constructed A_∞ (e.g., Joy: "If you keep filling it in, there's not going to be any white area"). We note that there was no discussion about A_n. However, they then held a long discussion about P_n (e.g., Joy: "So what counts as the perimeter?"; Carmen: "Is it cumulative perimeter?"). Our analysis resulted in the decision that while constructing P_n was under way, it had not yet been achieved. Next, they mentioned aspects of P_p (Joy: "So let's say the perimeter of this is three, we would add in… half of each. So, like, three… times the half") without completing a constructing process. They were reminded by the instructor of the area tending to zero, which brought tension with respect to the perimeter (e.g., Carmen: "if we keep zooming in, there's no area, so there can be no fence [perimeter]").

WCD 9 (the controversy) – This discussion in Group 1 prompted the instructor to ask for the teacher's permission to ask Carmen and Joy to present their controversy to the class. Carmen's argument (A2) used as data "there's no area" and claimed "there'd be no perimeter" with warrant "there's nothing to… nothing to put a fence around it". Joy, on the other hand, argued (A3) for the opposite, using as data "as you zoom in there's more and more to fence", supported by the warrant that one keeps putting in more fencing material. This brought about a suggestion that when one removed a triangle (i.e. colours it black), the perimeter of this black triangle is added to the existing perimeter. In argument (A4) the claim is that "the perimeter of the, the white is also the same as the perimeter of the, perimeter of the black part" (Curtis), and this is based on the data "When you shade it in, you're adding the perimeter of the black" (Kevin) with the warrant that "the fence is guarding both properties" (Carmen). When encouraged by the instructor to explain Carmen and Kevin's thinking, Soo built argument A5: Claim: "So you keep adding the numbers, right?"; Data: "So you have more areas"; Warrant: "You keep zooming in, you're going to get more triangles forming". Next, Mia argued (A6): Claim: "I see the perimeter increasing and then this, the unshaded area is what's left over, and that's constantly decreasing and going to zero"; Data: "You're going to have all these shaded triangles, with perimeters"; Warrant, upon Carmen's question "Is this a cumulative perimeter or a perimeter at a point in time?": "I see it the first way" (Mia). Several more arguments (A7, A8, A9) in this WCD focused on the area decreasing and tending to zero in an unending process of creation. The analysis of arguments in this WCD resulted in two ideas that FAIS:

FAIS A: *Perimeter of white is also perimeter of black*; this was a claim in argument A4 but a justification in argument A6; the justification was that the perimeters of the shaded (black) triangles cumulatively constitute the perimeter of the remaining, unshaded, white area. Hence this idea satisfies Criterion 2.

FAIS B: *The perimeter is cumulative*; this was a claim in A5 and a warrant in A6 (we will see it serving as justification again in A13), and hence also satisfies Criterion 2.

SGW 10 (connections) - Group 1 had a discussion of all four aspects of perimeter, completing the construction of, at least, P_n and P_p. They built their thinking on the fact that they used more and more ink at each stage to draw the additionally generated bits of perimeter, and concluded (Joy): "I thought it went infinitely, because if you zoom in, there's more fencing to put in. And if you zoom in there's more fence to put in". We have no evidence that they constructed P_l and P_∞. In fact, this is unlikely since they only completed constructing P_n and P_p toward the very end of the SGW. Moreover, Carmen, while admitting that the perimeter tends to infinity, insisted that intuitively, no area implies no perimeter.

Group 2 attempted to combine what they knew about the unending processes of area and perimeter. For example, Mia: "There's nothing for the area, but you're still… you're counting the perimeter of what you're taking out" and Kevin: "…as soon as you say - as *n* approached infinity, that means you're going to computation. So, I think what we want is something general, like - the area is getting smaller, but the perimeter is getting larger, and just leave it at that general statement". Our interpretation is that they may have started constructing A_∞ and P_∞ but are still far from completing these constructions.

The constructing processes resulting from the AiC analysis are summarised in Table 1. The table lists only constructs that we have evidence for; in other words, the fact that, for example, A_n does not appear does not mean they have not constructed A_n – it only means that A_n has not been discussed during SGW in a manner that lets us as researchers conclude that A_n has been constructed.

Group	SGW6	SGW8	SGW10

1	(A_p)	A_p A_l A_∞ (P_n) (P_p) (P_∞)	P_n P_p
2	A_p	(P_p) (P_l)	(A_∞) (P_∞)

Table 1. The constructs; parentheses mean (under construction)

WCD 11 (linking area and perimeter processes) - The instructor called on group after group to present their thinking, however tentative. Group 1 used an ink metaphore, the ink being used to draw the additional perimeter bits. In this way, they explained how cutting out further triangles reduces the area while using more ink – thus increasing the perimeter. Carmen, however, added, that for her "that's not resonating". The researchers identified two arguments while this group was reporting (A10, A11) and one (A12) while another group was reporting. During their report, Group 2 connected the perimeter process to the area process like Group 1. Elise built the following argument (A13): Claim: "The area is getting smaller and the perimeter is getting bigger". Data: "You're adding smaller and smaller pieces". Warrant: "…but you're adding those pieces to what you already have". Backing: "The perimeter is, like, all of this, combined with all of this, combined with all of this, combined…". We note that the argument focuses almost completely on perimeter, although the claim equally relates to area. Finally, the report of Group 4 included arguments A14 and A15. We only describe A15, produced by Sam: "So the area would be… go to zero… There would be limited amount of areas, so we're going to have a limited number of… perimeters. So, we don't have infinite number of perimeters". Sam's claim of a "not infinite perimeter" is based on the data that the area goes to zero and hence there is a limited amount of area, with the "limited number of perimeters" serving as warrant. Based on these arguments, we identified two more ideas that FAIS:

FAIS C: *Unending process of creation*; this meets Criterion 3: Continued use of an idea (e.g., *keep adding*) across multiple arguments to describe the process that is being analysed. This idea is related to potential infinity.

FAIS D: *Area going to zero*; this was repeatedly a claim, including in arguments A6, A7, A8, A9, A13 and became data in argument A15, hence satisfying Criterion 2.

The relationship between SGWs and WCDs

As a preliminary, we note the richness and diversity of students' ways of reasoning about area and perimeter, which in a less student-centred classroom might have been quickly undermined with an infinite geometric sequence that is decreasing ($r<1$) and hence tending to zero for area and an infinite geometric sequence that is increasing ($r>1$) and hence tending to infinity for perimeter. We note that the term geometric was once mentioned briefly with respect to area by Sam during WCD 11 ("it's geometric, so it's going to converge. So, the area would be… go to zero") but this was rather toward the end of class and was not picked up by any of the other students. Generally, students seem to have been satisfied by arguments of the type "the sequence is infinite and decreases, hence it tends to 0" (as in the discussion of Group

1 in SGW 8) or “the sequence is infinite and increases” (e.g., Mia and Elise of Group 2 in SGW 8).

The diversity manifests itself, among others, in the metaphors the students used (fence, ink), in a tendency to use, at least initially, numerical considerations for area and perimeter, in the attempt to link the area process with the perimeter process, and the related discussion about the nature of the perimeter as separating the region that belongs to the ST from the one that doesn’t. Little of this was initiated or suggested by the teacher or the worksheet.

This last issue appears as P_n in the AiC analysis of the SGWs and as FAIS A in the DCA analysis of the WCDs. Similarly, there are relationships between the other FAIS ideas and knowledge elements. Table 2 shows these relationships.

FAIS	A	B	C	D
Constructs	P_n	P_p	A_l, P_l	A_l, A_∞

Table 2. Relationship between FAIS and constructs

While FAIS idea A (*the perimeter of white is also perimeter of black*) is related to construct P_n, the relationship between the constructing process of P_n and the arguments establishing A as FAIS is complex. We don’t have evidence of P_n having been constructed in either of the two observed groups before the relevant WCD 9; and the first argument establishing idea A as a claim (A4) was initiated by Curtis but immediately supported by Kevin and Carmen. Moreover, the second argument, in which idea A became a justification, A6, was presented by Mia. This may indicate that the beginning P_n construction we identified in Group 1 was substantial, and maybe even that the discussion in Group 2, which on the face of it focused on P_p caused Kevin and Mia to think about P_n. Finally, we ask ourselves to what extent the constructing process of P_n continued during WCD 9 for Kevin, Carmen and Mia.

FAIS B (*the perimeter is cumulative*) – is similar to FAIS C (and intimately related to it from the point of view of the mathematical content). While we hesitated to claim that P_p has been constructed by Group 2, it is Mia from that group who produced A6 and Elise from that same group who produced A13, the two arguments where the element switched position to becoming a justification and thus allowed us, according to DCA to categorize this idea as FAIS.

FAIS C (*unending process of creation*) exhibits a case in which the two analyses connect rather smoothly. Group 1 constructed A_p and P_p, and Group 2 may be assumed to have implicitly constructed A_p and to be progressing in the constructing process of P_p. The frequent use of this knowledge element in many WCD arguments may indicate a similar situation in the other two groups. Soo from the Group 3 produced Argument A5; and Sam from the Group 4 produced Argument A9.

Finally, the relationship between FAIS D (*area going to zero*) and A_l seems obvious and needs little comment. When this idea functions as if shared in the classroom, it is possible that some of the students relate to construct A_∞ (as shown above for Carmen and Joy) and others think in terms of A_l or even in terms of A_p only. While to the expert, thinking in terms of A_l may be satisfactory at best, and thinking in terms of A_p may be insufficient, such differences are tended to be glossed over in this classroom with respect to the rather basic construct of area, and we may speculate that similar situations pertain to more complex constructs in this and other classrooms.

It becomes obvious that there are many ways, in which SGW and WCD can interact. The relationship is by no means unidirectional from constructing an idea in SGW to this idea FAIS in WCDs. Rather, constructing processes may well be continued or even initiated in WCDs. On the other hand, ideas may FAIS without having been constructed by all or even by a majority of the students in class. For example, we have no evidence for P_l having been constructed in either of the two observed groups, though Group 2 had started this constructing process; but C (*unending process of creation*) is anyway FAIS in relation to both area and perimeter. Of course, it could have been constructed in the groups we have no data on. This raises the question whether an idea can FAIS if it has not been constructed at least in some group. We can only say that we have no example for this having happened in these data.

On the other hand, notions can be constructed by some students, or in some groups, without ever functioning as if shared. In fact, some of the constructs may not have a chance to come up in any WCD. We have no unequivocal evidence for this happening but we do know that in the first few minutes of the next lesson (Day 10), which took place two days later, Kevin referred to the perimeter as an increasing and hence diverging geometric sequence. We could also point to the fact that A_p has been constructed by both analysed groups but does not appear in the lower row of Table 2. However, this argument is weak since A_p appears indirectly as a component of A_l.

CONCLUSION

The complexity of knowledge flow in the classroom, even based on this one class session, is far greater than one might imagine. Inquiry-based instruction features students' deep engagement in mathematics and peer to peer interaction. As such instruction increases at the university level, the field is in need of theoretically grounded approaches for analysing individual and collective mathematical progress. This paper makes a contribution in that direction. A strength of AiC is that the approach allows researchers to gain insight into the ideas that individuals or small groups of individuals construct, as long as the number of students remains small. DCA provides a complementary approach that provides researchers insight into the ideas and ways of reasoning that characterize the collective progress of the classroom community. In Tabach et al. (2017), we showed how the two approaches combine theoretically, and the present paper adds to this a detailed analysis of how an AiC analysis interacts with a DCA analysis to expresses the complexity of knowledge construction in the classroom. An open question, and one that we are currently pursuing with this data, is how to coordinate the small group and classroom level findings with individual interviews conducted shortly after such rich class sessions.

Finally, the analysis presented here also contributes to what we know about how students reason about area and perimeter in a paradoxical situation. In contrast to the findings of Wijeratne and Zazkis (2015), the students in our classroom found ways to profitably use contextual considerations, metaphors, numerical computations, and figural reasoning to support their endeavour to understand the apparent paradox.

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Analysing tasks for the Flipped Classroom from the perspective of Realistic Mathematical Education

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The advent of Flipped Classroom as a framework for organizing the teaching and learning of mathematics has the potential to revitalize the attention to tasks as a vehicle for meaningful learning in tertiary education. Flipped Classroom is based on the idea of student active learning under close guidance of the university teacher. Due to the possibility to engage the students in meaningful discovery of how mathematics can relate to real-life situations during in-class sessions, the tasks are seen to have a central role in a successful implementation of Flipped Classrooms. This paper explores the realistic mathematics education (RME) as a theoretical framework for task design and analysis in the context of in-class flipped engineering mathematics classrooms, and I seek evidence of knowledge construction through analysing students' work on modelling the height of a rider in a double Ferris wheel.

Keywords: Flipped Classroom, RME, Mathematics for engineers, the role of digital and other resources in university mathematics education

INTRODUCTION

One of the fundamental principles of Flipped Classroom (FC) approaches in education is the opportunity it provides for a more interactive and meaningful use of class time. (Bergmann & Sams, 2012). The idea is to use valuable classroom time for the more advanced learning processes like analysing, evaluating, and modelling in a cooperative environment consisting of peers and tutors. A contrast to this form of in-class student participation can be the traditional university teaching style where the lecturer is more or less seeking to "transfer" the mathematics in a monologue fashion (Sfard, 2014).

Moving the lecturing part out of the classroom as a preparatory activity can help remedy the problem lecturers often feel is present during the delivery of traditional lectures, the one of disengaged and passive students. Rather, the social setting of the classroom can be used to let the students discuss, evaluate, try out, and receive assistance with their ideas and conjectures. However, such learning processes need a structuring element, which can be the tasks that students work with, preferably in groups. The purpose of this paper is to advance knowledge on task design and analysis in connection with FC implementation, specifically considering the theoretical framework of RME. I attempt to do this through a case study of students' work with a modelling task that follows the heuristics of RME.

Previous research

FC is an innovative teaching approach, initially described by Lage, Platt, and Treglia (2000) that has many implementations. As the popularity of FC teaching has caught up, interest has been considerable also for tertiary educational settings (Wasserman, Quint, Norris, & Carr, 2017).

Various authors have done research on general issues of FC pedagogy (Abeysekera & Dawson, 2015; Wan, 2015). However, little research has emerged on analysing how task design should be addressed in connection to FC implementation. In particular, I was not able to find any research considering RME as a framework for analysing tasks in connection with FC in-class activities. Several articles seem to call for more research on providing insights on FC pedagogic heuristics (Song, Jong, Chang, & Chen, 2017; Wan, 2015). Task design is of particular importance for the in-class FC component, since the model of FC considers the classroom activities to be the arena for attaining the highest levels of skills and knowledge (Bergmann & Sams, 2012). The challenge is to bridge the out-of-class videos with these in-class activities in a meaningful way.

The particular task analysis performed in this paper is based upon students' group work with mathematizing the double Ferris wheel movement. A study using a similar type of problem is described in Sweeney and Rasmussen (2014). They found that the students' bodily engagement using gestures and measurements by fingers facilitated a link between the movement of the rider and the mathematical model.

THEORETICAL FOUNDATIONS OF RME

According to Van den Heuvel-Panhuizen and Drijvers (2014), RME is a domain-specific instruction theory for mathematics that focuses on rich, "realistic" situations serving to initiate development of mathematical concepts, tools and procedures. Core teaching heuristics give these directions:

- The activity principle emphasises students' direct participation throughout the learning process. Mathematics is considered to be best learned by doing mathematics, which is in accordance with student active learning that is considered a widely accepted principle of FC.
- The reality principle expresses the importance of presenting students with real-life situations and problems that they can imagine and mathematize upon.
- The level principle highlights the idea that students pass through various cognitive development phases, from informal context-related descriptions of the problem towards the use of more formal mathematical language. This process sets the stage for bridging student understanding that the *model of* the context-related situation at hand can become a *model for* similar kinds of problems.
- The intertwinement principle emphasises that mathematical content domain should not be considered as isolated fragments, but rather seen as a connected whole. This principle supports task designs facilitating open problems that stimulate students' own thinking and reasoning about which mathematical solution techniques and mediating artefacts to employ.
- The interactivity principle relates to the idea that learning mathematics is not purely an individual pursuit, but rather a social activity, where group work and

whole-class discussions should be orchestrated by the teacher. This principle aligns well with socio-cultural learning theories, as also emphasised by Cobb, Jana, and Visnovska (2008). They consider semiotic mediation and cultural tools as important means of conveying mathematical meaning in an RME setting.

- The guidance principle refers to the idea of "guided re-invention" of mathematics. Instructional sequences in task design can involve historical evolutionary steps in mathematics as inspiration for rich context problems (Gravemeijer, 1999).

Further, RME makes a distinction between horizontal and vertical mathematization. According to Van den Heuvel-Panhuizen and Drijvers (2014), students use mathematics as a tool to understand and organize problems in a real-life context, typically during task solution. This is called horizontal mathematization. Vertical mathematization refers to mathematizing one's own mathematical activity to reach a higher level of abstraction.

The research question explored in this paper can be framed in the theory of RME:

"*To what extent does RME task design facilitate students' modelling activities and knowledge construction in a FC context?*"

METHODOLOGY

The data for this paper was collected from video filming the work of two groups of computer engineering students in their first semester of study. These student groups were part of a larger research project conducted at a university campus in Norway. FC teaching was performed throughout the whole year of study. I focus on one in-class session where students were modelling the double Ferris wheel.

The Flipped Classroom setting

Before this in-class session, the students had watched an out-of-class session of videos that introduced the unit circle and how the y-component would map to the sine function as the angle rotated (Figure 1). In addition to this, there were videos showing examples on how to make mathematical models using this function, and how one could sketch a particular sine function based on textual information. Lastly, they were showed examples of how to determine the period, amplitude, baseline and phase shift by considering various graphs of this function. These videos were provided using English as the spoken and written language, while students were using Norwegian as working language in class.

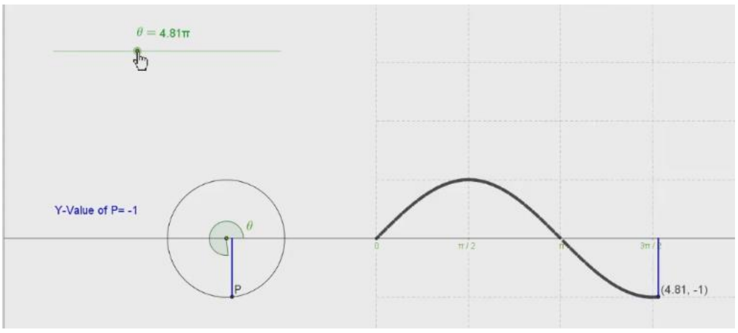

Figure 1: Mapping of unit circle y-component to the sine-function

The particular task that I refer to in this paper was given to the students in connection with the teaching of trigonometric functions during the second week of the semester. In this task, they were asked to create a model of the evolution of the height of a rider on a double Ferris wheel. The double Ferris wheel was represented in an applet available for the students to watch on their computer (Figure 2). The Ferris wheel consisted of two separate wheels rotating at the same speed and centred at the end of a larger rotating bar as seen on Figure 2. The students had the possibility to study each rotation separately by starting/stopping each of them. The height measurement of an imagined rider called Marit (marked with a dot on one of the wheels), was available as a readout at all time. The interested reader can study the applet at http://sigmaa.maa.org/rume/crume2017/Applet.html. The task presented for the students was the following:

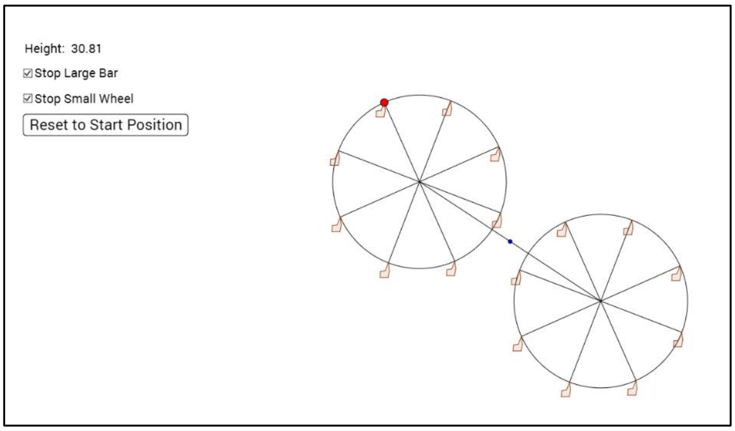

Figure 2: The double Ferris wheel simulation

1. Create a representation of your choice that illustrates Marits' ride on the double-Ferris wheel
2. Create a sketch/graph of Marits' height versus time as she ride on the double Ferris-wheel
3. Create a function for Marits' height versus time as she travels on the Double-Ferris wheel.
4. Graph the functions you found in the previous problem using a graphing utility like geogebra. Does the graph of your function make sense? Is it similar to the graph you sketched?

The design is seen to follow the principles of RME. Firstly, the purpose of the task is to engage students in exploring the dynamics of a familiar situation; the Ferris wheel ride. This meets the demands of the activity and reality principles. Furthermore, the sequence of the task, asking students to initially make a representation of the movement, followed by sketching a graph, and finally creating a function, attempts at making the students move from more informal to formal mathematizing, respecting the level principle. In setting the stage for employing trigonometric functions for mathematical modelling, the intertwinement principle is considered. Further, the interactivity principle is kept through the group work context of the task. The guidance principle, however, will not apply to this task design. For most themes in engineering mathematics, a reinvention can be a considerably challenge to embed in task designs.

Students were placed in groups of 3 or 4 when working with the task, and a 90 minutes session was spent on the task. The author of the paper was the teacher throughout this session[1]. The study was performed under a naturalistic research paradigm to ensure as realistic settings as possible for the observation of students' collaboration (Moschkovich & Brenner, 2000). The students were filmed using a high definition camera that followed the group work for the whole session. Prior to the filming, students had acknowledged their participation in the study through signing a letter of consent. The selected students had watched the videos beforehand. Statistics about which videos and how long the students had spent watching them were available in the tool I used for video-distribution to the students. I analysed each recording using descriptive accounts (Miles & Huberman, 1994), where sessions of data are broken into separate entities of activity. Each such entity were subsequently analysed for content informing us on the research question, and I report on a variety of such episodes in what follows. In the aftermath of the last case, I performed an interview with two selected students to obtain deeper insights about certain episodes during the work with the task and general qualities about videos and the session.

Furthermore, the study was performed on the cohort of 2017/2018, a class consisting of 20 students. The two groups that I report from had highly varied backgrounds, most of them from a pre-calculus course comprising two years of mathematics from upper secondary into 1 year of study at the university. They were acquainted with trigonometric functions through this pre-calculus course before arriving at the engineering study. However, as one of the students mentioned in an interview afterwards, he only remembered sparsely what the various mathematical terms was named by, but claimed to be able to use them anyway. I filmed two of the groups to be capture some of the variation taking place inside the single classroom environment. I will refer to these groups as group 1 and group 2.

I specifically looked for traces of content from the videos, that is, the out-of-class component of FC when analysing students' discourse. Such content can be word use, theoretical elaborations like definitions and theorems, visually mediated ideas and specific examples.

RESULTS

The two groups seemed to develop a rather different focus in terms of how to approach the task. One group spent much more time on the formal mathematizing part of the task, compared to the other one that wanted to understand the movement from a more informal, empirical viewpoint.

Group 1

The students in this group followed the layout of the task in the sense that they attempted to sketch a graph of the movement before moving on towards formal mathematical modelling. This seemed to set the scene for a discussion among the group

[1] A pilot-study in a very similar setting was performed on the previous cohort of 2016. This was part of a cooperation project with San Diego State University, where the research fellow Matt Voigt was responsible for the teaching material in addition to conducting the in-class teaching.

members on which physical properties were influencing the model. They constantly argued about where the "x-line/dotted line" should be, meaning the baseline, in addition to discussing how the various lengths of the rods and rotation times were influencing the amplitude and period. However, synthesizing the two movements into one seemed to form the biggest challenge for the group. For instance, the teacher was consulted on how one could decide the amplitude on a combined sine. After about 15 minutes of trying to comprehend this, the group moved on to a more formal mathematical phase of modelling. One of the students, let us call him Finnegan, appeared to get a breakthrough at about 17:40 (timestamp corresponds to video recording). Figure 3 show this situation where he is pointing to the printed unit circle that expresses various exact values of sine and cosine, while he states:

> "I suggest that we make two functions that are zero when… They are not zero, but $\frac{3\pi}{2}$ at 2π. Then you would have made a function that is zero when the period…Well you have to make two functions…"

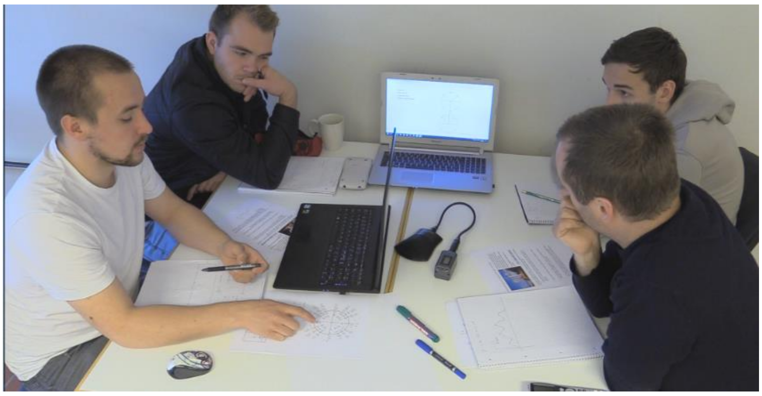

Figure 3: Phase shift of the sine mediated through the unit circle

After this episode, the students seemed to prefer working in two pairs to discuss further details of the functions. Both pairs of students agreed that the model should consist of a sum of two different sine functions.

Analysis

We can see that Finnegan struggles with the correct phrasing of what he is trying to express, but the visual connection between the Ferris wheel circle and the unit circle is guiding Finnegan towards an understanding of phase shifting. The printout of the unit circle became an important mediating artefact for Finnegan, although shared with students earlier for solving trigonometric equations, not for modelling. They spent much of the time getting accustomed to the mathematical terminology with amplitude, period and, as one sees, phase shifts. The informal part of the task where students were asked to elaborate on a sketch of the function seemed to facilitate these discussions, much in line with the interactivity and level principles of RME.

I was unable to make any clear connection to the videos from the discourse of these students. During the interview with one of the students in this group, it was pointed out that using English as the formal language in the videos was difficult to handle.

Group 2

The other group that was filmed during this session utilized a variety of means to depict the movement of the Ferris wheel. In Figure 4 we can see two of the students working in parallel, where one is considering using a graphing calculator, while the other is attempting to graph the model on GeoGebra. On the right PC the applet with the simulation is displayed. Another member in this group even mapped the movement from the PC screen to a sheet of paper, letting the pen trace the movement of a point on the Ferris wheel, similar to what was reported in Sweeney and Rasmussen (2014). The group also attempted other practical means of tracing the movement like taking screenshots in timed intervals and putting the variable height values displayed in the applet in a function table for later analysis. It was even suggested that they would let GeoGebra use these data-points to produce an expression for them using interpolation. After working with various such "empirical" approaches to the problem, mostly on an individual basis, one student arrived at an expression for the model at 49:16. He gave this explanation to the others in the group on how he was able to arrive at a summation of two sinusoidal functions:

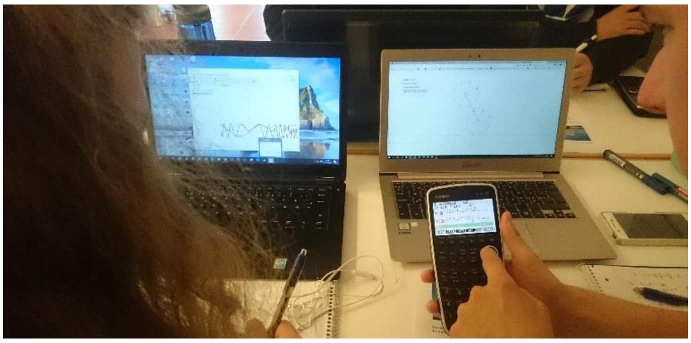

Figure 4: Utilizing various digital tools to model

> "Analyse the small circle, height over time, that became f(x), then we did the same with the large one, called this g(x), then we made a new one h(x) consisting of the two of them together"

Analysis

The big surprise in this session was that the one student that had not prepared watching the videos seemed to be the leading the group discussions. It might seem that RME can act to remedy group dynamics where certain individuals come unprepared. The reality principle in RME stresses the importance of using real-life situations which is likely to provoke engagement, discussions and active participation during group work.

In this particular data set, it seemed hard to find a direct link between not being prepared by the out-of-class session and poor participation in the group work although this was a prevalent finding in other analyses of group work (Fredriksen, Hadjerrouit, Monaghan, & Rensaa, 2017).

An important observation that was made for both these groups was the physical arrangement of four students placed face to face, which made it harder to collaborate on the task. This was a setup originally meant to spur more discussion since the students would face each other. However, it turned out to be a poor design in this particular setting due to the importance of using the laptops for controlling the visual simulation in addition to the actual modelling in a digital graphical environment. It became apparent that two students would look at one PC, while the adjacent students would look at an opposing one, effectively hindering much of the discussion across the group (Figure 3 and Figure 4) The teacher recognized this, trying to convince students to look at the same PC, but they quickly went back to work in pairs again. One of the students also confirmed this observation during the post-interview. Thus, it seems that this configuration interfered with the interactivity principle of RME for the groups.

DISCUSSION

In this study, I investigated to what extent RME task design facilitates students' modelling activities and knowledge construction in a FC context.

The analysis found students were making active use of computers, calculators, printouts of the unit circle, internet, and even their mobile phones to support their exploration of the task. This can be seen to support the *intertwinement* principle: students were actively using various tools, in addition to the mathematics introduced in the videos, to explore the problem.

The second group seemed to have problems progressing towards the formal mathematizing part of the task. An observation made from studying the group dynamics was a quite individualistic working attitude. There seemed to be little collaboration on working towards a common model for the group, effectively violating the *interactivity* principle of RME. During the post-interview with one of the students in the group, I asked him if he shared this impression, which he confirmed, stating that some kind of internal competition among the members evolved.

The *guidance* principle of RME refers to students' re-invention of mathematics. Although there are examples of RME facilitating such inductive discovery (Gravemeijer, 1999), we can conjecture that the FC out-of-class preparatory component turns this table. Through the videos, the students should have attained basic knowledge of mathematics that is to be articulated through the modelling activities in-class. Thus, instead of re-inventing the mathematics, they are rather re-employing it at a given situation and, through this, making conceptual ties towards it.

We can clearly see how the *level principle* of RME is at play in these observations. Initially, students are grappling with understanding the basics of the movement, using gestures, making sketches and talking to each other using informal language. The next stage of the task prompts students to express a formal mathematical description of the movement, which all students achieved to some extent. Although the students work was directed towards making a *model of* this particular movement, the whole-class discussion at the end of the session pointed towards similar situations that this could apply as a *model for*. This was also done in the videos prior to class.

The *reality* and *activity* principle seem to be well safeguarded in this task. This is also supported by the statements by one of the students during the informal interview conducted after the session:

> "To see the task in a physical real framing was motivating"

Throughout the session, students used the familiar terms and concepts related to the sine function. Words like amplitude, phase shift, period and baseline were all referred to, and the students seemed to be able to connect these to the real-life situation of the Ferris wheel. If this really stems from watching the videos or originates from previous experience with the sine function, I have no direct evidence. This of course influences the capacity to answer the research question. Light on this may be shed by statements from the other student interviewed:

> "It's very nice that you can see the videos, think about it for some days, get it refreshed in the lesson and then start working with tasks related to it. This has helped me a lot".

Although this is a statement about the general nature of the teaching layout, it nevertheless gives indication about FC as an instructional platform giving learners a potential for experiencing different motivations for mathematics.

CONCLUSION

When working with task design in mathematics, we employ various theoretical frameworks that has certain sets of design principles that the designer should adhere to (Kieran, Doorman, & Ohtani, 2015). We have seen that the Ferris wheel task design presented in this paper aligns well with the principles of RME for task design. However, designing such tasks for a FC pedagogic context will need an additional component, namely the adoption of the video preparation. Clearly, close attention to the pedagogical structure combining videos and tasks seem to be necessary for facilitating rich discussions, modelling activity and knowledge construction to take place in-class. This statement is supported by the cases that were analysed in this paper, where we saw students engaged in horizontal mathematization utilizing out-of-class preparation through videos and mathematical modelling of realistic situations.

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Engaging with feedback: How do students remediate errors on their weekly quiz

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Maths for Business is a first-year mathematics module for approximately 500 non-mathematics specialists. It has continuous assessment consisting of ten weekly quizzes, worth 40% of the final mark. In 2016/17, students who did not receive the maximum five marks on their weekly quiz were offered the opportunity to resubmit their quiz, with correction(s) and an explanation of their error(s), for one additional mark. We refer to this process as 'remediation'. In this paper, we examine how students remediate their errors in order to identify features of a 'good' remediation. These features are identification, description, and correction of errors. By analysing a subset of students (n=31), we observe that a student's quiz mark, and the cognitive level of the quiz question may impact the nature of the remediation provided.

Keywords: assessment practices in university mathematics education, feedback, remediation of errors, students' practices at university level.

INTRODUCTION

Maths for Business is a core first-year mathematics module for non-mathematics specialists enrolled on three business programmes in University College Dublin, Ireland. Topics from one- and two-variable Calculus are covered in the module and given the cumulative nature of the content, students should (ideally) achieve the learning outcomes for a topic before proceeding to the next. To encourage mastery of learning outcomes, the module has a continuous assessment component consisting of ten weekly quizzes, worth 40% of the final mark. A week after sitting a quiz, graded quizzes are returned to students with a mark out of five, and tutors provide oral feedback to each tutorial group highlighting the most common errors made. In addition, the lecturer provides an online video entitled "Most Common Errors" and posts a pdf of the quiz solutions online. With our focus on mastery, we believe that students who do not get full marks on a quiz should engage with the feedback to identify and remediate their errors in a timely manner. However, Gibbs and Simpson (2004) discuss how, even if timely and good quality feedback is provided to students, there is no guarantee they will engage with it. Therefore, to encourage this engagement, in 2016/17 we offered students who did not receive full marks on a quiz, one extra mark if they resubmitted their graded quiz one week after it was returned with error(s) identified and corrected. We refer to this process as "remediation".

Handley, Price and Millar (2011) propose a shift from examining feedback evaluation and attributes to investigating the process of students' engagement with feedback. To this end, we first wanted to explore: which students were most likely to participate in the remediation process; which feedback resources were they most likely to access; and, whether engagement in the process impacted academic achievement in the module. The analysis and findings from this part of the study are described in detail in Howard, Meehan and Parnell (under review). The main findings were that 70% of students who had the opportunity to remediate did so; the most accessed feedback resource was the pdf of the quiz solutions which were made available online; and, students who achieved an average quiz mark of 3-4 (excluding remediation marks) *and* who consistently engaged in the remediation process, exhibited the most learning gains as measured by their performance on the final examination. Secondly, we wanted to examine *how* students remediated quizzes in order to identify aspects of a "good" remediation, and from these findings, refine the instructions given to students at the start of the module on how to remediate their quizzes. We also want to identify and explore what factors might influence the nature of students' remediations. It is this second part of the study that we wish to focus on in this paper by addressing the following research questions:

1. What ways, in general, do students remediate their weekly quizzes?
2. What instructions would we give to future students to assist them in remediating their quizzes?
3. What factors may influence the nature of a student's remediation?

LITERATURE REVIEW

Assessment and feedback

There have been a number of in-depth reviews in the area of assessment and feedback (Bennett, 2011; Sadler, 1989) with some focusing specifically on higher education (Evans, 2013). Assessment is generally discussed under the headings of *formative assessment*, where the primary objective is to provide feedback to the student and evaluation of students' knowledge is secondary; and *summative assessment*, where the primary role is to evaluate students' knowledge and feedback is secondary. Ramaprasad (1983, p. 4) describes feedback as "information about the gap between the actual level and the reference level of a system parameter which is used to alter the gap in some way". Building on this description of feedback in terms of its effect rather than its content, Sadler (1989) argues that the learner must:

> (a) possess a concept of the *standard* (or goal of reference level) being aimed for, (b) compare the *actual* (or current) *level of performance* with the standard, and (c) engage in appropriate *action* which leads to some closure of the gap (p. 121, italics in original).

While most of the major studies on assessment and feedback relate generally to a variety of subjects, there have been calls for specific domain-focused research (Bennett, 2011). Specifically, in mathematics education at the university level the area of assessment and feedback seems to be under-researched. Of those studies conducted in this area, an emphasis on summative assessment and closed-book examinations has been noted (Iannone & Simpson, 2011; Iannone & Simpson, 2012; Trenholm, Alcock & Robinson, 2015). Underpinning the need to conduct discipline specific research in this area, Iannone and Simpson (2013) found that in contrast to the general literature on assessment, mathematics students prefer traditional closed-book examinations to more alternative assessment methods.

Engagement with feedback

There has been recognition that despite timely and informative feedback being provided to students, students may not take action on it (Gibbs & Simpson, 2004; Handley et al., 2011). Handley et al. (2011) emphasise the difference between the student who skims and bins the feedback to one who takes "responsibility for understanding, interpreting and applying assessment feedback" (p. 557). Price, Handley and Millar (2011) discuss how a student may reject feedback "due to lack of understanding, or based on identity or self-efficacy issues" (p. 892). They further state that students may need more support in taking action on feedback. Of course, feedback needs to be of an appropriate level to help the student. Similar to the remediation process, Covic and Jones (2008) provided psychology students with the opportunity to remediate corrected essay assignments. In this voluntary remediation, 48% of students opted to remediate for potentially higher marks. Students' feedback consisted of individual and group feedback on their corrected essays as well as an initial grade. We have been unable to find an equivalent study in a mathematics context.

MODULE CONTEXT AND DATA COLLECTION

For our analysis, we only considered students who were completing *Maths for Business* for the first time and who sat the final examination in the module (n=470). In *Maths for Business*, students have the choice of completing the module through using online videos or by attending lectures or a combination of both (Howard, Meehan & Parnell, 2017). The students are assigned to one of two lecture cohort groups and have three lectures weekly. The lectures are designed to be partly interactive with at least 15 minutes for in-class tasks. All students have access to 67 videos/screencasts which cover the entire module content and have an average length of 7 minutes each. These videos were designed and developed by one of the two module lecturers (and third author of the paper). There are no recommended textbooks for this module. Students also have access to the Maths Support Centre, and prior research has shown that students focus on using module resources with very little if any use of external resources such as websites. A student's final mark on the module consists of 40% continuous assessment and 60% for the final examination. To encourage consistent engagement with the module and mastery of learning outcomes, *Maths for Business* has ten fifteen-minute weekly quizzes, with each quiz usually consisting of two parts. Each quiz accounts for 5% of the final mark, directly relates to the module's content for the prior week, and is marked out of five. However, only students' best eight quizzes contribute towards their continuous assessment mark of 40%.

One week after completion of a quiz, tutors returned the graded quiz to each student with a mark for both parts of the quiz and the overall mark provided. Tutors also provided oral feedback to their tutorial class on the most common errors made in the quiz. There were approximately 50 students registered for each tutorial. On the university's Virtual Learning Environment (VLE) *Blackboard*, the lecturer provided a video entitled "Most Common Errors" and a pdf copy of the quiz solutions that also indicated the most relevant online videos from the module that a student may wish to revise. In Semester 1 of 2016/17, students who did not receive the full five marks on their quiz were given the opportunity to resubmit their remediated quizzes for one additional mark. The following instructions were provided by the lecturer to students:

- When your quiz is returned to you go over it and identify your errors. Write a sentence beside each error on the quiz sheet so that when you are revising the material again, you will have a note to yourself about where you went wrong.
- If it is the case that your errors were more than just a "slip", then you should write out the correct solution on the quiz sheet and write a sentence or two beside the solution summarising the method.
- You should use a different colour pen so that the tutor can clearly distinguish between what you wrote in the quiz and your remediation comments.
- Imagine you are correcting your friend's quiz and you are explaining to your friend where he/she went wrong.

The third author had difficulty in articulating these instructions, hence the second research question. To remediate a quiz, students were encouraged to use any of the resources available: "Most Common Errors" video; quiz solutions; relevant online videos; Maths Support Centre; tutor feedback; and, friends. From the VLE, we were able to record when a student accessed the first three resources listed, and from Maths Support Centre records we had information on who attended the centre for remediation purposes. We have no data for the number of students who sought help from their friends, or those who made use of the tutor's feedback comments.

Owing to the semester timetable, students could only remediate the first eight quizzes. In total 1,746 remediation marks were awarded. We collected the remediated quizzes, however, as some from the first quiz are missing, we only use remediated quizzes two to eight inclusive in our qualitative analysis (n=1,511).

QUALITATIVE METHODOLOGY

Qualitative analysis of the remediated quizzes loosely followed the stages of thematic analysis (Braun & Clarke, 2006). Initially, the first author examined the remediated quizzes several times in order to familiarise himself with the data. The remediated quizzes where students *only* provided a full, complete solution as remediation were removed as limited information could be obtained from examining them, especially since complete solutions were available as an online resource. This left 687 remediated quizzes where students had done something other than provide *only* a full solution. Guided by the instructions provided to students, the first author analysed each of the remaining 687 quizzes in order to determine:

1. Has the student successfully identified each error?
2. Has the student provided a solution for each error?
3. Has the student explained their error, and if so, how?

Most quizzes consisted of two questions, therefore each question was analysed separately. In order to ascertain whether a student had identified each of their errors, the first author identified each error on a student's quiz, and noted how many of these the student identified. In terms of examining whether the student had provided a solution for each error, two approaches taken by students were identified. Some students provided a full solution to the complete question even if the error made only related to part of it, whereas other students only wrote a solution for the specific error made. Finally, the first author analysed if, and how, students explained the errors made. Students seemed to vary in their approaches based on variables such as the nature of the question asked, for example procedural or conceptual, and the quiz mark received. We will elaborate on this further in the section below.

RESULTS

To allow for comparison between students with similar levels of achievement on the quizzes, we divided students into four groups based on their average continuous assessment mark (excluding remediation marks) received for the first eight quizzes (4-5, 3-4, 2-3 and 0-2). These groups were of sizes 103, 188, 115 and 64 respectively. Of the 470 students, 47% were female. The following findings are detailed in Howard et al. (under review): students who scored less than two on a quiz were less likely to remediate; students in the two lower groups showed a limited increase in final examination mark as a result of participating in the remediation process in comparison to their peers; and, students who averaged 3-4 on their continuous assessment, particularly benefited from participating in the remediation.

Students' approaches to remediation

After coding was completed on all remediated quizzes, it was possible to compare students' remediation styles within a single quiz as well as the individual progression through all submitted remediations. With regard to error identification, there was a notable difference between students with high and low quiz marks. For students with marks of 3 or 4 in a quiz, mistakes were usually simple calculation slips and thus many students had only one or two errors to identify. This resulted in the majority of students with these marks successfully identifying all their errors. However, for those with lower marks, there were more conceptual misunderstandings to identify as well as several calculation errors. It is not surprising that these students were less successful at identifying every error. For students who received low grades on average, this pattern was clearly evident, but in addition, higher grade students exhibited this style on quizzes where they obtained a lower mark. There was a clear contrast in the level of error identification between an individual students' highest and lowest scoring quiz. This supports our hypothesis that when students achieve low marks, they are less successful at identifying their errors. In terms of provision of solutions to errors, as noted above students either provided a complete solution to the question even if the error only related to part of the solution, or they provided a solution that related to a specific error.

In relation to how students explained their errors, three codes or types of explanations were identified. Based on our knowledge of feedback, we define these explanations as: diagnostic, instructional, and objective. Some students explained errors in the context of incorrect notions or ideas that lead to them making a mistake. We refer to this approach as being diagnostic for example, "I thought brackets implied find the product but I should have used the chain rule". Others focused on providing advice or helpful tips to themselves to help prevent mistakes in any similar questions they faced in future. We refer to this approach as instructional for example, "Add powers together as they have the same base (multiply rule). Finally take away the powers from each other (division/fraction rule)"). The final, and most common approach, was an objective explanation of the error, simply describing the specific error without referring to prior knowledge or providing instructions on how they might answer future questions on the topic for example, "I compounded continuously but the question asked for quarterly".

Additionally, there was a less frequent code for whether a student provided an incorrect statement, or identified their errors incorrectly. Having addressed the first research question, we propose that in future the following instructions be given to students:

1. With a different coloured pen to the one in which the quiz was completed, put an "X" beside each error or, where relevant, indicate an omission in your work.

2. Provide an explanation for each of your errors, either describing the error or if you can, elaborate on any incorrect notions you had which may have led to the error.
3. Correct each of your errors by writing the correct version beside each one. Avoid copying down the written solutions for each quiz without explicitly consulting your areas of error.
4. If you are still unsure how to remediate your quiz or need help with the questions, we suggest you visit the Maths Support Centre for help.

In addition to the above instructions, we would provide students with exemplars of "good" remediations.

In-depth analysis of specific students' remediations

We now turn to Research Question 3: What factors may influence the nature of a student's remediation? Owing to the large amount of data involved, we have chosen to address this research question by examining a specific subset of the data. It is our hope that this analysis will help us identify factors that may prove beneficial when analysing the larger data set. We consider a subset of students (n=31) who remediated at least six quizzes and achieved on average 3-4 quiz marks as these had particularly benefitted from the remediation (Howard, et al., under review) and we could investigate their style of remediating over several quizzes. These students attended between 1-26 lectures and accessed between 11-232 videos with students from the weaker mathematical backgrounds (based on the Irish State Examination Mathematics results) accessing more resources than the others. All of these students passed the end-of-semester examination achieving at least a B grade (60%) and 58% were female. Twelve students accessed the Maths Support Centre and all but one of these were female. Detailed records from centre show that at least five of these students received help from the tutors on remediating the quizzes, including quizzes where they received four out of five marks. Seeking help from the Maths Support Centre for remediating the quizzes was uncommon among the larger *Maths for Business* cohort. Overall, based on their module resource usage, these students seem to work consistently throughout the semester.

To investigate how these 31 students remediated, their coded, remediated quizzes were further analysed to examine any prevalent styles of remediation. Within this group some students remediated consistently in the same manner each week, while others exhibited various remediation styles. Due to the initial division made between students writing only a full solution as remediation, and those who articulated some form of further explanation, we decided to categorise a student's overall remediation style based on the amount of times a remediation consisting of just a solution was submitted. Students could be split into roughly equal groups based on whether they provided only full solution in every remediated quiz (n=10), more than half of their remediated quizzes (n=8) or in half or less of their remediated quizzes (n=13). Overall there was no obvious difference in the resources accessed (Maths Support Centre, quiz solutions etc.) between the three groups of students. In terms of error identification, these students were, in general, successful in identifying all their errors. A prevalent remediation style consisted of students identifying their errors, providing an objective explanation of the error, and writing a solution of the specific error. With regard to the three main approaches to error explanation, the objective explanation was the most common method. Interestingly, the diagnostic and instructional approaches were rarely used consistently in remediations by individual students.

The nature of the quiz question seemed to impact the remediation approach. The quiz questions could be considered under the headings: conceptual ('The following is a graph of the first derivative f'(x) of a function f(x). Your friend is attempting this problem...He asks you: "How can you tell that f has a minimum at x=2 just by looking at the graph?"'), procedural ('Find all first and second-order partial derivatives of z = f(x,y) = ...) and economic context ('Compute price elasticity of demand, E... In one sentence, explain what your answer means'). Notably, for these students, quiz questions that were more conceptual in nature, resulted in more "solution only" remediations. As full solutions were provided as a resource to students, it was possible to copy them and submit it as remediation. Thus, an increase in the number of solution-only remediations for a given quiz question may allude to a lack of student understanding of the module content. A number of quiz questions also resulted in more diagnostic remediation responses. These questions were more application based, and focused on application of techniques to economic contexts. The two quizzes that contained these applied questions had the highest average marks of all eight quizzes, with 21/31 quizzes obtaining full marks between the two. We believe the increased use of diagnostic remediations may indicate students find it easier to locate misconceptions in mathematical application than with more abstract questions.

DISCUSSION AND CONCLUSION

Semester 1 2016/17 was the first time we implemented the remediation process. Following from the definitions of Ramaprasad (1983) and Sadler (1989), students were provided with their actual level of understanding (mark out of five), the desired level of understanding (for example, pdf of quiz solutions), and were incentivised to engage with feedback for the intention of closing the feedback gap. Gibbs and Simpson (2004) propose that even if feedback is provided, students will not necessarily use it. Despite assessment marks being provided as an incentive, on average 70% of students engaged in the remediation process. Initially, we considered providing remediation marks on a sliding scale based on a student's initial quiz mark received. This method would reflect the additional work of remediating for lower-scoring students and offer them additional incentive over high-scoring students, however, this system was not implemented as it required additional time and effort from the tutors.

Qualitative analysis of the entire set of remediated quizzes allowed for the isolation of certain properties that indicate whether a student is engaging with their assessment feedback. We note that a substantial number of students provided the full solution only for their remediation, however we advocate the three-step approach - identifying, explaining and correcting errors encourages students to recognise their own standard and utilise feedback to help close the gap between their own level and the desired standard. It is for this reason that we suggest this style as a guideline for future processes similar to remediation.

Owing to the brevity of the explanations provided in this paper on the different remediation responses, it is pertinent to mention different avenues that will be investigated as this research progresses. The in-depth analysis of the 31 students in this paper suggests that task design and a student's initial grade influences remediation style. Different task types elicited different remediation responses, based on whether the question was conceptually or application based. Perhaps, owing to a lack of understanding of the content material or the inability to transfer knowledge from one context to another, students tended to provide full solutions to conceptual questions. In addition, quizzes where students received lower grades tended to result in more solution only remediations. As these solutions are available as resources, this may suggest that some students utilise solutions when they are unable to fully identify and understand their errors on a quiz. From this, one can hypothesise that the remediation process may have less benefit to weaker students as a level of baseline knowledge may be required in order to engage with the process. In subsequent research, task design and initial quiz mark will be examined further to discern any influence on remediation response across the full set of remediated quizzes.

One limitation of our study is the lack of student perspective on the remediation process. Also, while the remediation process was beneficial to the students of this large non-specialist mathematics module, an investigation of the benefits/drawbacks of the remediation process for mathematics modules for specialists would be a constructive contrast with this research.

This study was completed with approval from the University College Dublin ethics committee in accordance with ethics applications LS-E-17-20-Copeland-Meehan and LS-16-48-Howard-Meehan.

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Student Partners in Task Design in a computer medium to promote Foundation students' learning of mathematics

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A team consisting of three mathematics education teacher-researchers, four former Foundation students (called Student Partners, SPs), and two analytic assistants worked together to produce mathematical tasks in a computer medium for the mathematical learning of current Foundation students (FSs). We have explored the collaboration between the SPs and researchers, the processes and outcomes of task design, and the contribution of the collaboration to tutorial teaching of FSs. We seek insight into the learning of all concerned of mathematics, mathematics teaching, task design and personal-professional development. The project is ongoing. Here we introduce the project and present early findings – specifically related to task design and the contribution of SPs.

Keywords: Teachers' and students' practices at university level; Novel approaches to teaching; The role of digital and other resources in university mathematics education; Students as partners in task design.

INTRODUCTION TO THE CATALYST PROJECT

We report from an exciting new development project (2016-18: The Catalyst Project[i]) in which mathematics teacher-educators collaborate with former Foundation students as partners in designing computer-based mathematical tasks for current Foundation students. Our research explores the collaboration of participants, the design of tasks, teaching of students in tutorials, use of computer software for teaching and learning mathematics and the learning of all concerned.

The mathematics learning of students in our university Foundation Studies programme (henceforth Foundation Studies students – FSs) is the focus of this developmental project. These are students who need a higher-level qualification than they hold currently in order to be able to enter the first year of their desired undergraduate programme (e.g., programmes in engineering or science). For such programmes, mathematics is an essential component. All FSs are required to pass their year-long module in mathematics. It has been observed that teaching and learning in this module in the past has been rather procedurally based: students have been introduced to and expected to learn the application of procedures to mathematical problems and have been examined on their procedural competency. A current aim is to make the module more conceptually based, and to explore the use of computer-based tasks for this purpose. Our previous research has shown that:

a) The involvement of peer learning-teaching in a mathematics module has resulted in participating students gaining higher marks, with associated experiences reported as positive to their understanding of concepts; (SYMBoL, e.g., Duah, Croft & Inglis, 2013; Solomon, Croft, Duah, & Lawson, 2014);

b) The cultural differences between teacher-researchers designing an innovation in mathematics teaching-learning and students engaged in learning mathematics through the innovation have contributed positively to outcomes/higher marks. (ESUM, e.g., Jaworski, Robinson, Matthews, & Croft, 2012).[ii]

Beyond this activity, we have found very little other research involving students' engagement through partnerships in mathematics teaching and learning. There is relevant work in Higher Education more generally involving *Partnership Learning Communities* (Healey, Flint and Harrington, 2014), but this does not include mathematics specifically. A recent special issue of NoMAD, the Nordic Journal of research in mathematics education, included several papers in which teachers were involved in exploring their own practice. The reports address three themes, one being 'innovative approaches to teaching and learning, with emphasis on student participation in the educational process' (Goodchild & Jaworski, 2017). One paper, in particular, reported positively on students' activity in oral presentations as a tool for promoting metacognitive regulation in *Real Analysis* (Naalsund & Skogholt, 2014). Student engagement and understanding were seen to improve through their participative activity. Searches to date have revealed no other relevant work.

Building on our experiences in SYMBoL and ESUM, we sought to design mathematical tasks which would challenge the FSs in new ways, engage them visually and actively and promote the beginnings of a new learning culture. Recognising the value of *student design of resources* and *peer support* as demonstrated in the SYMBoL Project we built both of these aspects into our Catalyst project. One of the Catalyst aims was: To promote collaboration between staff and students that results in higher degrees of confidence, motivation and learning in mathematics and a new culture in the teaching-learning of mathematics (e.g., HEA, 2014). We hoped to learn from the various elements of this project in ways that would have relevance beyond mathematics, particularly in the inclusion of former FSs as Student Partners (SPs) in course design and teaching. Our own learning from working with students in these ways was also a central aim, with the intention to bring staff and student cultures into focus through our joint activity. The reflections of the students on their activity, both SPs and FSs, were seen as an important outcome of the project. Thus, our innovation in the Catalyst Project has two main areas of inquiry:

- The design of computer-based tasks in *matrices* and *complex numbers* for the FSs using *Autograph* software.
- The involvement of former FSs (the SPs) in the design of teaching and in the tutorial teaching of the current FSs.

According to its specification, the Foundation Studies programme provides "*An opportunity to make it onto a degree course at Loughborough University".* Within the programme there is a wide range of student experience in mathematics from GCSE grade C to A level grade A. We focus on a mathematics module called 'Applicable Mathematics' which prepares students to take up degree programmes in Science or Engineering. The two semesters focus on the following topics: *Semester 1:* Algebra, Logarithms, Inequalities, Functions, Trigonometry, Vectors, Differentiation, Integration, Sequences; *Semester 2*: Polynomials, Partial Fractions, Further Calculus, Conic Sections, Vectors**,** Matrices, Complex Numbers.

The project has focused on the teaching of *Matrices* and *Complex Numbers* in Semester 2 in 2017. The three project leaders (PLs) have worked with four SPs to design tasks using the computer software *Autograph* in the two topic areas. Tasks are for use in tutorials with Foundation Students (FSs). SPs are former FSs: in the previous year group they were successful in having achieved grades at the levels required for transition to programmes in Mechanical Engineering, Chemical Engineering, Physics and Chemistry. At the time of their recruitment, they were first year students in their current programmes. In addition, two doctoral students in Mathematics Education were recruited as "Analytical Assistants" to support data collection and analysis. Thus, nine participants have been involved in the project, with differing roles.

THEORETICAL BACKGROUND

We are concerned with *learning* at a number of levels.

- FSs learning of mathematics;
- SPs learning of mathematics, task design and participation with staff in preparing for undergraduate learning;
- Mathematics teachers and researchers learning about the design of teaching in partnership with students.

This learning is influenced by a wide range of factors which include the curriculum, and institutional settings within the broader sociocultural setting. Some of these factors we can seek to influence; others are less amenable to innovation. We take a fundamentally Vygotskian (e.g., Vygotsky, 1978; Wertsch, 1991) perspective recognising particularly mediation by people and tools that support learning; goal-directed activity and action related to learning and teaching; scientific concepts that require pedagogic mediation; and the zone of proximal development in which mediation fosters learning and development. We engage particularly with digital tools, their design and use, and the ways in which they mediate the learning process through both support and challenge for making sense of mathematical concepts.

An important theoretical concept is that of "partnership" between staff and SPs (e.g., Healey, Flint and Harrington, 2014). Relationships within the partnership have resulted in the design of mathematical tasks and their use with the FSs. The nature of this partnership is central to project outcomes, in terms of the designed tasks and their use. We see ourselves as having formed a 'Learning Community' in which *co-learning* is an important concept (Wagner, 1997), and which demonstrates tenets of a *Community of Practice,* such as *mutual engagement* and *joint enterprise* (Wenger 1998) and a *Community of Inquiry,* such as *critical alignment* (Jaworski 2006).

METHODOLOGY

We take a developmental research approach, consistent with our Vygotskian theory, which both studies project development and learning within the project and contributes to development and learning (Goodchild, 2008; Jaworski, 2003). Mediation and tool use, for example, can be seen in an interactive stance of reflection and negotiation in which participants engage together in activity and action with growth of mutual understanding and co-learning (Wagner, 1997). Analysis begins in questioning of what is done and achieved and is formalised through scientific inquiry addressing a range of data through recognised methods.

Research questions and data

Our *Research Questions* relevant to this paper are as follows:

- How have SPs engaged with task design and what has been the outcomes and issues arising?
- How have FSs worked with the designed tasks?
- What have we learned about the FSs' learning of mathematics with the designed tasks? What issues arise?

Data, which are being analysed to address these questions include:

- The involvement of SPs and staff in the design process as shown through recordings of project meetings, SP reflections/reports, documents (collected at the design meetings and from the SPs' own work).
- The tasks, and their use as seen through observation, screen capture and discussion. The teacher's narratives from her reviewing of tutorial data.

Analytical approach to date

Reflection and negotiation have taken place through meetings, discussion, sharing and review of designed tasks leading to increased awareness of issues in design and communication. This collaborative co-learning has involved a bringing to collective consciousness of key elements of the design process and issues to be resolved.

Analysis of collected data according to research questions has been qualitative, focusing on data from the design meetings, and from tutorials with FSs in which designed tasks have been used. A process of data reduction has summarised and coded recorded data, allowing initial identification of key elements and issues relating to research questions. The process is cyclic with developing depth of inquiry and insight to significant issues expressed through analytical memos. The tasks themselves have also been a focus of scrutiny which is ongoing. These analyses are as yet in their very early stages, so what we report below is tentative and indicative. Here, we discuss some emergent findings in task design and use of tasks with FSs.

EMERGENT FINDINGS

Task Design

The teacher/lecturer of the Foundation course provided course notes on the two topics, Complex Numbers (CN) and Matrices. An expert in Autograph gave the group an induction into its use and potential for mathematical representation and exploration. SPs were asked to review the notes and think about possible tasks using Autograph. Task design, in 2 SP pairs (one to each topic) took place over 6 weeks and across 4 meetings – finding times for these meetings, from timetables of 9 partners, over a short time period was challenging. At the meetings, SPs' presented their ideas to the whole team, hesitantly in the beginning but with growing confidence in response to expressed appreciation and suggestions from the team. The pair working on tasks in *complex numbers* were quick to provide examples and to modify them according to suggestions in meetings. The pair who worked on matrices found it harder to get going. Tasks in complex numbers appeared to be more readily achievable than in *matrices* where concepts seemed less amenable to digital representation/questioning – it became necessary both to identify the barriers and to find some resolution. Collaboration between the SP pair and the PLs, focusing on the mathematics of matrices and the learner difficulties suggested by SPs, resulted in a set of tasks on matrices. One of the PLs also designed a set of tasks in GeoGebra focusing on matrix arithmetic. The emerging 'raw' tasks, consisting of an Autograph (or GeoGebra) file with brief associated notes, were then 'prepared' by the FS lecturer to make them ready for FS use. We see two examples of the prepared tasks, one for each topic, in Figure 1. Certain characteristics, incorporated by the SPs into these tasks can be seen in the examples; FSs have to:

- ❖ use several display features of Autograph to “see” mathematical objects and relationships;
- ❖ undertake some associated calculation by hand;
- ❖ relate movements on their screen to the handwork and theory involved;
- ❖ reflect on specific results to develop a more general awareness of concepts.

Some of these demands turned out to be a challenge for many FSs as we see below.

Question 2: Open the Autograph File Task 2

There are three complex numbers labelled z_1, z_2 and z. z_1 is fixed while z_2 and z can be moved. Select z_2 and move it until z reaches the position 3 + i.

(a) What complex number is z_2?

(b) Right click and “Unhide All" to check your answer.

(c) What is the relationship between z_1, z_2 and z?

(d) Explore subtraction of complex numbers in Autograph.

(e) Now calculate by hand:

With $z_1 = -1 - 3i$ and $z = 3 + i$, find z_2 such that $z_2 - z_1 = z$.

(f) Draw (by hand) all three complex numbers on an Argand diagram.
Give a geometric interpretation of the relationship between z and z_1 and z_2.

Question 7: Open the Autograph File Matrices 5

On this page you see two straight lines. Their equations are $4x - y = 14$ and $7x + 4y = a$

(a) By hand, using matrices, calculate the value of a so that the solution to the simultaneous equations is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(b) When you have a solution, use the “constant controller” to vary the value of a until the point (3, 2) is clearly displayed.

(c) Select both lines by holding down the “Shift" key. Both lines should have changed colour.

Go to “Object" in the menu bar and choose “Solve f(x) = g(x)".

A point is displayed. To see what the co-ordinates of this solution are, go to “View" in the menu bar and choose “Results Box". Does your solution make Autograph show the intersection of the lines to be x = 3 and y = 2?

Figure 1: Examples of prepared tasks in complex numbers and matrices

Use of tasks with Foundation students

The tasks were used in timetabled tutorials with the FSs, who were asked also to comment on the tasks for the research. Those agreeing to participate were audio-recorded in conversation with the teacher and one researcher. From analysis to date we are gaining some insight into FS participation with the tasks. We have recorded instances of FS requesting help from the teacher, asking questions, explaining their solutions, revealing mathematical insight, surprise, or lack of understanding. In interactions with the FS, the teacher responds to students, asks questions, explains, and provides technical information. The dialogue in Figure 2 shows a teacher-student exchange, relating to Question 2 above, that reveals an issue in the student's engagement with the task that is typical of several such exchanges:

We see that the student had engaged with the task, moving z_2 as instructed. However, he did not know what is meant by "the relationship between z1, z2 and z" although he had written that "z_2-z_1 gives you z". It seems a case of not understanding the meaning of the word "relationship", although he had written the relationship. We are learning here about language issues in working between the visual mode of Autograph and the symbolic mode of expressing complex relationships.

T: What are you doing in Question 2? [She looks at what he has written] 3+i... Haa! What did you find out? What relationship?

FS: I don't know what it means.

T: You don't know what that means? Well they are connecting aren't they? You must be doing something with them. If they are moving together [z1, z2 and z are moving together] we do something to them and then you get the third.

FS: I wrote that z_2-z_1 gives you z.

T: So you did write it down. For [part] c, that is the relationship. Yes, that's what we meant.

FS: OK

T: So the relationship is subtraction. You are subtracting two complex numbers. How did you know that though? Did you know that from the picture? Or did you do something else?

FS: I worked it out.

T: Ah ok. So you actually did the calculation.   T=Teacher   FS= Foundation Student

result. Whether this is as well as discerning it from the movements in Autograph or instead of this, is not clear. Thus, it might be that the student had used Autograph effectively and made links with the symbolic forms. Or it may be that he had sidestepped interpretation of the visual and instead had worked out the result symbolically (the latter perhaps being a more familiar task).

The teacher's reflective narrative relating to the recordings from the second tutorial on CN reveals the following example [Teacher's written words are italicised]:

> *[Student] found the additive relationship by adding separately the real parts and then the imaginary parts. He says "is it bisecting the angle?" I reply "it's something to do with it".* [She asks about her related lecture presentation and mentions a comparison with *vectors*.] *The student correctly relates vector addition to "adding head to tail" and that this forms a "triangle".* [She indicates that he does not seem to understand what is meant by a "geometric interpretation".]

The teacher's reflection suggests a student engaging correctly with several concepts including complex addition and vector representation. Yet, again, we see a problem with language – the term "geometric interpretation" is unfamiliar to the student. One student, asked about his work on the Autograph tasks, replied that he found the questions "hard to read" and could not "understand the way they are worded".

The two examples quoted are typical of recorded exchanges. We see from these

a) Students not understanding the words or expression of ideas in the written questions (e.g., "relationship", "geometric interpretation")

b) Students not articulating explicit conclusions from what they see on screen – rather using the familiar forms of calculation to answer questions.

These observations lead us to question both the Autograph tasks and the wording of the tasks. How might we have worded the tasks differently so that students would engage with what they could see on screen and discern the mathematical relationships that the task was designed to reveal? How might we wish to modify the task itself so that students engage visually rather than depending on calculation? The challenge for the team here is twofold: to design a teaching approach that introduces the language we want students to use and enables students to become familiar with its use; and to design tasks that are revealing of concepts in and of themselves, so that students can see visually what they familiarly work out in calculation. These seem to be important elements of the learning culture we are trying to foster.

We are aware that many FSs come to their university course from school or college where their mathematical enculturation towards success in national final examinations may have encouraged a procedural perspective on learning mathematics (Minards, 2013) and that we see the results of this to some extent in their response to the designed tasks. As well as looking critically therefore at the design of the tasks, we have to consider the wider mathematical culture, the nature of teaching that seeks to interact with this culture and the ways in which the tasks can be incorporated within the teaching-learning interface.

CONCLUSION AND FURTHER RESEARCH

The developmental nature of the project can be seen through the development of mathematical tasks for Foundation students by SPs and PLs in partnership, the subsequent use of the tasks in tutorials with the FSs, and issues arising from task design and use revealed both in practice and in analysis of data from the various events. An aim of the project was to foster conceptual understanding of mathematics by the FSs. We see above some issues arising from the nature of the tasks and the ways in which they are written, and from the ways in which FSs' mathematical experience influences their engagement with the tasks.

Because analysis is in its very early stages, we are not yet in a position to report on many of the aspects of learning in the project (such as aspects of the learning of the SPs). However, already we can start to see indications of important learning and the feedback element of the developmental research. Co-learning has been demonstrated between the FSs and the teacher-researchers – FSs' learning of mathematics has raised issues for the intention and preparation of tasks which offer challenges to the researchers and for future work with the SPs. The tasks and their design have been mediational tools not only for the mathematical learning of the students but also for the awareness of the researchers about teaching-learning issues, not least the issue of language in which tasks are expressed. The project is ongoing, both in terms of teaching-learning development and of analysis of the data collected so far.

A major issue for the project has been the timescale as dictated by the funding body and university organisation of teaching. We had barely half a semester to recruit SPs, initiate the design process, hold 4 spaced meetings, prepare the tasks for FSs and hold the tutorials. The project end coincides with the time for the next cohort to reach the teaching of matrices and complex numbers, so we could not build this into the project. We expect to use the same tasks again with the new cohort and collect further data, informed by our experiences a year ago. Since our data is extensive, in depth analysis is ongoing from which more in-depth reporting should be possible. We expect to be reporting further on the many aspects of this project.

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[i] Supported by HEFCE (Higher Education Funding Council for England) Catalyst Fund: Innovations in learning and teaching, and addressing barriers to student success **A:** Small-scale, 'experimental' innovation in learning and teaching.

[ii] SYMBoL – Second Year Mathematics Beyond Lectures – was a project designed to support teaching in two second year mathematics modules, Vector Calculus and Complex Analysis. Students who had experienced these modules were employed over a summer period to design resources in collaboration with mathematics staff. The resources were used in subsequent delivery of the modules and a peer support system was initiated in which third year undergraduates supported their second year counterparts.

ESUM – Engineering Students Understanding Mathematics – was a developmental research project involving an innovation in mathematics teaching seeking to engage students more conceptually with mathematics through inquiry-based activity, a computer-based learning environment, small group tasks and an assessed small group project.

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Theoretical and empirical description of phases in the proving processes of undergraduates

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In the presented study we adopt a process-oriented perspective on proving in order to gain further insights into relevant actions and typical obstacles in undergraduates' approaches to proving. The primary aim is to theoretically and empirically describe different phases, understood as bunches of intentionally closely related actions of proving. Therefore, we suggest a theoretical model of the proving process and confirm empirically that it can be used as an analytical tool for proving approaches. Based on this model, several proving processes have been analysed. In this paper we present first findings regarding the contribution of each phase to proof construction as well as the general structure of the proving process.

Keywords: proof construction, proving process, phases, proving cycle.

THEORETICAL BACKGROUND

It is well known that undergraduates commonly have to deal with great difficulties in constructing proofs (e.g. Weber, 2001; Moore, 1999). In order to gain additional insights into the main obstacles in proving at university level, we focus on the processes of proof construction. Preparing further projects, the presented study serves as a pilot study, which initially aims at investigating the general structure of proving processes on a macroscopic level. Therefore, we first give an overview about the main characteristic of the proving process in contrast to its product. Based on this, we deduce a theoretical model from literature in order to use this model as an analytical tool for undergraduates' approaches to proving.

The process of proving

Talking about proof construction, there is a consensus that a proof, especially at university level, is a line of reasoning, which is strictly deductive and exclusively based on theorems and axioms. In contrast to its result, the process of proving contains not only deductive reasoning, but also includes inductive and explorative processes (e.g. Hersh, 1993). Thus, developing a key idea and transferring it into a formal proof is a complex and highly demanding process, which occurs in the tense atmosphere of conviction and explanation as well as intuition and formality (Hemmi, 2008). In the present paper we will discuss on a theoretical and empirical basis, how these different perspectives and processes interact in constructing proofs.

Mejia-Ramos and Inglis (2009) distinguish between three distinct types of construction activities, which emerge from different external conditions. Although all of these activities aim at constructing an appropriate proof, each is guided by a specific goal: *Exploration of a problem* consists of working on an open-ended question in order to discover and at least prove a new statement. However, *estimation of truth* starts with an already prepared conjecture and *justification* begins with a statement estimated to be true. In these cases, proof construction is rather aimed at determining or verifying the truth-value of a statement instead of inferring it. Meyer (2010) differentiates activities associated with mathematical proof construction in a similar way based on the theory of abduction, induction and deduction. According to his approach, each kind of inferring is related to a different type of construction activity. Thus, external conditions and, especially, task designs have a relevant influence on proving processes by stimulating different types of inferring and construction activities. Combining both frameworks, we assume that the activities described above are independent for the reason that they consist of different types of processes and pursue different goals. In particular, justification is not a part of exploration, which follows producing a conjecture, but requires specific cognitive processes.

In order to describe those cognitive processes, various models have been developed, which summarise the relevant actions and demands associated with proof construction. These models mainly focus on processes related to the exploration of a problem. As proving tasks at university level often consist of a statement estimated to be true, these models only seem to be partially suitable for analysing undergraduates' proving processes. Hence, we suggest a model of proving, which is mainly following existing models, but focusses on justification.

Models of proving processes

To analyse the cognitive processes of mathematical proof construction, there is a need for abstraction. The complex structure of the process has to be reduced to the relevant actions and, therefore, transferred onto a macroscopic level. Doing so, most process-oriented models use the unit of episodes or phases in order to subsume closely related actions in service of the same goal under a generic activity. Thus, phases are "macroscopic chunks of consistent behaviour" (Schoenfeld, 1985, p. 292), which summarise the relevant processes associated with proof construction.

Existing models of the proving process mainly differ in their amount of suggested phases. The most cited models have been presented by Stein (1984) and Boero (1999). Both models focus on the activity of problem exploration, which means that the proving process is based on an open-ended problem area. In the first phase of proving this problem area is explored regarding relevant conditions and regularities in order to produce a conjecture. When the conjecture has been formulated as a statement, the proving construction contains three phases: Identifying arguments for the correctness of the statement, linking them into a deductive chain and formulating an appropriate proof. As Stein's (1984) model focusses on proving approaches of students at secondary level and Boero (1999) describes the proof construction of mathematicians, quality and formalisation of the proofs intended in the models differ according to the mathematical standards shared in the particular context. Boero

(1999) even adds a further phase, in which mathematicians approach a formal proof. Apart from this last step, which primarily seems to be relevant for experts, the phases of proof construction described by Stein (1984) and Boero (1999) have been taken as a basis for several research projects in mathematics education (e.g. Reiss & Renkl, 2002). According to this, we suppose the phases described in both models to be relevant for undergraduates' proving processes as well. However, proving tasks in the initial phase of studies rather initiate justification instead of exploration. Due to this, we suggest the following variation of Stein's and Boero's model:

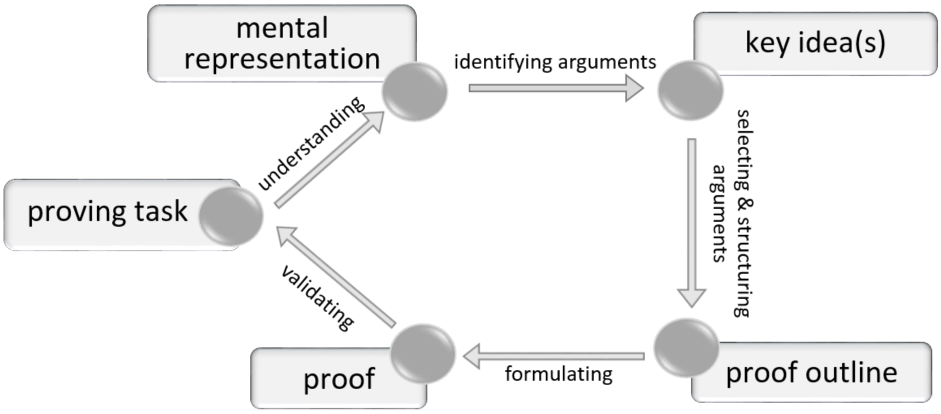

Figure 1: Proving cycle as a model for proof construction

In contrast to the existing models, the proving cycle starts with a proving task consisting of a statement estimated to be true. Although this statement can be similar to the produced conjecture in other models, it plays another role in the proving process. In case of problem exploration the statement is connected to insights from conjecturing and exploration, but in case of justification no further information about the statement is given. Hence, it is necessary to analyse the given statement, to clarify the terms and conditions and to access previous conceptual or strategic knowledge. Leaning on the approach of a situation model (Kintsch & Greeno, 1985; Reusser, 1990), we suppose undergraduates to develop a mental representation, which refers to their individual understanding of the given statement and guides further processes. Based on this mental representation we suggest similar phases as presented in the models of Stein (1984) and Boero (1999). The second phase aims at exploring the statement area and discovering key ideas. By (re-)constructing relations and objects or applying theorems reasons for the validity of the statement can be identified. If some key ideas have been found, it is necessary to select promising ideas, work out their details and structure single arguments in an appropriate deductive order. This phase results in a proof outline, which contains the main aspects of the proof, but can be fragmentary or difficult to understand for someone else. In order to prepare a final proof, which meets the mathematical standards of one's community, one has to fill gaps and revise the linguistic and formal arrangement of the proof. As a last phase the model includes validating activities, which can be compared to a certain extent to Pólya's stage of looking back

(Pólya, 1945). In this phase the final proof is reviewed regarding content, structure and linguistics. Beyond that, one can consider further (shorter or more elegant) proofs in this phase or reflect on the proving process, as, for example, thinking about key ideas, difficulties and their solutions.

The single phases in the model are arranged as a cycle. In accordance with Stein's (1984) and Boero's (1999) considerations, we assume that proof construction does not necessarily proceed in a linear way. Instead, the proving process is shaped by interruptions and revisions that cause transitions between different phases.

RESEARCH QUESTIONS

The primary aim of the presented study is to develop an analytic tool for proving processes, which makes it possible to describe and compare students' approaches to proof construction on an individual, macroscopic level. Therefore, a proving cycle has been derived from literature consisting of five phases estimated to occur in undergraduates' proving approaches. The unit of a phase seems to be well-suited to describe proving activities in a clear and abstract way without neglecting specific details and differences in the proving processes. Accordingly, the proving cycle may be used to analyse proving approaches by describing the frequency, the duration and the order of different phases occurring in the process. The presented study aims to provide evidence of the utility of the proving cycle as an analytical tool and – if so – to gain more detailed information about the process of proof construction at the level of phases. In detail, the aims of the study described above lead to the following research question:

1. Is the proving cycle an appropriate tool for analysing proving processes? That means, is it possible to reconstruct the different phases stated in the proving cycle empirically? Do further activities exist, which do not fit the theoretical description?
2. How can the process of proof construction be described in general? That means, which phases are taking a relevant share in the process? In which order do undergraduates go through the different phases? Can the cyclic nature of the proving process be confirmed?

METHOD

In accordance with the open-ended character of the research questions, an explorative laboratory study has been designed. In this study proving processes of undergraduates are initiated, observed and finally analysed. The concept of the study and its conditions are described in more detail below.

Sampling and data collection

The study focuses on undergraduates and pre-service mathematics teachers (high school) attending their first year of studies. Performing an informal unstructured interview, participants are encouraged to work on proving tasks in the field of real analysis. Doing this, they are told to prepare a joint solution that satisfies the requirements of a proof in the initial phase of studies and that would be accepted by a tutor or a lecturer. To encourage the participants to talk about their ideas and approaches, the working processes are organised in pairs. In order to secure the same conditions for all participants and, therefore, the comparability of the observed processes, the interviewer offers no support. Instead, a commonly used textbook of real analysis is provided. For investigation such proving tasks have been chosen that require a one- or two-step proof and can be solved by applying a prominent theorem of real analysis like the intermediate or mean value theorem. In order to explore the proving cycle's applicability, the proving tasks contain universal as well as existential quantifications and can be proved directly or by contradiction.

While working on the proving tasks, the participants are videotaped. That means, their proving approaches are recorded in sound and vision. Additionally, we collected the final solutions as well as the notes of each pair. During the pilot phase of the study seven pairs of undergraduates took part in the interviews and worked each on one or two proving tasks. Processing time for a single task varies from 30 to 75 minutes. However, not all proving approaches were successful. While some participants gave a proof, which was not completely correct, but contained useful approaches, other students could not achieve any solution.

Preparation and analysis of data

To prepare the observed proving approaches for analysis, we transcribed each videotape entirely. For precise investigations the transcripts of the dialog are expanded by further information like non-verbal activities and notes. Combining the transcription of natural conversation and written approaches, the protocols of the proving processes are finally encoded according to Mayring's (2014) structuring content analysis. In accordance with the research questions a deductive category formation with nominal categories has been chosen, which is closely related to the proving cycle described above. The aim of the coding is to identify changes in the participants' behaviour in order to describe the structure of their proving process as a sequence of transitions between different phases. Leaning on Schoenfeld's (1985) method of protocol analysis the coding consists of two steps: First, the proving process is parsed, that is, making decisions regarding dividing lines of phases. Once a proving process is partitioned into phases, each phase is characterised as one of the theoretical stated phases in the proving cycle. The coding results in a macroscopic description of the students' proving processes that combines closely related actions into phases and provides a summary of relevant activities.

RESULTS

In this section the results of analysing data from nine proving processes is presented. For the moment, a case study is introduced to demonstrate the methodological approach as well as possible results on an individual level. Based on this, findings regarding the proving cycle being an appropriate analytic tool are discussed in general. Beyond that, we present first assumptions concerning the frequency, the duration and the order of different phases in undergraduates' approaches to proving.

Case study of Michael and Leon

Michael and Leon are working on the following task, which can be solved by applying the intermediate value theorem:

> Let $f : [0,1] \to [0,1]$ be a continuous function. Show that f has a fixed point, that is, there exists a $x \in [0,1]$ with $f(x) = x$.

Partitioning their working process and characterising the identified phases, the proving process of Michael and Leon can be described by the sequence of phases presented in Figure 2.

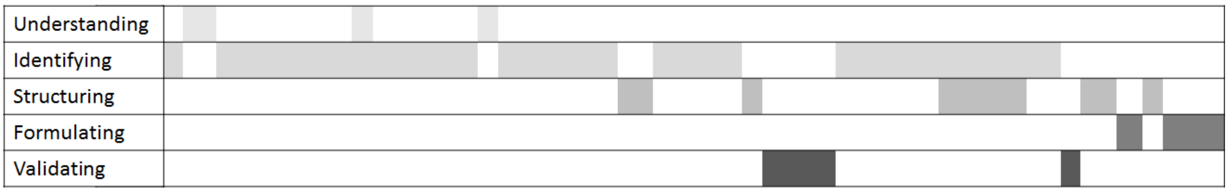

Figure 2: Parsing of the proving process of Michael and Leon

After reading the task, Michael and Leon start their working process with a brief brainstorming concerning useful theorems. Because the given function does not meet the required preconditions, they experience difficulties applying these theorems, which makes them reading the given statement a second time more carefully. Doing so, they express confusion about the term *fixed point* and, hence, try to clarify its meaning by looking it up and making a drawing. Based on an enhanced understanding, Michael and Leon review their ideas and add new ones by leafing through the book. Sooner or later, each of these ideas proves to be inadequate. While Michael is reading in the book searching for applicable theorems, Leon makes further drawings in order to visualise the preconditions and assertions given in the task. Thereby, he remembers a strategic approach used in a proof before. Leaning on this approach, he tries to construct an auxiliary function, which turns the original problem of the existence of a fixed point into a similar one of a zero. Michael and Leon work together on this approach until they recognise they have different concepts regarding the connection between the domain and the range of the given function. They resolve this disagreement by referring to the drawings and regarding some exemplary points of the function. For more than half an hour Michael and Leon have been working out the details of the proof now. At times they stop working on new ideas and summarise previous insights in order to structure their arguments and to identify gaps or inconsistencies in the proof outline. One of this structuring activities sends Michael having doubts about the correctness of their formulation.

Leon is able to convince Michael of their approach, but has to admit that some modifications are necessary like regarding an additional case. Later, another phase of validating occurs as a result of identifying arguments. This time they recognise that their auxiliary function is the identity function and that they can prepare their arguments in a more elegant way. Michael and Leon are now able to construct a satisfying proof by formulating and structuring alternately.

During the whole process of proof construction Michael and Leon are progressing continuously until they have achieved a satisfying proof. They only return to a previous phase, when their proving process comes to a temporary halt and they feel there is a need for reviewing some steps done before in order to specify or improve their mental representation, their key idea or the proof outline. Michael and Leon go through a phase of validating twice, because one of the student voices doubts about the previous considerations being correct. They do not review their final proof to check or improve details.

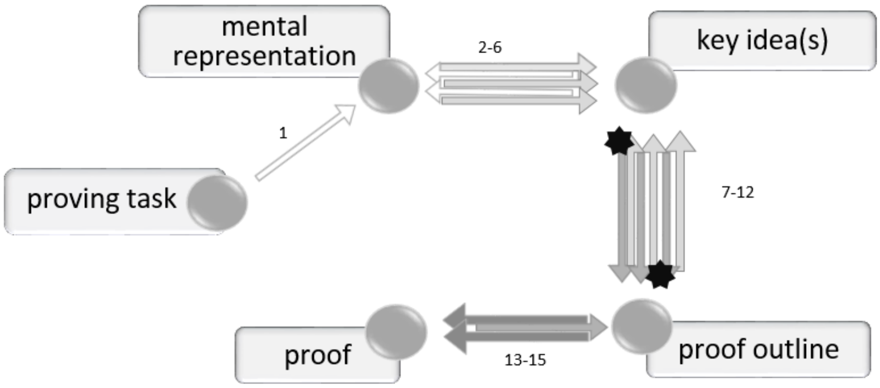

Figure 3: Proving process of Michael and Leon, validating activities are represented by stars

As displayed in figure 2, at times it has been inevitable to encode the same period of time with two different categories. This kind of double-coding is necessary, if both students work on their own in service of different goals or if some related activities from different phases are done contemporaneously. The analysis of the proving process presented in this section is quite typical for the sample. In the following section similarities and differences between the individual proving processes are discussed in general.

General observations

The analysis of data from nine interviews shows that the categories are applicable to the proving protocols in a satisfactorily objective and reliable way. Interrater reliability in coding is quite high (κ = .73-.93). Therefore, the proving cycle seems to be an appropriate tool for describing and analysing undergraduates' approaches to proof construction on a general macroscopic level. Regardless of the proving tasks and a direct or non-direct approach to proving, each of the suggested phases could be empirically confirmed in at least five of nine cases (Figure 4). In those cases, where the phases *identifying arguments* and *selecting and structuring arguments* are missing, participants have not been able to establish any serious approach due to a

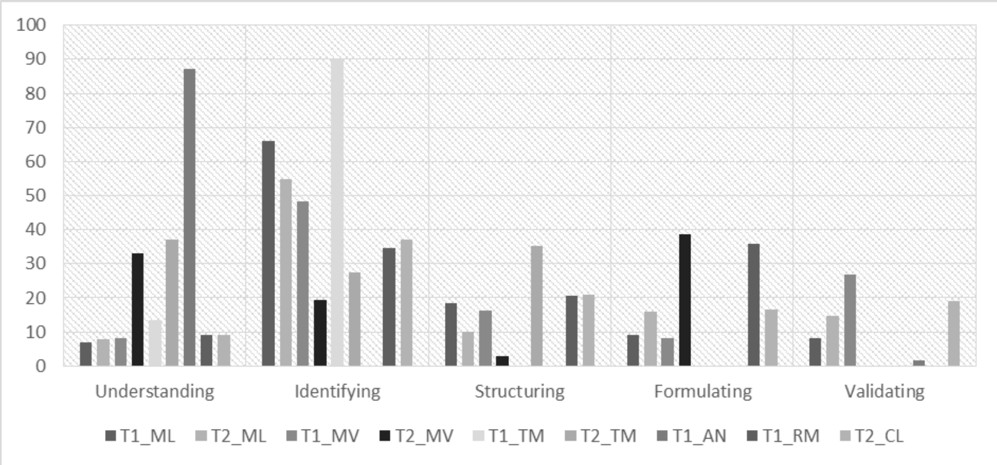

Figure 4: Percentages of the different phases occurring in proving processes

poor mental representation or a lack of key ideas. However, quitting the working process without any solution was only observed twice. More frequently it happens that even in successful proving processes the phases of formulating and validating are omitted. Instead of formulating a precise and clear proof, some participants content themselves with quick but fragmentary solutions. Here, formulating the proof according to mathematical standards does not seem to be as important as gaining insight into the key ideas of a proof and verifying the statement for oneself. This kind of view might be a general attitude towards mathematical formalism, but could also be caused by the laboratory setting. Analysing the phase of validating, we differentiate two kinds of action: Validating activities in service of reviewing and enhancing a final proof could only be observed in one single case. In contrast, activities like checking details and analysing suggestions occur continuously and are connected to several other activities in the proving process. Although validating activities are only listed in five of nine cases, it is reasonable that more validating takes place without being encoded on a macroscopic level because of no sufficient impact on the proving process. Comparing the percentages of different phases, it emerges that in nearly all cases the phase of identifying arguments as a highly creative and demanding action takes a large share in the proving process and, therefore, seems to be one of the most significant parts of mathematical proof construction. In contrast, the percentage of understanding and exploring the given assertion varies greatly. While some participants spend a lot of time on drawings or clarifications, others start with a brief glimpse on the task and continue with applying theorems immediately.

In regard to the composition of proving processes, it has been assumed that the underlying structure is a cycle. Due to that, there must be a high amount of transitions and revisions in the proving approaches. In fact, only one third of the encoded transitions is linear in that way that students move forward to the next phase in the proving cycle. Though, the proving processes does not proceed as cyclic as suggested. As illustrated in the case study of Michael and Leon a large shape of non-linear transitions is made by transitions between consecutive phases, that is, moving backwards to the phase before. Wider leaps from one phase to another are quite rare. Most of these transitions between non-consecutive phases contain an interaction with validating activities as students switch from any phase to validating and backwards. This finding supports the assumption that validating is an activity, which is closely related to other phases of the proving process. Only in a very few cases the key idea is rejected at some point in the proving process and the participants restart identifying arguments and exploring the given statement cyclically. Accordingly, we suggest proof construction to be less cyclic than a linear process, which is interrupted by several mini-cycles between consecutive phases.

DISCUSSION

In this paper a model for the process of mathematical justification at university level has been derived from literature to develop a macroscopic analytical tool, which describes a proving process as a sequence of phases. In a sample of nine processes the proving cycle has proved suitable for describing undergraduates' approaches to mathematical proof construction. Each of the suggested phases could be empirically confirmed. Analysing similarities and differences between individual proving processes, there have been two key findings: Undergraduates' proof construction mainly proceeds on a straight line basis, which is interrupted at times by transitions into immediately preceding phases in order to specify or improve considerations done before. An exception to this are validating activities. Questioning, reviewing and reflecting seem to be processes, which are rarely performed at the end of proof construction, but are closely connected to other phases of the proving cycle. Hence, initial results indicate that validating is not confined to the final proof, but relates to the mental representation as well as the key ideas and the proof outline.

The presented study prepares a larger project by providing an analytical framework for students' approaches on proof construction. Based on the proving cycle, we intend to analyse a larger sample including participants, who differ in progress and performance. Comparing first-year and advanced students as well as successful and non-successful proving processes might provide new ideas for fostering programs in the introductory phase of studies. On the one hand investigations will remain on the macroscopic level of phases in order to identify effective and less effective patterns of proving processes. Therefore, the occurrence and the duration of a phase in a proving process are compared with the quality of the corresponding proof. On the other hand, further investigations are intended, which gain more inductive insights on a microscopic level. Therefore, we aim at describing typical actions of each relevant phase in detail. By doing so, frequent difficulties and potential obstacles of an individual phase can be identified as well as effective and non-effective proving strategies to overcome these obstacles. This information can help arranging effective, process-oriented fostering programs.

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Student use of resources in Calculus and Linear Algebra

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In this study we have investigated the resources used by first year engineering students in a technical university in the Netherlands, for their learning of Calculus and Linear Algebra. Using a case study approach we have focused on how the resources and their use (a) differed from upper secondary school as compared to university, and (b) differed between the two university courses. The results indicate that, in terms of (a) students built on secondary school experiences and emulated these into their university courses, where some subsequently experienced difficulties. In terms of (b), we argue that the course organization and the alignment of curriculum materials with the learning goals had an impact on the students' choice and use of resources. Human resources played an important but varying role.

Keywords: Student use of resources, Case study, Transition from school to university, Calculus, Linear Algebra.

INTRODUCTION

At university level a large diversity of resources is currently made available for students learning mathematics. These include traditional curriculum resources (e.g. readers, textbooks); digital (curriculum) resources (e.g. YouTube, websites, apps); and also human resources (e.g. drop-in clinics run by tutors; setups for peer groups). The ways in which university mathematics teachers interact with various resources has been investigated by Gueudet (2017), for example, and several studies have been conducted related to university students and their use of particular resources to learn mathematics (Anastasakis, Robinson, & Lerman, 2017; Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016; Inglis, Palipana, Trenholm, & Ward, 2011). However, relatively little is known about how students of mathematics in their first year of university "cope with" the plethora of available resources available to them, and how they organise and coordinate them for their learning.

Using a case study approach, we have studied the resources in a Calculus (CS) and a Linear Algebra (LA) course, and their use by students, in the context of a first year Bachelor College programme at a technical university in the Netherlands. Moreover, we have investigated, retrospectively, which resources were used by the students, and how, in upper secondary school (as compared to university). Hence, we propose the following research question:

Which kinds of resources are used by the students, and how, in first year university Calculus and Linear Algebra courses, and how do these practices compare to students' experiences at upper secondary school?

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In this paper, first, we briefly outline selected insights from the relevant literature, including our theoretical frame of "resources" (and their use) as a lens to develop a better understanding of students' mathematics learning. Second, we describe our chosen methodology and data collection strategies, before we present our findings and discuss our results in the third section. Fourth, we present our conclusions and outline implications for the practice of university mathematics learning and teaching.

THEORETICAL FRAMEWORKS

Transition from secondary to tertiary education

In terms of mathematics learning, the transition from secondary school to university is challenging for many students (Pepin, 2014), as discontinuities exist between secondary and tertiary mathematics education. The literature reports numerous differences between studying mathematics at school as compared to university. It is said that in comparison to secondary education, at university: (a) the mathematical content is introduced at a higher speed; (b) more mathematical autonomy is expected; (c) the levels of generalization and abstraction are higher; (d) the approach is more formal with an increased emphasis on proof; and (e) the institutional cultures at the two institutions (secondary school, university) are different (Artigue, 2016; Gueudet, 2008). The ways the content is made available to students also differ between secondary and tertiary education (e.g. Corriveau & Bednarz, 2017). University students have to autonomously manage the various resources to learn mathematics, and it is argued that secondary school does not prepare them well for this task (Williams, Black, Davis, Pepin, & Wake, 2011). Thus, it can be expected that first year university students have to find new ways of working with the resources they have access to, and that are proposed to them in their courses.

Student use of resources

The use of resources by students has been the subject of relatively little research. Selected studies (e.g. Anastasakis, et al., 2017) indicate that students, in their selection of resources, have been predominantly motivated by the goal to be successful in examinations (and to obtain high grades). The authors of this study made an inventory of the resources used by students when studying for mathematics modules, and explicitly related these to their learning goals. The most widely used resources were those that the university provided for the students, and their own notes. The use of particular resources, for example mathematics textbooks, was specifically linked to the study of worked examples, which were said to help students to prepare for examinations; albeit this often lead to emphasise the surface aspects of the examples (Biza, et al., 2016). In their review study Biza, et al. (2016) identified several limitations of tertiary mathematics textbooks, in particular the emphasis on formal aspects of mathematics, at the cost of opportunities to develop intuitive meanings and understandings. Relating the use of particular resources to examination grades, a study by Inglis, et al. (2011) found that students who attended lectures or used the university's mathematics support centres had higher grades than students who often watched online lectures. The authors suggest that students might need explicit guidance on how to combine the use of various resources into an effective learning strategy. Before this guidance can be given, or be reified in a blended learning environment, more in-depth information on the actual use of resources by students is needed.

The lens of resources

In this study we use the notion of “re-source/s” that students have access to and interact with in/for their learning. We assume that the ways university students learn mathematics is influenced/shaped by their use of the various resources at their disposal. By “use of resources” we denote, for example, which resources students choose (amongst the many on offer) and for what purpose (e.g. revision); the ways they align them (e.g. first lecture then checking the textbook, etc.); which ones seem central to achieve particular learning goals (e.g. for weekly course work, examinations, for their engineering topic area). However, we do not address the specific learning of CS and LA, that is how students interact with particular (e.g. cognitive) resources to learn particular topic areas in CS and/or in LA.

Gueudet and Pepin (in press) have defined student resources as anything likely to re-source (“to source again or differently”) students’ mathematical practice, leaning on Adler’s (2000) definition of mathematics “re-sources” (in Adler’s case used by teachers). In this study we distinguish between (1) material resources, and (2) human resources. (1) For material resources a further distinction has been made between (a) curriculum resources (those resources proposed to students and aligned with the course curriculum), and general resources (which students might find/access randomly on the web). Curriculum resources are developed, proposed and used by teachers and students for the learning (and teaching) of the course mathematics, inside and outside the classroom (Pepin & Gueudet, 2014). They can include text resources, such as textbooks, readers, websites and computer software, but also feedback on written work. General resources are the non-curricular material resources mobilized by students, such as general websites (e.g. Wikipedia, YouTube). (2) In terms of human resources we refer to formal or casual human interactions, such as conversations with friends, peers or tutors.

METHOD

Context

The study took place at a university of technology in the Netherlands, with a student body of approximately 13000 engineering students. The university offers 15 bachelor courses related to technology and engineering.

We selected two first year courses in the first term of the 2016-2017 academic year: Calculus (CS); and Linear Algebra (LA). We purposefully chose these courses, as they were different in size and target group: the CS course was obligatory for all first year engineering students, approximately 2000 students, whereas the LA course was targeted at “applied mathematics and physics” engineering students only, approximately 130 students. The CS course was organized by the mathematics department, differentiated at three levels (A, B, and C), according to perceived level of difficulty and with varying level of emphasis on formal aspects of mathematics (e.g. proof).

In CS, six hours of lectures were organised each week, and one hour of tutorials in groups of eight students. In the course catalogue, and this was supported by lecturers, the aim of CS was to give engineering students a “basis” to be able to “calculate correctly”. It appeared that the aims of the CS course were to provide students with a basic set of mathematical/computational tools they could subsequently use in their engineering studies and in their future work as engineers.

In LA, four hours of lectures were organised each week, and four hours of tutorials, in groups of approximately 30 students. As in CS, the LA learning aims were described as the acquisition of mathematical skills. Moreover, aims of the course were to help students develop the skills and realize the importance of correct mathematical communication, including writing formal proofs. Completing a mathematical writing assignment was part of the course requirements to reach this aim. It appeared that the purpose of LA was to prepare students for higher mathematics (used in the mathematics and physics courses).

Participants

In total, 24 students participated in the study: 18 CS students (involved in nine different engineering programs and all taking the B level CS course); 1 CS student who dropped out of university; 5 LA students (all studying for the ‘applied mathematics’ engineering course). In terms of background, of the interviewed CS students 15 came from secondary schools in the Netherlands, three came from other educational systems. For the Dutch students the CS content was partly familiar, in particular for those who took “strong mathematics” courses (Wiskunde D) at secondary school. Four of the five interviewed LA students came from secondary schools in the Netherlands, one student had attended secondary school in Belgium.

Data collection strategies

Data collection strategies included the following:

(1) Student interviews: The CS students were interviewed in four focus groups, and one individual interview. During the interviews students were asked to make a drawing of the resources they used for their mathematics course (Schematic Representation of Resource System, SRRS - Pepin, Xu, Trouche, & Wang, 2017). These helped the interviewer to understand the ways the resources were used, and for which purpose. The LA students were interviewed in two groups of two, and one individual interview.

(2) Documents/curriculum resources: Relevant curriculum materials and documents (digital and text materials) were collected and analyzed. These materials were provided by the university for the students (e.g. examples of examinations, LA syllabus, LA

study guide, LA assignments, CS study guide, the CS textbook, course summaries in the university's course catalogue, video clips, videos of the lectures).

(3) Teacher interviews: Interviews with two CS lecturers and one CS tutor were conducted, as well as one LA lecturer and one LA tutor.

For analysis, the interviews were transcribed and interview quotations were coded using ATLAS-ti software. The codes were based on our knowledge from the literature concerning the different curriculum resources and their use. In the next step of the analysis the findings from CS and LA were compared, and subsequently these with those from upper secondary school.

RESULTS

Resources at secondary school

In terms of curriculum materials/material resources the textbook was an important resource for most secondary school students, and so were the graphical calculator (also used in examinations) "to quickly plot graphs" (interview reference: CSS01), and past examination papers for revision and practice to prepare for the national examinations. The textbook was seen as the main source of exercises, which were done in class or at home. Regarding homework one student remarked:

> At school I didn't do my homework. There was homework but yeah, if you worked on it during the class … you would get halfway and then at home I was like oh, I get it. I don't have to do the remaining exercises (LAS01).

Online general resources (e.g. YouTube; Kahn academy) were hardly mentioned in relation to secondary school.

In terms of human resources the teacher and classmates were mentioned as an important support for secondary school students. Interestingly, teachers' explanations of the mathematical concepts were not important for all students: in some schools students apparently worked largely independently (with the textbook) and the teacher was only occasionally consulted.

In short, to learn the mathematics and pass the examinations, students reported that it was sufficient to follow the teachers' explanations, do (all) the exercises in the textbook, and practice with the past examination papers. There were few resources, and the ones provided could be straightforwardly accessed and used for solving the problems posed.

Resources for CS

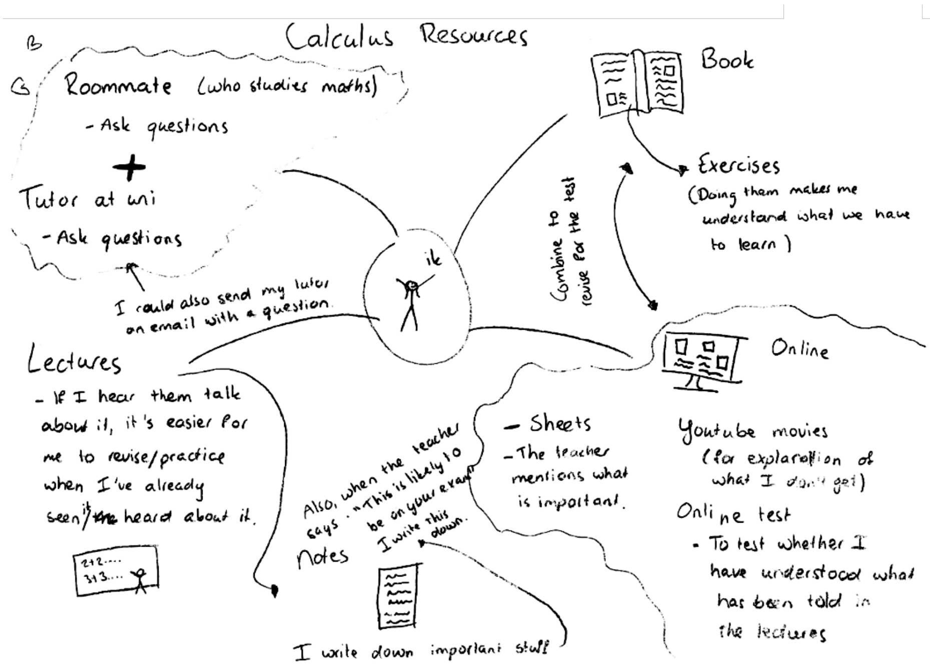
Figure 1 shows the typical resources used by a CS student. The lecture appeared to be an important starting point for many students, albeit not for all. They provided an orientation on the subject ("it's easier for me to revise/practice when I have already seen/heard about it"; Figure 1), and to some extent an enculturation into the world of mathematical concepts and their usages. A lecturer said:

> I do the historical aspect too. Some things have been known for 3000 years, so you have to know that too. So I also do applications and add historical things (..) I just want them to be excited about the subject (CSLI03).

CS was supported by an almost 1100 page general CS textbook (authored by an "external" author) chosen from commonly used university CS textbooks. It contained the essential theory and part of the homework exercises (explained by someone "outside" the students' environment), but its content was not specifically aligned with the lectures and the final examinations. Curriculum resources, such as the textbook, were mainly used for exercises, whereas lecture notes (by the teacher) seemed important for knowing about the content to be learnt. Students' own notes were often used for orientation ("I write down important stuff; also when the teacher says 'this is likely to be on your exam', I write this down."; Figure 1). Selected students used the textbook for additional/different purposes: to read the theory in the textbook to prepare in advance for the lecture.

At the same time digital resources, such as general online videos (e.g. YouTube; Kahn academy) and video-recorded lectures (whole lectures or clips of particular moments), were said to provide additional explanations. The course's online tests and the weekly coursework were used to check whether one had a good (basic) understanding of the content. Moreover, students also mentioned human resources, such as the lecturer/tutor and the roommate "to ask questions" (Figure 1). Friendship groups were important for many students, and for some they were their first line of support (before the tutor or peers in the tutorial/s), to work with on the coursework or to consult about difficult theory or exercises.

Figure 1. Purpose and coordination of resources by a CS student



Resources for LA

In terms of material resources LA was supported by a 200 page course-specific reader, authored by the lecturer and developed and improved over the years. It contained the essential theory and the exercises required to prepare for the examinations - this was "the backbone" of the course, according to the lecturer and the tutors. Other resources, such as the lectures, the homework/coursework exercises, the lecture notes and the videos with worked examples were all aligned with the reader, to become a comprehensive and complete set of resources for the students. The importance of using these resources to individually make sense of the mathematical content, and doing lots of practice exercises "at home", was emphasised by a student:

> But now you have like a huge amount of homework and then you also have workgroups where you can work on it, but then you don't get very far. (…) And if you don't do it at home, you just won't get it and you won't make your tests really good. So you really have to do a lot at home (LAS01).

As in the CS course, the lectures appeared a starting point for many students; they provided an orientation on the subject, and an enculturation into the world of mathematics (e.g. when mathematical proof was explained as an essential mathematical thinking process). Moreover, in terms of human resources, students relied on peer groups (e.g. they collaboratively solved problems during the weekly tutorials), and on the tutors to provide help- this was an important support in the LA course. Tutors were considered more approachable than the lecturers, although students were generally positive about the possibilities to ask questions to lecturers.

Comparing resources and their use

School – university: Whilst selected resources (e.g. textbooks, past examination papers) appeared to be part of the "staple diet" for every student at university or school level, at university students tended to use more, and more varied resources than at upper secondary school, including online lecture videos, video clips (of "difficult" notions), online texts. In addition, selected resources, such as lecture notes, were not mentioned in the secondary school context, where the theory would be taught by the teacher who aligned his/her lessons with the book. Some of the additional resources, such as video lectures, teachers' lecture notes, readers on specific topics, and online tests, were part of the curriculum resources made available by the university. Other resources, such as online applets and videos, were identified by the students themselves.

In terms of human resources there were also differences: at secondary school for most students the teacher provided practically all of the necessary guidance for learning the mathematical topic. At university the lecturer provided the theory, an overview of what was important for their learning of the topic area (and also for the examinations), and selected worked examples. The practical guidance, i.e. how to solve particular mathematical problems, was mainly provided in the tutorials and the exercises/coursework accompanying the lectures. Hence, students had to find their own learning/peer groups and supports for learning, as on their own it was not possible to manage the amount of work and the pace it was taught. This situation was exacerbated in the CS course, as only one hour of tutorial group work was offered, and students had to work collaboratively outside this hour for completion of their tasks. Hence, many CS students organized and coordinated their own support to work with their peers on the coursework, or to consult about difficult theory or exercises.

The students reported that, compared to secondary school, at university: (a) the pace was faster, (b) the content was more difficult to understand, and (c) the mathematical content was offered in larger steps/sections. The interview data suggest that the role and importance of resources changed as a result of this, as students needed more support structures and feedback on their work. This was particularly pertinent with one (autistic) student, who had dropped out of university. He claimed that he had done all possible CS textbook exercises and interim tests – a practice he had succeeded with at school, but he could not make sense of the questions when he sat the final examination. He was lost in the immensity of resources on offer, which he could not possibly all trial out and use for his learning. And he clearly missed the guidance and support given by his schoolteacher, practices which had provided him with confidence for his learning, and success.

CS- LA: Amongst the university curriculum resources, the student usages of the LA reader and CS textbook differed. To come to understand the topic/s, most LA students reported reading the reader, or the lecture notes, which were aligned with the reader. CS students mentioned the textbook as one of their resources, mainly used for worked examples and exercises. In CS, lecture notes and online resources were considered practically as important as the textbook, as part of the provided resource system. This can be understood in the light of the fact that the LA reader was very different, in relation to the course, to the CS textbook: the LA reader was a "book" prepared by the lecturer to align with his lecture, hence further lecture notes or online resources became secondary/ complementary. The reader contained all information for students to pass the examinations, and all other resources were related to/in line with the reader. In contrast, the CS book was only a backup for the lecture notes (which provided the essential notions to learn and study for the examinations), and students were only expected to "dip into" it for clarification, explanation and/or further exercises. Hence, the textbook did not provide a succinct support for CS students (e.g. to pass the CS examinations).

An important difference between CS and LA was due to the different organization of the tutor hours: in LA- 4 tutor hours/week, tutor groups of ca. 30 students/ tutor; in CS- 1 tutor hour/week, tutor groups of ca. 9 students/tutor. This meant that CS students had to work on practically all of the problems by themselves, as there was less support from the tutor. The fact that a wider variety of resources (used) was reported by the CS students, can in part be understood in the light of the different course organisation: in the LA case they adhere to the resources provided by the lecturer; in the CS case the students had to identify and organize their own support (e.g. online resources, friendship groups).

CONCLUSION

The results of this study have shown that the students built on secondary school experiences and they took these as default positions into their courses. However, learning mathematics at university was for most students different from learning mathematics at secondary school. At secondary school the resources (text book, past examinations, teacher) were well aligned and the teacher provided guidance and support. At university more difficult content had to be understood in a shorter period of time; and students had to identify and coordinate the relevant resources, and organise their own support system (including human support such as friendship learning groups), in particular in CS.

When comparing the two courses, the results indicate that (a) the course organization and (b) the provisions and organisation of the curriculum materials (in line with the learning goals) had an impact on which resources students used, and how they used them. In the LA course, with aligned curriculum resources, four weekly tutorial hours and group work, the use of resources largely corresponded to the intentions of use by the university teachers. In the CS course resources were not clearly aligned (although selected resources were recommended); it seemed that students were provided with a "bag of tools" to choose from. Moreover, students had only one weekly tutorial hour (plus six hours of lecture). This meant that students had (a) to identify which were the relevant resources for their individual needs, and (b) to find and navigate their own path through these resources, in order to work efficiently (with regards to examinations) and effectively (with regards to the learning of the mathematics). In both courses human resource, such as lecturers and tutors, peers and friends, played important albeit changing roles for orientation and help seeking.

The results of this exploratory study indicate that in particular large courses, such as CS (> 2000 students), could become better manageable for the students, if they were supported and coached in their resource choice and organisation/management, so that they can cater for their individual needs and preferences. This finding was less visible in a smaller course, where less resources were on offer, and where resources and resource use were more prescriptive and well aligned with the learning goals (e.g. the LA course). However, when particular educational reforms are implemented (e.g. towards more blended learning, with an abundance of digital learning tools on offer), students need to be supported in their "use" of these resources, in particular at transition from school to university. Course designers would also need to take this into consideration.

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Comparing male and female students’ self-efficacy and self-regulation skills in two undergraduate mathematics course contexts

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Students’ efficacy beliefs have a positive influence on students’ academic achievement and retention, especially for female students. These beliefs are closely linked to students’ ability to regulate their learning. In this quantitative study, students’ self-efficacy and self-regulation of learning are compared in two university mathematics courses that differ in content but also in their pedagogical setting: one is a more traditional lecture-based course, and the other course is taught with the Extreme Apprenticeship (XA) method. The analysis is based on the same cohort of students in the two contexts (N=91). The results suggest that students have higher self-efficacy levels in the course using the XA method. Also, the XA course seems to diminish some gender differences present in the more traditional course setting.

Keywords: teachers’ and students’ practices at university level, novel approaches to teaching, self-efficacy beliefs, self-regulation, instructional design

INTRODUCTION

According to Lave and Wenger’s (1991) view of situated learning, a set of skills as well as a set of values and perspectives are needed for a holistic understanding of a topic. Collins, Brown, and Holum (1991) suggest that the process of acquiring this kind of understanding occurs most naturally within the community possessing this knowledge. In this light, it is not irrelevant what kind of instructional practices are implemented to teach university mathematics, and how do these practices offer students opportunities to participate. As Greeno (1997, p. 9) states, “methods of instruction are not only instruments for acquiring skills; they also are practices in which students learn to participate”. Therefore, to enhance students’ learning, instructional designs used to teach university mathematics should offer students both an opportunity to acquire knowledge and an opportunity to become part of a community.

Partly to answer to this need, lots of effort has been put into developing the educational setting at the Department of Mathematics and Statistics at the University of Helsinki (see eg. Oikkonen, 2009; Rämö, Oinonen, & Vikberg, 2015). The department’s teaching has gone through major changes during the past few years as many of the undergraduate level courses are now using the Extreme Apprenticeship (XA) method as their pedagogical framework. In addition, some of the courses working within the traditional lecture-based framework have been developed towards a more interactive direction.

This paper approaches the development of university mathematics education from the perspective of efficacy beliefs and self-regulation of learning. As a part of a larger research project aiming at comparing university mathematics teaching practices and transferring knowledge from research into practice, this paper elaborates on students' experiences of different instructional designs with the focus on their efficacy beliefs and self-regulation of learning.

THEORETICAL FRAMEWORK

Self-efficacy is a person's belief about how well they can perform a specific task in a specific context; these beliefs determine how people feel, think, motivate themselves and behave (Bandura, 1994). Self-efficacy beliefs play a crucial role in learning mathematics as self-efficacy enhances academic achievement (Pajares, 1996; Peters, 2013), especially in female students (Raelin et al., 2014). The gender aspect of self-efficacy is relevant as it affects female students' career choices and increases their retention in STEM fields (Pajares, 1996; Raelin et al., 2014). In terms of instructional design, Peters (2013) shows that students' self-efficacy is higher in teacher-centred than in learner-centred classroom. However, Kogan and Laursen (2014) argue that female students obtain affective gain from student-centred courses as they are more confident in their mathematical abilities in these kinds of contexts.

The notion of self-regulation characterises how students regulate their cognition, behaviour, motivation and emotions to enhance their personal learning processes (Pintrich, 2004). Students are expected to learn self-regulation skills during their university studies and therefore instructional designs should support the development of these skills (Coertjens et al., 2013). The self-regulation process is cyclical in nature as feedback from prior performance is used to adjust future learning performances (Pintrich, 2004; Zimmerman, 2000). Consequently, the quality of self-regulated learning is supported by motivational beliefs, such as self-efficacy beliefs (Heikkilä, & Lonka, 2006; Pajares, 1996). As the social aspect of learning has a significant role in learning self-regulation skills (Volet, Vauras, & Salonen, 2009), instructional designs should also encourage student collaboration.

Aims and research questions

The aim of the paper is to compare students' self-efficacy and self-regulation of learning in two different course contexts. The further analysis focuses on gender as previous research shows that especially female students benefit from more student-centred course designs. The research questions are:

1. How do self-efficacy and self-regulation of learning differ between the two course settings?

2. How is the course setting related to male and female students' self-efficacy and self-regulation of learning?

METHODOLOGY

This study approaches the research questions with a quantitative analysis of students' self-efficacy and self-regulation of learning in two course contexts. The following subsections move on to describe the context, the data collection procedure and data analysis in greater detail.

Context

The research was conducted in a research-intensive university in Finland. Data was collected from two different courses that students usually take during the first semester of their university mathematics studies. Both courses are proof-based, six-week and five-credit (ECTS) courses with over 200 students. In addition to mathematics majors, the courses are taken by many students studying mathematics as their minor subject; these students are usually majoring in physics, computer science, chemistry or education. The two courses, course A and course XA, are implemented in accordance with different pedagogical frameworks. The main difference in the course implementations are the role of lectures, design of the tasks, and the form of support given to the students by the teaching assistants.

Course A is an analysis course. The main content of the course includes limit of a function, continuity, derivative, and its applications. It is necessary to point out that the course is an analysis course rather than a calculus course as exact definitions and proof construction are emphasised. The course functions within the traditional lecture-based setting. However, it has been developed over a decade towards a more interactive direction to respond to students' challenges in the beginning of their university mathematics studies.

The course A consists of four hours of lectures and four hours of small group sessions per week. The lectures are focusing on the main content of the course and aim at creating deep understanding behind those concepts. Inspired by Tall's *three worlds of mathematics* (see e.g. Tall, 2014), the lectures are an active interplay between the human and formal sides of mathematics. The small group sessions are led by a teaching assistant, who is usually an older mathematics student. There are two different kinds of small group sessions. The other one is allocated to the problems students have solved prior to the class. The other small group session is allocated to solving a new set of problems during the session together with other students and with the help of the teaching assistant.

Course XA is a linear algebra and matrices course. The main content of the course includes general vector spaces, subspaces, linear mappings, and scalar products. In addition to mathematical content, the course emphasises skills such as reading mathematical text, oral and written communication, and proof construction. The course is taught with the XA method. The XA method is a student-centred educational method developed in the Department of Mathematics and Statistics and the Department of Computer Science at the University of Helsinki. The method emphasises learning by doing, personalised scaffolding and continuous feedback, and the core idea is to support students in becoming experts in their field by having them participate in activities that resemble those carried out by professionals (see eg. Rämö et al., 2015). The XA method is constructed upon the ancient process of apprenticeship, where a skilled master supervises a novice apprentice, and its theoretical background is in situated view on learning and Cognitive Apprenticeship (Collins et al., 1991; Rämö et al., 2015).

In the XA method, students learn skills and gain knowledge by working on tasks that have been divided into smaller and approachable goals, which are then merged together as the students start to master a topic. The main method of teaching is instructional scaffolding, and it is accompanied with continuous, bi-directional feedback. Further, it supports students to establish relations within the communities of practice which enhances the students' integration into the community (Lave & Wenger, 1991).

In practice, the teaching of the course consists of weekly problems, course material, guidance and three hours of lectures per week. There is a flipped learning approach as students start studying a new topic by solving a set of problems. These topics have not yet been discussed during the lectures, so students need to read the course material to complete the tasks. However, the tasks are designed to be approachable and there are teaching assistants specifically to this course helping the students in solving the problems. The teaching assistants guide the students in a learning space in the middle of the department in drop-in basis approximately six hours a day. Student collaboration is encouraged in the learning space. Students return written solutions to the problems every week. Few problems are selected for inspection and students get feedback for their solutions. The feedback focuses on solutions' logical structure, but also readability and language are evaluated, and students' have the possibility to improve and resubmit their solutions. Students are prepared when they come to lectures as they have done pre-lecture tasks. Lectures focus on active interaction as various small group activities are implemented and students' active participation encouraged. The aim is to form links between the topics and enhance holistic understanding. After the lectures students get more challenging problems on the topic.

Data collection

Quantitative data was collected on a five-point Likert scale (1=completely disagree, 5=completely agree) from students attending both courses. The questionnaire included items measuring students' approaches to learning, their experiences of the teaching-learning environment, self-efficacy and self-regulation of learning. In this paper, the analysis includes items measuring self-efficacy and self-regulation of learning from students who answered the questionnaire for both course contexts (N=91).

There are five items measuring self-efficacy. The items are slightly modified from Pintrich (1991) and they are validated and highly used across disciplines in the Finnish context (see e.g. Parpala & Lindblom-Ylänne, 2012). The 15 items measuring self-regulation of learning are originally from the Inventory of Learning Styles (ILS, Vermunt, 1994) and they have been modified to Finnish context (Heikkilä, & Lonka, 2006). There self-regulation of learning is measured in four scales: self-regulation of process, self-regulation of content, external regulation, and lack of regulation. Self-regulation of process refers to a student's ability to regulate their own learning when facing challenges. Self-regulation of content measures student's seeking of additional literature beyond the course material. External regulation measures to what extent the lecturer regulates student's learning. Lack of regulation refers to possible problems in regulation of learning, such as not knowing how to proceed in the learning process or having challenges in finding ways to cover the course content.

Data analysis

The data analysis is conducted by using IBM Statistics 24. The data in this paper is a part of a larger data set. The results reported here are from the factor analyses computed for the larger data set.

At first, exploratory factor analysis (EFA) was conducted with principal axis factoring and a direct oblimin rotation. Based on the exploratory factor analysis, there are four factors measuring self-regulation of learning. This is in accordance with previous research (see eg. Heikkilä, & Lonka, 2006). Similarly, the factor structure of the self-efficacy scale is like in previous studies (see e.g. Parpala, & Lindblom-Ylänne, 2012) forming one factor. Boundaries used for Kaiser-Meyer-Olkin measure of sampling adequacy (KMO) was 0.7, and for Bartlett's test of sphericity $p<0.001$. One item measuring self-regulation of content was excluded from the factor based on a low communality, a mixed factor loading and deviant skewness and kurtosis. Every factor was then checked for internal consistency: the Cronbach's Alpha is 0.905 for the self-efficacy factor, 0.681 for the self-regulation of process factor, 0.671 for the self-regulation of content factor, 0.708 for the external regulation factor, and 0.661 for the lack of regulation factor. The reliabilities are above the 0.65 level which can be considered acceptable.

As the current study follows a repeated measures design, the data was analysed by using two-tailed paired samples t-test and Cohen's effect size d. In addition, one-way MANOVA with Wilk's Lambda was used to analyse the interaction of independent variables (gender) on the dependent variables (different course settings).

RESULTS

The data consists of 91 students (46 male, 45 female). These students attended both course A and course XA. Students' scores on self-efficacy and self-regulation scales in both course contexts are presented in Table 1 with means, standard deviations, mean differences, paired-samples t-test for statistical significance, and Cohen's d for effect size.

The biggest differences between the two course contexts lie in the self-efficacy and lack of regulation factors. Students report statistically significantly higher self-efficacy levels in course XA compared to course A (MD=0.58, t(90)=6.226, p<0.001). This means that students are more confident in their abilities to succeed in course XA compared to course A. The effect size (Cohen's d=0.62) implies a moderate role for the course context when measuring self-efficacy. A similar phenomenon occurs in the lack of regulation factor; students report statistically significantly less lack of regulation in course XA compared to course A (MD=0.48, t(90)=6.987, p<0.001). In practice this means that on average, students report that they lack regulation of learning more often in course A compared to course XA. In other words, it was easier for students to find ways to handle large quantities of content, self-evaluate their learning, and to meet the learning goals in course XA compared to course A. The effect size (Cohen's d=0.63) implies a moderate role for the course context when measuring lack of regulation.

There are also smaller mean differences in the self-regulation of process and self-regulation of content factors between the two course contexts (MD=0.14 and MD=-0.19 respectively). These differences are statistically significant on a 0.05 level (t(90)=2.189, p<0.05 and t(90)=-2.383, p<0.05 respectively). The results indicate that on average, students in course XA seek more actively additional literature beyond the course material and do more work than expected when compared to course A (self-regulation of content). In addition, an average student in course XA reports that they are more capable of regulating their learning processes when facing challenges when compared to course A. However, one must notice that the effect sizes are below 0.2 suggesting an insignificant role of the course contexts.

There is no statistically significant difference in the external regulation factor (MD= 0.06, t(90)=0.941, p=0.35). This means that on average, students report that the lecturers' instruction on how and in what order to proceed in learning the content influences their learning similarly in both course contexts.

	Course A		Course XA			
Variable	Mean	SD	Mean	SD	Mean difference (XA-A)	Effect size
Self-efficacy	3.09	0.96	3.67	0.86	0.58***	0.62
Self-regulation of	2.83	0.82	2.97	0.80	0.14*	0.17

process						
Self-regulation of content	2.37	1.08	2.19	1.07	-0.19*	0.17
External regulation	3.59	0.78	3.65	0.72	0.06	0.08
Lack of regulation	3.28	0.78	2.80	0.75	-0.48***	0.63

Table 1: Students' scores, mean differences and effect sizes on the self-efficacy and self-regulation factors in courses A and XA, as determined by two-tailed paired samples t-test (* for p<0.05 and * for p<0.001 significance levels) and Cohen's d.**

Let's now move on to analyse both male and female students in one course context at a time. In course A context, male and female students differ statistically significantly in the self-efficacy factor (MD=-0.54, $p<0.01$, $F(1,89)=6.602$, partial $\eta^2=0.079$). This difference is not present in course XA context (MD=0.04, $p=0.821$, $F(1,89)=0.038$, partial $\eta^2=0.001$). These results indicate that female students have statistically significantly lower self-efficacy in course A compared to male students; however, this difference between genders is not present in course XA context.

There are statistically significant differences also in the external regulation factor. In course A context, the mean difference between male and female students is not statistically significant (MD=0.31, $p=0.054$, $F(1,89)=3.827$, partial $\eta^2=0.041$). However, the p-value is very close to the 0.05-significance level. In course XA context, male and female students differ statistically significantly in the external regulation factor (MD=0.51, $p<0.001$, $F(1,89)=12.947$, partial $\eta^2=0.127$). This means that, in course XA contexts, an average female student applies more external regulation compared to an average male student. In other words, female students report that the lecturers' instructions influence their learning more when compared to male students.

There are no statistically significant differences between male and female students in the two course contexts in self-regulation of process, self-regulation of content, and lack of regulation factors.

		Male		Female		
Factor	Course	Mean	SD	Mean	SD	Mean difference
Self-efficacy	A	3.36	0.96	2.82	0.89	-0.54**
	XA	3.65	0.96	3.69	0.72	0.04
External regulation	A	3.43	0.80	3.74	0.74	0.31(*)
	XA	3.40	0.72	3.91	0.61	0.51***

Table 2: Differences between courses A and XA based on students' gender, as determined by one-way MANOVA (* for p<0.05, ** for p<0.01, and * for p<0.001 significance levels).**

DISCUSSION

There are statistically significant differences between course A and course XA in relation to the self-efficacy and self-regulation scales. The biggest differences are in the self-efficacy and lack of regulation factors. The results show that an average student has higher self-efficacy levels in course XA than in course A, and that an average student lacks regulatory skills more often in course A than in course XA. This is supported by the effect sizes implying a moderate role for the course contexts when measuring self-efficacy and lack of regulation. The results bear significance as self-efficacy has a strong positive and lack of regulation a strong negative relation to academic performance (Pajares, 1996; Peters, 2013; Vermunt, 2005).

Gender has a statistically significant interaction with the factor measuring self-efficacy. On average, female students report lower self-efficacy levels in course A compared to male students. In contrast, the self-efficacy levels are very similar for male and female students in course XA. In practice, an average female student is less confident in her abilities to succeed in the course A context when compared to an average male student. However, it seems that the change in course context diminishes the difference as there is no statistically significant difference present between genders in the self-efficacy factor in course XA context.

Female students report more external regulation compared to male students in both course contexts. The results are statistically significant only in course XA context, but the p-value is very close to the 0.05-significance level also in course A context. This means that the lecturers instructions have more influence on female students' learning processes than on male students' learning processes. This is supported by prior research (Vermunt, 2005), although the current study does give any explanations to this phenomenon. However, in Vermunt's (2005) study there was no consistent interaction between students' gender and their learning patterns, and external regulation did not relate negatively to academic achievement. Further research is needed to understand the motivations behind this phenomenon, as well as its implications to instructional practices.

One of the major limitations of this study is that the two courses differ in content. The limitation is caused by the choice of research design as it was not possible to attain the same cohort of students in two different pedagogical settings with the same course content. However, the different course contents do not fully provide an explanation for the result that male and female students' ability to regulate and reflect on their learning processes is dependent on the course context. One can argue that the gender differences are not caused by the characteristics of the mathematics studied but by the characteristics of the learning environment used to study the mathematics. The results of this study may also be affected by the fact that some students are more capable to adopt themselves into new instructional designs. As argued by Kogan and Laursen (2014), student-centred course settings often feature collaborative work, problem-solving and communication, aspects known to be effective for female students. Also, Vermunt (2005) states that female students like cooperative learning more compared to male students. In addition, students who have high confidence in collaboration seek more likely help from other students; these help-seeking students then perform better in a flipped mathematics classroom compared to students seeking less help (Sun, Xie, & Anderman, 2018).

Despite of the limitations, the findings of this study are supported by prior research. To conclude, the results propose that student-centred course designs support students' self-efficacy and self-regulation, especially in female students. More thorough analysis should be completed to understand the mechanisms and motivations behind the differences in these two course contexts and to draw more general conclusions regarding instructional designs used in teaching university mathematics.

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Situational interest in university mathematics courses: similar for real-world problems, calculations, and proofs?

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In the educational context where this study was conducted, the transition from school to university is associated with changes of the learning domain mathematics. In high school, solving real-world problems and performing calculations are dominating practices, whereas in university, the main focus is on proving. Successful learning processes are associated with appropriate prerequisites, e. g. interest. The question is which component of interest concerning which practice is important for successful processes. We developed an instrument to differentiate these facets of students' situational interest. A study with 339 first-semester students in mathematics partially confirms the expected factorial structure of this instrument. Precise information concerning learners' interest may help us to support students at this challenging transition.

Key words: Students' interest; Feeling- resp. value-related component; Mathematical practices; Transition to and across university mathematics; Teachers' and students' practices at university level.

INTRODUCTION

The transition from school into a university mathematics programme is a challenging phase for many students. Reports mentioning high dropout rates in academic study programmes with a focus on mathematics (OECD, 2010) illustrate this fact. Reasons for students' problems in the first year of university study are primarily attributed to the changes of the learning domain: while the school subject mathematics strongly relies on performing calculations and solving real-world problems, mathematics in university is presented as a scientific discipline with a focus on proving (Rach, Heinze, & Ufer, 2014). It is yet not clear which of the individual prerequisites, that students bring from school, support successful learning processes in university courses. Many researchers assume that subject-specific interest, in the sense of a person-object or a person-situation relationship, is an important learning prerequisite in general (Wigfield & Cambria, 2010). Common instruments to survey interest in mathematics are questionnaires. Mostly, these use the word "mathematics" to describe the objects of interest. At the transition from school to university, however it is not clear if students refer to school or university mathematics when reporting their interest in this questionnaires. To obtain a more differentiated insight into students' interest with a stronger relation to concrete learning situations, we develop a task-based instrument to measure students' interest concerning different mathematical practices. In this contribution, we present the conceptualisations of this instrument in detail and report results of an empirical study with 339 first-semester students that investigates the factorial structure of the instrument.

THEORETICAL BACKGROUND

In the first section, we give an overview about the motivational construct “interest” by addressing common definitions and results concerning its role in learning processes. In the second section, we summarize ideas to the objects of interest, mathematics.

Interest in mathematics learning processes

Researchers agree that interest is an important motivational variable. Its role as a learning prerequisite, a learning processes measure, and as a learning outcome has been put forward repeatedly (Rotgans & Schmidt, 2017; Sonnert & Sadler, 2015; Wigfield & Cambria, 2010). Interest is defined as a specific relationship between a person (here: a student) and an object or a situation (here: mathematics resp. mathematical practices) (Krapp, 2002; Schiefele, 2009). The objects of interests may be concrete objects, topics, or school subjects. Whereas Schiefele, Krapp, and Winteler (1992) conceptualize interest mainly as a relatively stable trait: “*Individual interest* is conceptualized as a relatively stable affective-evaluative orientation toward certain subject areas or objects” (Schiefele, 2009, pp. 198), Linnenbrink-Garcia et al. (2010, 2013), in contrast, see interest as specific to a situation: “*Situational interest* is a temporary state aroused by specific features of a situation, task, or object (e. g., vividness of a text passage)” (Schiefele, 2009, pp. 197–198). Situational interest is limited at a point of time, e. g., in a certain learning situation and may or may not develop into individual interest (Hidi & Renninger, 2006). Schiefele (2009) distinguishes between two components of interest: (1) a feeling-related component, related to fun or other positively experienced emotions, and (2) a value-related component, related to a high importance of the objects of interest for oneself (adapted from Linnenbrink-Garcia et al., 2010, 2013). Linnenbrink-Garcia and colleagues (2010, 2013) could confirm in empirical studies the two-factor structure of (maintained) situational interest and separate the two components from each other. The separation of interest in a feeling-related and a value-related component is also in line with expectancy-value models that distinguish between an intrinsic and a utility value (Eccles & Wigfield, 2002).

Several researchers assume that interest can trigger successful learning processes: interest can lead to more engagement and to an enhanced use of deeper learning strategies which then result in better learning achievement (Hidi & Renninger, 2006). However, empirical studies could not confirm this assumption for learning processes in undergraduate mathematics courses (Rach & Heinze, 2017). The conceptualisation and operationalisation of interest in mathematics may be one reason for these conflicting findings: firstly, the interest object “mathematics” changes its character at the transition from school to university. So the objects of reported interest may differ from the learning object in the first semester (see next section). Secondly, many of the used instruments measure the individual interest rather than the situational interest, although the situational interest is a more proximal variable to the specific learning situation.

Mathematical practices at the transition from school to university

In the literature (Gueudet, 2008; Engelbrecht, 2010), changes in two relevant aspects of the learning environment at this transition are described: a shift in the character of the learning domain, mathematics, and a change in the learning opportunities and their use. As interest is a person-object relationship resp. a person-situation relationship, the change of the learning domain is important for the conceptualisation of interest. The specific differences of mathematics at school and at university might vary between countries due to traditions concerning learning goals etc. The subsequent presentation refers to the situation in Germany which was analysed in empirical studies (Rach et al., 2014) and which is relevant for the empirical study we present below. Nevertheless, several of the described features might hold for other countries as well. One central goal of teaching and learning at school is to apply mathematics for solving real-world problems (CCSSI, 2010; OECD, 2016). Thus, describing realistic situations mathematically, performing computations, and applying mathematical procedures are central. On the contrary, university mathematics is usually taught as a scientific discipline based on formal definitions of concepts and formal-deductive proofs (Gueudet, 2008; Nardi, 1996). Mathematics at university is often presented in a DTP (Definition-Theorem-Proof) structure to emphasise its logical rigidity (Engelbrecht, 2010; Weber, 2015). These conclusions are mainly based on observations of lectures (Weber, 2004) or of tutorials (Nardi, 1996), interviews with involved parties (Nardi, 2008), or analyses of tasks (Gueudet, 2008). In general, mathematical tasks are often used to describe and examine learning situations and their cognitive demands (e. g., Stein & Lane, 1996).

Theoretical analyses and anecdotal evidence from students and teaching staff have described a change of the learning domain (e.g., Engelbrecht, 2010; Gueudet, 2008; Thomas & Klymchuk, 2012). However, empirical studies supporting the role of this shift and its effects on student learning are scarce. In particular, reliable evidence on the role of students' motivational variables, especially of students' interest, is scarce.

The SISMa project

The goal of the SISMa project ("Self-concept and Interest when Studying Mathematics") is to contribute evidence concerning students' interest and self-concept with regard to certain mathematical practices (Ufer, Rach & Kosiol, 2017). As a first step of the project, we developed interest and self-concept measures that focus central mathematical practices. Using these measures, the development of these variables during students' learning processes and their effect on student learning in undergraduate mathematics programmes are investigated.

In this contribution, we focus on the measurement of situational interest. We developed a questionnaire of situational interest that is based on the conceptualisation of interest as a person-object (situation) relationship, and on the assumption that interest concerning a specific task is an indicator of situational interest. Here, we follow the idea of Schukajlow and colleagues (2012) who operationalize the two components of situational interest with the constructs enjoyment and value (see also Schukajlow & Krug, 2014). In this questionnaire, mathematics tasks are presented to students and they are asked to imagine solving this task (not to solve the task), and then to state their anticipated enjoyment and value appraisals when working on the task. For our questionnaire, we designed mathematical tasks concerning the topic derivatives that each prototypically represent one of the three practices "solving real-world problems" (resp. applying mathematics, sample item: "Using metal, you should produce a cylindrical can with a prescribed volume. For which radius is the material consumption minimal?"), "performing complex calculations" (sample item: "Let f be $f(x) = \frac{\sqrt{1+x} \cdot e^x}{4+x^2} - 1$. Calculate the extrema of the function f."), "and "proving" (sample item: "Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that f is continuous."). After a pilot study (see Rach et al., 2014), we used 12 tasks. While the first and second practice are considered central in mathematical lessons at school, the last practice predominates in university courses. After reading each task, students rate their agreement to one statement concerning the feeling-related component (item: "It would be fun to me to work on this task.") and one statement concerning the value-related component of interest (item: "Even if the task is not part of an exam, it is important to be able to solve the task."). So in sum, situational interest is divided into six subscales – one for each component and each mathematical practice (see table 1).

	Solving real-world problems, applying mathematics (4 tasks)	*Performing complex calculations (4 tasks)*	*Proving (4 tasks)*
Feeling related	Feel Apply	Feel Calc	Feel Proof
Value related	Value Apply	Value Calc	Value Proof

Table 1: Two components of situational interest and three practices.

RESEARCH QUESTIONS

The aim of this contribution is to investigate the structure of students' situational interest as measured with our instrument. To achieve this, we applied the instrument to students from a first semester mathematics course. The questions focused in this contribution address students' reported levels of interest and the empirical structure of the questionnaire:

1. What level of feeling- and value-related situational interest do students from a first semester mathematics course report concerning the three practices?
   Since prior research has shown similar trends, we expected that students would report lower interest levels for each component and practice after six weeks, compared to the first day of their studies. One reason might be that high demands in the first semester lead to a decrease in interest.
2. Is the theoretical structure of subscales, that guided their development, reflected in the factorial structure of the newly developed instruments?

We expected that the two components of situational interest, the feeling- and the value-related component, are reflected in the empirical data. Moreover, we expected that subscales concerning the three different practices can be separated from each other.

METHOD

Design and sample

We present data from two measurements, at the beginning of the first semester (T1) and in the middle of the first semester (T2). Our sample consists of 339 mathematics students (162 female) of one German university of the course "Analysis I" which is compulsory for first-semester mathematics students. In this course, mathematics is presented as a scientific discipline with a strong focus on formal concept definitions and deductive proofs. The students were enrolled in the bachelor's programmes "mathematics" ($n = 90$), "business mathematics" ($n = 91$), or a mathematics teacher education programme for the highest attaining secondary school track in Germany ($n = 104$) – for the remaining students, we have no information about their study programme. The participation in the study was voluntary.

Instruments

The mathematical tasks were arranged in a fixed, random order. The questionnaire was submitted to the students with the following instructions: "Imagine how you solve these tasks. Do not solve the tasks, but report your agreement to the following statements". Students rated each statement on a four-point likert scale from 0 (disagree) to 3 (agree). The individual mean value of a single student on a scale was computed if this student had answered at least half of the items of the scale.

RESULTS

Table 2 shows the means, standard deviations, and Cronbach's Alpha of the six scales at the first and second measurement point.

	Beginning of the first semester (T1)			Middle of the first semester (T2)		
	N	*M* (*SD*)	α	*N*	*M* (*SD*)	α
Feel Apply	323	2.22 (0.61)	.73	226	2.02 (0.67)	.76
Feel Calc	323	2.16 (0.71)	.79	230	2.01 (0.74)	.80
Feel Proof	331	2.18 (0.61)	.77	237	2.00 (0.61)	.82
Value Apply	323	2.25 (0.64)	.82	224	2.04 (0.71)	.82
Value Calc	325	2.26 (0.70)	.86	232	2.00 (0.76)	.86
Value Proof	331	2.31 (0.61)	.88	237	2.06 (0.67)	.86

Table 2: Means, standard deviations, and Cronbach's Alpha of the interest scales. Likert scale from 0 (disagree) to 3 (agree).

Concerning all 24 items (3 practices, 4 tasks, 2 components), there are no floor or ceiling effects. Reliability analyses underpin the internal consistency of the six subscales. For T1 (N = 323–331), Cronbach's alpha of each scale ranges from .73 to .88, for T2 (N = 224–237), from .76 to .86. Indeed, students reported higher value and feeling ratings concerning all practices in the first as compared to the second measurement (cf. table 2; t(194–197) = 5.42–6.83; p < .001; d = 0.39–0.49; measured with students who were present at both measurements). Levels of value- and feeling-related interest were relatively similar between the three practices.

Correlation analyses (see table 3) show that, as expected, the feeling-related and value-related scales concerning each practice strongly relate to each other (Solving real-world problems: T1: r = .55, T2: r = .65, Performing complex calculations: T1: r = .49, T2: r = .53, Proving: T1: r = .53, T2: r = .60). Moreover, the three feeling-related scales (r = .43–.60) resp. value-related scales (r = .59–.79) correlate strongly. The correlations of feeling-related scales with value-related scales for different practices are moderate (T1: r = .28–.44, T2: r = .31–.39).

	Feel Apply	Feel Calc	Feel Proof	Value Apply	Value Calc	Value Proof
Feel Apply		.47	.43	.55	.31	.31
Feel Calc	.47		.60	.28	.49	.32
Feel Proof	.44	.54		.38	.44	.53
Value Apply	.65	.31	.31		.66	.68
Value Calc	.30	.53	.39	.59		.79
Value Proof	.37	.37	.60	.66	.79	

Table 3: Correlations between the interest scales. Over the diagonal T1 (*N* = 334–339), under the diagonal T2 (*N* = 239–243). All correlations significant with *p* < .01.

As some of the situational interest scales correlate strongly, we investigated whether our expected scale structure would be replicated by exploratory factor analyses. The results of Principal Component Analysis with Varimax rotation for every measurement point partially support our expected structure of our scales. For T1, results indicate four factors that explain nearly 63% of the variance. Table 4 shows the factor loadings of the items on the four identified factors.

As expected, the feeling-related items concerning each of the three practices load strongly on one of the factors two to four each. Contrary to the theoretical construction of our questionnaire, the first factor includes the value items for all three practices. Some value-related items have cross-loadings on the feeling-related scales for the same practice. For T2, the factor analysis shows similar results, with the four factors explaining 65% of the variance. Thus, it also seems to be possible to combine all value items into one value scale. This value-related scale with twelve items has an excellent reliability of α = .93 (N = 317, T1) resp. α = .92 (N = 223, T2).

	Factor 1	Factor 2	Factor 3	Factor 4
Feel Apply 1		.35	.36	**.42**
Feel Apply 2		.31		**.72**
Feel Apply 3			.44	**.64**
Feel Apply 4				**.81**
Feel Calc 1		.50	**.41**	
Feel Calc 2		.38	**.67**	
Feel Calc 3			**.78**	
Feel Calc 4		.35	**.74**	
Feel Proof 1		**.71**		
Feel Proof 2		**.62**		
Feel Proof 3		**.65**		
Feel Proof 4		**.70**		
Value Apply 1	**.70**			.44
Value Apply 2	**.61**			.50
Value Apply 3	**.60**			.56
Value Apply 4	**.55**			
Value Calc 1	**.75**			
Value Calc 2	**.75**		.32	
Value Calc 3	**.69**		.46	
Value Calc 4	**.76**		.35	
Value Proof 1	**.73**	.35		
Value Proof 2	**.82**			
Value Proof 3	**.71**	.32		
Value Proof 4	**.76**	.35		

Table 4: Factor loading on the four extracted factors (T1, *N* = 313). Loadings under .30 not shown. The expected loadings on a factor are bolded.

DISCUSSION

Students' interest is an important variable in successful learning processes, including the transition to university mathematics. Based on prior research, we argue that to measure students' interest validly, one requires situated, e.g. task-related, measures, to cover feeling- and value-related components of interest and to take into account the changing role of different mathematical practices during this transition (Rach & Heinze, 2017). Contrary to the findings of Schukajlow et al. (2012), the results of our empirical study show that the relations between the feeling-related components for different practices are relatively low as compared to the respective correlations between the value-related components. This partially supports the expected six-factor model (c. f. Schiefele, 2009), and underpins the different roles the three practices play during the transition in the educational context in Germany. Beyond this, factor analyses indicate that the value-appraisals seem to be fairly consistent over all three practices and the exploratory factor analyses also allow a model assuming only one value factor over all practices. This observation applies to data from the first day of university study, but also to data collected after six weeks into the semester. After six weeks, students report across all scales lower approvals. Interestingly, even though students seem to differentiate their interest reports by practices, we find almost no differences in their mean levels of interest between the different interest scales.

Of course, the results of our study rely on students' self-reports about their anticipated situational interest in a set of specific tasks. Even though the differences between school and university mathematics have been described internationally, it might be interesting to replicate the studies in other educational systems. In future studies, the relation between the anticipated situational interest and the actual engagement in learning situations should be considered. However, Schukajlow and Krug (2014) found only slight differences in prospective and retrospective ratings of interest in working with mathematical tasks. In sum, the newly developed instrument may provide more differentiated insights into students' interest concerning different mathematical practices. Further research should investigate which facets of situational interest indeed go along with learning gain in the study entry phase. In particular, it is an open question if interest in practices that are typical for university mathematics, such as proving, are more important for learning gain than other interest facets. In the future, this instrument may help to evaluate support courses in the study entrance phase, to analyse the development of students' interest (c. f. Hidi & Renninger, 2006), and to investigate the impact of different interest facets on study success.

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Determining your own grade – Student perceptions on self-assessment in a large mathematics course

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In our poster, we report on students' perceptions on self-assessment in large undergraduate course context taught with the DISA model (Digital Self-Assessment) in which the final exam is replaced by self-assessment. The students found that practising self-assessment skills was valuable and encouraged them to study for themselves. At the same time, self-assessment was considered unfamiliar and therefore difficult. We argue that the DISA model encouraged students to take more responsibility on their own learning.

Keywords: Assessment practices in university mathematics education, Novel approaches to teaching, Self-assessment, Student perceptions, Ownership of learning.

INTRODUCTION AND THEORETICAL BACKGROUND

Self-assessment skills are often mentioned as an important ingredient in becoming an effective life-long learner (Boud, 2000). Good self-assessment skills are vital also during studies, as they form a key part of self-regulation (Zimmerman, 2002). In university mathematics, self-assessment practices have been shown to promote learner autonomy and mathematical communication skills (Stallings & Tascione, 1996). In a study for engineering students, self-assessment was linked to better time management and more effective learning (Friess and Davis, 2016). In earlier studies, the use of self-assessment is usually limited to low-stakes training exercises or controlled self-grading of the final exam. We, however, do not want merely to introduce a reflection tool, but instead question thoroughly the traditional concept of external assessment in mathematics. The aim of our study is to develop the theory of self-assessment in the specific field of mathematics.

In the DISA model, the traditional end exam is removed, and final course grades are awarded by students' self-assessment based on a learning objectives matrix. We use Yan and Brown's (2017) model of cyclical self-assessment process as our theoretical framework. In the DISA model, the development of self-assessment skills is supported with various feedback methods throughout the course. The students' reflective skills are formatively practised with several self- and peer-assessment exercises. In an earlier study (Nieminen, Rämö, Häsä, & Tuohilampi, 2017), we found that mathematics students connected the self-assessment training and the lack of exam with deep learning approach. The purpose of this poster is to present students' perceptions on self-assessment in the DISA model.

DATA COLLECTION AND ANALYSIS

The participants of the study were students of a linear algebra course taught with the DISA model in a Finnish university in autumn 2017. Data was collected after the course with a questionnaire with open ended questions (n=113). A qualitative content analysis was conducted to study the student perceptions on self-assessment.

RESULTS

Data analysis of qualitative survey data resulted in two main categories. One category was “Ownership of learning”, containing a theme of positive perceptions on the self-assessment method and another theme concerning developing one’s own reflection skills. The other category was “New kind of learning culture”. Students’ comments in this category were either “micro level“ comments concerning the course arrangements or “macro level” comments concerning self-assessment as part of a new kind of learning culture. Our results indicate that the theory of cyclical self-assessment can be applied to the field of mathematics. Furthermore, we argue that digital learning environments can be used in large mathematics courses to practice self-assessment skills.

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Seminar groups as part of first-semester mathematics teaching: What did the students learn?

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In this study, the learning encouraged by teaching activities in a small-group setting was investigated through the analysis of students' responses to survey and interview questions. The results indicate that the students perceived an increased ability to communicate mathematics in written form, but to a lesser extent developed their ability to discuss mathematics and build conceptual understanding.

Keywords: Teachers' and students' practices at university level, teaching and learning of specific topics in university mathematics, communication, small-group teaching.

Empirical research on mathematics teaching at tertiary level is an area of sustained growth and interest (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). Besides studies on the discourse of mathematics teaching in lectures (e.g. Viirman, 2015), there are studies focusing on small-group tutorial sessions (e.g. Mali, Biza, & Jaworski, 2014). Inquiry-based small-group activities were found to have reasonable effect in a project aiming to improve the conceptual understanding of engineering students (Jaworski, Robinson, Matthews, & Croft, 2012). However, there is still a need for research on the use of small-group teaching as a complement to lectures across different national and institutional practices. In the study presented here, we analysed students' experiences of learning from teaching activities in a small-group setting at a large Swedish university. Our research question is: How do the students perceive their mathematical abilities to have developed through the small-group seminars?

In the first-semester mathematics course, besides whole-class lectures, students once a week took part in so-called seminars with 10–15 students in each group. The stated aim of the seminars was "to learn to discuss mathematics, and to present mathematics orally and in written form", but also to "support the learning of algebra and calculus". Each seminar group was led by a lecturer. Before each seminar, students handed in a written solution to one or two tasks. During the seminars, they presented the solutions orally and received comments from the seminar leader on both their oral and their written presentations. Furthermore, the students discussed conceptual tasks given by the seminar leader or discussed questions posed by the students themselves.

A survey was conducted in a lecture after 10 of the 13 seminars had been held. Of the 49 students present, 42 completed the questionnaire. Students were asked to indicate on a Likert scale the extent to which the seminars had facilitated their learning of presenting and discussing mathematics, solving problems, and understanding concepts.

Seven students volunteered for individual interviews undertaken three weeks after the survey. The focus was on students' experiences from the seminars and questions were posed about their learning. In the analysis, we searched for utterances where students' perceived learning was pronounced. These utterances were then grouped together in categories due to the abilities mentioned. This was a deductive analysis drawing on the notion of competencies (e.g. Niss & Jensen, 2002). For each category we then interpreted how the students perceived that their mathematical abilities had developed.

The analysis of data is ongoing. However, the preliminary analyses indicate that students perceived that they developed their ability to communicate mathematics in written form, but to a lesser extent developed their ability to solve problems and discuss mathematics. In the survey, students marked high on the Likert scale for learning to make written presentations (mean 3.5, scale 1–4 with 4 as 'a lot'), a little lower on solving problems (3.1), understanding concepts (3.0), oral presentations (2.9) and discussions (2.9).

The categories revealed by the interviews were: understanding of concepts, communication, solving problems, procedural knowledge, and reasoning. Analyses of the interviews confirmed the results from the survey; the perceived learning from the seminars largely concerned the written presentations. While some students talked about their aim to build conceptual understanding, they mentioned also how the focus on details and technical aspects in the written communication distracted them from this aim, a finding in line with the distracting role of aspects such as examination forms found by Jaworski et al. (2012). In the interviews, we also met descriptions of different ways in which seminar leaders had chosen to use representations and problem solving (cf. Mali et al., 2014). Further data collection is planned to answer questions on the role of these characteristics and the seminar leaders' mathematical discourse (cf. Viirman, 2015).

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Students' usage patterns of video recorded lectures in an undergraduate mathematics course

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It is customary for some universities to offer recordings of live lectures to students. Whether this improves learning and academic performance is debated in the literature. As with most technology, correct usage can lead to increased academic performance, but there are also usage patterns that can be considered counter productive, especially for learning mathematics. In order to investigate patterns in students' usage of such online recorded live lectures, we analyze log files from a server holding the recordings of an undergraduate mathematics course. This poster presents results from the statistical analysis and discusses some of the usage patterns found in light of Moore's theory of transactional distance.

Keywords: teachers' and students' practices at university level, the role of digital and other resources in university mathematics education, video recorded lectures, streaming media.

INTRODUCTION AND THEORETICAL FRAMEWORK

It is becoming customary for some universities to stream lectures and make recordings of lectures available online, in particular for large enrolment courses. Students appreciate the flexibility that access to recorded live lectures give (Yoon & Sneddon, 2011), but it is unclear whether it improves learning and academic achievement, particularly in subjects such as mathematics where conceptual understanding is more important than rote learning. Several studies indicate a weak association between frequent use of online recorded lectures and poorer performance (Brooks, Erickson, Greer, Gutwin, 2014; Howard, Mehan, & Parnell, 2017; Inglis, Palipana, Trenholm, & Ward, 2011, Trenholm, Alcock, & Robinson, 2012 and references therein). Yoon & Sneddon (2011) suggests that easy accessibility of recorded lectures may give some students a false sense of security, resulting in procrastination and missing the lecture completely in the end. One of the three variables in Moore's theory of transactional distance (Moore, 1993) is *autonomy*, the ability a student has to manage their own learning (the other two variables are structure and dialogue). Within this theory, students with a high degree of autonomy would be able to utilize recorded lectures better than students on the other end of the autonomy-spectrum.

RESEARCH TOPIC AND RESEARCH QUESTIONS

The research topic of this poster is students' usage patterns of online recorded lectures in an undergraduate mathematics course. Data are collected from log files of viewings of recorded lectures in a first year (second semester) course in linear algebra and differential equations at the Department of Engineering Sciences at the University of Agder held in spring 2016, with a total of 381 enroled students. We analyze the data to answer the following research questions: How many students watch the recordings? Do students watch the recordings regularly or in a more random fashion? How much of each recorded lecture is watched? When do students watch the recorded lectures? What is the delay between when the lecture was given and when the recording is watched?

RESULTS

We present the results of the analysis in this poster in terms of histograms and bar plots with accompanying texts. In short, our results are as follows: Out of 381 enroled students, approximately 80 students (21%) use the recordings on a regular basis, with a marked increase in viewings around sports- and Easter holiday. Closer analysis shows that typically all lecture content in the videos are watched, suggesting that most students watch the whole recording instead of just smaller parts of it. The maximum number of viewings on a day happens on the same day as the live lecture, and the viewings then drop during the following two-three weeks after a lecture. There are relatively few viewings before the exam suggesting that students do not use the recordings for exam preparations.

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TWG 5: Teachers' practices

Helping lecturers address and *formulate* teaching challenges: an exploratory study

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In this paper we present an exploratory study on the kind of questions or difficulties lecturers point out at the beginning of an educational course – to be addressed in it. These questions happen to be very general and are poorly connected to the knowledge to be taught. We suggest a twofold interpretation of this phenomenon. On the one hand, and in the line of the didactic transposition theory, teachers do not allow themselves to raise questions about the knowledge that is supposed to be their main field of expertise. On the other hand, the prevailing institutional pedagogy does not provide teachers with a fruitful enough conceptual frame to formulate this kind of questions. From the experience of several lecturer education courses, we postulate that didactics can help university teachers better interpret their practice and question it in a more productive way.

Key-words: university teacher education, pedagogy, knowledge to be taught, scale of levels of codeterminacy, teacher problems.

INTRODUCTION: THE PROBLEMS TEACHERS FACE

Since 2009, our research group has designed and implemented two different courses to provide university lecturers and research assistants with educational tools enabling them to better design, implement and analyse teaching and learning processes. The first course took place at IQS – Universitat Ramon Llull in Barcelona, an institution offering degrees and master programmes in engineering and management. This first course was addressed to PhD students teaching at the institution or planning to teach soon. The PhD students' research domains and subjects taught were diverse and included econometry, finance, mathematics and engineering, among others. The course was structured into thirteen 2-hour sessions and lasted 4 academic years. In the first session, the participants were asked to raise teaching questions they would like to address in the course. The collected questions were then classified according to the level of co-determinacy they affected (Chevallard, 2002, we will come back to this notion later). The subsequent sessions were each devoted to addressing the questions that belonged to one of the levels, starting from the general ones (Civilisation, Society) and finishing with the content-specific ones (Domain, Sector, Theme, Question). At the end of the course students were asked to design a teaching project for a subject of their specialty, including a syllabus, the planning of learning goals and a detailed description of three teaching activities: a lecture, a student-centred task and an autonomous out-of-class activity.

The second course was held at EUSS-UAB in Barcelona, an engineering school offering Mechanical, Electronic, Electrical and Management engineering degrees. The participants were in-service teachers with different educational backgrounds and research fields. The course was organised in six 2-hour sessions. It was based on a *study and research path for teacher education*, an inquiry-based teaching format focused on the study of a professional teaching question (Florensa, Bosch, & Gascón, 2017). The question addressed was “Could modelling be the main motivation of my subject?” It was approached through different phases where participants experienced an inquiry study process in the position of the students, then analysed the process experienced and finished by designing an inquiry study process for their subjects.

Both courses started by asking the participants to provide a list of questions or difficulties they would like to address with the help of the educators. In all the cases, we were surprised to find there was only a small number of questions that dealt specifically with the knowledge to be taught. Teachers mainly mentioned general issues related to assessment, class management, coordination or student characteristics (diversity, lack of motivation, the role of mathematics in their subject, etc.). They rarely included their subjects in the questions and, when they did, the problems formulated were very general.

We compared this result with an investigation research carried out by Cirade (2006) in pre-service teacher education in France within the anthropological theory of the didactic (ATD). In this research, during 3 editions of a 25-week course in 3 academic years, the participants who were doing an internship in secondary schools were asked to formulate a question every week. These “questions of the week” constituted the basis of the course, despite the fact that only a small sample of them could be addressed – all in all, more than 7,000 questions were collected. Cirade provides a systematic gathering and analysis of the teacher-students’ spontaneous questions and uses them to identify the mathematical difficulties teachers encountered and their trouble in making them explicit. The kind of questions raised at the beginning of the course – which coincided with the beginning of the academic year – were initially very general, and were related to how to behave in class, how to manage the students’ behaviour, what to do in a meeting with parents, etc. Then, as the teacher education course progressed and certain tools coming from the field of didactics of mathematics were introduced, teacher-students became more and more able to state questions related to the knowledge to be taught. In a sense, we can say that they stopped taking the knowledge to be taught as a given and dared to state questions about their own field of expertise. For instance, they ended up asking questions such as “How to justify the need of sketching functions given their analytical expression?”, or “Why do we need to measure angles in radians in addition to degrees?”, etc.

Following Cirade (Chevallard & Cirade, 2010; Cirade, 2006), we postulate that educational courses for university teachers cannot ignore the way teachers problematize their professional practice and teachers should take their concerns and difficulties as the starting point of educational processes. Besides, as researchers in mathematics education, we also agree with the importance of approaching these questions from a discipline-based level. As stated by Berthiaume (2009, p. 215):

> For some time now, educational researchers have investigated the idea that, in order to be effective, higher education teaching may have to be 'discipline-specific'. In other words, teaching in higher education has to take into account the specific characteristics of the discipline being taught. This means that developing an understanding of teaching and learning is not sufficient to become an effective teacher in higher education. Rather, one must also develop an understanding of the teaching and learning requirements of one's own discipline. This has been termed 'discipline-specific pedagogical knowledge'.

We consider essential for university teachers to be able to formulate their difficulties, not only as general issues concerning students and class management, but also including the knowledge to be taught as a key element. Even if teaching problems are initially perceived as general in their manifestation, the way to address them will necessarily involve knowledge-based activities. From the perspective of the ATD, taking the knowledge to be taught into account means more than simply including it as a variable or parameter of the problem formulation. It also means considering it as an institutional construction, questioning its current shape and searching for possible new reorganisations, taking into account – without assuming – the epistemologies and pedagogies prevailing at the university (Barquero, Bosch, & Gascón, 2013).

The aim of our study, which is still at an exploratory stage, is to analyse the kind of questions university teachers are able to state at the beginning of an educational course – as the ones we implemented – and locate their questions at different levels of specificity/generality regarding the knowledge to be taught. We postulate that knowledge in didactics is important to provide university teachers with conceptual and methodological tools not only to improve their professional practice, but also to describe, interpret, conceive and question it in a more productive way. The first step to make progress in this direction is to start understanding how lecturers spontaneously formulate the challenges faced during their daily practice.

WHAT PROBLEMS DO LECTURERS SET FORTH?

We collected a total of 143 questions from the 4 courses, 35-40 per course, each of which was attended by 10-15 participants. In all of the cases, teachers attending the course were asked the following: "Write down two or three problems, difficulties or doubts that you find, or you think you may find during your teaching practice." There was a lot of redundancy in the questions, so we eliminated repetitions even if the phrasing was different. We are presenting this selection according to the questions' generality, using the scale of levels of didactic co-determinacy. This tool was introduced by Chevallard (2002) in the didactic analysis to include aspects of the institutional organisation of teaching and learning processes that are usually taken for granted (Artigue & Winsløw

2010; Chevallard & Sensevy, 2014). It helps distinguish the conditions and constraints affecting teaching and learning processes that are originated within the discipline, and the generic levels common to the teaching of any discipline:

Upper levels	*Lower levels*
Humanity	↓↑
↓↑	Discipline
Civilisation	↓↑
↓↑	Domain
Society	↓↑
↓↑	Sector
School	↓↑
↓↑	Theme
Pedagogy	↓↑
↓↑	Question

Figure 1. Scale of levels of didactic co-determinacy

Civilisation and Society

The upper levels of the scale refer to the conditions that are set up by our society or, when these are common to several societies, by the civilisation they belong to. We identified the following questions at this level:

- What to do in a culture in which effort and reward are no longer related?
- How does a social situation influence the effectiveness of a course or a teaching format? For example, the students' attitude seems different in times of crisis…
- To what extent should study plans be aligned with the labour market?
- What is "academic freedom" and what are its limits?
- Where do competencies come from? How are they established?
- Clashing of (generational or social?) values: sometimes it is difficult to act as a teacher, a guide or a referent when our own values seem to be obsolete (or to strongly contrast with those of our students). For example: the value of effort, the gratification of work well done, the fact that money cannot buy everything or that not everything is on the web, the importance of culture, of thinking, that there are things that are "right" and others that are "wrong" (e.g. cheating in an exam is "wrong"), etc.
- What to do with students who act as "clients"?
- The application of Bologna is an adaptation of the learning process and an evolution or change: more participative students, more teacher-student interaction, etc. Adaptability is therefore considered a consequence of an evolution, but if we analyse it, we are giving the same classes, in the same environment, with the same student profile. Can we do anything to make the context change?
- How is the knowledge of the different subjects selected and what criteria are used?
- How far should we, as teachers, arrive in our role in and out of class? What are teachers educated for?

As we can see, all these questions refer to dimensions or difficulties related to university teaching that do not depend on the specific institution considered – many of them can be extended to any kind of teaching and to other educational levels. What is questioned is the way our societies – more or less explicitly – conceive, organise, and manage the dissemination of knowledge, and the general roles assigned to teachers as guides, leaders or knowledge disseminators.

School (here University)

The School level includes the conditions and constraints that depend on the specific teaching institution considered, in our case, the University with its own specificities:

- Is the number of students per class important in terms of effectiveness of the teaching? Is there an optimal number? Are there exceptions?
- How are decisions regarding time-schedule, session duration, etc. taken?
- To what extent are university facilities important? Are there optimal premises? How to adapt to the ones available?
- How to respond to the pressure of introducing ICT in the classroom? Is it used because of real educational reasons or is it cost saving? Is it just a trend?
- How to ensure a good coordination between teachers of the same subject? What happens when they have a different conception of the subject to teach?
- How to establish more synergies between colleagues, sharing methodologies?
- How to ensure a coherent programme? What relationships exist between subjects?
- How to integrate the different subjects to obtain a more global education?

These questions also reveal the aspects teachers think can be changed and the ones they take for granted, not even considering them questionable. For instance, in the fifth question, only the coordination with teachers of the same subject is considered, according to the traditional compartmentalization of knowledge in higher education. Together with the sixth question, they reveal the lack of a professional culture that might include coordination between teachers. However, the sixth question seems to consider that this coordination only affects "methodologies", which again appears to be a vague and general dimension of teaching. The last two questions are content-related, but only with respect to the relationships between subjects, as these are considered to be previously determined – and, therefore, untouchable.

Pedagogy

The level of pedagogy is common to the different subjects or disciplines that can be taught. It includes all the resources, formats, and strategies teachers and students activate – many times without even noticing it – for teaching and learning processes to take place. We gathered numerous questions that can be located at this level of the scale and organised them into two main blocks: students and lecturers.

Students

- How to manage long projects, where students slack off and decrease their work intensity? How to reach a balance between establishing milestones and letting students work independently?
- What to do with the students that chat, are unfocused, use their mobile phone, do other things than what is requested of them?
- How to deal with student diversity?
- How to arouse the students' interest in subjects that are not at the core of the degree?
- How to motivate students beyond the minimum required "pass" grade?
- How to encourage students to participate in a large group?
- Should students be monitored closely or should they work more independently?
- How to motivate students to behave in class?
- How to encourage students to be more competitive through the intrinsic values of the subject that is taught?
- Students are not previously taught how "to learn". How will this affect our job?
- How to teach students to listen and maintain their attention?

Lecturers

- How to improve oral and body expression?
- How to organise assessment in a fair and impartial way?
- How to assess core (non-disciplinary) competencies?
- Is it better to use final examinations or continuous assessment? How to measure long-term student learning outcomes?
- How to design contents, planning and methodologies of the subject that take into account the student diversity?
- How to reach all the students and not only those who have more knowledge, excluding the ones that got lost? How to find the balance between maximising student learning and the amount of information provided?
- How to ensure an individualised methodology considering the time limitations?
- How to become the best teacher for each student?
- How to deal with the so-called "decline in student knowledge"?
- Does the decline in student knowledge correspond to something real or is it just what each generation says about the previous one?
- Does it make sense to give lectures nowadays?
- How to improve teaching resources and methodologies using ICT?
- What to do after the class? How to analyse what happened and what the teacher did? How to assess teacher performance?

As can be seen from the questions above, most of them focus on specific teaching practices, but they do not refer to the difficulties of the corresponding subjects. The questions are mainly related to what the teachers can or might do, and they are very general. Only two of them refer to specific teaching formats: projects and lectures. There is no mention of the activities organised (labs, tutorials, problem solving or case study sessions, outdoor activities, etc.) and the way to better implement them.

The need to implement new kinds of activities is not mentioned. The questions mainly have to do with the teacher and the teacher's actions. For instance: assessment is always considered as a lecturer's task; "motivation" is assumed to be generated (only) by the teacher. The questions thus reveal many features of the traditional pedagogical contract, which seems to be fully assumed by the lecturers.

Discipline

As said before, we were astonished to find so few questions at this level, which corresponds to the conditions and constraints directly linked to the content taught and learned. They can be related to the general discipline the content belongs to (Mathematics, Engineering, Economics, Management, etc.) or to the different components of the discipline, according to the way it is structured or delimitated in the considered institution. The general terms used to specify these levels are: domains, sectors, themes, tasks or questions. The divisions and boundaries established in a discipline or field of knowledge are institutional constructions. They vary from one institution to another and from one historical moment to another. The collected questions at this level remain very general; none of them specifies the difficulties related to the teaching or learning of a given piece of content. The first one, for instance, is very similar to those located at the School level: it depends on whether we interpret the question as affecting the design of an entire programme or the possible actions in one discipline:

- How to better connect the different subjects of the programme?
- How to highlight the multi-disciplinary nature of the subjects?
- How to select the learning goals of the subject? What content should be included?
- What should the level of the learning goals be?
- How to relate the subject with the real world?
- How to balance learning goals between specialisation and generalization?

WHY THESE QUESTIONS? AN INTERPRETATION FROM THE ATD

The assumed educational contract between lecturers and educators

The first reason that came to our mind when trying to understand why lecturers did not ask any content-related question is the kind of implicit didactic contract that was assumed by them at the beginning of the course. Given the fact that the course was about university teaching, they might have expected to learn certain generic tools to help them in their teaching practice; not something related to their specific subject. The educators were seen as specialists in teaching processes and the questions were stated at this general level. Either way, this shows a first important phenomenon: lecturers expect to receive help with general teaching practices that are only a part of their daily practice. A lot of their teaching work (elaboration of the syllabus; choice of textbooks, reference books and other kind of resources or materials; selection, design and organisation of activities, cases or problems; decisions about the kind of in-class and out-of-class activities students should carry out; renewal of the subject matter; etc.) does not seem to have been included in the objectives of the course.

A problem of legitimacy

The second reason we put forward is related to what we call a problem of legitimacy. University teachers are often also researchers or at least experts in the subject they teach. Therefore, they may be reluctant to accept the idea that their teaching difficulties might come from problems with the subject matter they are supposed to master. Their lack of expertise can only be attributable to what is external to the discipline they teach. This reinforces the previous reason about their expectations from the educational course.

The divide between pedagogy and didactics

There is another important and more general factor that may explain the lack of content-related questions. It corresponds to the dominant interpretation of teaching and learning phenomena that has been called "pedagogical generalism" (Gascón & Bosch, 2007) or the "didactic divide" between pedagogical and subject-matter knowledge (Bergsten & Grevholm, 2004). It tends to introduce a strict separation between instructional processes and the "content" of these processes, that is, using the scale of didactic co-determinacy, between the level of Discipline and the level of Pedagogy. The main point in crossing the boundary between the two levels is the way knowledge is conceived in the considered teaching process or, in other words, which aspects of the subject-matter are questioned and which ones are assumed as a given.

When a teacher – or a lecturer – is asked to teach a given piece of knowledge *k*, the first question she will first ask herself is "what should I do to teach *k*?", not "what is this *k* I should teach?" What the theory of the didactic transposition (Chevallard, 1985; Chevallard & Bosch, 2014) states is that instructional processes rely on the fiction that there is only one way to define *k* and that this is the *k* that is taught and learned. Questioning the knowledge to be taught, asking about its origin, selecting and applying a given instructional process rarely occur. This is why it is normal the participants of the course did not set forth questions of that kind. In the questions stated, knowledge always appears as a given, not as a variable.

The "pedagogical generalism" that is found in the teachers' questions is not an isolated fact. If we look at the teaching support some universities offer their (new) faculty, we see that only the Pedagogical level is addressed, and possibly some aspects of the School level. For instance, in the *Teacher Training in Higher Education* (FDES)[1] programme proposed by the Autonomous University of Barcelona, the structure of the programme is presented as follows:

[1]http://www.uab.cat/web/personal-uab/personal-uab/personal-academic-i-investigador/formacio-i-innovacio-docent/programa-fdes/estructura-1345703511726.html

Activity 1. Teaching in the new context of learning and teaching
Activity 2. Practicing oral discourse
Activity 3. How to assess university students' learning?
Activity 4. Experiences in educational innovation
Activity 5. Observation in the classroom
Activity 6. Teaching planning: from study programmes to syllabus
Activity 7. Teacher's portfolio

Similar programmes can be found at other universities. For example, some years ago, the *Teaching Engagement Program* of the University of Oregon posted a list of frequently asked questions (FAQs) organised according to the following headings: "Getting ready to teach; Presenting and facilitating information; Motivating students; Questions of respect; Assessment; Managing the classroom climate". None of the questions was content-related. It seemed as if, once certain answers were provided to the pedagogical issues, their specification to the subject-matter was considered evident or, at least, non-problematic.

CONCLUSION: A LACK OF TOOLS AND NOTIONS

One of the consequences of "pedagogical generalism", that can partially be seen in the questions stated by university teachers, is the lack of terms and concepts to go below the level of Pedagogy and start questioning the levels of Discipline. University teachers develop their professional activity at institutions where little is said about the way knowledge should be selected, arranged, updated, organised, "elementarised", put-into-practice, problematized, etc. in order to teach it or to help students to learn it. This is a crucial aspect in which the results obtained from research in Didactics of Mathematics, both practical and theoretical, can assume an important function.

From the experience of the courses here presented, we have seen how introducing certain elements of the ATD (the notions of praxeology, didactic contract, didactic moments, Herbartian schema, media-milieu dialectics, didactic ecology, etc.) provides lecturers with a productive enough framework to talk about and start questioning a larger part of their teaching activities (Florensa, Bosch, Gascón, & Ruiz-Munzon, 2017). The more is said about didactic processes, the more dimensions of these processes can be questioned and tentatively changed. Our hope is that the education of lecturers, as is the case with primary and secondary school teacher education, will have the power to make this state of things evolve. Our experience with the courses presented lets us be moderately optimistic in this respect.

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An Analysis of University Mathematics Teaching using the Knowledge Quartet

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We analyse accounts written by three mathematics lecturers on their practice using the Knowledge Quartet framework. This framework has been used to study how a teacher's knowledge of mathematics and mathematics pedagogy influences his/her actions in the classroom at both the primary and secondary level. We consider how the framework could be used to study university level teaching, and we report on the dimensions of teacher knowledge that were made visible by this framework.
Keywords: Knowledge Quartet, teacher knowledge, university mathematics teaching.

INTRODUCTION

The first three authors of this paper are mathematics lecturers at three universities in Ireland, who also engage in mathematics education research. Over the course of two years, they wrote accounts of incidents which occurred during their teaching as part of a professional development project using the Discipline of Noticing (Mason, 2002). In this paper, we report on our more recent use of a different theoretical framework, the Knowledge Quartet framework (Rowland, Huckstep & Thwaites, 2005), to analyse these accounts. The Knowledge Quartet categorizes situations from classrooms where mathematical knowledge surfaces in teaching. There has been one previous attempt to use the framework to analyse university mathematics teaching (Rowland, 2009). The focus of that paper was the knowledge-grounded foundation beliefs of the university lecturer, about mathematics and about teaching and learning.

The purpose of our current study is twofold. Firstly, we are interested in whether the Knowledge Quartet framework could be applied to study teaching at university level. Secondly, we would like to know what features of university teaching are highlighted when our set of accounts are analysed using the Knowledge Quartet. Previously, the first three authors had analysed their accounts to study the many decision points that arose while teaching and in O'Shea, Breen and Meehan (2017) these decision points and their triggers were categorised. We were interested to see if using the Knowledge Quartet framework would draw attention to other aspects of teaching in the accounts.

In this article, we will first of all consider the literature on teacher knowledge, especially at university level. We will then expand on the Knowledge Quartet framework, and give some results from our analysis using this lens. Finally, we will discuss our findings and suggest some future avenues for research.

LITERATURE REVIEW

The Knowledge Quartet is a theoretical tool for observing, analysing and reflecting on actual mathematics teaching. Ball, Thames and Phelps (2008) also studied mathematics classrooms to develop a theory of mathematical knowledge for teaching that built on the work of Shulman (1987). This resulted in the identification of an important subdomain of content knowledge - ‘specialized content knowledge’. This is distinct from ‘common content knowledge’ and is unique to the work of teaching.

Independently, Ainley and Luntley (2007) suggested that experienced teachers draw on ‘attention-dependent knowledge’ in addition to subject knowledge and pedagogical knowledge (both general and subject-specific). Few research studies have been concerned with the knowledge employed in university mathematics teaching. McAlpine and Weston (2000) conducted a research study with six professors considered exemplary in their teaching and found that all the professors drew on pedagogical knowledge, pedagogical content knowledge, content knowledge and knowledge of learners (following Shulman (1987)) while monitoring their own actions and making decisions during lectures. This was despite the fact that three of the professors were mathematicians who had no pedagogical training (while the remaining three were mathematics educators or trained teachers). McAlpine and Weston (2000) hypothesised that the mathematicians constructed this knowledge largely through experience and reflection, and that their lack of training led them to depend more on their experience than the mathematics educators did.

On the other hand, Wagner, Speer and Rossa (2007) examined the knowledge, other than content knowledge, required by a mathematician teaching an undergraduate course. They reported that he was unable to anticipate how students would respond to particular activities and how the content or sequence of individual classes contributed to the instructional goals of the entire course. The authors claim these findings lend support to the assertion that there is knowledge particular to teaching that is distinct from, and not easily constructed from, knowledge of content.

Speer and Wagner (2009) focussed on whole-class discussions and examined the nature of the knowledge that a mathematician could employ to make effective use of undergraduates' mathematical contributions in a way that furthered the goals for the class. Their analysis focussed on the role of (a lack of) pedagogical content knowledge and specialized content knowledge in the difficulties experienced by the instructor in scaffolding student learning while orchestrating such discussions.

THEORETICAL FRAMEWORK

The Knowledge Quartet

The Knowledge Quartet is a ‘theory’ in the sense that it proposes a way of thinking about mathematics teaching in the usual institutional settings (lessons/classes), with a focus on the disciplinary content (mathematics) of the lesson.

The Knowledge Quartet (KQ) was the outcome of empirical research at the University of Cambridge, UK (Rowland et al., 2005), in which 24 mathematics lessons were videotaped and scrutinised. The research team identified aspects of the teachers' actions in the classroom that could be construed as being informed by their mathematics subject matter knowledge or pedagogical content knowledge. This inductive process initially generated a set of 18 codes (later expanded to 21), subsequently grouped into four broad, super-ordinate categories or dimensions.

The first dimension of the KQ, *foundation*, consists of teachers' mathematics-related knowledge, beliefs and understanding. The second dimension, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. The third dimension, *connection*, concerns the ways by which the teacher achieves coherence within and between sessions. The final dimension, *contingency*, is witnessed in classroom events that were not envisaged in the teachers' planning. Essentially, it is the ability to "think on one's feet".

Conceptualising the Knowledge Quartet

The concise conceptualisation of the KQ which now follows is a synthesis of the characteristics of its four dimensions.

Foundation

The first member of the KQ is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned in preparation (intentionally or otherwise) for their role in the classroom. The key components of this theoretical background are: knowledge and understanding of mathematics *per se*; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt. The remaining three categories focus on knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself.

Transformation

At the heart of the second member of the KQ is Shulman's observation that the knowledge base for teaching is distinguished by " … the capacity of a teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (Shulman, 1987, p. 15, emphasis added). This dimension picks out behaviour that is directed towards a student (or a group of students), and which follows from deliberation and judgement informed by foundation knowledge. The choice and *use of examples* has emerged as a rich vein for reflection and critique, and one of the most prevalent codes observed in practice (Rowland, 2008).

Connection

The next dimension concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Our conception of connection includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics, but also awareness of the relative cognitive demands of different topics and tasks, and the implementation of strategies to remove (or lessen) obstacles to learning.

Contingency

Our final dimension concerns the teacher's response to classroom events that were not anticipated in the planning. This dimension of the KQ is about the ability to 'think on one's feet': it is about *contingent action.* Whilst the teacher's intended actions can be planned, the students' responses cannot. The teachers' response to students' unexpected contributions is one of the most low-inference codes of the KQ.

Many moments or episodes within a session can be understood in terms of two or more of the four units; for example, a **contingent** response to a student's suggestion might helpfully **connect** with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on **foundational** knowledge.

The KQ is a lens through which the observer 'sees' classroom mathematics instruction. It offers a four-dimensional framework against which mathematics lessons can be discussed, with a focus on their subject-matter content, and the teacher's related knowledge and beliefs.

This framework has been used in different contexts and for different purposes. For instance, Rowland (2012) used the KQ to examine situations in which mathematical knowledge surfaces in primary and secondary mathematics. He concludes that elementary mathematics teaching poses challenges which are qualitatively different from those confronting secondary mathematics teachers. However, the mathematics knowledge primary mathematics teachers must possess is neither less profound nor easier to acquire than that of secondary teachers. Turner and Rowland (2011) describe a project in which the framework was used to guide pre-service teachers in a process of personal reflection on their teaching. The participants found that the KQ helped them to focus more effectively on the mathematical content of their lessons and its enhancement. The authors reported that this enhanced focus on mathematical content knowledge had a positive influence on its further development. There was also evidence that the KQ helped the participants to develop a more learner-centred view of teaching and one in which conceptual understanding rather than procedural fluency was emphasised. Other recent studies using the KQ have focussed on contingent moments in the classroom (e.g. Rowland & Zazkis, 2013).

METHODOLOGY

The accounts which form the data for this study were written using the Discipline of Noticing (Mason, 2002). This advocates that practitioners write 'brief-but-vivid' accounts of incidents that they have noticed in their practice. Mason (2002) defines a brief-but-vivid account as

> one which readers readily find relates to their experience. Brevity is obtained by omitting details which divert attention away from the main issue. The aim is to locate a phenomenon, so the less particular the description, the easier this is, without becoming so general as to be of no value….Thus description is as factual as possible. (p.57)

He advises that these accounts should also avoid justification of incidents or actions, and should therefore be 'accounts of' rather than 'accounting for' a particular situation. The first three authors of this paper had written brief-but-vivid accounts of their teaching over a two-year period. These focused on notable incidents that occurred while they were teaching, but are not reflections or descriptions of a whole lecture. For more details, see O'Shea, Breen and Meehan (2017).

For this paper each of the three lecturers chose one of their modules; only the accounts relating to that module which contained references to mathematical knowledge were analysed (20 accounts for Lecturer 1, 29 for Lecturer 2, and 38 for Lecturer 3). Lecturer 1 chose a one-semester Introduction to Analysis module for 27 second-year students (this module was delivered separately to 7 Pure Maths students and 20 Science students), Lecturer 2 also chose a one-semester Introduction to Analysis module for a group of 75 second-year students, while Lecturer 3 chose a year-long Differential Calculus module for a group of 49 first-year students. All three lecturers aimed to foster dialogue in their classrooms, perhaps because of their interest in educational research and the relatively small class sizes in these modules.

When coding the data we compared our accounts with the descriptions of each of the 21 codes associated to the KQ framework, with reference to the examples available at www.knowledgequartet.org. We began the coding process by first coding a small set of accounts together. Then each lecturer coded her own set of accounts and passed on her analysis to the other two lecturers in turn. They coded the accounts independently before comparing their analysis with that of the original instructor. All three discussed any discrepancies and agreed on the final coding.

During the coding process, we felt that the names of a few of the codes did not fully reflect the terminology used in teaching mathematics at the university level. We interpreted the code *Teacher Demonstration (to explain a procedure)* to also encompass teacher demonstration to explain a proof. We chose to use the code *Choice of Example* (CE) to include particular instances of an abstract concept or a general procedure; and, as the rehearsal of a procedure or 'exercise' (Rowland, 2008), and also for non-routine tasks. We also applied the code *Responding to Students' Ideas* (RSI) from the Contingency Dimension to encompass instances where the lecturer had to respond to a *lack* of students' ideas.

RESULTS

A summary of the number and percentage of codes found in each of the four categories of the KQ for each author is given in Table 1 below. While a number of codes could be applied to some events, the one which we judged to be predominant was what was counted in this table.

KQ Dimension	Lecturer 1	Lecturer 2	Lecturer 3
Foundation	10 (20.83%)	4 (11.11%)	41 (34.74%)
Transformation	14 (29.17%)	17 (47.22%)	32 (27.12%)
Connection	7 (14.58%)	8 (22.22%)	24 (20.34%)
Contingency	17 (35.42%)	7 (19.45%)	21 (17.8%)
Total	48 (100%)	36 (100%)	118 (100%)

Table 1. Number and percentage of codes in each KQ dimension for each lecturer

On coding the accounts it became apparent that all three lecturers frequently wrote accounts about giving a task to the class or instigating a whole class discussion around a task, recording some students' responses (or lack of responses) in relation to the task, and noting what the lecturer thought or learned about student thinking. Therefore it is not surprising that the code *Choice of Example* (CE) in the Transformation Dimension was the most frequently occurring code for all three lecturers, while *Responding to Students' Ideas* in the Contingency Dimension was in each of their top three most frequently occurring codes. In many accounts, the lecturer contrasted student learning on a task with learning on the same task the previous year or with students in a different class, often noting what students found easy or difficult. The tasks were usually designed and planned by the lecturer with specific aims for student learning in mind, thus the codes *Anticipation of Complexity* (AC) in the Connection Dimension and *Awareness of Purpose* (AP) in the Foundation Dimension frequently appear for all three lecturers. Lecturer 3, who was simultaneously conducting a research project on mathematical tasks, was often explicit about the pedagogical rationale behind a given task. Consequently, another significant code for her accounts was *Theoretical Underpinning of Pedagogy* (TUP) in the Foundation Dimension.

By way of example, the following is an account from Lecturer 1, coded as RSI. She struggles to understand what the student is asking but still feels she has to respond:

> A student asked a question in the middle of a complicated proof. I didn't understand the question and asked him to ask it again. He tried but I still couldn't understand. So I explained the proof again as best I could paying attention to what I thought he had had problems with. However I realised I had made a choice. I could have continued probing until I figured out what he was asking. I decided not to do that so as not to embarrass him, but maybe I didn't really answer his question in the end.

While the majority of accounts were on incidents during lectures, some relate to preparation of tasks and lessons, or conversations with students outside of class. The following is an example of an account by Lecturer 2 coded as CE, which describes a task given to students to work on during the second lecture of the semester.

> I handed out the first Inclass Exercise of the module. It contained the following statement: There exists a university in the world, where every Analysis student achieves a final mark of

> at least 90% in the module. The instructions were as follows: Write down what you would need to do to prove that Statement B is *false*. At the end of the class, a student came up to me and said that suppose there were infinitely many universities in the world, then you couldn't actually disprove the statement because you wouldn't be able to get around to all of them to check the Analysis grades. I was impressed with how he extended the statement.

Given that CE was the most frequently occurring code for all three lecturers, the following account provides another example of a task given, this time by Lecturer 1, to help students propose conjectures about the relationship between bounded and convergent sequences.

> I was talking about bounded sequences with the class today. I got them to come up with some bounded and some unbounded sequences. I tried to get the class to make conjectures by asking them to guess what the next theorem would be, or what it definitely wouldn't be. They immediately realised that there would be no theorem that said that every bounded sequence converges and then conjectured that every convergent sequence is bounded. They seemed to enjoy the process.

Next we present an account from Lecturer 3, coded as AP. She is explicit in her intentions to engage students in mathematical sense-making and on challenging students' views of mathematics as a set of rules to be learned and applied.

> Today I continued with sketching graphs of functions and asked the students to draw the graph of f(x)=1/x on its natural domain, among others. I circulated the room as they were doing this and noticed that a number of what I had considered to be the more able students were drawing the graph incorrectly (possibly confusing f(x)=1/x and g(x)=1/x^2). I have been trying to put across the idea of Calculus as a 'science' from the point of view that 'experiments/trials' can be undertaken to check 'hypotheses', results can be 'replicated' and so on, but it appears some students are disregarding this and still regard it as a collection of facts to be learnt and remembered.

Finally, we present an account from Lecturer 3. Her pedagogy is underpinned by having students take a guided-discovery approach as a classroom community (TUP).

> I tried to use a 'guided-discovery' approach to facilitate students' realization that the graph of a function and its inverse are mirror images of each other in the line y=x. However, each step of this took a lot longer than I envisaged. Moreover, I wasn't convinced at the end that the students would retain this particular piece of information longer or understand it better for having discovered it themselves as a class community.

DISCUSSION

In this paper we have used the KQ to analyse a set of accounts written as part of a professional development project that involved engaging with the Discipline of Noticing (Mason 2002). This is not the usual type of data that has been used in previous KQ studies. Typically, the researchers in those studies had access to classrooms (of either pre-service or experienced teachers), and have been able to record and analyse entire lessons. Our data is different in two key ways. Firstly we do not have recordings of entire lessons but the brief-but-vivid accounts of the instructor herself on some aspect of the class, which was memorable to her. This is a limitation because we may have chosen not to include some relevant aspects of our classes, or of our students' experience and reactions, but the accounts do shed some light on the 'attention-dependent knowledge' of the instructor (Ainley & Luntley, 2007). We did not write our accounts in order to give a representative view of our teaching, rather we concentrated on aspects which were troublesome to us. However, we do have accounts from almost every lecture in the modules considered whereas previous studies have data only from a very small number of classes with a given teacher.

In his KQ analysis of university mathematics teaching, Rowland (2009) refers to only one lecture. The analysis homes in on the foundation dimension and in particular on the beliefs of the lecturer (about mathematics and pedagogy), but does not explore the other three dimensions. Our analysis has shown that all four dimensions were present in our data. It should be noted that all three lecturers pursued an interactive approach in their classes, and perhaps the same spread of codes would not be present in an analysis of a more stereotypical university lecture.

On the other hand, the prevalence of the use of the *responding to student ideas* code for the accounts discussed here suggests that the traditional image of a lecture (in which a lecturer delivers from a pre-prepared script, rarely deviating from it, and interacts minimally with students) is not always accurate and highlighted this element of our practice for us.

In addition, given our previous focus on decision points in these accounts (O'Shea, Breen & Meehan, 2017), we may have expected the contingency dimension to be dominant but this was not the case. The KQ highlighted the importance of the other three categories in our accounts, especially the transformation dimension in the *choice of examples*. We found the framework provided a lens through which the *knowledge* brought to bear in the preparation and teaching of lessons could be viewed in a coherent and comprehensive manner.

Each of the first three authors is a mathematician and while none has any formal pedagogical training, all three conduct research in mathematics education. Many of the accounts suggested an *awareness of purpose* on the lecturers' behalf or a *theoretical underpinning to the pedagogy* used when teaching. Perhaps this is a consequence of their familiarity with the research literature. However, the fact that the instructors often contrasted student learning in the lectures for which accounts were written with that of other cohorts lends some support to the hypothesis of McAlpine and Weston (2000) that a teaching mathematician can construct knowledge of learners and pedagogy through experience and reflection.

In several accounts the three lecturers highlighted what they noticed about student thinking on a given task and reflected on this after the lecture. These reflections could be said to *inform* their knowledge about mathematics pedagogy, particularly their knowledge of content and students and knowledge of content and teaching (Ball et al., 2008), which is a component of the Foundation Dimension. Although we chose not to code these reflections since they occurred after lectures, they point to a growth of knowledge as a consequence of reflection on teaching. It seems that use of the KQ as a reflection tool could afford mathematicians (with no formal pedagogical training) an opportunity to develop pedagogical knowledge. It is interesting to compare this with Turner and Rowland's (2011) finding that the KQ afforded preservice primary teachers (typically non-specialists in mathematics) an opportunity to develop mathematical content knowledge, illustrating the usefulness of KQ to mathematics teachers of a variety of backgrounds.

Some KQ codes did not appear in our analysis. For example there were no accounts coded as displaying behaviour such as *adherence to a textbook* or *concentration on procedures*. This may be because of the nature of the mathematics taught in the given modules. We also found very few references to *use of mathematical terminology* and *overt display of subject knowledge*, which is not to say that the lecturers did not use terminology or show their subject knowledge during classes but that they did not talk about it in their accounts (perhaps because it was normal and not problematic). We used the code *identifying student errors* sparingly, even though many accounts contained instances of a lecturer noticing a problem with student understanding. In our accounts the lecturers seemed to focus more on how to respond to a student rather than being able to tell when a piece of mathematics was wrong, and so we coded these episodes using the *responding to student ideas* code. We also used this code when the lecturer was faced with a lack of student ideas, for instance when she asked a question but received no replies. It may be that this is a situation that occurs more frequently in university than in school, where the size of classes can result in unwillingness to take part in discussions. We found that the type of specialist knowledge required to teach abstract mathematics at university was accounted for in the KQ with many of the codes mentioned earlier as well as others such as *choice of representation, recognition of conceptual appropriateness* and *making connections between concepts or representations*.

Even though there are some differences in the prevalence of codes at school and university level, we believe that the KQ offers a useful lens with which to study undergraduate teaching. It has drawn our attention to the importance of different facets of lecturers' mathematical knowledge which we may otherwise have overlooked. It would be interesting to explore the relationships between the four dimensions of the Quartet, for example how the underpinning dimension of foundation knowledge influences the lecturers' choices made in the other three dimensions, and how it is in turn influenced by knowledge generated by the lecturer in a contingent moment. We used the KQ to code reflective accounts written by mathematics lecturers as they reflected on their teaching. However, we suggest it could also be used to guide the reflective process and the writing of the accounts. It would be interesting to explore whether such an approach would lead to a change in the lecturers' perspectives on teaching similar to those described by Turner and Rowland (2011).

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Knowledge of the Practice in Mathematics in University Teachers

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This research is part of the study of university teachers' knowledge that has emerged as a new line of research, which aims to understand the components of this knowledge, its development, and how it is reflected in university teachers' teaching practice. This study, using the Mathematics Teacher's Specialized Knowledge model, seeks to characterize a university teacher's knowledge of practice in mathematics in the content area of mathematical analysis. Based on an instrumental case study, through classroom observation, we provide indicators of the teacher's knowledge of ways of reasoning, validating, and proceeding in mathematics, contributing to the understanding of the nature of this teacher knowledge.

Keywords: teachers' and students' practices at university level, teaching and learning of analysis and calculus, university teachers' knowledge, knowledge of the practices in mathematics, Mathematics Teacher's Specialized Knowledge model.

INTRODUCTION

Research on mathematics teachers' knowledge began to be carried out in the nineties (e.g. Fennema & Franke, 1992; Broome, 1994) and currently continues to be developed with great force. Studies in this line of research have focused on teachers' knowledge of various concepts, such as fractions (Llinares & Sanchez, 1991) and functions (Even & Markovits, 1991), and in different mathematical domains, such as algebra (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) and geometry (Herbst & Kosko, 2012). We also find proposals of models of teacher knowledge such as the Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005), Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) and, more recently, Mathematics Teacher's Specialized Knowledge (Carrillo, Climent, Contreras, & Muñoz-Catalan, 2013). These models have been used for studying mathematics teachers' knowledge, principally in primary and secondary education, with scarce accounts of studies of university teachers, as noted by Speer, King, & Howell (2014).

Currently, it can be said that research in higher education has gone from being centered on students to having a more balanced interest in both, students and teachers (Artigue, 2016), to the point that research on university teachers' knowledge has emerged as a new line of research from which it is asked what is understood as knowledge, how this knowledge is developed, and how it is reflected in the teaching practice of university teachers (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016).

In agreement with the above and taking into account that the Mathematics Teacher's Specialized Knowledge model (MTSK) has been shown to be useful for studying university teachers' knowledge (e.g. Vasco, 2015), we carry out our research based on the MTSK with the aim of characterizing the knowledge of a university mathematics teacher who teaches content in the area of mathematical analysis. In this text, some results are presented in relation to knowledge of the practice in mathematics, one of the sub-domains of knowledge considered in the model.

MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE

Mathematics Teacher's Specialized Knowledge (MTSK) is an analytical model for understanding mathematics teachers' knowledge and at the same time a methodological tool that allows analyzing teachers' teaching practices (Carrillo et al., 2013). The model was developed based on a theoretical, empirical, and reflective work proposed by Shulman (1986) regarding foundational knowledge for teaching and the refining of Mathematical Knowledge for Teaching (Ball et al., 2008).

In MTSK, two domains of teacher knowledge are distinguished, mathematical knowledge (MK) and pedagogical content knowledge (PCK), considering that all of this knowledge is specialized, that is, it derives from the teaching profession and is conditioned by mathematics itself. Consequently, MTSK does not include, for example, general psycho-pedagogical knowledge included in Shulman's works. Also, the model takes into account that teacher's beliefs and conceptions about mathematics, its teaching, and its learning permeate the organization and the use of knowledge (Carrillo et al., 2013).

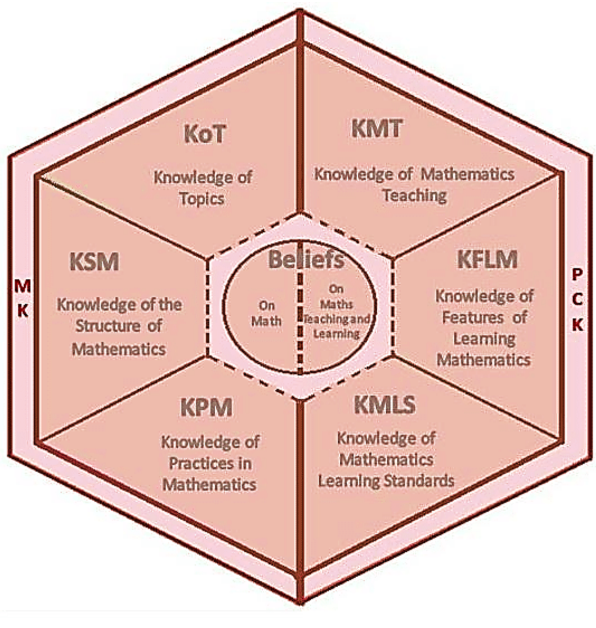

Figure 1: Mathematics Teacher's Specialized Knowledge (Carrillo et al., 2013).

Mathematical Knowledge comprises three sub-domains, knowledge of topics (KoT), knowledge of the structure of mathematics (KSM), and knowledge of the practice in mathematics (KPM).

KoT includes knowledge of concepts, properties, procedures, classifications, formulas, and algorithms, with their respective meanings and foundations. For example, knowing the topological property, density of the rational numbers in **R**, lies in this sub-domain. KSM includes knowledge of the interconceptual connections that can be established among mathematical concepts. So, knowledge of relationships between infinity and the Archimedean property of the real numbers belongs to this sub-domain. KPM includes knowledge of how to proceed, reason, and establish validity in mathematics. Knowledge of how proofs are made using different methods is part of teachers' KPM.

Pedagogical Content Knowledge contains three sub-domains: knowledge of mathematics teaching (KMT), knowledge of features of learning mathematics (KFLM), and knowledge of mathematics learning standards (KMLS).

KMT includes knowledge about didactic resources, strategies, tasks, and examples for making mathematical contents understandable. A teacher using an example to illustrate the meaning of a necessary condition forms part of his or her KMT. KFLM addresses the teacher's knowledge about mathematical contents as an object of learning, for example, teacher's knowledge about students' difficulties in understanding proofs belongs to this sub-domain. KMLS describes what students should achieve in a given course, conceptual and procedural capacities and mathematical reasoning that are promoted in given educational moments. The sequencing of the topics completeness theorem, characterization of the greatest element, and the Archimedean property of the real numbers is an example of KMLS.

KNOWLEDGE OF THE PRACTICE IN MATHEMATICS

The idea of knowledge of the practice in mathematics (KPM) comes from the works of Schwab (1978), Ball (1990), and Ball & Bass (2009) regarding syntactic knowledge of mathematics, which implies that the teacher should know how to reason mathematically, know different kinds of reasoning, and know in which mathematical contexts a particular kind of reasoning is more adequate than others. In this regard, within KPM, the importance of the teacher not only knowing established mathematical results, but also how to proceed and think in mathematics to arrive at these results is highlighted.

Knowledge that makes up part of this sub-domain is propositional logic, mathematical language and its precision, how definitions are made and used in the construction of mathematical knowledge, knowledge about different kinds of proof and their internal logics, the role that examples and counterexamples play in proofs, different kinds of heuristic reasoning, how knowledge is created in mathematics, how it is validated, reasoned, and generalized, and the role of mathematical conventions and symbols.

Although this knowledge is important, research carried out in MTSK with teachers at different educational levels reports scarce evidence of KPM in their mathematics classes (e.g. Montes, 2014; Vasco, 2015). In this regard, this investigation contributes to the understanding of the nature of this knowledge in university teachers.

METHODOLOGY

In this investigation, based on an interpretive paradigm and a qualitative methodology (e.g. Denzin & Lincoln, 2000), we carry out an exploratory study such that our results correspond to a first approximation for characterizing a university teacher's knowledge of the practice in mathematics (KPM). Our work is supported by an instrumental case study (Stake, 1995) for which we chose as an informant a teacher who taught a real analysis course for prospective teachers of mathematics during one semester in a Chilean university.

The teacher, who we will call Diego, is a mathematics researcher with a Ph.D., with more than 20 years of teaching experience, and this is the sixth time in recent years that he has taught the real analysis course. These academic characteristics of Diego make it likely that he possesses plentiful knowledge in elements of KPM.

The data was obtained through video recording while Diego taught the system of real numbers. The video recordings were transcribed and later subjected to content analysis (Bardin, 1997), identifying the units of analysis associated with the KPM sub-domain and considering the differentiation between evidence and indication of knowledge (Moriel-Junior & Carrillo, 2014). An evidence is an element that supports the presence of teacher knowledge, while an indication provides suspicion of the existence of knowledge but requires additional information in order to be confirmed as evidence.

RESULTS

In this section, we present episodes from one of Diego's classes on the properties of real numbers that allow us to observe his knowledge of the practice in mathematics (KPM).

Diego begins the class enunciating the property of the density of **Q** and **R\Q** in **R**. For the proof of this property, Diego takes an interval $[a, b]$ with a and b in **R** and considers the case $a = 0$, then he generalizes it for any positive a. In fact, he takes a and b positive numbers and indicates that with this supposition there is no loss of generality.

Teacher: We are going to assume that a and b are two positive numbers, which is not a great assumption. If they were negative, for example, I work with $-a$ and $-b$.
If I have a and b here [indicating on the number line], If I take $-a$ and $-b$ then they are on the other side of the zero.

And if I find a rational number here [indicating on the number line between a and b], then its opposite, will also be rational.
And if I find an irrational number, its opposite will be irrational.
And if one number is positive and the other is negative, then zero is rational, and it is between them, and the irrational will come from a proposition.

In the previous episode, Diego's knowledge of the different cases that should be taken into account to make a proof whose hypothesis possesses an implicit or explicit disjunction is observed, as is his knowledge that a particular case can be sufficient for showing the behavior of other cases in the proposition. Following this, the teacher's knowledge about the consideration of cases to particularize or generalize, is a *way of proceeding* in mathematics and can be considered as an indicator of KPM. The teacher's knowledge that the proof goes beyond the example of a concrete case and addresses all the possible cases the statement can include (e.g. Brodie, 2010; Montes, 2014) is part of his KPM.

Additionally, Diego emphasizes that in mathematics for a fact to be considered valid it must be proven:

Teacher: Now, a question, do I know of any irrational number that is less than 1 and greater than 0?
Student: The square root of square root of 2.
Teacher: Ah! Why is it less than 1? We have to prove that it exists, and, wait, we have to prove that the root exists and that it is a number less than 1. And prove something else, because they didn't ask us for a number between 0 and 1, but rather for an irrational number between 0 and 1.
So, you can take a real number, prove that its root exits, but you still don't know if it is irrational or not.

In the prior episode, Diego's knowledge of proof as a *way of validating* in mathematics is shown. As Brodie (2010) maintains, in addition to knowing that a kind of proof exists that confirms the truth of a statement, it is necessary to know how this type of proof works. Furthermore, Diego points out the importance of this role of proofs (de Villiers, 1990), as he is in front of a course for prospective mathematics teachers and after proving the density property he says:

Teacher: So, now you can say to your students with certainty that between any two rational numbers there is always an irrational number.

The comment above shows as indicator of KPM, the teacher´s knowledge of the necessity and importance of proofs as a *way of validating* in mathematics (e.g. Balacheff, 2000).

Continuing with the density proof, Diego discusses with the students about taking the maximum or minimum of a set of natural numbers. In the following episode, we observe Diego's knowledge about the convenience of selecting a certain element to develop an argument in a proof.

Teacher:	The issue is that if I take either of the two [the maximum or minimum] I can argue that there is a rational number in between them. What I have to guarantee before choosing one of the two is that in fact there is something to choose, so, which is easier to guarantee? The smallest or the largest?
Student:	The smallest.
Teacher:	Why?
Student:	Because of the well-ordered principle.
Teacher:	Of course, the minimum exists by the well-order principle.

Nevertheless, on occasions, rather than choose, is necessary construct an element that allows developing an argument. Diego also gives evidence of this knowledge when discusses with the students about irrationality of a number that belongs to the interval they are working on. He uses $\sqrt{2}$ to construct $\sqrt{2}/m$ as shown below.

Teacher:	I know that $\sqrt{2}$ does not belong to the rational numbers. We proved it. It is real, it exists, and it is not rational. If I now divide this number by a whole number m, will it continue to be not rational? will it become a rational number? That is, this is not rational, but is this [pointing to $\sqrt{2}/m$] not rational either?

According to above episodes, an indicator of KPM is the teacher´s knowledge of the construction or selection of elements for developing an argument in a proof as a *way of proceeding* in mathematics.

Regarding to $\sqrt{2}/m$, Diego prove its irrationality and establishes this affirmation as a lemma that he uses in different moments in the proof of the density property:

Teacher:	So, if I take the Archimedean property for $\delta = \varepsilon/\sqrt{2}$, then exists an $m \in N$ such that $0 < 1/m < \delta$, and this implies that, $\sqrt{2}/m$ is smaller than ε and does not belong to the rational numbers. So, between 0 and ε, no matter how small ε is, there is always a rational number, and there is always an irrational number. Ok, so this affirmation that I just wrote, we're going to write it as a lemma.

This episode gives us evidence that the teacher knows that establishing preliminary results is a *way of proceeding* in mathematics that facilitates the development and the communication of a long and/or complex proof.

On the other hand, regarding the existential quantifier present in the Archimedean property and the well-ordered principle, Diego expresses that the existential quantifier only gives information about the characteristics or properties of an element, but does not say what the element is like or which element it is, only that it exists

Teacher: When the *for all* quantifier is given, I can always choose an element that works best for me, but when it is *existence*, it's "whatever you've gotten", no more. Then as *exists* gives you "whatever you've gotten", you have to do some work so that what you get, is what you need.

An indicator of KPM in this episode is the teacher's knowledge of the role and the meaning of the quantifiers when they are found in the hypothesis or in the conclusion of a proposition.

Summarizing, indicators of the teacher's knowledge of the practice in mathematics related to ways of reasoning, validating, and proceeding in mathematics has been evidenced. In the exposed episodes, the teacher teaches mathematical reasoning in order to give meaning to the mathematical activity (e.g. Brodie, 2010), not only for students to understand and acquire sensibility regarding how to establish truth in mathematics, but also, they find meaning in the need to do so (Montes, 2014).

CONCLUSIONS

A mathematics teacher is a professional whose knowledge of the discipline he or she teaches has a level of deepening, organization, and structure that is greater than what the students are going to receive (Ma, 1999). In this regard, KPM is necessary knowledge for the teacher, as it provides logical thinking structures that help to understand the function of diverse aspects of mathematics (Flores-Medrano, Escudero-Avila, Montes, Aguilar, & Carrillo, 2014). As observed in the case studied, the teacher's knowledge of processes of particularization-generalization, of the necessity and importance of proof for validation, and the knowledge of different ways of proceeding in mathematics are closely linked to the transition to advanced knowledge (e.g. Pino-Fan, Godino, Castro, & Font, 2012) and are related to the particular way of mathematical work.

In line with the ideas above, we consider that KPM allows the teacher to promote in students the construction of mathematical knowledge and the acquisition of abilities for reasoning, proof, and problem solving that are considered important for learning mathematics at all levels of education (e.g. Flores-Medrano et al., 2014).

With reference to the indicators of KPM obtained in this investigation regarding ways of reasoning, validating, and proceeding in mathematics, we agree with Sosa, Flores-Medrano, & Carrillo (2015) that indicators shown directly in empirical data can contribute to identifying, understanding, and analyzing teachers' knowledge in their discipline. Given that teacher knowledge is a complex and multidimensional construct, more studies are necessary to deepen the understanding of its different components, in order to advance in interrogations proposed regarding how this knowledge is developed and how it is reflected in the teaching practice of university teachers (e.g. Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016).

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Exploration of new post-secondary mathematics teachers' experiences: preliminary results of a narrative inquiry

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This paper reports on a pilot study that has taken place during the winter semester of 2017, in the context of a larger project whose goal is to contribute to studying the transitions from being a university mathematics student to becoming a post-secondary mathematics teacher. With very scarce literature on these transitions at this specific level, this pilot study acts as an exploration into new teachers' significant experiences that may be involved in shaping their relationships with mathematics, and its teaching and learning. We conducted a narrative inquiry with three new post-secondary mathematics teachers who were interviewed on a regular basis during a semester. Those interviews provided an insight into the new teachers' experiences by pointing out themes that are relevant to them. We conclude with a discussion on what remains to be achieved to conduct this research.

Keywords: Teachers' and students' practices at university level, Preparation and training of university mathematics teachers, Narrative Inquiry, Becoming a teacher.

INTRODUCTION

This paper presents a pilot study that took place during the winter semester of 2017. It addresses the transition from being a university mathematics student to becoming/being a post-secondary mathematics teacher. More precisely, the context is set in cegep institutions (general and vocational colleges), the first step in post-secondary education in the province of Quebec, Canada. The focus of this study is on new cegep mathematics teachers who, with an education mainly – sometimes exclusively – in mathematics, negotiate the transition from being a mathematics student (undergraduate or graduate) to teaching at post-secondary level. We start with a brief description of cegep institutions, followed by a discussion on the literature on becoming a teacher at post-secondary level. The theoretical framework and the object of study are then described followed by the methodology and the main results. We conclude with the next steps in the study and our expectations for the results.

A Few Words on the Context of Study

Cegeps are general and vocational colleges. Two-year pre-university programs are offered in sciences, arts or social sciences; three-year technical programs, such as nursing, computer science, building engineering technology, etc., are also offered. All programs include compulsory general courses, such as philosophy, French, English and physical education and multiple programs offer mathematics courses. Some technical programs include mathematics courses that are specific to the field of study. Those courses are offered mostly by mathematics instructors, even if they are not very familiar with the profession their students are aspiring to do. Pre-university programs in sciences and social sciences offer two calculus courses (differential and integral calculus) and one linear algebra course, slightly different from one program to the other, depending on the institution. Social science students are required to take a quantitative methods course. Some institutions also offer optional mathematics courses for science students, such as multivariable calculus, probability and statistics or discrete mathematics. Therefore, from advanced calculus for science students to quantitative methods for social science students, as well as mathematics applied to computer science, cegep mathematics teachers are required to teach a wide array of courses, to a wide range of programs and students.

To teach in a cegep institution, individuals are (officially) required a 3-year undergraduate degree in mathematics or connected field (Conseil Supérieur de l'Éducation, CSÉ, 2000). However, each institution can add other requirements, such as a master or a doctorate in mathematics. A handful of universities in the province offer a one-year graduate certificate in cegep teaching, unique for all disciplines taught in cegep. Institutions will sometimes see it as an asset when hiring although cegep instructors who have completed one of these certificates qualify them as far from the reality of cegep teaching (CSÉ, 2000).

This context is quite different than that of elementary and secondary institutions. To teach at these institutions individuals are officially required a 4-year degree in education, which sometimes include minimal formal mathematics training. Also, cegep institutions are often a place of transition from secondary to university education; students are introduced to more formal mathematics and some level of autonomy is expected from them (schedule less structured, no mandatory attendance to classes).

Teaching at Post-Secondary Level

Little is known about becoming a mathematics teacher at post-secondary level (e.g. Speer & Hald, 2008). More and more research about post-secondary mathematics education is being conducted, arguing this level is key to the training not only of future teachers but also scientists, engineers and mathematicians (Hodgson, 2001). However, many recognize the need for change in the way the mathematics are taught at this level. Most research emphasizes the mistakes and negative traits of university teaching and claim the inadequacy of the existing training programs on creating change and improving post-secondary teaching, while only few seem to offer concrete solutions (e.g. Beisiegel, 2009; Belnap, 2005; Speer, 2001; Speer, Gutmann & Murphy, 2005). About this, Beisiegel's work (2009) focused on how graduate students "developed a sense of themselves […] as post-secondary teachers of mathematics" (p. 2) during a teaching assistantship. The author looks closely into the graduate students' lives and their journey into becoming teachers. She concludes that existing training is inadequate and puts forward the need to investigate closely the *experiences* of these students becoming new teachers in order to effectively study post-secondary teaching and training (2009, p. 25). This recommendation is the starting point for our work with new cegep teachers who were not so long-ago university mathematics students. Our goal is to better understand their experiences in their life as new mathematics teachers and in the transition from being a mathematics student to becoming a post-secondary teacher.

The challenge – and partially the trigger for our pilot study – is that the literature on post-secondary teaching in general, and in cegep in particular, is very scarce. Looking at research on elementary and high school teachers, we see that it is heavily based on the education they received, namely a degree that aims at training them as *teachers* (e.g. Ambrose, 2004; Ensor, 2001; Franke & al., 1998). However, post-secondary mathematics teachers are mainly, if not solely, trained to become *mathematicians*. Therefore, we cannot transpose or extend the results encountered in the literature about teachers' training for elementary and secondary levels to the group we are interested in. With this lack of background on the transition under study, we chose to conduct a pilot study to inform our main research project.

THEORETICAL FRAMEWORK AND OBJECT OF STUDY

The focus on experience of this study led us to consider Dewey's philosophy (1938) to frame it. For Dewey, one learns through experience and experiences shape how one goes about the world and about new experiences (1938, p. 35). Dewey also sees being faced with challenging experiences as a key aspect of life, growth and change: "growth depends upon the presence of difficulty to be overcome by the exercise of intelligence" (1938, p. 79). Indeed, faced with a familiar context, one can have an idea of how to act, and the consequences of those actions, based on knowledge acquired through past experiences. However, more reflection is needed in an unfamiliar context, where one could have to connect many different, apparently unrelated, experiences in order to know how to act and the related consequences (1938, p. 68). This results in new knowledge that could be applied to one's action per respect to future experiences.

Because of the importance of *new and challenging experiences* in life, growth and change, we chose them to be the focus of our study. In the context of our research, becoming a mathematics teacher at post-secondary level is seen as a trigger for new and possibly challenging experiences. And because a situation may be new and challenging for some but not for others depending on one's past experiences, it is important to put the focus on the experiences lived by the new teachers that are challenging and educational for *them*, and not all experiences that are lived or that might be assumed challenging and educational by others. We call those *significant experiences*.

However, this approach opens the door to a broad and varied spectrum of experiences that may be significant in the transition under study. In our project, we focus on significant experiences that may shape new teachers' ***r****elationships* ***w****ith* ***m****athematics and its* ***t****eaching and* ***l****earning*. We will use the abbreviation RWMTL for the remaining of this paper to refer to the relationships under study. These RWMTL entail the visions, opinions, beliefs, attitudes and feelings about mathematics, and its teaching and learning.

In this sense, we hypothesize that new teachers possess a RWMTL, developed through years of doing, learning, being taught, and sometimes teaching mathematics, and that these RWMTL play an important role in their becoming post-secondary mathematics teachers. This hypothesis is supported by the work of Speer (2001) and Beisiegel (2009). Speer's work emphasizes that graduate mathematics students have beliefs about mathematics and mathematics education, and that their RWMTL play a role on how new teachers will act in their new profession, how they will deal with, interpret and react to what happens to them. In particular, she claims that teaching assistants' beliefs, especially about mathematics and undergraduate students, have a significant impact on how they teach and interact with students. These RWMTL are influenced by the education they received in university and by the implicit teacher training they received during their time as students (Beisiegel, 2009, p. 42). It creates a relationship, conscious or not, with education through their experiences as students. Indeed, Beisiegel (2009) claims that graduate mathematics students' life experiences in a mathematics department, shape their "views of the discipline and teaching" (p. 43), as well as how they view what and how they should become. She argues that the experiences those students have can play an important role into how they define their role as mathematics teachers: "it appears that in the lives of mathematics graduate students there exists a complex and intricate interplay between the structures that they encounter, their feelings about mathematics and themselves and their ideas of their future role as mathematics instructors or professors" (p. 43). This last quote reinforces our hypothesis that the RWMTL play an important role in individuals becoming postsecondary mathematics teachers. In general terms, the goal of our main study is to investigate these RWMTL; in particular, our goal is to identify the nature of the significant experiences that shaped new teachers' RWMTL. Incidentally, the goals of the pilot study this paper is reporting on were to come up with a list of themes relevant to, or in relation with, the significant experiences of the new mathematics cegep teachers, in order to support and guide the main study later on.

METHODOLOGY

Our methodology is based on ideas brought forward by *narrative inquiry* ("NI", Clandinin & Connelly, 2000). NI argues that the essence of human experience happens narratively and choosing this method means acknowledging that people make sense and give meaning to their lives narratively – *they lead storied lives* (Clandinin, 2013, p. 13). Furthermore, NI takes into account the wholeness of someone's life while allowing the researcher *and* the subject to collaboratively investigate and distinguish what makes it unique and specific.

NI was developed specifically to understand and inquire into experiences, in relation to the people who have them and the physical and social context where they are

(Clandinin, 2013). This is to emphasize how past experiences and social and individual matters influence how someone lives an experience. In our case and for the research project, we use NI to determine what significant experiences shape new teachers' RWMTL. In the context of the pilot study we are reporting on, we use NI to shed a first light into these significant experiences; collaboratively working with the subjects in thematising them, their nature and their shaping role.

With this in mind, we built open and semi-structured interviews with broad questions, so teachers would account for what is actually relevant or important in their journey of becoming teachers. This gives a chance for the unexpected to arise.

The Process of Investigation

Weekly meetings were planned with three cegep teachers during a whole semester (January to May 2017), but we ended up meeting with each of them 15, 10 and 4 times respectively over the whole semester. The meetings, which lasted between 30 minutes and 90 minutes, were audio-recorded. The first teacher was starting his third year as a teacher, had completed a master in mathematics and a one-year certificate in pedagogy. The second participant was starting his second year of teaching, had completed a master in mathematics and had also completed a one-year certificate in pedagogy. The third participant was teaching his second course while pursuing their second year of Ph.D. in mathematics. They were asked to share stories of events they lived in the week prior to the meeting, which they identified as significant for their RWMTL, and of reflections they made about these events. After each meeting, an account was written in the form of a story of the meeting, using as much as possible the words of the teacher, with only slight changes for clarity purposes. Those accounts constitute the data that was analyzed.

RESULTS

Emerging themes

The pilot study served as a way for us to learn about significant experiences of new teachers in relation with mathematics, teaching and learning. Our goal was to circumscribe areas, contexts or topics, we call them *themes* to be concise, which seemed to play an important part into their lives, thoughts and reflections. We wanted to find themes that seemed to foster experiences that were significant for them. This list is not meant as "a list of understandings" but rather as a list of words to help us think to understand the stories (Clandinin, 2013, p. 39). Of course, those ideas will not be final as we do want to stay open to what will come up as we meet new people next year, for the main study, and learn about their experiences.

First, as we considered the teachers' relationship with **teaching**, two categories arose: "being a teacher" (1) and "teaching to students" (2). The former (1) addresses the very personal aspects of the journey into becoming a teacher. It includes themes such as the *new teachers' expectations of the profession of teaching mathematics in a* *cegep institution*. This unfolds in reflections in relation to whether or not they attended cegep as students, the lifestyle they expect to have while holding such a position and the expectations arising from their memories of their own teachers. Another theme that came up is about *being part of an institution and being part of a community of teaching*, who respectively involves reflection on the role of cegeps in the society, and integrating in a team of established teachers. Finally, *the ability to teach*, which includes their reflections in relation to their own and others' abilities, and the *initial training* they should have or want to have, were topics that played a role in their everyday life and reflections.

The latter subdivision (2) of the themes in relation with teaching that we found addresses themselves as teachers in relation with their students. This includes the *role they (should, can) have in their classroom*, whether it is to pass on knowledge, make it interesting or getting them prepared for a job or university. They also reflected on the *assessment of the level of difficulty* of concepts or problems as very central in their daily life, whether it comes from their own judgment or from a formal source such as ministerial specifications. In the same line of thought, reflections on the *assessments* were playing a huge role in their experiences with questions such as how many, when, weight of each, and also how to prepare the students for them and how to assess exactly what needs to be assessed. Finally, their *teaching method*, using technology for example, and the *ability to adjust to the students*, to their ways of thinking and being, were present in the reflections expressed.

As we considered the teachers' relationship with **learning**, the following themes emerged. First, the *expectations of the level of the students*, as far as what they should know from high school or from previous cegep courses, and the *expectations in relation to the students fulfilling their role in the classroom*, such as participating in class and doing their homework, were central topics for the teachers. On another note, the teachers found challenging to manage the *students' expectations in regard of their teacher*, as far as the act of teaching and the level of the material they were expecting. Finally, the teachers mentioned choosing the *attitude towards the students* as being a challenge, especially when it comes to differences between countries and provinces, since many teachers are not from Quebec, where the relationship between a teacher and its students might be different according to different cultural norms.

As far as the teacher's relationship with **mathematics**, we did not come up with a list of themes as we found that our data did not allow us to do so. Indeed, as we met the participants over and over, we realized that it was not spontaneous for the new teachers to talk about mathematics when asked about their daily lives. They favoured talking about teaching and their students. We conjecture that they have a number of things to think about other than the mathematics, such as the themes mentioned above, which seem to take much more work to be mastered or managed than to master the mathematics they are teaching. In other words, it made us realize that for (these) new teachers, it is difficult to talk about mathematics spontaneously when questioned about their experiences as new teachers; considerations in relation to teaching and learning seem to take all their time. Among the three participants, only one would address mathematics when discussing their everyday activities; the individual still in the midst of completing their Ph.D. in mathematics. This made us think that the new teachers that were full-time, not actively conducting research in mathematics nor taking mathematics courses, seemed to have in the front of their mind matters that were farther from mathematics and closer to teaching and (their students') learning. The two other participants, when asked directly, struggled to talk about mathematics for more than a few moments.

Finally, we found multiple themes what were present **across mathematics, teaching and learning**. Indeed, it seemed that some institutional aspects specific to cegep were having an impact on the teachers experience at multiple levels. First, as mentioned in the introduction of this paper, cegep institutions offer science, social science and technical programs. The teachers expressed multiple times that there were challenges on multiple levels to teaching to different audiences. For example, in a differential calculus class for science, a teacher has to teach to future engineers and future doctors, two groups who do not have the same interests and goals. Also, this same teacher may teach in the same semester differential calculus for social science students (the course covers almost the same material), who have again very different interests, goals and aspiration, to study psychology or economics for example, in relation with mathematics and, for some of our participants, very different ways of thinking. Finally, that same teacher may have to teach mathematics to technical programs, where the goal is for the students to learn the mathematics they would need to apply in their field. Therefore, the teachers we met expressed how difficult it was to navigate teaching to those different audiences: how to teach them according to their expectations and aspirations, how they learn best, and to determine what kind of mathematics, and what in the mathematics, they really needed.

Another key aspect across mathematics, teaching and learning is the fact that cegeps offer courses during the day and courses at night as part of continuing education. Indeed, teaching in continuous education meant that the teacher was not assigned an office space, and was not active in the department (they were not necessarily invited to department meetings). This also meant that they were often left on their own to discover the ways of the institution, would rarely come across other instructors and seemed to lack opportunities for asking advice. It is also important to mention that new cegep teachers often have to teach at multiple institutions during the first few years, sometimes even during a semester, before they can be guaranteed work at one place. This isolation and constant change seemed to be heavy for teachers who enjoy working with others, reflect through discussions on teaching, get advice from more experienced colleagues or just be active in their workplace. The students are also different, as most day-students are young adults who go to school full-time and, night-students are older and work full-time during the day. And again, to try to adapt to all those aspects turned out to foster a number of significant experiences for our participants.

CONCLUSION: WHAT'S NEXT

This pilot study helped us understand and circumscribe themes that were at the heart of becoming a teacher in cegep institutions.

However, through the 29 interviews we conducted, very few experiences were complete enough to be talked about in terms of how they played a role in the new teachers' lives, mainly because the reflection in relation to an experience was often missing or incomplete. Indeed, from our data, we found that the reflective experience can take a long time, longer than the academic term in which the experience happened. This made us realize that our goal of understanding changes in the RWMTL could not be tracked over this short period of time. Indeed, our pilot left us believing that reflection on recent experiences is often not mature enough for teachers to be comfortable verbalizing it and sharing it with us. Also, we were unable to know if the recent experiences would play a significant role in the transition under study on a longer term. Therefore, to be able to hear about *significant experiences*, we need to go another route. Our conjecture is that by introducing a theme first and asking participants to share past experiences in relation to this theme, we could be able to grasp some of these significant experiences and the role they play in shaping new teachers' RWMTL.

Indeed, by asking the participants to talk about experiences that have shaped their RWMTL in relation to a specific area, context or topic, we hope to hear about the experience *and* the reflection associated to it. As Dewey (1938) would say, the quality of an experience comes mostly from what it opens to, i.e. the resulting tools and how they help understanding and acting towards new experiences (p. 27). This takes time, since we think about our past experiences differently with time, reflection and probably other experiences. We therefore want to understand how the new teachers evolved in this transition, how their way of living and acting towards new experiences evolved with time and new experiences. As Clandinin (2013) said, we, the participants and the researchers, "are always interpreting [our] pasts from [our] present vantage points" (p. 46).

The pilot interviews also revealed that the participants had a hard time focusing on experiences specifically related to mathematics and often tended to discuss experiences related to class management, institutional constraints, etc. This lead us to design new protocols with the hope of instilling the norm that the goal is to talk about experiences with *mathematics*, and its teaching and learning.

To conclude, the pilot study helped us in creating protocols for the main study that would guide our endeavour to understanding some of the experiences that shape new cegep teachers' RWMTL, and more precisely, what kinds of experiences are significant as they transition from being mathematics students to cegep teachers, all while staying open to the new lives we will meet as the main data collection unfolds.

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Inquiry-based learning and pre-service teachers education

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We use Hintikka's analysis of the notion of inquiry *to point out some obstacles that the implementation of inquiry-based learning might find, and to suggest a way to overcome these difficulties via the use of the Reference Epistemological Models developed in the framework of the Anthropological Theory of the Didactic.*

Keywords: inquiry-based learning; teacher's and students' practices; novel approaches to teaching

INQUIRY-BASED LEARNING

Inquiry-based learning (IBL), roughly conceived as a teaching method in which students are invited to learn in a way similar to that of a scientist or a mathematician, has been suggested, with different levels of precision, by several researchers, institutions and pedagogical and didactic approaches. Artigue and Blomhøj (2013) include as instances of IBL the teaching proposals made from the following approaches in Mathematics Education: problem-solving, theory of didactical situations, realistic mathematics education, modelling perspectives, anthropological theory of the didactic, and dialogical and critical approaches. For its part, European Union, through reports prepared by experts (see, for instance, (Rocard, Csermely, Jorde, Lenzen, Walberg-Henriksson, & Hemmo, 2007)) and projects (see, for instance, PRIMAS, http://www.primas-project.eu), has supported also the implementation of IBL in educative european institutions.

It would be incorrect to believe that these IBL proposals have been suggested as a means to reach the very same educational end. Let us mention some proposal aiming different ends. The Theory of Didactical Situations proposes the IBL, embodied in the notion of *situation*, as a means to achieve a *real* knowledge of mathematics (Brousseau, 1997, p. 22). For its part, the Anthropological Theory of the Didactic proposes the IBL, via the notion of *Study and Research Path*, as a means to transform the cognitive *ethos* of our society, that is to say, to provide new attitudes and habits with respect to the acquisition of knowledge (Chevallard, 2015). European Union, in reports and projects like the ones mentioned above, stands up for IBL as a means to remedy the declining interest of youth in science, confirmed by some OECD reports, and the subsequent lack of technological innovation in Europe. According to this point of view, IBL would contribute to change science and mathematics learning into a motivating activity. On the other hand, Europen Union also promotes IBL, for instance in the PRIMAS project, as a means to prepare students for a future in which "it is no longer sufficient (…) to learn facts" but "to be able to solve non-routine problems, to analyse data, to discuss with colleagues, to communicate their result and to work autonomously" (Maaß & Reitz-Koncebovski, 2013, p. 10). Therefore, the ends to which different proposers aim to arrive through IBL are as varied as the very ends of regulated education: the acquisition of knowledge, habits, attitudes, values, etc.

KINDS OF KNOWLEDGE

To analyse possible uses of IBL in the acquisition of knowledge we have to wonder: are there different kinds of knowledge? In Epistemology it is customary to distinguish between three kinds of knowledge (Ichikawa, Steup, 2017; Fantl, 2017): knowledge by acquaintance, knowledge how and propositional knowledge. *Knowledge by acquaintance* is the kind of knowledge you have when you can identify something or someone (for instance, the label *criterion of divisibility by 3*) and the corresponding name, description, formulation, etc. (in this case: a natural number is divisible by 3 if and only if the result of adding all its digits is divisible by 3). *Knowledge-how* is the kind of knowledge you have when you carry out a series of intentional actions (for example, to apply the criterion of divisibility by 3) towards the attainment of an end (to know whether a given number is divisible by 3, in this case). Finally, *propositional knowledge* is the kind of knowledge you have when you know *why* a certain proposition is *true* (for example, when you know why it is true the criterion of divisibilty by 3). Notice that in the knowledge by acquaintance you also seem to know the truth of a proposition (for instance, the proposition asserted by the statement "The criterion of divisibility by 3 is the statement *A natural number is divisible by 3 if and only if the result of adding all its digits is divisible by 3*"). But in this case this is just a contingent truth, existing by convention.

INQUIRY AND PROPOSITIONAL KNOWLEDGE

In order to analyse possible obstacles to the implementation of IBL, we shall use a theoretical model of the notion of inquiry, namely, the one developed by the logician and philosopher Jakko Hintikka and collaborators in several works. Of course, one can also find other theoretical models of the notion of inquiry in some approaches to Mathematics Education. Anyway, here we choose the one provided by Hintikka due to its enlightening use of Logic, which seems to be an essential ingredient for a true comprehension of an inquiry process.

Interrogative Model of Inquiry

To get a better understanding of the concept *inquiry*, as claimed by Hintikka (1982),

> is not enough to study individual acts instantiating (…) whatever the concept in question may be. We also have to study the more complicated rule-governed behavioral complexes in which their "logical home" is.

For this, Hintikka has developed a game-theoretical model, the so-called *Interrogative Model of Inquiry* (IMI), which presents a formal approach to inquiry. A good reference for this model, which has been explained here and there, is (Hintikka, Halonen, & Mutanenet, 2002). In few words, inquiry is regarded as a game with two players: *Inquirer* and *Nature*. The game starts with a pair, (T, Q), where T is a given theoretical premise, and Q is a question. The game finishes when the Inquirer finds an answer to Q. Along the game the Inquirer is allowed to make two kinds of moves: *questions* to Nature, and *deductions*. The only moves of Nature are *answers* to Inquirer's questions. We will not explain here all the details of Hintikka's IMI, but at least we will point out two of its main features:

1) The principal goal of an inquiry is the acquisition of propositional knowledge.
2) Inquiry is a process in which a complex dialectics between deductions and questions takes place.

We emphasize here these two aspects because, as we hope to show below, they seem to be crucial for practical didactic considerations about IBL.

The goal of inquiry is to acquire propositional knowledge

The inquiry finishes if we can *deduce* (based on the premise and on Nature's answers to our questions) an answer A to the initial question Q (Hintikka et al., 2002, Theorem 1). Of course, the deduction procedure, which is carried out through a series of familiar rules (Hintikka et al., 2002, § 2), has to do with semantics. Indeed, these rules are such that if the premise T and Nature's answers are *true* in a given *model* M, then the answer A is also true in this model M. This is why we can read in (Hintikka, 1996, p. 38) that what the winning of an "interrogative game" shows is "knowledge of truths".

Later, we will speak of the implications of the presence of the model M in the use of IBL for teaching. For the moment, I would like to stress the fact that, according to Hintikka's IMI, the result of an inquiry is a proposition. Inquiry is a quest for propositional knowledge, which is the kind of knowledge linked to non-contingent truth and falsehood. This does not prevent, in the course of an inquiry, the acquisition of knowledge-how and knowledge by acquaintance. But we would not consider an inquiry finished if, along this inquiry, we would have found a successful technique (knowledge-how), whose success remains unexplained. In other words, in the course of an inquiry we can get knowledge-how, but we then would pursue the corresponding propositional knowledge able to explain the success (and even its limitations, portability to other contexts, etc.) of this knowledge-how.

Dialectics between deductions and questions

In a game of inquiry not all the questions can be asked at any moment. On the contrary, one is forced to pay attention to the *presuppositions* of that questions. As explained by Cross and Roelofsen (2016), there are different *logical* kinds of questions: whether-questions, which-questions, why-questions, etc. Each kind of questions determines its own kind of presuppositions. For example, the presuppositions of *which*-questions (e.g., "Which is the smallest prime number bigger than 7?") are existential statements (e.g., "There exists a smallest prime number bigger than 7").

According to Genot and Gulz (2015), based in turn on (Hintikka & Hintikka, 1989), the *Inquirer's range of attention*, at a given moment of inquiry, is the set of questions such that:

- The corresponding presuppositions are already available or can be obtained by a deduction move.
- Inquirer is aware of those presuppositions.

The main problem of interrogative inquiry is the mismatch between Inquirer's range of attention and available Nature's answers, since sometimes Inquirer might ask a question which has no answer, or fails to ask a relevant question whose answer is available.

In the literature on IBL it is usual to distinguish between several kinds of inquiries depending on two parameters: how open is the initial question Q, and how guided is the inquiry (PRIMAS, 2011, pp. 11-12).

Concerning the first parameter, a serious study of types of questions and degrees of openness in relation to the development of the subsequent inquiry is still to be done. Our point is that IMI provides a suitable theoretical framework to carry out such a study.

Concerning the second parameter, we refer to (Hintikka, 1982) in order to learn about all the possible moves the teacher can do in a game broader than an inquiry-game, in which the Student-Inquirer and Nature are but two players among many others (the Teacher being one of them). In particular, a kind of move the Teacher can do in order to guide the inquiry is to manipulate Inquirer's range of attention so that to avoid the mentioned mismatch. Among the possible kinds of manipulation one finds: to ask whether we can deduce a certain proposition from the available information (in order to use it further as a presupposition of a question), to attract attention to a certain (already proved) proposition and to incite to consider it as a presupposition of a question, etc. One might wonder to what extent the Teacher can avoid this manipulation in a IBL process channeled to the acquisition of some propositional knowledge. Concerning this, Genot and Gulz (2015) proved that "a trade-off between success [of the inquiry] and autonomy is unavoidable" (p. 1). Indeed, in one hand, with no manipulation of Inquirer's range of attention success might not be achieved. On the other hand

> The IMI [Interrogative Model of Inquiry] does however warrant the following conclusion: a guaranty that an inquiry learner will be able to solve interrogatively a problem can always be obtained by manipulating the learner's range of attention. (Genot & Gulz, 2015, p. 18)

OBSTACLES TO IBL

Admittidly, to plan for and support IBL is difficult due to the presence of obstacles of different nature: political, cultural, concerning teacher's view of her own profession, concerning teacher's training, epistemological, etc. Let us use the IMI as a microscope to inspect here two of them.

Propositional knowledge and models

According to (PRIMAS, 2011, p. 22):

> (…) many teachers find that IBL comes into conflict with the way they learnt science and mathematics in school and at the university, and even with the way they have been teaching science and mathematics for many years. That is, with their beliefs about the nature of mathematics/science and/or their beliefs about teaching of mathematics/science. Probably this will be one of major obstacle you will find.

This is strongly related to the fact that content taught in formal education is, typically, knowledge by acquaintance and knowledge-how, with a bleak lack of genuine propositional knowledge. Therefore, in agreement with the analysis of inquiry previously exposed, this content, not being propositional knowledge, can not be learnt through a proper inquiry. This is the case of the knowledge by acquaintance consisting in knowing the statement of theorems (Pythagorean theorem, Thales theorem, etc.) without proof. It is also the case of the following examples of knowledge-how, which lives just in the realm of syntax, of manipulation of symbols, without being supported by meanings: all kind of *algorithms* of addition, subtraction, multiplication and division of several kind of numbers, algorithms to calculate the greatest common divisor and least common multiple, divisibility criteria for natural numbers, etc. *Summarizing, for a piece of mathematics to be learnt in an inquiry-based fashion, it must be presented as propositional knowledge and, insofar as this kind of knowledge is concerned with true propositions, the presence of models fixing meanings and guiding the inquiry is unavoidable. In other words, without models there are no meanings, without meanings there are no truths, without truths there is no propositional knowledge, and so there is no inquiry.*

One might argue that, after all, these mathematical objects do have a meaning. For instance, the fraction 2/3, applied to an object, refers to any portion of this object equivalent (with respect to a certain, previously fixed, magnitude: volume, mass, area, etc.) to the one obtained after performing the following steps:

i) Split this object into three parts so that they were equivalent with respect to the fixed magnitude.

ii) Take any two of these three parts.

We would eventually agree that some objects, like natural numbers, positive fractions, etc., are attached to a meaning in regular teaching. But I claim that this meaning is not used later to support techniques. This makes the difference: whereas an unexplained technique (for example, to multiply fractions) is just mechanical knowledge-how, a justified one becomes propositional knowledge. Let us illustrate my claim with the case of the product of fractions.

Typically, we say that the product of two fractions, a/b and c/d, is the fraction (a·c)/(b·d), whose numerator (respectively, denominator) is the product of the two numerators (respectively, denominators). Notice that this mirrors the common definition in formal mathematics, where a fraction is defined as an ordered pair (a, b) of integers, with b different from zero, and the product of two fractions, (a, b) and (c, d), is (a·c, b·d). This formal definition, taking part in the praiseworthy human enterprise of founding Mathematics in Set Theory, does not need any further proof. Actually, being a definition, it can not be proved. But, for the existence of a proof, there is a more important obstacle other than the fact that it is a definition: there is no model-interpretation neither of the term *fraction* nor of the term *product* referring to fractions.

Concerning this we would like to underly two points:

- Once you have a model-interpretation of these terms you can (at least) try 'to prove your definition', namely, you can try to ask the question "Is it true that, according to the fixed meaning of the terms *fraction* and *product*, the product of two fractions is calculated by following the former procedure?"
- The way you answer the question strongly relies on the model-interpretation of the terms.

Imagine, for instance, that your interpretation of the term *fraction* is the one above: the denominator indicates the number of equivalent (according to a fixed magnitude) parts into which a given object has been split, and the numerator indicates the number of these parts you are considering. You still have to give an interpretation of the term *product*. This is a hard task. To beging with, we can say that a *product* is the result of a *multiplication*. Now, what is a *multiplication*? In natural numbers, a multiplication is what we do to calculate an amount of magnitude which has been expressed as *a whole amount of a whole amount*, for instance, to calculate the cardinal of a set which results from the union of 27 sets, each of which has 63 elements. Similarly, we can say that the multiplication of fractions is what we do to calculate an amount of magnitude which has been expressed as a *fraction amount of a fraction amount*. For instance, to know which is the total fraction we are considering when we calculate 2/3 of 4/5 of some amount of a given magnitude? I would know how to answer it if I knew how to answer in the case of 1/3 of 4/5. Similarly, this will not be a problem if I knew how to calculate 1/3 of 1/5. But it is not difficult to calculate that if each of the 5 fifths were divided into 3 parts, then the initial amount would be divided into 15 parts. Thus, 1/3 of 1/5 is 1/15. Now, since 1/3 of 4/5 is 1/3 of 4 times 1/5, we get 4 times 1/15, which is 4/15. And 2/3 of 4/5 is 2 times 1/3 of 4/5, that is to say, 2 times 4/15, which is 8/15. One can see, in this and other examples, that the numerator (respectively, the denominator) of the final fraction can be directly obtained from the first two fractions just by multiplying their numerators (respectively, denominators).

These considerations show that only after having an interpretation of the term *product* (again, it is the result of a *multiplication*, and a *multiplication* is what you do to calculate an amount of magnitude which has been expressed as a *fraction amount of a fraction amount*) you can *prove* the truth of the following proposition: "the product of the fractions a/b and c/d is (a·c)/(b·d)". The good news is that models allow to go from knowledge-how to propositional knowledge. The bad news is that the acquisition of this propositional knowledge strongly relies on the chosen model. This is not a minor issue. What to do if we have many possible interpretations of our theoretical terms? Which of them should be considered? Is there any didactic criterion (for instance, to overcome some learning obstacle) to choose among the different possible interpretations? For instance, the interpretation given above to the term *fraction* is the most usual, but it is not the only one. Should we also consider the others interpretations of the term *fraction*? If so, how to prove the multiplication formula for these interpretations? We face the problem to choose criteria for attaching meanings such that: 1) they help us to fight against undesirable didactic phenomena specific to the usual teaching of fractions; 2) they are compatible with the meanings attached to other numerical fields, like negative or real numbers.

How to plan and support inquiries for students?

Even if all the questions raised in the previous paragraph about the implementation of semantics in mathematics syntax are answered, we still have to deal with further difficult problems.

A very first one is: what could be the initial question Q of the inquiry? This question seems to be very difficult to find as it is intended to initiate an inquiry through which many kinds of mathematic entities would appear. Among them, notably, concepts. In this direction, we find the following teacher's claim which figures in (Maaß & Reitz-Koncebovski, 2013, p. 12):

> What about conceptual knowledge -surely students cannot be expected to reinvent mathematical or scientific concepts for themselves?

In terms of Hintikka's IMI, to search for Q amounts to looking for a question such that, together with the premises T (student's previous knowledge) and Nature's answers (derived from mathematical examples possibly examined by students), allows to deduce (as a final product but also as something obtained in the course of inquiry) the aimed mathematical propositions.

But still, even if Q is already clear, as it is said in Anderson's study (as cited in PRIMAS, 2011, p. 20), teachers have difficulties with managing "the challenges of new teacher roles and new student roles". According to Walker's work (cited in PRIMAS, 2011, pp. 20 - 22),

> Teacher loses control: although it depends on the degree of freedom teacher gives to students, it is clear that in IBL students should take control of the lesson.

Also, although the next quotation refers to IBL of science, it is perfectly translatable to mathematics:

> Inquiry based lessons might not "work": there is the risk that experiments do not work, that students collect wrong data and that they will get a wrong idea. In the classical use of experiments, these are carefully planned so that they always work and offer the right exemplification of the phenomena that is at stake.

Therefore, teachers have questions concerning their role in and control of students' inquiries. In terms of Hintikka's IMI, the question is: which would be the dialectics between deductions and questions in the expected inquiry? To have this dialectics relatively planned contributes to prepare the teacher in her duty of manipulating student's range of attention, which, in turn, grants the success of the inquiry in order to provide the acquisition of the aimed propositional knowledge.

PROPOSAL FOR TEACHING IBL TO PRE-SERVICE TEACHERS

In order to solve the problem of finding, given a piece of mathematics to be taught, a good model for the theoretical terms of this content, a good initial question Q and a way to handle the corresponding inquiry, PRIMAS has the so-called *professional development modules* [1]. There is, for instance, a module with examples of questions to initiate an inquiry. There is a module with strategies to promote students' questioning. There is, also, a module with examples of ways of acquiring concepts. But still there is a need of *complete* examples, including all together an initial question, the relevant moves students should do in the corresponding inquiry game, and moments in which the teacher could manipulate well enough student's range of attention in this very game.

As we said at the beginning of this work, the Anthropological Theory of the Didactic (ATD) is one of the proposers of IBL. ATD suggests the implementation of the so-called *paradigm of questioning the world* (Chevallard, 2015). According to it, formal education would be carried out by means of *study and research paths* (SRP). Succinctly, a SRP is the process you follow to find the answer A to a question Q. ATD emphasizes that, along this process, you are involved in different kinds of activities: studying possible (perhaps partial) answers to Q, formulating new auxiliar questions, etc. Although less philosophically informed, ATD's analysis of the notion of *inquiry*, via the notion of SRP, runs almost parallel with that of Hintikka, via the IMI.

There is a continuous spectrum of types of SRP, the extremes of which are what we could call *open* and *closed* SRP. The open ones are those in which the teacher is not specially interested in leading the students towards a particular piece of knowledge O. In contrast, the closed ones are those in which the question has been selected with the intention of leading to the natural emergence, in the course of the SRP, of a certain piece of knowledge O previously selected.

In formal mathematics this knowledge O is not expressed as the output of an inquiry, but as a series of axioms, definitions, theorems, examples and standard techniques. Therefore, closed SRP demand, at least, to reorganise the piece of knowledge O to be found along the inquiry. The corresponding reorganisations are what ATD calls *Reference Epistemological Models* (REM) (see, for instance, Sierra, 2006).

Typically a REM is expressed in terms of *praxeologies* (Chevallard, 2006), that is to say, in terms of: *types of tasks*, *techniques* devoted to face these types of tasks, a *techno-logical* considerations about each technique (a detailed description, a justification, a study of its scope and reliability, possible enhacements) and, possibly, also some *theoretical* considerations about the situations under study (our metaphysical description of them: basic entities, basic properties, etc.). It is a key feature of a REM that essentially everything in it appears motivated by the study of the types of tasks. It is worth mentioning that normally the construction of a REM on a piece of knowledge O is not only guided by the aim of using it as a basis of a SRP, but also by the intention of counteracting some undesirable didactic phenomena specific to the usual teaching of O (Gascón, Nicolás, in press).

It is still an open question, but our point is that each REM [2] *implicitly* provides a *complete* example of a possible inquiry, and so a solution to the obstacles to the implemenation of the IBL previously mentioned, namely, the need of interpretations-meanings, the need of a good initial question and the need of knowledge about when and how to guide the students in their inquiry. More precisely, we think each REM *implicitly* proposes: a *model* for the theoretical terms and syntax appearing in the mathematical content O to be studied, a *question* Q to initiate an inquiry, a *proof* (based on Logic and Game Theory) of the fact that the corresponding inquiry would fully cover O, and a set of *moments* of the inquiry at which teacher should evaluate, and possibly manipulate, Inquirer's range of attention. In future works we will try to provide evidences for this point via a logical analysis (IMI-like, in terms of deductions, questions and answers) of some of the published REM.

NOTES

1. Available at http://www.primas-project.eu/artikel/en/1221/Professional+development+modules/view.do

2. For the moment, there are, among others, published REM on natural numbers, integer numbers, decimal numbers, proportionality, algebra and differential calculus. See http://www.atd-tad.org/grupo-tad/

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A model for instructional design in mathematics implemented in teacher education

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This paper presents a model for instructional design in mathematics and reports from student teachers' experimentation with it in secondary school. The model is framed within the theory of didactical situations in mathematics. It contributes to the research field through the notion of an epistemological model, which provides an enhanced understanding of the stage of 'conception and a priori analysis' of didactical engineering. Findings show how the model is instrumental in creating student teachers' awareness of the impact of the milieu on the nature of the knowledge developed by the pupils.

Keywords: teachers' and students' practices at university level, novel teaching approaches, epistemological model, adidactical situation, milieu.

INTRODUCTION

The mathematical performances that represent the ambitions of the mathematics classroom and curriculum are constituted through teacher and student participation in activities stimulated by mathematical tasks designed (or selected) by the teacher for the realisation of an instructional purpose (Clarke, Strømskag, Johnson, Bikner-Ahsbahs, & Gardner, 2014). The centrality of tasks as instruments in mathematics classrooms is reported in the TIMSS 1999 Video Study: in the eighth-grade classrooms that were investigated (in seven countries), at least 80% of lesson time, on average, was spent on solving mathematical tasks (Hiebert et al. 2003). In this paper, I present a model for instructional design in mathematics, where tasks are embedded in *situations* that preserve meaning for the target knowledge. Further, I report from student teachers' utilising the model to teach an optional piece of mathematical knowledge in secondary school. The research question addressed in the paper is: *What do student teachers learn from using the model for instructional design?*

The model for instructional design is applicable when there is an intention of teaching someone some particular mathematical knowledge. It is rooted in the theory of didactical situations in mathematics, TDS (Brousseau, 1997), the main concepts of which I present briefly in the next section.

A BRIEF INTRODUCTION TO TDS

TDS provides a systemic framework for investigating teaching and learning processes and for supporting didactical design in mathematics, where the particularity of the knowledge taught plays a significant role. In the following I explain concepts of TDS (based on Brousseau, 1997) that are central to the model for instructional design.

The *milieu* represents the elements of the material and intellectual reality with which the students interact when solving a problem. An *adidactical situation* is a situation in which the students as a group try to solve the problem given to them on the basis of features of the milieu, without significant help from the teacher. The milieu of an adidactical situation is called an *adidactical milieu*. An appropriate adidactical milieu provides feedback to the students, whether their responses are adequate with respect to the knowledge at stake. The teacher has three main roles in the broader didactical situation: first is the *devolution* of an adidactical situation to the students, which means to present the adidactical situation and the problem to be solved and make the students accept this transfer of ownership; a second role is the *regulation,* which means to handle the evolution of the adidactical situation and its milieu; and, a third role is the *institutionalisation* of the knowledge developed in the adidactical situation, which means to transform the contextualised responses produced by the students into scholarly knowledge aimed at by the institution.

The *didactical contract* refers to the phenomenon that the interaction between the teacher and students in a didactical situation is regulated by rules related to the knowledge at stake. These rules form a set of implicit reciprocal obligations and mutual expectations. In devolution, the teacher (implicitly) negotiates a contract that involves a temporary transfer of responsibility for the knowledge at stake, from the teacher to the students.

After devolution, four situations follow where the role of the teacher and the status of knowledge change: Situations of action, formulation, and validation are (intentionally) adidactical, whereas the situation of institutionalisation is didactical. The adidactical situations are designed with milieus that are supposed to give feedback to the students, as mentioned above. The situation of *action* is where the students engage in the situation on the basis of its milieu without the teacher's involvement—that is, they construct an implicit solution (a strategy) that guides them in their decisions. The situation of *formulation* is where the students' formulations are useful in order to indirectly solve the problem—that is, formulation of an explicit solution that enables somebody else to operate on the material milieu. Here, the teacher's role is to make different formulations "visible" in the classroom. The situation of *validation* is where the students try to explain a phenomenon or verify a conjecture. Here, the teacher's role is to lead a whole-class discussion and trying to make the students use precise mathematical notions. The situation of *institutionalisation* is where the teacher's role is to connect the contextualised knowledge built by the students with scholarly forms of knowledge.

A MODEL FOR INSTRUCTIONAL DESIGN IN MATHEMATICS

The model for instructional design introduced here is based on the theoretical analysis presented in (Strømskag, 2017). It contains four phases: epistemological analysis; development of an epistemological model; implementation; and, institutionalisation. They correspond to design and implementation phases of didactical engineering (Artigue, 2015): epistemological analysis corresponds to ‘preliminary analyses’; development of an epistemological model corresponds to ‘conception and *a priori* analysis’; and implementation followed by institutionalisation corresponds to ‘realization’. The model contributes to the research field through the notion of an *epistemological model*, which provides an enhanced understanding of the phase of ‘conception and a priori analysis’ of didactical engineering. Its elements are illustrated in Figure 1 (the dotted curve signifies that the epistemological analysis informs the institutionalisation).

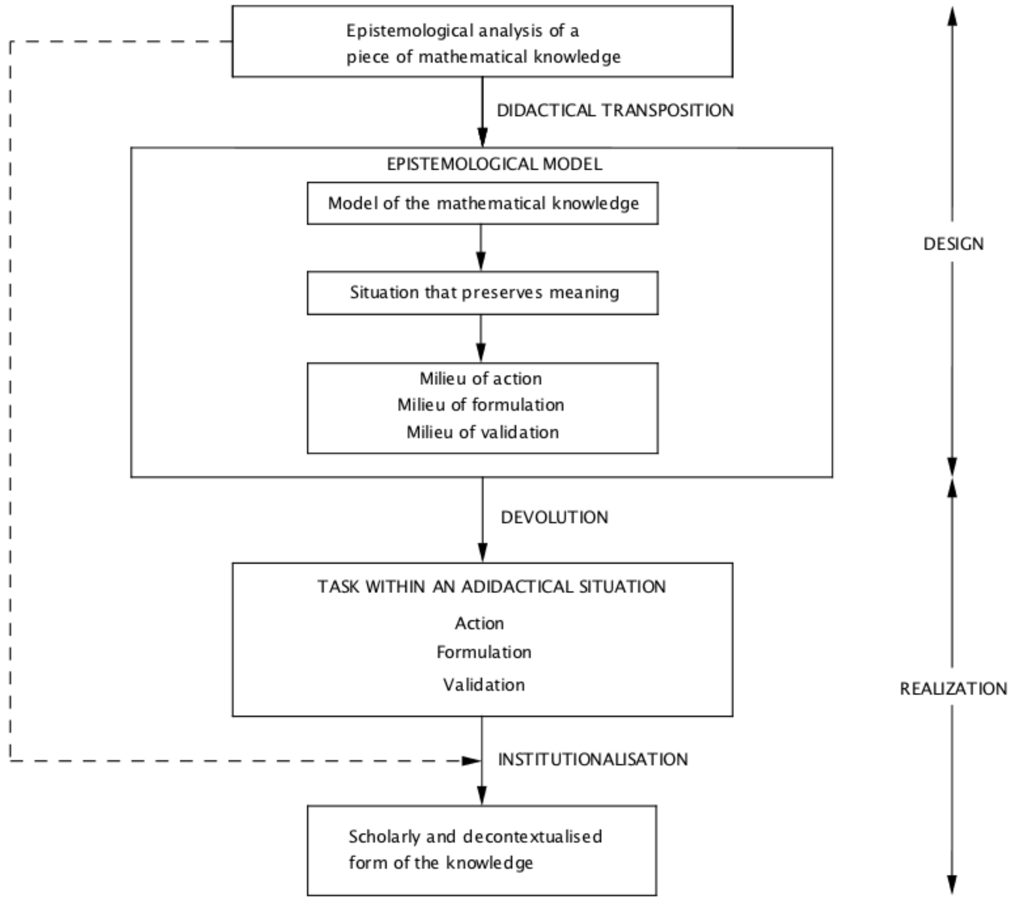

Figure 1. A model for instructional design (reproduced from Strømskag, 2017, p. 912)

Figure 1 shows how TDS concepts are constituents in design and implementation of a didactical situation that aims at some particular mathematical knowledge. For an account of its methodology, see (Strømskag, 2017). The phases of the model are described briefly in the following sections.

Epistemological analysis

Epistemological analysis of the knowledge at stake involves two components: an analysis of the knowledge itself, and a didactical analysis. The aim of analysing the knowledge is to identify possible epistemological obstacles, to find out what this knowledge is *for*, what questions that motivated its genesis, and how its validity can be justified. A didactical analysis aims at surveying research on teaching and learning of the knowledge at stake. The purpose of the epistemological analysis is to figure out what *conditions* must be fulfilled for a situation to implement the target knowledge (i.e., what problem would have the target knowledge as its solution), and by this inform the next phase, development of an epistemological model.

An epistemological model

An epistemological model of the knowledge at stake is a construct that consists of three components: first, a model of the target knowledge—possibly an iconic representation; second, a *situation* that preserves meaning—involving a problem that can be solved in an optimal manner using the target knowledge; and, third, milieus of situations of action, formulation and validation—designed so as to make students' knowledge progress towards gradually more explicit and mathematical forms, based on an image of students' adaptations to the milieus. The second and third component together can be considered a model of the generic and epistemic student's intended learning—it is developed according to the conditions that must be fulfilled for a situation to implement the knowledge it defines. The point is to design *milieus* so that the responses produced by the students will become gradually more explicit and mathematical, and ultimately can be institutionalised to become the scholarly knowledge aimed at by the teacher.

Implementation

The third phase involves implementation in the classroom, where the epistemological model is the basis for the teacher's devolution and regulation of an adidactical situation (including a problem) aiming at students' interaction with the milieus of situations of action, formulation and validation. Students' learning during implementation is understood as *independent adaptation*.

Institutionalisation

The fourth phase involves institutionalisation of the solution to the problem into scholarly and decontextualised forms of knowledge. Students' learning in this phase is understood as *acculturation*, which is intended to enable them to know the place, importance, and future of the mathematical knowledge reached.

The two processes of learning—adaptation and acculturation—are governed by the didactical contract, and the relationship between them is understood as devolution and institutionalisation. The model presented in Figure 1 displays the teacher's roles in the design and realization of a didactical situation: the process of *didactical transposition* (Chevallard, 1989) transforms the knowledge at stake into an epistemological model; the process of *devolution* transforms the epistemological model into a problem embedded in a situation; and, the process of *institutionalisation* transforms the situated knowledge used to solve the problem into scholarly forms of knowledge aimed at by the institution.

AN EPISTEMOLOGICAL MODEL EXEMPLIFIED

I give here a brief presentation of an epistemological model, where the target knowledge is the general numerical statement expressing that the sum of the first *n* odd numbers is equivalent to the square of *n*, possibly represented by $\sum_{i=1}^{n}(2i-1)=n^2$. I have implemented the epistemological model exemplified here in a teacher education programme, a detailed account of which is given in (Strømskag, 2017).

A model of the theorem $\sum_{i=1}^{n}(2i-1)=n^2$

A model of the target knowledge is created using a dissection of a square into L-forms consisting of consecutive odd numbers of unit squares (where 1 is represented by one unit square, hence a degenerated L). A generic example is given in Figure 2, illustrating that $\sum_{i=1}^{4}(2i-1)=4^2$. It is made of a dissection of the fourth square into the first four odd numbers. The model of the target knowledge is not to be shown to the students. It is a tool for developing a model of the students' intended learning, parts of which I describe below.

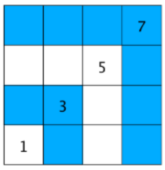

Figure 2. A model of the target knowledge

A situation that preserves meaning of the target knowledge

I invented a situation based on an imaginary company called TILEL selling a special kind of tile formations that can be used to cover squares. The tile formations have shapes as L-forms, and consist of an odd number of unit squares. The idea of this situation is derived from the model of the target knowledge, shown in Figure 2.

The task has two main parts:

- Finding a method for building a square of side length a natural number, using L-forms from TILEL.
- Explaining why the method will work for *any* natural number.

The milieu

The milieu of action is the *material milieu* on which the students are supposed to operate. The material milieu consists of ten paper cut-outs with $1, 3, 5, \cdots, 19$ unit squares that represent the first ten odd numbers, as illustrated in Figure 3. The task in the situation of action is to *find a method of building a square of side length a natural number up to ten, using L-forms of different sizes*. The features of the milieu (common for action, formulation, and validation) are the following:

- The material milieu does provide feedback: it is visible for the students whether or not they succeed in building a square of the intended size, using the L-forms.

- There is an obligation to use L-forms of different sizes. This is part of the didactical contract, and is to ensure that the students engage with odd numbers.
- It is only the size of the resulting square that matters. This is part of the didactical contract, and means that different configurations of L-forms should not be distinguished (i.e., it is not a combinatorial task).

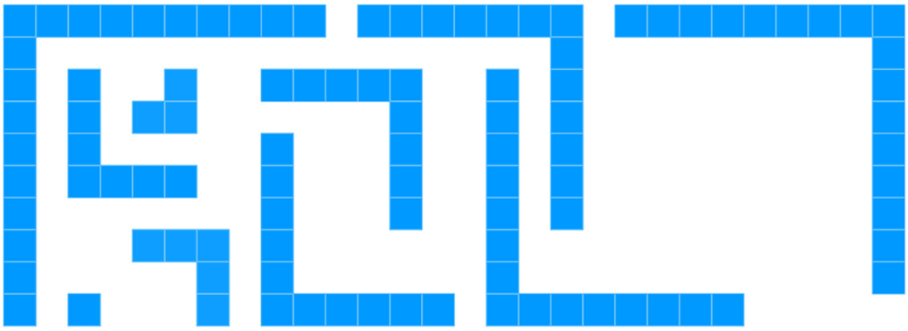

Figure 3. The material milieu for the intended theorem (paper cut-outs)

For an account of the milieus of formulation and validation, and the intended knowledge progress based on the TILEL-situation, see (Strømskag, 2017).

STUDENT TEACHERS' EXPERIMENTS IN SECONDARY SCHOOL

Methodical approach

In 2017, within a master's course in mathematics education, I gave a group of 13 student teachers (hereafter 'students') the task of using the presented model to design and implement (with pupils in secondary school) a situation that preserved meaning for a chosen piece of mathematical knowledge. The students were in the 4th or 5th year in a 5-year teacher education programme in mathematics and natural sciences.

The preparation at campus for the experiment (spanning 4 lessons, á 90 minutes) entailed first, my introduction of necessary concepts from TDS and presentation of the model itself. Then the students worked on a task on percentages that I had designed (using the design principles of the model) to let them experience and solve a problem embedded in a TDS situation. The next step was my comparison of the *a priori* and *a posteriori* analyses of the TILEL-situation, implemented with 20 students in a teacher education programme (this experiment is reported in Strømskag, 2017). This was followed by a brief presentation of two other models of mathematical knowledge: models of convergent geometric series, shown in Figures 4 and 5. The students discussed what series they were models of, and what the limit of each of them was.

The last lesson comprised students' work in four groups on an assignment that involved using the model for instructional design to design and implement a teaching situation aiming at some particular mathematical knowledge (suitable for the class in which they were to do the experiment). I was present for guidance during the initial design part, where they created an adidactical situation with a milieu and a problem.

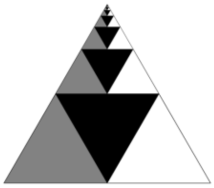	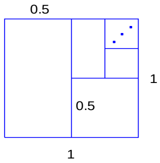
Figure 4. $\sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i = \frac{1}{3}$	**Figure 5.** $\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i = 1$

Data used to explore the utility of the proposed model consist of students' reports from the experiments, which I received their consent to use (texts from the reports are translated into English by H. S.). Even if the students worked in groups to plan and conduct the experiment, they wrote individual reports. A demand was that the report contained a comparison between *a priori* and *a posteriori* analyses of the implemented situation—that is, how the situation was suitable to develop the target knowledge.

Results

A general finding from the 13 reports is that the implemented situations did not have a satisfactory adidactical potential. The students described how they—contrary to the ideal roles of the teacher in TDS situations—had to intervene during the pupils' work. This was explained by too little attention or effort to create an appropriate milieu for the intended knowledge. In the following, I present how three students'—through their comparison of *a priori* and *a posteriori* analyses—in each case, reflected on limitations of the milieu of the implemented situation, and suggested improvements that might possibly strengthen its adidactical functioning. In the last case, the strength of the designed situation is also accounted for.

The target knowledge for Uma's pupils (upper secondary first year) was a formula for the number of components of the *n*-th element of a shape pattern, $F_n = n^2 + 2(n+1)$. This knowledge was the solution to a problem embedded in a situation with a barbeque area of variable size with fixed shape, covered by tiles as illustrated by the shape pattern in Figure 6.

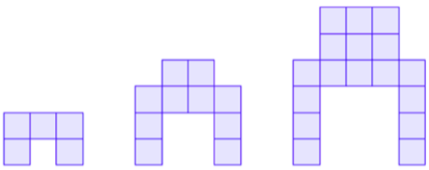

Figure 6. The material milieu for Uma's barbeque area problem (iconic representations)

Uma described how emphasising the *structure* of the elements of the pattern by, for example, using different colours, might have made it easier for the pupils to implement the knowledge aimed, without the teacher's guidance and help. This idea of Uma can be seen as stimulating algebraic thinking. She also commented on the importance of the devolution for the adidactical functioning of the milieu.

The target knowledge for Tracy's pupils (upper secondary second year) was the theorem expressing that the sum of two consecutive triangular numbers is equivalent to the square number of rank the same as the largest of the triangular numbers. The theorem was symbolised as $T_{n-1} + T_n = n \times n$, where $T_n = 1 + 2 + 3 + \cdots + n$. Tracey presented a model of the theorem, a generic example illustrating that $T_3 + T_4 = 4^2$, as shown in Figure 7. The target knowledge was the solution to a problem embedded in a situation with production of quadratic birthday cards, made by two staircase figures (representing triangular numbers) as shown in Figure 8.

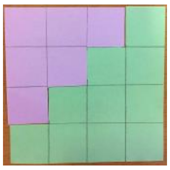

Figure 7. Tracey's model of the target knowledge

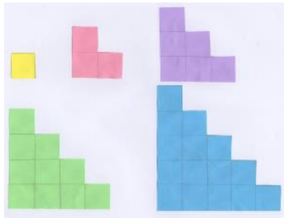

Figure 8. The material milieu for the birthday card problem (paper cut-outs)

Tracey explained that the colours of the material milieu were an obstacle to pupils' engagement with the knowledge at stake; the pupils referred to staircase figures by their colours instead of by their structure (sum of natural numbers) and rank. Further, generalisation of the birthday card problem was supposed to be brought into effect by the task: "Formulate a method to make a birthday card of an arbitrary size, and argue why the method works". This part was problematic because the pupils understood the concept *arbitrary* (in Norwegian: "vilkårlig") in the meaning of any birthday card made by the figures available in the milieu, instead of any birthday card made by imaginary staircase figures of any size (as intended).

The target knowledge for Tina's pupils (grade 8) was an explicit formula for the number of components of the *n*-th figure of a shape pattern (shown in Figure 9). This knowledge was the solution to a problem at a four-way crossing, where a car in the middle blocked the crossing and caused one car at each of the four ways to be stuck every second. The situation after two seconds is shown in Figure 10 (this was the material milieu along with a pile of paper cut-outs of cars).

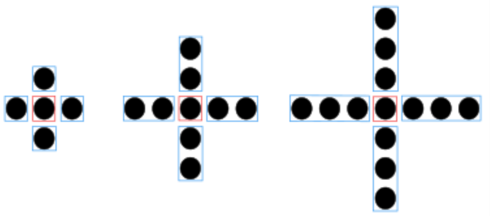

Figure 9. Tina's model of the target knowledge

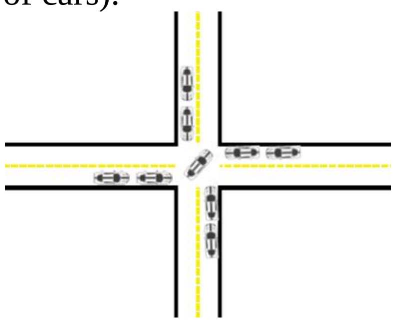

Figure 10. Material milieu for the crossing situation (paper cut-outs)

For small numbers of seconds, Tina expected the pupils to use a recursive approach. Hence, she tried to motivate for an explicit formula by asking for the number of cars after 1 minute. Some pupils, however, used a strategy where they treated the situation as a direct proportionality (representing the four cars added each second) and added 1 in the end for the car in the middle. One pupil reasoned that since there were 21 cars after 5 seconds, he multiplied 20 by 2, then multiplied 40 by 6, and added 1 in the end. This gave the right answer, but it was not what Tina was after. She was after a functional relationship between the seconds that had passed and the total number of cars. Tina explained that in planning it was not anticipated that anyone would use the mentioned method, so the milieu did not have adequate feedback to handle it.

In conclusion, Tina wrote: "The results show that the work in advance of implementation is very important regarding the knowledge actually developed by the pupils […] The results show, however, that feedback from peers has been instrumental in helping pupils to correct misunderstandings in adidactical situations. This I see as a strength of the designed milieu." Another strength reported by Tina is that in institutionalisation, the pupils were exposed to a new shape pattern, the relationships of which they were able to represent through algebraic symbols. Tina claimed that this was evidence that the knowledge developed through the experiment had in fact been generalised beyond the designed situation.

DISCUSSION

There were institutional constraints on the study reported here: the time available for instruction at campus was only 4 lessons (a modifiable condition); and the students had no knowledge of the pupils before the experiment (an unmodifiable condition). The fact that the students—contrary to their intention—had to intervene during the pupils' work is not unique to the model for instructional design. What the *methodology* of the model offers, however, is validation of an implemented teaching situation based on comparison between *a priori* and *a posteriori* analyses; this is similar to the methodology of didactical engineering (Artigue, 2015). In the model for instructional design, the *a priori* analysis is constituted by an epistemological analysis and development of an epistemological model, whereas the *a posteriori* analysis is an analysis of the realisation of the designed situation.

The answer to the research question posed—about what the students learned from using the model for instructional design—is that the students were able to identify relationships between the adidactical milieu and the knowledge progress in each case. The points addressed in the students' reports were about features of the milieu that constituted *limitations* on generalisation processes: In Uma's case it was about the devolution, and a material milieu that failed to focus on structure and algebraic thinking. In Tracey's case it was about features of the material milieu that suppressed focus on structure, and pupils' not knowing the meaning of the term "arbitrary". In Tina's case it was about a milieu without feedback for a linear generalisation method used by many of the pupils. However, Tina also emphasised a quality of the milieu: how pupils' progress in adidactical situations were enabled by *feedback from peers*.

Such comparisons between *a priori* and *a posteriori* analyses will identify relationships between the nature of the milieu and the knowledge progress in adidactical situations, and thereby create awareness of what enables and what prevents pupils from reaching the target knowledge. In this way, the model has potential to inform future design (aiming at the same knowledge): it might provide deeper insight into the nature of the knowledge at stake, and a better understanding of necessary conditions on the milieu concerning its adidactical functioning.

In conclusion, the impact of the study on my practice as a teacher educator is that the students' experiments are used as objects of study (with the same or other students), where the task is to revise the experimented situations and their milieu to better match them with their original didactic intention, and subsequently, to implement them with pupils to validate them. In this way, the experimented situations are considered material milieus for instructional design. This represents a meta-level perspective on instructional design in mathematics teacher education.

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University Teachers-Researchers' Practices: the Case of Teaching Discrete Mathematics

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Our work is part of a thesis which deals with the teaching practices of university mathematics professors. Practices of university professors concur specificities: an example is the articulation between teaching and research. We seek to characterize the research impact in the teaching practices at university. We choose discrete mathematics as an object of study. It is an area whose didactic transposition is not fully achieved while the links with other mathematical fields remain undefined. These elements make the choice of resources to be mobilized and conceived quite complex. Our exploratory study draws on interviews with university professors aiming to characterize the interaction with resources; which can help us clarify the research activities impact on teaching practices in the case of discrete mathematics.

Keywords: Teachers' practices at university level, discrete mathematics, undergraduate education, impact of research, resources.

INTRODUCTION

Our work is part of an ongoing thesis which deals with the practices of university mathematics professors. The study belongs to the growing body of research on university professors' practices (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). At this level of our study, we designate by "University professors" the university mathematics teachers who occupy teaching/research positions.

While we are interested in teaching practices at university, we try to characterize the different factors that impact them (institutional, didactic and epistemological), in particular, the impact of the research activity on the teaching practices. Throughout the text, we will be relying on the following definitions:

- "Teaching practices" and "university teachers' practices" to refer to the "teaching" aspect of the university professors' work;
- "Research activity" to refer to the research aspect of the profession.

Our study focuses on the place of discrete mathematics in undergraduate level. Discrete mathematics holds an epistemological importance, and it is a branch whose topics are not stabilized in mathematics curricula (Heinze, Anderson, & Reiss, 2004; Hart & Sandefur, in press). Hence, it is particularly interesting to study university mathematics professors' practices in the field of discrete mathematics, a choice that we will justify later in the document.

In a study of ICME [1], Biza et al. (2016) conducted a survey of existing research on mathematics education at university. Drawing on journal publications and conference proceedings on university mathematics education published since 2014, Biza et al.

identified the current trends and the most recent advances in the field. They classified the existing work according to the topic of research. Among these, the influence of teachers' research activity on their teaching approaches in particular contexts: the use of examples in mathematics tutorials and the use of graphic representations. The study highlights the emerging interest in taking into consideration resources in teaching at university, in particular, interactions with resources for mathematics education and their impact on teachers' professional development. The study points as well the need for further research on practices of university mathematics teachers and the possible impact of their research activity on their teaching practices.

The work that we present in this text is a contribution to the growing body of research in mathematics education on the practices of mathematics teachers at university level. We will present first the context of the study and the research questions. Then, we will describe the first steps in the construction of our theoretical framework and the collection of data. Finally, we will present the methodology we used in our study as well as some extracts, along with some preliminary results and perspectives.

CONTEXT AND RESEARCH QUESTIONS

The study of the practices of university professors is part of a field of research currently in development, but still largely unexplored. There are similarities in many aspects of the practices of secondary and university teachers: the preparation of the courses and tutorials, the design of instruction, the conception of evaluations, the classroom management, and the interactions with students. However, the practices of university professors conquer specificities, from an institutional point of view and given their training and their academic background.

University professors have some freedom in the design of their courses and the choice of contents to be taught, and in the selection and development of resources. Moreover, their research activity can impact (or not) different aspects of their teaching practices (Biza et al, 2016; Mali, 2015) such as the instructional approach, the form of interaction with students, and the choice of new knowledge to be taught.

University professors, in their teaching practices, maintain a constant interaction with various resources (Gueudet, 2017) derived from their teaching practices as well as their research activity (old resources, software, online resources, videos and podcasts, manuals, research articles, etc.). The interaction with the resources can take place at different moments of their teaching practices: in the design of the sessions, the choice of contents to be taught, the teaching in the classroom, and the evaluation of learning. In the working environment of a university professor, the same resources can be used in his teaching practices and his research activity (numerical computation software for example). Other interactions between teaching and research can take place, but in a less tangible way. The study of the university professors' interactions with the resources could be a first step in clarifying the nature of the relationship between their teaching practices and their activity of research: in the choice of contents to be taught, in the teaching mode of different mathematical contents, and in the learning they wish to develop with students (reasoning, application of properties, etc.).

In mathematics education, teaching practices are not considered independently of the contents taught. The relationship between university teachers' practices and their research activity may depend on several parameters, including the mathematical branches addressed in teaching and research; discrete mathematics, in particular, offer an interesting experience. Discrete mathematics have a real epistemological importance as they unfold in different mathematical domains; the objects are easy to access and can contribute to the understanding of other branches of mathematics (Grenier & Payan, 1998, Maurer, 1997). They provide an introduction to modeling, optimization, operational research, and experimental mathematics (Grenier & Payan, 1998, Maurer, 1997). They promote the learning of proof and the development of heuristic processes in students (DeBellis & Rosenstein, 2004, Goldin, 2004).

Discrete mathematics is a field in expansion, with significant achievements in society. Attempts to integrate discrete objects in the curricula appear in several countries at the international level. In France, the place of discrete mathematics in the curriculum is still not sufficiently stabilized and takes various forms depending on the educational context (Ouvrier-Buffet, 2014); which makes this branch very interesting to study the relationship between research activity and teaching practices of university professors. Hence the research questions can be formulated as follows:

- How do university professors interact with the resources in / for the teaching of discrete mathematics?

- How can we characterize the relationship between the teaching practices of discrete mathematics at the tertiary level and research activities in the same field?

Trying to answer these questions requires specific theoretical and methodological developments.

TOWARDS THE CONSTRUCTION OF A THEORETICAL FRAMEWORK

We conducted a first theoretical choice: *the documentational approach* (Gueudet & Trouche, 2009). We will justify the choice in what follows. We note that the choice of the documentational approach constitutes a tool for a first exploration of the field.

Gueudet (2017) based her work on the documentational approach (Gueudet & Trouche, 2009) to analyze the interactions of university professors with *resources* derived from their teaching practices. In the documentational approach, there is a distinction between *resources* and *documents*. University professors select, mobilize and use various resources (contents to be taught in class, written evaluations, old resources, etc.). This interaction with the resources generates a document, which is the association of resources and a *scheme of use* of these resources. A scheme is used here as it was defined by Vergnaud (2009) as the invariant organization of conduct for a set of situations having the same aim. According to Vergnaud (1998), a scheme is a dynamic that has four interacting components:

- An *aim* that can be easily identified and that indicates intentionality in the organization of the activity;

- *Rules of actions* which are the ways of acting generated by the scheme in order to achieve a specific aim;

- *Operational invariants* that influence the rules of action. They can be *theorems-in-action* (propositions considered as true by the subject, but may be true or false) or *concepts-in-action* (considered as relevant in a given situation);

- Possibilities of *inferences,* that are the adaptations that the subject can bring to his activity in order to respond to the specificities of a situation corresponding to an aim.

A scheme developed by a subject is associated with a class of situations (Vergnaud, 2009). A class of situations includes all situations having the same aim. In her work, Gueudet (2017) considered classes of situations for specific aims (for example, preparing an assessment in linear algebra) and thus corresponding to a single document, and larger classes of situations that are independent of the mathematical content (for example, preparing an assessment). She made these choices in order to observe the organization of the resources of university mathematics professors globally, and to analyze the operational invariants and the rules of actions associated to mathematical contents in more restricted classes of situations. Although Gueudet (2017) recognizes the impact that university professors' research activity can have on their teaching practices, she did not focus on this aspect in her study.

We rely on the methodology developed by Gueudet (2017). We seek to analyze the choices, of contents and resources, of university professors in their teaching of discrete mathematics. Furthermore, we are interested in the impact of the research activity on the teaching practices. Therefore, we will adapt and develop Gueudet's methodology (2017) in a way to be able to consider the particular field of discrete mathematics and to take into consideration the research activity's impact on teaching.

ELABORATION OF INTERVIEW GUIDELINES AND DATA COLLECTION

The methodology of our exploratory study is based on interviews. We consider university professors whose field of research is discrete mathematics in Lebanon and in France. The interviews, of approximately 90 minutes, took place in workplaces of the Lebanese professors in order to have access to their resources. For French professors, the interviews were conducted via Skype.

We have developed the interview guidelines in order to characterize the interactions of university professors with various resources, resulting from their teaching practice and activity of research. We asked questions such as "What are the main resources you use for your teaching? How do you select the contents of your courses?" these questions intended to give a panoramic view of the resources selected, mobilized and used in the different classes of situations (preparation of lectures and tutorials, evaluation); We believe that the type and nature of the resources (resources resulting from research or teaching) can inform us about the relations between the research activity and the teaching practices of university professors; furthermore, the choice of contents (lectures, tutorials and evaluations) can give us insight about the different rules of action and the operational invariants corresponding to the different classes of situations. Another question was "What are the conditions and constraints that guide your different choices?" Learning about the institutional conditions and constraints (related to the educational institution or to the research institution) can contribute in characterizing the research activity's impact on teaching practices. Other questions were about the links with other mathematical fields, the collective work, the experimentation in classrooms, etc.

We conducted pilot interviews with two discrete mathematics university professors, one in Lebanon and the other in France, to test the interview guidelines. We recorded and transcribed the interviews. To keep the anonymity of the professors interviewed, we will call them Michel (Lebanon) and Bertrand (France) throughout the document.

THE METHODOLOGY OF ANALYSIS

Our methodology of analysis consists of two steps which we will be presenting next.

The first step is analyzing the practices of university professors in terms of their interactions with resources. For that, we detected the aims, mentioned by the university professors in the interviews, related to their teaching practices. In other words, we identified possible classes of situations based on statements, such as "preparing a lecture" or "preparing an assessment". For each aim, we identified the associated resources (manuals, books, scientific articles, computers, online resources etc.) that were explicitly mentioned in the interview. For the rules of action, we relied on the university professors' declarations; they described the ways they behaved in order to achieve an aim, and the adaptations they brought to their actions in accordance with the peculiarities of each situation. These are the conditions and actions that can be expressed in statements of the form "if (condition) ... then (rules of action) ..." or "for (aim) … I (action) ...". The regular ways of acting in specific situations reveal the presence of operational invariants that university professors do not always express in their discourse. To detect them, we identified in the interviews, the statements justifying the rules of actions, the propositions held to be true by the university professors (theorems-in-action) and the logical reasoning underlying the choice of actions to be conducted according to the specificities of each situation.

We constructed a table for the documents of each university professor; horizontally, in each table, we come across a document associated with a given class of situations (the resources used and the scheme of use). Vertically, we find the sets of the resources used, the rules of action and the operational invariants.

The second step consists in studying the impact of the research activity on the teaching practices in classes of situations related to teaching. For each class of situations, first we choose to detect, in the set of resources used, the presence / absence of an impact of the research activity (for example, the choice of contents to be taught in class might be influenced by the research activity of the university professors or not). Then, we will define a typology of operational invariants; it would allow us to classify the reasons behind some rules of actions of the university professors and behind the "transfer" (use, selection and modification) of some resources from their research activity to their teaching: beliefs developed by the activity of research, gestures acquired with professional experience or institutional constraints and conditions. We will eventually search for additional tools for a deeper study of the impact of research activity on teaching practices, following the results of the pilot interviews.

ANALYSIS OF THE FIRST PILOT INTERVIEW

Our first interview was with Michel (Lebanon), whose research domain is graph theory. Michel teaches graph theory for students in masters' degree and discrete mathematics for students majoring in mathematics in year 2. We detected, in Michel's declarations, the resources produced and / or used in his teaching practice.

Aims	Resources	Rules of actions	Operational invariants
Prepare a course of discrete mathematics in second year	well-known books in the world Scientific publications Typical texts	He tries to convey the basic ideas in discrete mathematics. He uses many examples. He chooses contents that allow students to discover ideas.	The contents must be aligned with the official instructions of the major. Definitions in discrete mathematics are simple, there is a need for applications to initiate the work. Understanding the ideas is very important in discrete mathematics.
Implementing a discrete mathematics course in class	well-known books in the world Typical texts	He engages students in writing. He encourages students to read typical texts. Before writing the proof of a theorem,	It is important that students practice writing. It takes examples and applications to familiarize students with discrete mathematics. It is very important to illustrate

		he illustrates the ideas.	the ideas in discrete mathematics.

Table 1- *Extract of Michel's documents table from his teaching practices (aims, resources, rules of action and operational invariants)*

The operational invariants that justify the actions of Michel can be classified as follows. From a didactical point of view, the importance of understanding and writing ideas, as well as illustration; and the importance of examples and applications in familiarizing students with the field of discrete mathematics. From an epistemological point of view, the characteristics of proofs in discrete mathematics such as the accessibility of discrete objects and the simplicity of definitions. From an institutional point of view, the alignment of contents to ensure a smooth progressivity.

In developing the contents of his courses, Michel uses examples and "technical proofs". He believes that students need time to get used to reasoning in discrete mathematics. He tries to convey the basic ideas in the theory of graphs such as the definitions and characteristics of objects; a condition that guides his choice of contents is the compliance with the instructions of the major.

In teaching, Michel stresses the importance of proof and illustration, and engages his students in reading and writing proofs. He teaches discrete mathematics in a theoretical way without real life applications; there is a lack of experimentation in his courses. He recognizes the importance of computers for research (for expanding the empirical space), but he does not use them for teaching in his classes.

The interview with Michel was set up to test the guidelines. We have identified areas for improvement and adapted the guideline before the second interview.

ANALYSIS OF THE SECOND PILOT INTERVIEW

The second interview was with Bertrand. Bertrand (France) is a researcher in graph theory and a research director at the National Center of Scientific Research. He teaches by choice: an optional course of combinatory games and mathematical reasoning to students in first and second year and a course of graph theory for master's degree students.

We are particularly interested in his teaching practices at university. Therefore, we will identify the different classes of situations related to his teaching practices. We present in Table 2, an extract from the table of documents of Bertrand.

Aims	Resources	Rules of actions	Operational invariants
Develop a course of combinatory games and	Research problems such as hunting the	He selects problems arising from research situations.	It is important that students experience research problems. Problems in discrete mathematics are easily

mathematical reasoning for students in first and second year of university	beast and problems issued from situations proposed in workshops.	He makes choices that can put emphasis in his teaching on proofs.	accessible. Discrete mathematics makes it possible to teach mathematical knowledge and knowledge related to the proof.
Implement a discrete mathematics course in class	Research problems such as hunting the beast and problems issued from situations proposed in workshops.	He focuses more on proofs than on mathematical objects. He puts the students in the position of researchers. He uses material objects for experimentation. He does not use computers in teaching.	Doing mathematics means working on proofs. It is important that students experience research problems in mathematics. In discrete mathematics, it is important to experiment with material objects. A computer can be interesting for research purposes, but does not contribute in the teaching of discrete mathematics.

***Table 2** - Extract of Bertrand's documents table from his teaching practices (aims, resources, rules of action and operational invariants)*

The operational invariants that justify Bertrand's actions can be classified as follows. From a didactical point of view, the role of discrete mathematics in developing reasoning skills, the importance of experiencing research moments for the students. From an epistemological point of view, the importance of proofs, the characteristics of problems in discrete mathematics (fun and accessible). From an institutional point of view, the freedom of a researcher in the choice of contents of his courses.

To develop a course in discrete mathematics, Bertrand selects research problems and problems arising from situations experienced in workshops. The course of discrete mathematics to students in first and second year of university is optional and students are not all in scientific majors. Bertrand has some freedom in the choice of contents. He is convinced that the course should develop students' reasoning and knowledge about proofs, as well as knowledge related to discrete mathematics; these convictions guide his choice of contents. He focuses in his teaching on the activity of proof.

Bertrand's classroom management is based on putting students in situations of research. He does not provide the answers, he believes that they must experience research situations like real scientists. He engages students in experimentations with games such as hunting the beast. He thinks that computers are not useful in class.

RESULTS AND DISCUSSION

First results at the interview grid and difficulties encountered in the analysis

In the analyses, we realized the impact of the level of education (Bachelor or Master) on the interactions of university professors with the resources in their teaching practices and on the relationship between teaching practice and research activity.

We relied on the methodology developed by Gueudet (2017) to study the interactions of university professors with resources derived from their teaching practices. It seemed to us that the institutional context has a large impact on teaching practices in university (distinction between Bachelor/Master, progressivity in certain fields, training of non-scientists, etc.). Taking into account the institutional context implies new theoretical and methodological choices.

First results at the research questions

Our first research question is "How do university professors interact with the resources in / for the teaching of discrete mathematics?" There is a big difference between Michel and Bertrand. Michel's course of discrete mathematics in the second year of university aims to prepare the students for a Master's degree in graph theory; a constraint on the choice of contents is the compliance with the instructions of the major. On the other hand, Bertrand's course is optional, which allows him a larger degree of freedom in the selection of contents. Bertrand uses, in teaching, resources derived from his research activity, which may be due to his involvement in research.

Both professors insist on the importance of reasoning and proofs, therefore they select contents that contribute to develop students' ability to write proofs. Bertrand uses games and material objects for experimentation while Michel's approach focuses more on theories. Both professors state that the use of software cannot contribute to reach the teaching objectives in class, although it can be useful for research purposes.

The second research question is "How can we characterize the relationship between the teaching practices of discrete mathematics at the tertiary level and research activities in the same field?" Bertrand holds a position of research. He adopts a researcher's attitude in class; he engages students in research and adapts the plan of his course according to their results. On the other hand, Michel follows a stiffer plan in his teaching. Quoting Bertrand "A teacher has to have the knowledge and pass it on, a researcher does not know everything, seeking answers is part of the research".

The analysis of the pilot interviews shows that the choice to consider the interaction with the resources is revealing for the study of the relations between teaching practices and research activity. The operational invariants identified are based on experience in teaching as well as experience in research. It seems interesting to define, according to the interviews, a typology of operational invariants (for classes of teaching situations). We plan to complete the data collection with a questionnaire that will be disseminated in universities in Lebanon and France for an analysis of the institutional context of discrete mathematics education. The results of this questionnaire will give us a map that will allow us to better study the case.

[1] International Congress on Mathematical education

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Characteristics of participation - A mathematician and a mathematics educator collaborating on a developmental research project

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In this paper, a developmental research project involving offering mathematical modelling (MM) activities to university biology students is presented, and a particular aspect is studied, namely the project as a collaboration between mathematicians and mathematics educators. The aim of the paper is to investigate what characterizes their participation in the project, and how the characteristics of the project and its development might influence this participation. Interview data as well as observation data from the MM sessions are analysed, and findings show that the mathematics educator served as a broker influencing the practice of the mathematician. It is hoped that the findings of the study can be of use when planning future collaboration between mathematicians and mathematics educators.

Keywords: teaching and learning of mathematics in other fields, teachers' and students practices at university level, cross-disciplinary collaboration, mathematical modelling, communities of practice.

Recent developments in science as well as in higher education have led to a greater need for cross-disciplinary collaboration. For instance, the changing needs of a biological science more and more dependent on mathematical methods has led several authors to suggest a closer integration of mathematics and biology in the education of future biologists (e.g. Brewer & Smith, 2011). This was part of the motivation for a recent collaborative developmental research project between two Norwegian centres of excellence in higher education (The Centre for Research, Innovation and Coordination of Mathematics Teaching, MatRIC; and the Centre for Excellence in Biology Teaching, bioCEED) in which biology-related mathematical modelling (MM) activities were introduced to undergraduate biology students as a means to increase their appreciation for, and competence in, mathematics. In this paper, however, I will use the project as a case for studying a different kind of cross-disciplinary collaboration, namely between the mathematicians and mathematics educators developing and conducting the project.

MATHEMATICIANS AND MATHEMATICS EDUCATORS

There is a close relationship between mathematics and mathematics education – as remarked by Kilpatrick (1998, p. 36), "neither can exist without the other". However, collaboration between mathematicians and mathematics educators is still relatively rare, although it has received attention in the literature (e.g. Fried & Dreyfus, 2014), and its importance and relevance to both communities have been emphasized (e.g.

Nardi, 2008). Dörfler (2003) has pointed out several obstacles to collaboration. For instance, the communities of mathematics and mathematics education are often quite separated: mathematicians and mathematics educators work in different departments, teach different subjects and have different educational backgrounds. Furthermore, there are few arenas where they can meet professionally: they mostly publish in different journals and attend different conferences. Dörfler also highlights prejudices about the other field as a possible obstacle to collaboration, something also discussed by other researchers (e.g. Ralston, 2004). On a related note, the mathematician in Nardi's (2008) book states the need for mathematics educators to "be able to talk to mathematicians about mathematics", for there to be a basis for collaboration (p. 270). At the same time, "there is no such thing as *the* mathematician, *the* mathematics educator, or *the* mathematics teacher" (Thompson, 2014, p. 319). What is described above is only the general picture – there are numerous examples of functioning collaborative relationships. Most of these, however, take place on the individual rather than the organizational level. Indeed, the importance of individual relationships and trust is pointed out repeatedly in the literature. As Thompson (2014) puts it: "successful collaboration requires mutual trust and respect among collaborators in the context of a shared commitment to solving a problem" (p. 331).

UNIVERSITY MATHEMATICS TEACHING AS SOCIAL PRACTICE

The MM activities forming the basis of the project were developed and conducted by a MatRIC team consisting of a mathematician and three mathematics education researchers, and in order to study what characterized this collaboration I will adopt a Communities of Practice perspective (Wenger, 1998). A *community* is "a group of individuals identifiable by who they are in terms of how they relate to each other, their common activities and ways of thinking, and their beliefs and values" (Biza, Jaworski & Hemmi, 2014, p. 162), and a *community of practice (CoP)* is a community characterized by *mutual engagement*, *joint enterprise* and *shared repertoire* (Wenger, 1998, p. 73). Mutual engagement concerns, for instance, norms and social relationships within the community; joint enterprise refers to common understandings of the aims and ideals of the practice; and shared repertoire concerns what and how various resources are used in the practice (Biza et al., 2014, p. 163). Participation in a CoP can vary from the central participation of an experienced "old timer" to the more peripheral participation of a newcomer (ibid, p. 162). An individual's sense of belonging to a CoP involves *engagement* – active involvement in mutual negotiation of meaning; *imagination* – extrapolating from your own experiences to form an image of your own place within the CoP; and *alignment* – coordinating your activities to fit within the structures of the CoP and contribute to the enterprise (Wenger, 1998, p. 173-174).

The CoP perspective has been used successfully in research on university mathematics teaching (Biza et al., 2014). For example, Jaworski and Matthews (2011), in a study of university teachers' lecturing, found little indication of a joint enterprise of teaching. Also originating in research using elements of social practice theory to investigate university mathematics teaching is the Spectrum of Pedagogical Awareness (SPA) (Nardi, Jaworski, & Hegedus, 2005). The SPA provides a means of describing and reflecting upon university mathematics teachers' pedagogical thinking and practices, and consists of four levels ranging from *Naïve and Dismissive*, through *Intuitive and Questioning* and *Reflective and Analytic* to *Confident and Articulate* (ibid, p. 293). There are also several case studies using socio-cultural perspectives to investigate university mathematics teachers' teaching practices (e.g. Treffert-Thomas, 2015). However, there is little research on the type of situation investigated in the present paper. Here, a team of mathematicians and mathematics educators collaborate on developing and conducting teaching activities at the university level, not primarily aimed at introducing new mathematics but rather at getting students to apply the mathematics they already know in new contexts. Of relevance for its focus on the collaborative aspect is the study by Cooper and Zaslavsky (2017), investigating a mathematician/mathematics educator co-teacher partnership in a course on mathematical proof.

The present study seeks to answer the following questions: What characterizes the participation of the mathematician and mathematics educator in the project? How might the characteristics of the project and its development influence this participation?

THE PROJECT – METHODS, DATA AND PARTICIPANTS

As mentioned above, the project involves MM activities aimed at improving biology students' motivation for, interest in, and perceived relevance of mathematics in biological studies. The planning and teaching was carried out by a MatRIC team consisting of one mathematician and three mathematics educators. The mathematician, Yuriy Rogovchenko, is a professor of mathematics with extensive experience of MM both in teaching and in research. He is also coordinator of the MatRIC network on MM. Yuriy had the main responsibility for planning the mathematical content of the sessions and selecting tasks. The mathematics educators were: Simon Goodchild, professor of mathematics education and leader of MatRIC, with extensive experience of teaching mathematics in secondary school, but whose teaching experience at the university level mostly consists of courses in mathematics education; the author, a post-doctoral researcher in mathematics education with a background in mathematics, whose role in the project was mostly data collection and research support to Yuriy; and a doctoral student in mathematics education who was mostly acting as an observer and research assistant in this iteration, in preparation for the second iteration which would form a major part of his doctoral project. Since writing about their collaboration at the level of detail necessary for this paper would make anonymization practically impossible, professors Goodchild and Rogovchenko kindly agreed not only to participate in the research, but also to make their identities known. For simplicity and brevity, however, in what follows they will be referred to as ME and M, respectively. The project was conducted at a well-regarded Norwegian university where biology students take one compulsory mathematics course, taught in the first semester, designed not specifically for the biology undergraduate program but for students from about twenty different natural science programs. Hence, the course provides little opportunity for focusing on issues specific to biology.

So far, the project has been through two cycles of development; however, in this paper only the first iteration is considered. It consisted of a pilot, where the team met with 10 volunteer students for one three-hour session, a regular sequence of four three-hour sessions with a different group of 11 volunteer students concurrently with their compulsory mathematics course, and a follow-up meeting with the second group of students the following semester. All sessions were taught in English, but student group-work was conducted in Norwegian. The basic structure of the sessions was similar throughout, with an introductory lecture conducted by M introducing some ideas and tools of MM, followed by small-group work on various MM tasks set in a biological context, with whole-group follow-up, led by M but with some input from ME. Examples of tasks included estimating the density of a rabbit population based on the number of roadkill rabbits, and investigating the growth of a yeast culture in a petri dish. All sessions were video recorded. Data analysis is still ongoing, but initial results have been presented elsewhere (Viirman & Nardi, 2017).

The analysis presented in the present paper draws on the video-recorded data from the four regular sessions in the first cycle of the project. To provide further insight into the collaborative process involved in developing the project, after the conclusion of the second iteration of the project the author of the present paper conducted an audio-recorded, semi-structured group interview with the two principal members of the team, M and ME. It was decided not to include the fourth member of the team (the doctoral student) in the interview, since his involvement in the first iteration of the project was peripheral. A thematic content analysis was then conducted, of the interview data and the video data from the whole-class sessions, using the CoP framework as a tool for structuring themes, focusing on signs of mutual engagement, shared enterprise and joint repertoire, central and peripheral participation, and possible obstacles to a CoP developing. For instance, concerning shared enterprise, I looked at what M and ME said about their aims when developing the project, and then examined the video-recordings of the sessions looking at how these aims translated into what was emphasised in their teaching practice, comparing and contrasting the practices of M and ME. Some consideration of my own role in the project is appropriate. Although I was involved in the project from an early stage, and am well acquainted with its aims, my role has been that of the researcher. I did not participate in the planning of the content of the sessions, and my involvement in the sessions was for the most part restricted to data collection.

RESULTS

Looking at the video recordings from the sessions, certain patterns in the behavior of M and ME could be discerned, where M often had a very clear focus on the mathematics and the tasks, while ME assumed responsibility for the students' wellbeing and the nature of their learning. For instance, in the second session work on the first task had taken longer than expected. Still, M, enthusiastic about the mathematics and eager to get through all that he had planned for the session, begins introducing the next task:

M:	You will be solving an important medical problem. You are
ME:	Yuriy?
M:	Yes?
ME:	Are you aware of the time?
(…)	
ME:	I'm talking to our student friends. Are you ready to go on and have a look at this task, or are you saying "Hey, hang on a minute, it's five o'clock?"

Similar situations occurred on several occasions during the sessions, with M getting carried away by the mathematics, and ME intervening on the students' behalf. At the same time, this pattern was not entirely consistent. There were occasions where M showed a clear concern about the students' wellbeing and enjoyment, telling jokes and striving to make the students feel at ease. Similarly, there were occasions where ME got to present his solutions to some of the tasks, displaying an obvious enthusiasm for and a deep engagement with the mathematics. Still, the overall tendency was relatively clear.

Another more distinctive difference between M and ME concerned their engagement with student contributions to the solution of the MM tasks. Again, M had a strong focus on the mathematics, to the extent that he displayed signs of what Ralston (2004) calls the One Right Answer Syndrome, clearly having one particular solution in mind. This affected the way he lead discussions. For instance, when students came up with solutions or suggestions that fit with this expected solution, they were received as being the "correct" ones, as in these examples from sessions 1, 2 and 3, respectively:

M:	Any other assumptions you were using in your work?
S:	Yeah, that 97 dead rabbits were per 24 hours
M:	OK, that's a good one
S:	Because it said that they were easily recognizable
M:	Correct, you read exactly what was meant there
(…)	

M: I will show you how the reasoning goes

M: When we look at the answers, the second group gave the correct one

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M: This is the perfect one (…) the only thing we need to check is, how far is this from my computation

Particularly the last quote displays how M used his own solution as the yardstick against which student solutions were measured. On the other hand, when student contributions did not resemble what he had in mind, M often did not comment on them at all, or used a different terminology: “Well, that was a very constructive way to solve the problem.” This contrasts with how ME talked about the students’ work on the tasks, as in this example from session 3:

ME: We really, really, REALLY would like you to share with us solutions that you have worked on within groups on those tasks. Now, it doesn’t matter at all whether they are good or wonderful solutions, perfectly correct solutions, whatever perfectly correct means, whether they look like his [points at M] solutions or look like my solutions, which will be quite different – it really doesn’t matter. What matters is the engagement, and the thinking processes, and sharing with each other, and with us, the thinking processes.

In line with this, on the few occasions during the sessions where ME conducted discussions, he instead let the students lead. He interrupted students’ presentations less frequently, and was not so quick to evaluate their contributions. When presenting solutions to tasks he emphasized the process of doing mathematics over the result, presenting his mistakes and dead-end attempts, seeing these as opportunities for students’ learning.

Still, there were signs of M gaining an increased understanding of students’ needs and potential for contribution as the project progressed. Preparing the last session, he decided to involve the other team members more in the session, providing support to the groups in their work, stating: “We should have done this in the first session”. He also expressed this willingness to reflect on his own practice in the interview:

M: Feedback was essential, so it was like my teaching was shaped during those four sessions towards some goals – I was experimenting, I was changing – partly I was listening to what you [ME and the author] were saying, partly I was trying to do what I felt was appropriate (…) I had very useful advice which might have affected my way of teaching.

Interviewer: In what way?

M: I probably more often look at what I’m doing from the other side. From the students’ perspective.

At this point, ME commented that this was "very much the didactician's view", and that it showed "a readiness to reflect" upon his own teaching. This readiness to reflect and change was further emphasized by how M later in the interview talked about his plans for a new course he would be teaching, where he intended to incorporate more student-centred teaching methods.

In the interview, when discussing the collaboration M emphasized the particular qualities of himself and ME: "We are both weird people". When asked what this "weirdness" consisted in, he alluded to the stereotypical views mentioned by Dörfler (2003), saying that he has a (for mathematicians) uncommon interest in teaching, partly due to the influence of ME, while ME is (for a mathematics educator) uncommonly knowledgeable in mathematics. Furthermore, both M and ME strongly emphasized the value they place on their professional relationship – that two people from adjacent, but still different fields can work so productively together. M added that in no department where he had worked (and he has worked in numerous places, in many different countries) had educational matters been on the agenda. The collaboration was always on the individual level, never on the departmental, and as ME added, this holds true even at their current university, where mathematicians and mathematics educators work in the same department. At the same time, M expressed the view that the differences between mathematicians and mathematics educators had helped improve teamwork within the current project, but emphasized how this was dependent upon all participants being able to engage with the mathematics. Concerning other aspects of successful collaboration, ME highlighted the need for mutual motivation: "Maybe this interdepartmental work has worked so well because bioCEED wants something, and we [MatRIC] want something, and those two wants coincided." When asked what the mutual motivation was that enabled M and ME to work together, ME highlighted the will to see MatRIC succeed, and how this success was in part dependent on the collaboration between ME and M. M struggled with formulating an answer, but emphasized interest in and curiosity about mathematics.

The main obstacle to collaboration mentioned in the interview was lack of time for preparation:

> ME: In terms of preparation, one of the things that I (…) found a bit frustrating, was that the preparation was very often done between three and four o'clock in the morning on the day of the session, and therefore there was very little opportunity for discussion between the three of us [M, ME and the author] about what was going to happen in that session.

ME attributed this to differences in individual styles of working, but still pointed out that because of this and other time constraints, discussions about the mathematical and didactical aspects of the sessions rarely managed to go below the surface level. In response to this, M stated that his initial plan was to have the whole sequence of activities pre-planned, but that the feedback he got while conducting the sessions led to continuous refinement and change. Still, he agreed that providing the rest of the team with the tasks further in advance would have been preferable. Generally, both M and ME stated that the main difficulty with the project was the lack of time – everyone involved were so busy with other things that they found it difficult to engage with the project as deeply as should have been needed.

CONCLUSIONS AND DISCUSSION

In the findings from the actual teaching sessions, M was seen to a large extent emphasising the mathematics and ME showing more concern with the students, their wellbeing and learning experiences. These characteristics of the participation of M and ME in the sessions align well with observations made by Cooper and Zaslavsky (2017), where a similar pattern could be seen. At the same time, these characteristics were not uniform throughout the sessions, and the overall picture of their engagement with the content and the students is not quite as simplistic as that. Still, the differences are discernible and should not be downplayed.

Furthermore, one might ask whether a CoP developed around the project. From what M and ME expressed in the interview there are definite signs of *mutual engagement* – they both emphasize the close professional relationship, and there is a dedicated support between the members of the project in making it work. There is however, less evidence when it comes to the *joint enterprise* and *shared repertoire*. In particular, practical circumstances regarding the project – the late preparation, the lack of time for discussion and reflection – made the development of a shared repertoire difficult. The responsibility for developing activities and tasks fell mainly on M, and the ways of interacting with students differed quite substantially, as the findings above show. It is also difficult to find evidence of joint enterprise. On a surface level, this could of course be said to be the project itself, but looking at what M and ME say about the aims and motivations behind the project, the existence of a joint enterprise is less clear. For ME, as leader of MatRIC, it is about the success of the centre, whereas for M it has more to do with engagement in mathematics at a personal level. Hence, it is difficult to claim that a CoP has developed.

Rather, what happened was that ME, occupying a more central position within the mathematics education community, acted as a broker (Wenger, 1998, p. 105), introducing elements of mathematics education practice and thereby contributing to changing M’s practice. In this way, M displays the beginnings of a trajectory from a peripheral position towards a more central. In the interview, M clearly states how participation in the project has changed the way he views teaching – he talks about “seeing things through students’ eyes”, described by ME as “a didactician’s view”. These changes are perhaps less clearly visible in the video recordings from the sessions, but as mentioned above there are signs of an increased awareness of students’ needs as the project progressed. This could also be understood in terms of the Spectrum of Pedagogical Awareness. When M talked about reflecting about his own practice, and “seeing through students’ eyes”, this fits well with the *Reflective* *and Analytic* level of the spectrum, which is characterized in part by “awareness in starting to articulate pedagogical approaches and of reflection enabling making strategies explicit” (Nardi, Jaworski & Hegedus, 2005, p. 293). At the same time, much of what M did in the sessions would rather fit with the *Intuitive and Questioning* level of the spectrum, with a less explicitly articulated pedagogical thinking and a more intuitive recognition of students’ difficulties and needs (ibid, p. 293). Hence, a movement along the spectrum could be discerned as the project progressed, and there is some support for claiming that M’s pedagogical awareness has increased as a consequence of the project.

Regarding what factors contribute to functioning collaboration between mathematicians and mathematics educators, what has been reported here is only a single case, and one should be wary of drawing strong conclusions. Still, it is worthy of note how well findings from literature, about mathematicians demanding that mathematics educators know the mathematics (Nardi, 2008) and the need for individual relationships and trust (Thompson, 2014) resonate with what M and ME say in the interview. Both point out these factors as crucial to their collaboration, while lack of time for joint preparation and reflection is pointed out as the main obstacle. An awareness of the importance of these factors will be highly useful when planning and conducting further collaborative efforts involving mathematicians and mathematics educators. Indeed, providing institutional means and resources for preparing and developing projects of the kind discussed in this paper might be one way of moving parts of the responsibility for establishing collaboration from the individual to the institutional level, something which is necessary if such collaboration is to be sustainable.

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A case study of a university teacher of Calculus 1

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This case study of a university teacher of calculus1, based on data from a questionnaire, semi-structured interview, and observation, illustrates how the teacher used their knowledge of calculus teaching to sequence the building blocks of mathematical theories (BBMT) of the concepts of calculus.

Keywords: teachers' practices, university level, teaching calculus.

INTRODUCTION

Calculus is important at the university level because understanding calculus is an essential step in understanding how the world works. It is a foundation on which other skills can be built and is often a requirement or prerequisite for STEM programmes. Nevertheless, it is a challenging area regardless of which educational institution is offering the instruction (Petropoulou et al, 2016). As such, while there is much research on calculus learning and teaching, Rasmussen et al (2014) identify teacher knowledge as an under-developed topic. In this study our research questions are; how does a calculus teacher demonstrate their pedagogical content knowledge (PCK), to achieve their aims, to develop instructional strategies (ISs), and to assess students' understanding? Here we present elements of a single case study.

RESEARCH ON PCK OF UNIVERSITY TEACHERS

Numerous studies, from Shulman (1987) to Khakbaz (2015), have indicated that teachers require different types of knowledge in the classroom. PCK was proposed by Shulman (1987) as an essential component of teacher knowledge, defined as a "special amalgam of content and pedagogy that is … understanding" (p.8). Later work by, for example, Khakbaz (2015), confirms that it is necessary to go beyond the subject (e.g. Calculus) and examine how the teacher interprets the subject matter (e.g. the interpretation of calculus) and how this is linked to their role in facilitating learning in the classroom. Research on PCK of university teachers remains limited.

THEORICAL FRAMEWORK AND METHODOLOGY

For the purpose of this study, we adopted Lessing's (2016) PCK framework. Based on this, we take PCK as a combination of knowledge of purposes of teaching calculus, of the building blocks of mathematical theories (BBMT) and of ISs, while taking into account knowledge of learners' conceptions and difficulties.

A mixed methods multiple case study design was adopted for this research. Qualitative data was gathered by semi-structured interview and observation of eight taught sessions of the module from 15 April to 15 July 2017 to explore the ways in which calculus teachers implemented their PCK in the classroom setting. Quantitative data was gathered using a questionnaire (multiple choice) on their knowledge of ISs, learners' conceptions in teaching and learning calculus, and learning difficulties. The qualitative data was analysed by coding and categorising according to the theme in order to identify the ways that the participating teachers demonstrated their PCK during their calculus teaching.

THE CASE OF TEACHER JOHN

John is a mid-career mathematician with four years of experience of teaching calculus1 at a university. The calculus teaching strategies employed by John were a pattern of topic-specific ISs based on using the BBMT in teaching calculus. From our data, it was clear that John approached each calculus topic through the BBMT using axioms, definitions, theorems and proof. Therefore, the lecture structure chosen by John often aligned with the mathematical concept used in the topic. Here, John often started the lecture with a definition, then theorem, and then sometimes the proof and then examples illustrated with graphs. On one occasion, John asked the students to read a proof. In an interview, John explained that, in the lectures, students are sometimes asked to read a proof because John wants to understand the students' conceptions, and misconceptions, of proofs. On one occasion, in a part of the topic 'continuous functions on an interval', John gave some examples and asked students to give a definition. At other times, John used ISs such as diagnostic techniques, (through class discussion, etc.), reviewing previous lessons as a way to introduce subsequent lessons, and using various mathematical representations.

CONCLUSION

The analysis of John illustrates how knowledge of calculus teaching (i.e. calculus PCK) was used in sequencing the BBMT of the concepts of calculus. The next stage of the research is documenting other cases as part of the multiple case study.

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Who Becomes U.S. Mathematics Teachers: A Longitudinal Analysis of First-time Praxis® II Mathematics Content Knowledge Examinees

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The U.S. uses the Praxis® tests to measure academic skills and the subject-specific content knowledge needed to be a teacher. This study examined Praxis® Mathematics Content Knowledge scores from 2006 to 2016 to describe patterns in who are becoming certified to teach mathematics in the country (n=89,693). Longitudinal analyses were used to discern patterns in the demographics of examinees and trends in exam performance across several demographic characteristics. The results reveal substantial differences in performance and pass rates between examinees of different genders, races, undergraduate majors, undergraduate GPAs, and census regions. From our analyses, we suggest several measures for the improvement of mathematics teacher preparation.

Keywords: assessment practices, longitudinal analysis, mathematical content knowledge, teacher preparation.

DESCRIPTION OF POSTER CONTENT

This poster will provide information on the U.S. *Praxis®* (ETS, 2017) tests to an international audience. A small section will describe the theoretical perspective (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008; Shulman, 1986) and how it was used in this study. Our research questions will be included on the poster, as well as how data were collected and analyzed to answer these questions. The middle portion of the poster will include several graphics depicting the model resulting from the analysis, specifically the statistically significant factors in whether test takers pass the exam and the demographic characteristics of those who pass. These figures will illustrate the substantial differences in performance and pass rates between examinees of different genders, races, undergraduate majors, undergraduate GPAs, and census regions.

We will offer several measures for the improvement of the mathematics teaching workforce and establish potential leaks in the teacher pipeline that may impact the quality and diversity of U.S. mathematics teachers (Gitomer, Brown, & Bonett, 2011). Specifically mathematics education faculty would do well to recognize that students who have sat in their classrooms and lecture halls are represented in this sample and that these scores are at least partially reflective of what they have retained from their courses (Kleickmann, 2013). The adoption of instructional strategies that promote students' long-term conceptual understanding over those that validate rote memorization may be a critical choice on the part of mathematics faculty in improving the preparedness of future mathematics educators.

NOTES

1. In full disclosure, the U.S. Educational Testing Service reviewed this paper, though the analyses were conducted by our research team.

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Cross-linking maths – using keynotes to structure a curriculum for future teachers

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The paper gives a frame for a concept for university maths teacher education, that is based on three keynotes: central scientific notions, history and language. Amongst other benefits, the keynotes serve as cross-links between the different courses the students go through in their studies.

Curricular and institutional issues concerning the teaching of mathematics at university level, Teaching and learning of specific topics in university mathematics, teacher education, history of mathematics, language education.

INITIAL CONDITIONS AND REQUIREMENTS

Subject-specific content knowledge is an essential component of professional teacher knowledge (for the concept of professional knowledge see for example Schwarz (2013)). It forms a basis for the ability to judge a specific topic's significance for an entire subject and thus forms an important requisite for didactical reflections. Considering this, mathematics teachers' content knowledge must comprise not only advanced insight into singular topics but also a netlike overview of mathematics as one, consisting of notions and relations between them as well as methods and basic principles. To achieve this, a university curriculum for future teachers demands cross-links between the single courses. Basic notions (such as sets, functions, algorithms…) that are found in diverse mathematical subdomains seem suitable for serving as cross-links between scientific lectures, both, in a horizontal and a vertical way. This would transfer Bruner's (1969) concept of fundamental ideas and a spiral curriculum onto teacher education.

Basic ideas and fundamental notions can be found when one looks into the history of a subject as they should appear throughout time (Schweiger, 1982), though maybe in different disguises. Apart from that aspect, including the history of mathematics into teacher education has been suggested long-since, presumably bearing a whole lot of further benefits (Schubring, 2000; Katz, 2000; Jankvist et. al., 2016), for example experiencing mathematics as a process conducted by human beings rather than as a mere product.

Generation, clarification and precision of mathematical concepts and notions require the use of language (e.g. Morgan et al., 2014). Without language, definitions and propositions could not be worded and not be taught either, so language forms another essential part of content knowledge, concerning scientific as well as educational knowledge. Due to increasing heterogeneity in pupils' language abilities, it is necessary that future teachers learn to illustrate subject matters on differing language levels and that they become able to vary by those levels.

THE HILDESHEIM CONCEPT OF LEARNING ALONG KEYNOTES

We believe that integrating the three keynotes – basic notions, history and language of mathematics – into a teachers' curriculum and thus cross-linking the courses supports the construction of a professional teacher content knowledge as required. Therefore, they should become a constitutional part of all scientific lectures throughout studies. Our poster presents a yet draft concept of how the three keynotes might constitute a scaffolding for a spiral curriculum, where historical and language elements serve as enrichment as well as embedding of basic contentual and methodical concepts. The concept seems convenient for linking subdomains as well as different levels of challenge throughout a curriculum that is adapted to the specific needs of future teachers. A research frame for evaluating efficacy of the concept has not yet been developed.

Examples of implementation by emphasis on basic ideas in the lectures, integration of historical tasks in practices as well as inclusion of training of language skills (for more details on this see the poster presented by Schmidt-Thieme at INDRUM on the project SuM_MaSt) will be presented on the poster.

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BiMathTutor – The program for further training of new tutors in University Mathematics

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In the Faculty of Mathematics at Bielefeld University, new tutors have to attend the further training BiMathTutor, which took place for the first time in the winter term 2017. At this program, they learn about the essentials of teaching. This program is being researched to the effects on tutors and their students. The poster presents the theoretical framework, the concept of BiMathTutor, the research design and first results.

Keywords:1. Teacher's and student's practices at university level, 7. Preparation and training of university teachers, 4. Novel approaches to teaching.

INTRODUCTION

In this paper, we focus on the development of a further training for tutors. The main aim of a further training for tutors is to improve teaching skills and students' learning. In most cases, the tutors are experienced students who are interested in passing down their knowledge and, of course, earning some money to support themselves. These teaching skills should be based on a qualification program to establish and maintain standards of teaching and learning.

In the Faculty of Mathematics at Bielefeld University, starting in October 2017, new tutors have to attend the program for further training called "BiMathTutor" (Bielefeld Mathematics Tutor Program). This program has been reformed over the past year and evaluation is happening under different aspects, namely, the change of attitude towards teaching, the expectations of the tutors of what is important for the tutorials and the effects of "BiMathTutor" on the exam results of the students attending the tutorials (see figure 1).

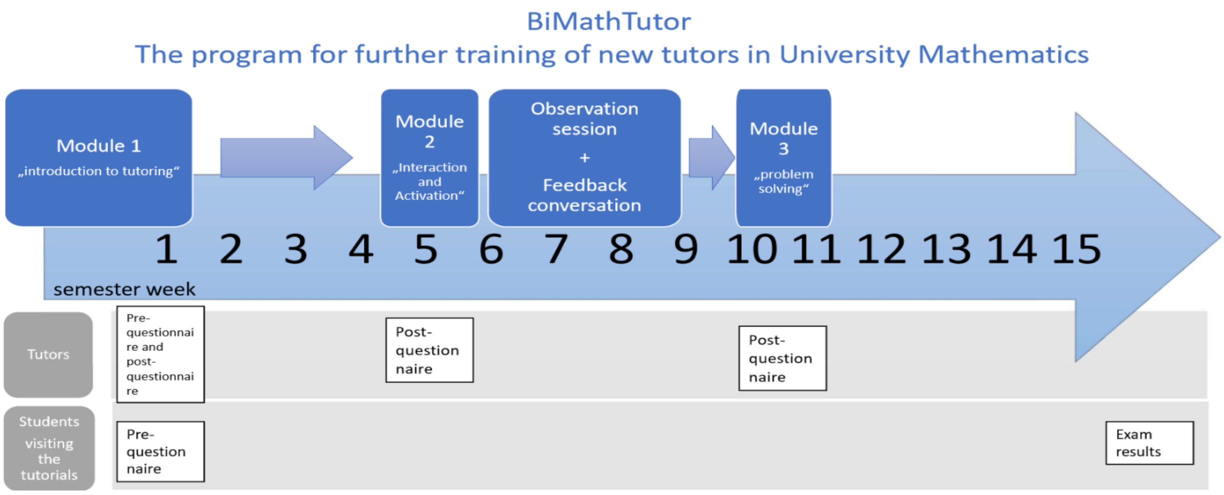

Figure 1: Time table of BiMathTutor

RESEARCH QUESTIONS

The object of research are the effects of is the tutor program. Hence, the research questions for the survey are the following:

1. How do new tutors assess the further training?
2. What expectations do students visiting the tutoring class have?
3. What is the grade of effectiveness of the further training on the students' exam performance?

The first hypothesis is that new tutors will know the typical process and the organisational aspects of a tutoring class, will be able to write and apply a horizon of expectation, will execute different methods to engage students in a learning process, will establish characteristics of a good tutoring class, will be aware of students' difficulties dealing with university mathematics and will investigate their own strengths and weaknesses. On the second research question, one will expect that students have the goal to pass the exam. They will not be aware of the learning process involved and the possible simplification by a tutor, who attended BiMathTutor. The third hypothesis is that the further training has an effect on the students' exam result.

THEORETICAL FRAMEWORK

The theoretical framework is still in process of elaboration. The design of BiMathTutor was made by Dee Fink's theory "Designing courses for significant learning" (2003). The key components "Learning goals", "Teaching and Learning activities" "Feedback and Assessment" have to be in unison. Additionally, the situational factors must be considered.

DATA COLLECTION

The flowsheet (see figure 1) shows the different elements of the intended surveys. Module 1-3 and the observation session plus feedback conversation is stated on the top. The semester week shows a typical semester at Bielefeld University. There are two general cohorts, one being the tutors and the other being students visiting the tutorials. We have several times of measurement indicated by black bordered boxes. The study uses a combination of quantitative and qualitative methods. The sample size of the tutor survey is N1=24. The sample size of the student survey is N2= 597. The test instruments are in form of questionnaires and exam results.

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Getting language awareness: A curriculum for language and language teaching for pre-service studies for teachers of mathematics

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As language is an important medium as well as an aim of mathematics lessons at school, teachers of mathematics have to achieve the required competencies in language and language teaching. Therefore, we designed a curriculum, which offers students opportunities to get these competencies throughout their academic studies. Some examples and first results will be shown.

Keywords: Teaching and learning of specific topics, Curricular issues, Language awareness, Special language; Explaining.

LANGUAGE AND LANGUAGE TEACHING COMPETENCIES AS PART OF TEACHER PROFESSIONAL KNOWLEDGE

Teaching and learning mathematics is based on the use of language: for introducing and defining new mathematical objects, discussing different ways of calculating, documenting the results of a proof or explaining how to handle with teaching materials, different representations are used, but almost always accompanied by language. Language as a medium and an aim of mathematics lessons at school is a well-known object of didactic research and laid down in several publications (see Morgan, Craig, Schuette, & Wagner, 2014). But in the last years one can observe a focus being laid on differentiations and transitions between different languages or registers of one language used in mathematics (Duval 2006). On the other side, the professional knowledge of teachers, and questions on how this can be developed in pre-service and in-service lessons gained considerable interest (Gniffka & Roelcke, 2016; Kunter et al., 2013). This leads to following questions: (1) Which language-related competencies a teacher has to have? How do they interact with other parts of his professional knowledge? (2) When, where and how can he achieve these competencies? For this in the last years several universities in Germany designed and developed exemplary trails in pre-service teacher-studies. In most of the case these are extra, but obligatory courses on “German as second language”. Another approach is followed for example by a project called “Umbrüche gestalten” (http://www.sprachen-bilden-niedersachsen.de/-index.php/projekt.html).

A “LANGUAGE CURRICULUM” (SUM_MAST)

Following the approach of this project we developed a pre-service “language curriculum” for teachers of mathematics (based on experiences out of the implementation of language studies in teacher studies above mentioned) following four requirements for the learning opportunities (used to guide the curriculum): they should

(a) be spread from the beginning to the end of the academic studies, like a vertical spiral curriculum; (b) be integrated – besides explicit language related courses - in mathematical lessons, for the technical language of mathematics is best learned by doing mathematics; (c) evoke an active and reflective handling with language in learning situations and be applied and tested in authentic situations; (d) include individual feedback and allow some comparative measurements. (And in addition: It should be transferable to other designs of academic studies for teachers at other universities.) Our suggestions is, that a curriculum following these leads to a connection of linguistic and mathematical knowledge as a theoretical base for teaching practise.

After some pre-studies the design based research project SuM_MaSt started fully 2016 at the University of Hildesheim. Besides schemas of linguistic requirements of mathematics lessons (in multilingual classrooms) and the language-related competencies of teachers of mathematics (as a part of PCK), the poster will give an overview over the types of “linguistic tasks” (theory/practice, real/simulated, ...) and the curriculum implemented at the University of Hildesheim. Four examples will clarify the tasks, the type of language competence, which can be achieved, and show some first results of the evaluation of the above mentioned suggestion :

(a) linguistic tasks in mathematical lectures (examples: geometry, arithmetic, number theory; evaluation (qual.): correctness of the mathematical content and adequacy of language),

(b) input and tasks on “Explaining” in the lecture “arithmetic” (evaluation (qual.): describing the development of linguistic competencies),

(c) contents and tasks of an explicit course “Language and Mathematics” (evaluation (qual.): analysis of the products during the lessons),

(d) questionnaire for reflecting and self-assessment of the competence (evaluation (quant.): questionnaire).

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A university lecturer's role and interactions in a flipped classroom

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This poster explores the role of the lecturer in a university level mathematics class utilizing the flipped classroom approach to teaching. The role of the lecturer is investigated using the Systematic Classroom Analysis Notation (SCAN) tool and the theoretical framework of instrumental orchestration.

Keywords: Novel approaches to teaching, lecturers' practices at university level, flipped classrooms, instrumental orchestration, analysis of lecturers' actions

NEW RULES OF INTERACTION IN FLIPPED CLASSROOMS

University lecturers traditionally presents new mathematical topics in class, while students work on related tasks at home or in colloquiums arranged by the university. The Flipped Classroom approach (FC) flips this around, having the students watch pre-recorded videos at home, while working in a more collaborative and student-centred manner while in class. This flipping of the classroom changes the rules of the classroom drastically, as the lecturer now has to prepare videos for the students' out-of-class to "prime" them for an in-class active phase of learning (Fredriksen, Hadjerrouit, Monaghan & Rensaa, 2017). To explore the new role of the lecturer in a FC, I will look at a lecturer's role using the framework of *instrumental orchestration*, and analysing the in-class interactions of the lecturer using a modified version of the Systematic Classroom Analysis Notation (SCAN) (Beeby, Burkhardt & Fraser, 1979).

THEORETICAL FRAMEWORK

To conceptualize the lecturers' new role as a result of flipping the classroom I use the framework of *instrumental orchestration* (Trouche, 2004). The term is defined as the lecturer's intentional and systematic organization and use of the various available artifacts in a learning situation – in this case both out-of-class and in-class – in order to facilitate students' *instrumental genesis* – the process during which an artifact or object is transformed into the psychological construct of an *instrument*, the combination of artifacts and schemes to be used for different specific types of task.

Within instrumental orchestration it is possible to distinguish between three elements: a didactic configuration, an exploitation mode and a didactical performance (Drijvers, 2010). The didactic configuration is the arrangement of the teaching setting and the artifacts involved in it. The exploitation mode is the manner in which the lecturer uses a given didactic configuration for the benefit of his/her didactic intentions. The didactic performance takes into consideration the decisions the lecturer must make on the fly while performing in the given didactic configuration with its didactic intentions. Using these three elements of orchestration, it should possible to explore the lecturers' new role as a result of FC.

RESEARCH QUESTIONS

The research questions address both the in-class and out-of-class aspects of the lecturer's new role in a FC.

1. How does the lecturer in a FC interact with the students in the lesson?

2. How does the lecturer perceive his new role, and how does this compare to his observed interactions while in-class?

METHODOLOGY

To look into these questions, I have observed and interviewed a lecturer utilizing the FC approach in a mathematics course for engineers. During the course of a week, four 90 minute in-class sessions were videotaped. These sessions were part of a module on mathematical series. Short videos, out-of-class preparation for problem-solving activities in-class, facilitated the learning of Taylor series and its applications and tests of series convergence. The study employs an interpretative research paradigm with qualitative research methods. The videotapes of sessions will be analyzed utilizing SCAN (Beeby et al., 1979) to code the teacher's interactions with students. SCAN is based on "time-slicing", and works on three different time-scales – "event", "episode" and "activity". A prominent feature is a set of qualifiers for each activity, which evaluates the depth of demand and level of guidance in lecturer interactions. The interview will be analyzed using open coding.

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Programming in mathematics teacher education – a collaborative teaching approach

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Keywords: novel approaches to teaching, teaching and learning of mathematics in other fields, team teaching, algorithmic thinking, programming.

The digitalization of modern society is having a major impact on schools and schooling, and in recent years, programming and algorithmic thinking is increasingly seen as topics to be included in curricula at all levels. In Sweden, for instance, the National Agency for Education have instigated a revision of the national curricula to emphasize what they denote as “digital competence” (Skolverket, 2017a, b). In particular, the mathematics curriculum now includes the use of algorithmic thinking and programming as tools for problem solving. Since these topics have not been a mandatory part of the education of mathematics teachers in Sweden, this creates a need for such courses aimed both at practicing and prospective teachers. Given the direction taken in the revised curriculum, these courses will need to take an integrated approach, focusing on how programming and algorithmic thinking can be used to learn mathematics and to solve mathematical problems. This places particular demands on the instructors teaching these courses. Not only do they need to have knowledge of programming, but also of mathematics and mathematics education. Most probably, courses in programming aimed at practicing mathematics teachers will attract teachers from all levels of the educational system, placing additional demands on the mathematical competence of the instructors. In the light of these challenges, we are considering the idea of collaborative, or team, teaching. Having mathematicians and/or mathematics educators collaborating with computer scientists can provide different viewpoints on the topics considered and problems posed in the course, thus serving to more closely connect the programming to the mathematics. Moreover, having two teachers in the classroom naturally leads to discussions, showing by example the type of questions experts pose when engaging with a topic, and hopefully fostering a classroom climate where students also engage in discussion. Before outlining the planned project, we sketch what team teaching is about, and exemplify how it has been used in university mathematics education.

Various forms of collaborative teaching are already used in schools, where teachers collaborate on course preparation, implementation, and assessment. There are several models for collaborative teaching (Friend, Cook, Hurley-Chamberlain & Shamberger, 2010, p. 12), ranging from “one teacher, one assistant” models, via parallel and alternative teaching through to team teaching proper, with two teachers in the classroom together, taking shared responsibility for content. The research literature on team teaching of mathematics at the university level mostly consists of case studies in the context of teacher education. For instance, Clarke and Kinuthia

(2009) describe a project where two lecturers collaborated on the planning of courses on mathematical methods and instructional technology, emphasising the cross-disciplinary character of the courses. However, one teacher for each course did the actual teaching. What we aim to do is slightly different. A team of instructors with backgrounds in mathematics/mathematics education and computer science will co-teach a course on programming in a mathematical context, sharing responsibility for content, planning and assessment. For part of the course, two teachers will lecture jointly, presenting and discussing content together, providing different perspectives on programming in school mathematics. For instance, in the context of probability theory, the computer scientist can show how to program simulations of probabilistic experiments, with the mathematician pointing out how the underlying mathematical ideas influence the design of the simulation. The mathematician can then go deeper into the mathematical theory, with the computer scientist highlighting the algorithmic aspects. In this way, we aim to provide students with opportunities to develop a deeper understanding of how to integrate programming and computational methods in their mathematics classes. The programming course is still in its planning stages, but we are currently piloting the team teaching approach in the context of a first-year calculus course within the engineering programs.

From a research perspective, taking a discursive view on learning (Sfard, 2008) we wish to investigate to what extent the team teaching might support students' active participation in mathematical and computational discourse. We are also interested in how the discourse of the students develops through participation in such a course, particularly concerning the interplay of mathematical and programming discourse. Furthermore, we view team teaching as having great potential for teacher development, with participating teachers being able to learn from one another as well as jointly developing innovative teaching practices.

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TWG 6: Transition to and across university

Different goals for pre-university mathematical bridging courses – Comparative evaluations, instruments and selected results

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To ease difficulties in the transition from school to university, bridging courses are implemented at many German universities. In this paper, we present instruments we have developed for evaluating those bridging courses. We also show selected results from six bridging courses at five German universities, comparing their different goals and achievements.

Keywords: mathematical bridging courses, instruments for evaluation, transition to and across university mathematics, teachers' and students' practices at university level

INTRODUCTION

The transition from school to university is a big challenge for many students, especially in mathematics (Biehler, Hochmuth, Fischer, & Wassong, 2011; Gueudet, 2008). Several supportive measures such as pre-university bridging courses or mathematical support centres are implemented at German universities to ease students' difficulties in the transition phase (Hoppenbrock, Biehler, Hochmuth, & Rück, 2016). But we often do not know how effective these supportive measures are as detailed studies on the effects and success conditions are missing.

THE WIGEMATH-PROJECT

At this point, the ongoing WiGeMath project (Effects and success conditions of mathematics learning support in the introductory study phase), a joint project of the universities of Hannover and Paderborn (Liebendörfer et al., 2017) in collaboration with 14 universities, comes in. We distinguish four types of support: pre-university bridging courses, mathematics support centres, newly designed bridging lectures in the first semester, and support systems accompanying traditional lectures such as e-learning material or extra tutorials. The WiGeMath project's goals are developing a theoretical framework in order to be able to describe, analyse and compare support measures, investigating effects and success conditions and elaborating recommendations for effective designs for mathematical support measures in the introductory study phase. The theoretical framework for the examinations is the 3P model of Thumser-Dauth (2007). It describes a programme evaluation for higher education measures based on Chen's theory-driven evaluation approach (Chen, 1990). Based on transition literature from mathematics education, we refined this framework to make it content-specific. Interviews with our collaborating universities and document analysis were used to locate the specific measures in the framework (Liebendörfer et al., 2017). The reconstructed programme theories contain goals, procedures, circumstances and expected effects of the measures. Based on the theoretical framework, instruments were developed for evaluating the success of the measures.

Bridging courses in Germany

Most German universities provide bridging courses in mathematics for various kinds of beginning students shortly before the first semester. They differ in length, structure, amount of e-learning, content, audience, and goals. Some courses focus on the repetition of school mathematics while others aim at introducing students to university mathematics content and working methods (Bausch et al., 2014; Biehler & Hochmuth, 2017).

One main aim is to evaluate the success of bridging courses by assessing short term and medium term effects on attitudes and mathematical knowledge of the students. Therefore, we questioned the students at the beginning of the course, immediately after the course and after two months in the first semester.

Sample: Selected Bridging courses in the WiGeMath study

The following six bridging courses at five German universities are included in the analysis.

University	Online (O) or Attendance (A)	Duration in weeks	Aimed at…
A	O	5	Math., Comp. Sci., Engineering, teacher ed.
B	A	2	Engineering
C	A	2	Math., Physics, teacher ed.
D_A	A	4	Math., Comp. Sci., teacher ed.
D_O	O	4	All math programs (except econ. and physics)
E	A	2	Math., teacher ed.

Table 1: Overview over the investigated bridging courses

RESEARCH QUESTIONS

In this paper, we will focus on the post-test directly after the course. Apart from evaluating different instruments used in the post-test questionnaire the research questions are:

1. Which goals do lecturers of bridging courses set for their courses? How can the profiles of the courses be compared and located in the WiGEMath framework?
2. To which extend do students think they achieved explicit or implicit goals of their bridging course?
3. How much do the results of two different instruments measuring to which extend the students think they achieved different goals in the bridging course differ?
4. How do the (theoretical) profiles set up by the lecturer differ from the empirical profiles of the course?

METHODS AND INSTRUMENTS FOR EVALUATING BRIDGING COURSES

Instruments based on the WiGeMath Framework

The 13-pages questionnaire for the post-test contains about 205 Items – usually 6-level Likert-scale from "strongly agree" to "strongly disagree". The following tables illustrate the scales with exemplary items. Most of our scales had a reliability above 0.6 in the majority of cases at all locations.

Category of goals	**Scale name**	**Example item**
School math. knowledge and competencies	Identifying and overcoming deficiencies in school mathematics.	"I got to know my individual deficiencies in school mathematics."
	Recapitulating and elaborating school mathematics	"School mathematical topics were repeated."
University math. knowledge and competencies	University mathematics knowledge and competencies	"I learned new mathematical topics."
Mathematical terminology	Mathematical terminology	"I have learned new mathematical symbols"

Table 2.1 Knowledge goals

Category of goals	**Scale name**	**Example item (of 2 to 4 per scale)**
Mathematical modes of working	Process-related competences concerning math. texts	"I have learned how to read mathematical texts."
	Metaknowl. for math. modes of working	"I know how to recapitulate a mathe-

		matical lecture."
	Working autonomously on math. tasks	"I can work on mathematical tasks and topics on my own for some hours. "
University modes of working (*)	Organizing university routine	"I learned how to organise my daily routine at university on my own."
Learning strategies (*)	New ways for learning mathematics	"I learned about new ways to study mathematics."
Learning and working behaviour	Study groups (*)	"I learned to work in study groups."
	Knowledge about digital tools and how to use them	"I know the digital learning platforms used at my university."

Table 2.2 Behavioural (action-oriented) goals. (*) only one item

Category of goals	**Scale name**	**Example item (of 3 to 7 per scale)**
Beliefs	Metaknowl. and beliefs concerning higher maths.	"In the course, I recognised the role of proofs in higher mathematics."
Relevance of school maths. for future studies	Estimating how relevant school mathematics are for future studies and later profession	"In the course, I became aware that school mathematics provides a basis for my further studies."

Table 2.3 Attitudinal goals

Category of goals	**Scale name**	**Example item (of 2 to 6 per scale)**
Social contacts	Social contacts between students	"I met fellow students."
	Perceived social integration (Rakoczy, Buff, & Lipowsky, 2005)	"I think the other students of the course would help me, if necessary."
	Studying together with fellow students (Liebendörfer et al., 2014)	"If I have an idea for a solution, I will discuss it with other students."
Making uni-	Gaining insight into university	"I gained insight into higher

versity study demands transparent	learning/teaching methods regarding mathematics	mathematics learning and teaching methods at university.”
	Getting to know possible difficulties at the beginning of university and how to solve them	“I heard about possible difficulties at the beginning of my studies.”

Table 2.4 System-related goals

Additionally, we asked the participants about some affective characteristics (such as mathematical fear and self-regulation), these items are not used in the analysis for this paper.

Instruments adapted to the explicit goals of the course

The WiGeMath instruments are based on a comprehensive framework of potential goals of a bridging course. As a supplement we used a learning outcome oriented evaluation system, called BiLOE, proposed by Frank and Kaduk (2015). For the BiLOE, each lecturer is asked to specify her/his three to six major learning goals in his/her own words. Additionally, the lecturers had to specify up to seven study activities that should help the students to achieve these goals. Students are asked to evaluate these goals and activities. An important further element of the BiLOE is that the students have to state their personal goals for the course and are asked to which extend they think they achieved them. Those students who did not believe they achieved a learning goal were asked to give reasons for this at the end of the questionnaire. The BiLOE also requests the students to evaluate the relevance of the lecturer’s goals and how much certain activities helped them to achieve those goals.

SELECTED RESULTS

RQ 1: The different profiles of the investigated bridging courses

We categorized the major goals provided by the lecturers in the BiLOE from the perspective of the WiGeMath framework. The results can be found in table 3. Quite different profiles become visible.

Category	A	B	C	D_A	D_O	E
School math. knowledge and competencies	1	1	0	0	1	0
University math. knowledge and competen-	1	1	1	1	1	3
Mathematical terminology	1	0	0	0	0	1
Mathematical modes of working	0	0	0	0	0	2
University modes of working	0	0	1	0	1	0
Learning strategies	0	0	0	0	0	0
Learning and working behaviour	0	2	1	0	2	0
Social contacts	1	0	0	1	0	0

Making university demands transparent	0	0	0	2	0	0

Table 3: Number of learning objectives in the respective category mentioned by the lecturers

It is striking that no attitudinal goals were mentioned among the major goals, neither beliefs, nor affective features, nor mathematical enculturation. Likewise, none of the lecturers mentioned teaching learning strategies as a goal of their bridging course.

RQ 2: Goal achievements

The results of the WiGeMath and the BiLOE instruments provide valuable information for every single lecturer. The broader spectrum of the WiGeMath results will moreover provide information on the effects of the course from the perspective of its participants that the course lecturer may not have explicitly thought of in the selected major goals. This analysis provides empirical profiles and assesses the success of the various bridging courses.

In all cases, we calculated the percentage of students who *rather agree* up to *fully agree* (meaning greater than 3 in Likert scales with 4 steps or greater than 4 in Likert scales with 6 steps, respectively).

The following tables show the percentage of participants agreeing to the WiGeMath scales concerning the respective categories.

Category of goals / Scale	A	B	C	D_A	D_O	E
School math. knowledge and competencies						
Identifying and overcoming deficiencies in school math.	59	81	58	33	71	34
Recapitulating and elaborating school math.	86	89	69	34	87	35
University math. knowledge and competencies						
University mathematics knowledge and competencies	60	89	100	95	73	95
Mathematical terminology						
Math. terminology	63	86	99	97	74	96

Table 4: Results knowledge goals: Rounded percentage of participants agreeing to the WiGeMath scales (n=651)

Category of goals / Scale	A	B	C	D_A	D_O	E
Mathematical modes of working						
Process-related competences regarding math. texts	36	42	74	46	31	56
Meta knowledge for mathematical modes of working	34	72	70	66	46	65
Working autonomously on mathematical tasks	70	73	76	65	64	71

University modes of working						
Organizing university routine	45	58	42	41	46	49
Learning strategies						
New ways for learning mathematics	55	58	74	51	44	69
Learning and working behaviour						
Study groups	25	49	88	47	17	82
Knowledge about digital tools and how to use them	58	80	54	67	77	45

Table 5: Results action-related goals: Rounded percentage of participants agreeing to the WiGeMath scales (n=651)

Category of goals / Scale	A	B	C	D_A	D_O	E
Beliefs						
Meta knowledge and beliefs towards higher mathematics	50	70	94	90	59	90
Relevance for eventual profession and for subsequent studies						
Estimating how relevant school mathematic is for university and profession	63	74	63	54	71	44

Table 6: Results attitudinal goals: Rounded percentage of participants agreeing to the WiGeMath scales (n=651)

Category of goals / Scale	A	B	C	D_A	D_O	E
Social contacts						
Social contacts between students	34	84	98	76	64	86
Perceived social integration	46	82	93	88	74	89
Studying together with fellow students	36	71	81	66	57	88
Making university demands transparent						
Gaining insight in university learning/ teaching methods regarding mathematics	34	82	95	85	44	91
Getting to know possible difficulties at the beginning of university and how to solve them	31	67	81	42	62	63

Table 7: Results system-related goals: Rounded percentage of participants agreeing to the WiGeMath scales (n=651)

RQ 3: Differences between the two evaluation tools

To compare BiLOE and WiGeMath data, we first matched the learning goals given by the lecturers with the framework categories. The BiLOE results are mostly similar to the WiGeMath results. There are only six cases with differences of more than 15 percentage points. We reported back the interesting differences to the respective lecturers but these are relevant only for the individual and provide the general insight that the WiGeMath framework is sufficient for the evaluation.

RQ 4: Comparison of theoretical and empirical profiles

With these empirical results, a re-evaluation of the profiles based on the formulated learning goals of the respective bridging courses is possible. We will evaluate in which categories the percentage of agreeing participants are high or low and compare these results to the theoretical profiles. Here, "high" means an agreement to the WiGeMath scales of more than 80% of the participants and "low" is an agreement of less than 40%.

Course A. Based on the formulated learning goals, bridging course A has many goals. School mathematics, university mathematics, mathematical modes of operation, and social contacts are aimed at equally strongly. The empirical results differ: The only category with high agreement is school mathematics. There are some categories with low agreement, including social contacts, which was originally formulated as a goal by the lecturer.

Course B. Goals in various categories were stated as well. The empirical profile is similar but even broader: high agreement is reached in school mathematics, university mathematics, mathematical terminology, social contacts, and gaining insight in university learning and teaching methods. No goals concerning the last three categories were stated by the lecturer.

Course C formulated various goals. The empirical results show that there is high agreement in the categories university mathematics and study groups. There is also high agreement in the categories mathematical terminology, meta knowledge and beliefs towards high mathematics, social contacts and making university demands transparent.

Course D_A's empirical results also fit the theoretical classification very well. Additionally, high agreement is reached for mathematical terminology and meta knowledge and beliefs towards high mathematics. The only category with a low percentage is school mathematics, which was not an explicit learning goal, however.

Course D_O. The empirical profile of this bridging course differs significantly from the profile based on learning goals. The only category with a high percentage of agreement is school mathematics. Therefore, the focus of the course seems to be more on school mathematics than on university mathematics. Based on the learning goals, both could have been seen as equally strong.

Course E was the only one with a clear profile based on the formulated learning goals which was on university mathematics (including mathematical terminology). This is reflected in the empirical results, which however show a broader spectrum. Additionally, there is also a high percentage in the categories studying in study groups, meta knowledge and beliefs towards high mathematics, social contacts, and gaining insight in university learning and teaching methods. Low agreement was found concerning school mathematics, which was not a formulated learning goal.

SUMMARY AND DISCUSSION

The presented results are an intermediate step in communicating back to those who were responsible for the respective bridging courses with two goals. The immediate goal is to give feedback in order to improve and change the profile of the course – if desired. The second goal is to redesign our instruments so that the future instrument combines scales from the WiGeMath framework and more specific goals of the lecturers. The lecturers' goals given are quite diverse but all goals could be classified into the theoretical framework of the WiGeMath project. Some WiGeMath categories remained empty, however, e. g. learning strategies. We asked for the most important 5 goals, so it may be the case that our lecturers regarded them as minor ones. Additionally, some lecturers stated more specific goals, while other stated general ones. This may be due to the lack of experience with formulating learning goals as most of the lecturers do not work in the field of didactics. It seems necessary and valuable to extend the phase of specifying BiLOE goals by informing the lecturer in more depth about the WiGeMath framework as a supportive frame for specifying their own goals.

The students in the different courses differ when referring to their achievement of certain goals and categories. This is no surprise. For example, an online bridging course will not provide as much social contact to other students as an attendance based bridging course. It is important to mention that all answers are based on the students' self-assessment. The instruments developed (termed WiGeMath scales) and the BiLOE mostly yield similar results, sometimes the results differ. That can be explained by the BiLOE items being more specific or some of the learning goals only having a corresponding category but no perfectly fitting scale was found, as the questionnaire was already up to 13 pages long. The BiLOE is limited by the number of goals a lecturer can state, while the developed instruments of the WiGeMath projects allow providing a general survey of the bridging course. Additionally, the (theoretical) profiles set up by the lecturer differ from the empirical profiles of the course. For example, no lecturer mentioned an attitudinal goal. Nevertheless, the WiGeMath scales show that there is very high agreement in this category in relation to courses for students in mathematics and mathematics teacher education. In these courses the focus explicitly lies on university mathematics and not on the recapitulation of school mathematics.

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From single to multi-variable Calculus: a transition?

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We recently used the notion of praxeology from the Anthropological Theory of the Didactic to model the knowledge that is necessary for students to learn in order to succeed in an undergraduate multivariable Calculus course. We considered the presence and absence of elements of the knowledge to be taught, as proposed by curricular documents, in the knowledge to be learned, as indicated by final exams. Our results indicate that the mathematical activities expected of students at this level align with the activities observed in differential and integral Calculus, where exercise-driven assessments set students' work mainly in the recognition of types of tasks and recollection of appropriate techniques.

Keywords: transition to and across university mathematics, assessment practices in university mathematics education, teaching and learning of analysis and calculus, Anthropological Theory of the Didactic, praxeology.

INTRODUCTION

So far, research on the teaching and learning of Calculus has focused on *single-variable* Calculus. Cognitive and epistemological obstacles have been illustrated against students' learning of Calculus (Tall & Vinner, 1981; Sierpinska, 1994) and an institutional perspective has also been taken to study the influence of institutional practices on students' learning of Calculus (Barbé, Bosch, Espinoza, & Gascón, 2005; Hardy, 2009). There's a pattern that indicates Calculus students mostly engage in procedural work that requires only a superficial grasp of the underlying concepts (Hardy, 2009; Lithner, 2004; Selden, Selden, Hauk, & Mason, 1999).

We recently undertook a study that shifts the focus to *multivariable* Calculus courses (Brandes, 2017). Our goal was to determine the knowledge that is *essential* for students to learn in order to provide acceptable solutions on the final exam of an undergraduate multivariable Calculus course. To this end, we used the notion of *praxeology* from the Anthropological Theory of the Didactic (Chevallard, 2002) to model the knowledge students are expected to learn and the knowledge to be taught. We present our operationalization of this concept in the first part of this paper. In second stage, we discuss a partial result of our study that places this multivariable Calculus course along the transitions that university mathematics students undergo in their engagement with mathematics (Winsløw, Barquero, de Vleeschouwer, & Hardy, 2014).

THE EDUCATIONAL SYSTEM

We studied a 'Multivariable Calculus I' course offered to students in two mathematics programs at a large North-American university. One of the programs is for those who plan to join the workforce after graduation; the other aims to prepare students for graduate studies in mathematics. Students in either stream will have completed one-variable differential and integral Calculus and an introductory Linear Algebra course on matrix and vector algebra. The multivariable Calculus course and its sequel ('Multivariable Calculus II') are prerequisite to most of the courses in the program geared towards graduate studies. Students usually complete Multivariable Calculus I and II within the first year of their degree.

In any given term, the course is split into two sections per program, with about 70 students per section. The course is heavily coordinated across sections and terms through a strict curriculum, course examiner, and common assessments. The course outline specifies what to teach every week along with exercises from the textbook. The course examiner writes common assessments for students in all sections. A student's grade is obtained from the highest of the following: 10% assignments, 30% midterm, 60% final exam, or 10% assignments and 90% final exam. Finals exams are therefore the crux of a student's performance; in turn, the exams are consistent from term to term in both format and content. Past exams are readily available to students, and concern with their reactions prevents changes being made to the final exams.

ROUTINE PROBLEMS IN SINGLE-VARIABLE CALCULUS

We are interested in the mathematical activities with which students of a multivariable Calculus course are expected to engage. We focus on the types of problems that typify the learning of *multivariable* Calculus; a wealth of studies do so for *single-variable* Calculus (Hardy, 2009; Lithner, 2004; Selden et al., 1999). These studies emphasize the exercise-driven quality of the course assessments, in the sense of Selden et al.'s (1999) *routine* problems, which "mimic sample problems found in the text or lectures, except for minor changes in wording, notation, coefficients, constants, or functions" and "can be solved by well-practiced methods" (p.18).

The exercise-driven quality of the course assessments extends to elements of the curricula (Lithner, 2004). Calculus textbooks traditionally adhere to a definition-theorem-example-exercise format, wherein the exercises repeat the problematics of the examples and algorithms outlined in the text. Lithner (2004) measured the extent to which intrinsic mathematical properties play a role in the minimal reasoning required to solve routine tasks in traditional Calculus textbooks. Lithner's classification of reasoning types runs along a scale of how big a role is played by the mathematical properties intrinsic to the problem versus the reapplication of known algorithms; this scale runs parallel to Selden et al.'s (1999) spectrum of problems from *very routine* to *very non-routine*, which vary based on how familiar the solver is with the given problem. The more routine the problem, the less interaction is required of the solver with the mathematics specific to that problem.

Assessments in North-American Calculus courses are largely drawn from the course textbook, which Lithner (2004) showed to be steeped in routine problems. Accordingly, he found students' strategies to be anchored in what they recall superficially rather than in the mathematics specific to a problem. This correlates with Calculus students' failure to complete non-routine problems (Selden et al., 1999; Hardy, 2009). If textbook exercises can mostly be solved by identifying superficial similarities with a known example (Lithner, 2004), then students' non-reliance on intrinsic mathematical properties and over-reliance on the recall of algorithms may have roots in their learning environment. We follow this view by framing our study within the Anthropological Theory of the Didactic and focusing on elements of students' learning environment: curricular and assessment documents.

ANTHROPOLOGICAL THEORY OF THE DIDACTIC (ATD)

Framework

From the perspective of the ATD, knowledge does not exist in a vacuum, rather, it is bound to the institution in which it is shared and somehow connected to the knowledge shared in related institutions; such connection is called *transposition* and is of a didactic nature in the context of educational institutions (Chevallard, 1985). Didactic transpositions take place along a spectrum of knowledge in which *scholarly mathematics* (the knowledge developed, shared, and used by the experts – the mathematicians) is transposed into knowledge to be taught in a given institution, up to a transposition into knowledge actually learned by the students. This transformation of mathematical knowledge takes form in several stages: scholarly knowledge, knowledge to be taught, knowledge actually taught, knowledge to be learned, and knowledge actually learned.

An essential feature of the ATD is an epistemological model called *praxeology*. It allows the researcher to model knowledge at any stage of a didactic transposition. The notion of *praxeology* is based in the assumption that any human activity consists of a practical block (*praxis*) and a theoretical block (*logos*). The *praxis* is made up of tasks T to be accomplished and techniques τ with which to accomplish them; the *logos* is the discourse that produces, justifies, and explains the techniques in the practical block. Chevallard (1999) specifies two components of a theoretical block: technology θ, the discourse that produces and justifies the techniques in the practical block, and theory Θ that justifies the technology.

In light of these theoretical considerations, and given our goal of finding the minimal core of knowledge that students must learn in order to succeed in their multivariable Calculus course, we treated three instances of didactic transposition. We created a model of the knowledge to be learned, as determined by the final examinations; to this end, we needed a model of the knowledge to be taught, as indicated by the curricular documents. In order to familiarize ourselves with the mathematics prior to these two instances of didactic transposition, we also created a reference model based on the scholarly multivariable Calculus knowledge to be transposed. Before we present our praxeological models of the knowledge to be taught and to be learned, we review some of the literature about mathematics students' praxeologies.

Transitions in students' praxeologies

Winsløw et al. (2014) explain that students, at the pre-university level and in some cases at the university level, tend to have a praxeology defined mostly by practice. This is especially the case in differential and integral Calculus courses where assessment is concerned mostly with the practical block and does not address the ways in which the theoretical maintains the practical. This may have a precedent in the way knowledge is taught in the classroom, as teachers may not have time to justify tasks and techniques, given often-hefty curricula to deliver. Students, for their part, tacitly accept the existence of a theoretical discourse supporting the practical without concerning themselves with it (Hardy, 2009; Winsløw et al., 2014). Their work is mainly in recognizing types of tasks and identifying a suitable technique (Hardy, 2009; Winsløw et al., 2014), much as in Lithner's *identification of similarities* reasoning (2004) and Selden et al.'s *routine problems* (1999).

As students progress in university mathematics, they undergo two transitions. Where once they might have ignored theoretical blocks and worked exclusively within the practical block of a praxeology, they increasingly have to engage with theory and technology in their completion of tasks. Winsløw et al. (2014) call the transition from praxeologies that are purely practical to praxeologies that include a theoretical *and* a practical block a *first transition* of university mathematical praxeologies (p.101). For example, prior to the first transition, students complete tasks such as using derivative rules to find the derivative of a function. Here, differentiability is an always-met *condition* of the functions upon which students act in the tasks they do. Prior to the *first transition*, it is sufficient for students to attend only to the practical block of the mathematical knowledge; at the other end of this transition, students are required to acknowledge the theoretical block as the justification for the techniques they use for accomplishing a task. For instance, students may have to address the differentiability condition of a function before engaging in finding its derivative.

A *second transition* occurs when students reach courses whose curricula and assessment prioritize what once may have been the theoretical block of a praxeology; as students transition into proof-making and validating, theoretical blocks of the past become their practical blocks. For instance, the second transition will have occurred in a student who knows to use the definition or theorems about continuity to prove that, *if a function is continuous*, then some property of that function is true. The characteristics of a second transition are that students explicitly acknowledge and use the theoretical block to *generate* a technique for achieving a task.

KNOWLEDGE TO BE TAUGHT

The textbook of the multivariable Calculus course is typical of those used in North-American Calculus courses and follows the usual definition-theorem-example-exercise format. The course outline lists the textbook sections to be covered each week and a choice of end-of-section exercises. By *knowledge to be taught* (KT) we mean the mathematical knowledge in the sections and exercises listed on the outline. To model the KT, we identified the praxeologies of which it consists.

In the case of the knowledge to be delivered in this course, we found that technology and theory can be taken as one. There is no clear distinction between the two in the textbook; the discourse throughout is set in the geometry and algebra of three-dimensional space organized in the Cartesian system, and at times in Euclidean metric spaces. However, the theory is not made explicit and tends to be woven into the technology. Further, we found that the focus of the KT is mainly in the practical blocks. For the purpose of this study, then, it was sufficient to compile a list of items (definitions, theorems, etc.) that form the theoretical blocks of the praxeologies of KT without distinguishing theory from technology.

This tended to the theoretical block of the praxeology that modelled each section of the textbook on the course outline. To identify the tasks to be accomplished, we considered the examples and the end-of-section exercises listed in the outline. To describe the associated techniques, we consulted the examples and discussion portions (theorems, explanations) of the text. To account for the build-up of knowledge between sections (e.g. the notion of derivative of a vector function is defined in one section and reused in later sections), we cross-referenced across theoretical blocks and across and within practical blocks.

KNOWLEDGE TO BE LEARNED

In an operational sense, we define *knowledge to be learned* (KL) as the subset of the KT which students need to know in order to provide solutions on final exams. This operationalization was necessary from a methodological perspective: while the questions in the final exams indicate the tasks to be accomplished, in most cases there is no indication as to the expected technique or theoretical justifications. The model of KT was therefore necessary to identify these elements of a mathematical activity. In this sense, the main purpose of the model of KT was to model the KL. Our operationalization, although useful to describe and characterize the KL, does not properly reflect the fact that a transposition takes place and that some of the praxeological elements (likely, elements of the theoretical block) are more likely ill-defined than well-defined subsets of the praxeological elements of the KT. While the KL may borrow elements of the KT praxeologies, the discourse that unifies the two blocks of a praxeology might be distorted in the transposition.

Our model of KL is based on twelve final exams given recently within a span of three years. We described the solution to each exam question in terms of KT task-technique pairs that occur in the solution. Here is an instance of this work. Consider the following item from one of the exams:

> Find the tangent plane T that touches S at $(x, y) = (2,1)$, where the surface S is given by $z = f(x, y) = 1 - e^{-\left(\frac{1}{4}x^2 + y^2\right)}$.

We recorded this as "to find the tangent plane to a surface at a point." This task corresponds identically to task $T_{19.1}$ from the KT model; in turn, the technique for this task requires the completion of $T_{18.2}$: to find the value of the partial derivative of a function at a point. Thus, we associated to this task the KT sequence $[(T_{18.2}, \tau_{18.2}), (T_{19.1}, \tau_{19.1})]$. This particular exam task corresponded identically to a KT task; this was not always the case. Nevertheless, apart from a handful of cases, we were able to identify sequences of task-technique pairs that would form complete solutions to the exam questions; this methodological affordance may attest to the routine quality (Selden et al., 1999) of the tasks students are expected to accomplish.

Next, we grouped tasks of the same type so as to reflect praxeologies that occur in the KT. For example, the following tasks come up in solutions to exam questions:

> To find the first partial derivatives of a function $(T_{18.1}, \tau_{18.1})$
>
> To find the first partial derivatives of a two-variable function defined implicitly $[(T_{18.1}, \tau_{18.1}), (T_{18.1}, \tau_{18.1.1})]$
>
> To verify that a two-variable function satisfies a partial differential equation $[(T_{18.1}, \tau_{18.1}), (T_{18.5}, \tau_{18.5})$

This cluster of tasks is drawn from the praxeology of KT specific to partial derivatives. Altogether, we partitioned the model that captures KL about partial derivatives and surfaces into groups of tasks that match up with these praxeologies of KT: the above cluster specific to partial derivatives, along with tasks that draw from KT praxeologies specific to functions of several variables, the chain rule, tangent planes and linear approximations, directional derivatives and the gradient vector, extreme values, and Lagrange multipliers. Organizing the model of KL in parallel to the model of KT facilitated our analysis of the structure of the KL.

STRUCTURE OF THE KNOWLEDGE TO BE LEARNED

The KL has to do with partial derivatives and surfaces; space curves and vector functions; equations of lines and planes and distance in R^3; limits of rational functions; polar curves; and Taylor Series. Let's call 'ideal student' one who has the requisite knowledge to write acceptable solutions in a final exam. How might we characterize the praxeologies of an ideal student in this course? Below, we consider which parts of the KT praxeologies are to be learned and characterize them in the language of Lithner (2004) and Selden et al. (1999).

Knowledge from all KT praxeologies occurs as knowledge to be learned. Thus, the KL is not necessarily a subset of KT in the sense that some praxeologies are to be learned while others are not. Rather, we found that the KL is a subset in the sense of what's *left* of the KT praxeologies after the didactic transposition of KT into KL.

First, the practical blocks of the KT praxeologies are downsized in this transposition. For instance, consider the praxeology of KT about polar coordinates. The ideal student can convert polar equations into Cartesian equations and sketch the curve – *given the following curves* (up to a change in constants and functions sine or cosine):

$$\gamma_1:\; r = 2 + \sin\theta, \quad 0 \le \theta \le 2\pi; \quad \gamma_2:\; r = 3\sin\theta, \quad 0 \le \theta \le \pi$$

The algebraic manipulations specific to converting these types of polar equations into Cartesian equations are in examples from the textbook, as is the technique for sketching them. The ideal student's *topos* ('action space') (Chevallard, 2002) does not need to extend beyond the point praxeology (a praxeology of knowledge that is particular to a single type of task) specific to these functions. We found many of the practical blocks of KT praxeologies to be reduced in this way to point praxeologies.

Most of the praxeologies of KT are downsized in another sense: their theoretical block is removed following the transposition from KT to KL. For instance, consider the praxeologies that constitute the knowledge to be taught about partial derivatives. We found that the practical blocks are reduced to computational tasks where the ideal student needs to apply the appropriate differentiation algorithm; the geometric interpretation of partial derivatives as slopes is unneeded and the ideal student does not need to know any of the theory or technology at the backbone of the procedures. The ideal student does not need to know the limit-based definition of partial derivative nor the definitions and roles of limits, continuity, and differentiability in the concepts of derivative, gradient, and extrema of a function. The theoretical blocks of these praxeologies vanish in the transposition from KT to KL. In general, it seems that the ideal student needs to be fluent in the algorithms prescribed by praxeologies of the KT but doesn't need to justify or explain them.

The absence of theoretical blocks in the ideal student's praxeology is manifested in several ways: first, the student needn't justify the validity or choice of technique (e.g. by verifying or stating that the chain rule is applicable, since the functions in the exams are always differentiable); second, the exam questions do not require students to interpret any results (e.g. by making a sketch of a surface near a point where some geometric properties of the surface were computed); and finally, it suffices to have a superficial grasp of the concepts in the theoretical blocks in order to accomplish the types of tasks in the final exams. We expand on this point.

In general, the ideal student can recognize task types and identify the appropriate technique, in reasoning similar to Lithner's (2004) *identification of similarities* (IS), whereby a strategy for tackling a problem is chosen based on the similarities of certain surface properties between the new problem and a known problem (e.g. given a limit-finding problem, note whether the limit is taken at a numerical value or infinity and identify the type of function involved). For instance, the exam questions specific to limits of multivariable functions are as follows: find the limit of a function $f(x, y)$ at the origin, if it exists, or show that it does not exist. The function f given in the exams is either an odd rational function with no limit at the origin or it involves a trigonometric component which could be rid of to reduce f to a rational function in the process of an $\varepsilon - \delta$ argument (in these cases, the exam functions invariably have limit 0). This task occurs in examples in the textbook and exercises in these students' assignments. In general, tasks required by the exam questions were similar to those in the KT, so that students could rely on IS reasoning rather than on the underlying mathematics in order to choose the appropriate technique. This course is therefore in line with students' pre-university mathematics, where much of their responsibility is in recognizing types of tasks and choosing an appropriate known technique (Winsløw et al., 2004).

IS reasoning is characterized as requiring little reflection on the intrinsic mathematical properties of the problem at hand (Lithner, 2004). To successfully implement IS, the ideal student needs to recognize terms in the question statements (arc length, curvature, normal plane, binormal vector…) and the formulas for deriving them. But the ideal student is not tested on the meaning of these quantities and geometric properties as they relate to a curve at a point (e.g. a student might need to find the equation for an osculating plane, but does not need to explain what the osculating plane describes). The irrelevance of intrinsic mathematical properties to the tasks students need to achieve suggests that the theoretical block of KT praxeologies need not be present in the ideal student of either course.

Theoretical blocks are missing from the ideal student's *topos* in a few senses: the student is not required to justify or explain the techniques chosen to complete a task, and at times is even told which technique to use (e.g. via instructions to 'use Lagrange Multipliers' or 'use the chain rule'). The ideal student is not required to interpret the numerical or algebraic results of their calculus in any way; and it suffices to learn the components of the theoretical blocks only superficially. In all, this multivariable Calculus course seems to follow in the pre-university mathematics tradition whereby students need not link the practical and theoretical blocks of a praxeology (Winsløw et al., 2014). Further, the components of the practical blocks themselves are discrete, as the ideal student does not need to combine tasks in any way – for instance, the ideal student must know how to find invariant quantities of a curve, but needn't provide a local description of a curve based on its invariant quantities. This may be called the "*compartmentalization* of knowledge in calculus courses" described by Winsløw et al. (2014, p.104).

On the whole, it appears that only a surface version of the KT theoretical blocks is *essential* for the ideal student to learn: they need to know terms and associated formulas, in some cases have some intuitive image of certain concepts, and be fluent in the algorithms described by the technologies. This surface acquisition of the theoretical block serves to recognize routine tasks and identify a suitable technique.

In light of the absence of theoretical blocks in the minimal core of knowledge that is *essential* for students to learn in order to provide solutions to exam questions, we conclude that the KL cannot be described by actual praxeologies (made up of a *praxis* and a *logos*). Rather, the KL is an amalgamation of practical blocks. This places this university-level multivariable Calculus course in the stage prior to the *first transition* in university mathematics education previously discussed:

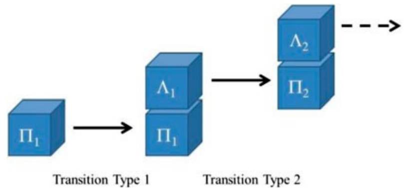

Figure 1. Transitions in university mathematics education (Winsløw et al., 2014, p.101)

where Π refers to the practical block of a praxeology and Λ to its theoretical block. Winsløw et al. (2014) explain that this *first transition* occurs when students no longer work strictly within the practical block of a praxeology and begin to incorporate a theoretical block, perhaps by using it to justify or produce a technique; a *second transition* occurs when students' past theoretical blocks turn into their current practical blocks, as when they start making and validating proofs in Analysis.

CONCLUSIONS

The aim of our study was to determine the minimal core of knowledge that is necessary for students to learn in a multivariable Calculus course in order to provide acceptable solutions on their final exam. We found that the exercise-driven quality of the course assessments makes it *essential* for students to recognize certain types of tasks and to identify the appropriate technique, but does not require students to learn the theoretical block that maintains these tasks and techniques.

Historically, the studied educational system introduced the multivariable Calculus course as a prerequisite to Analysis in an effort to help students adapt to university mathematics in the first year of their studies. It seems, however, that the mathematical activities expected of students in this bridge between pre-university and university courses are of the type expected in past Calculus courses: students' action space is fully within the practical block of the praxeologies that model the knowledge to be taught. As a result, students are no more required to engage with the theoretical in this Calculus course than they previously were. Meanwhile, the mathematical activities in Analysis courses are two steps ahead, after the *second transition* described by Winsløw et al. (2014), where students must work within what once were the theoretical blocks that backed the practical of Calculus. The question therefore remains: what course would make it *essential* for students to incorporate the theoretical blocks of a praxeology into the work they do in a practical block?

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A study of transitions in an undergraduate mathematics program

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In this paper, we introduce an in-progress study of the transitions students face as they advance in their mathematics courses. Previous work has discussed the changes that occur in the transition from high school to university. With regards to the knowledge students are expected to learn, however, significant similarities have been noted: to do well in introductory university courses, students can learn to solve a particular subset of tasks through routinized techniques, with limited awareness of the supporting mathematical theory. In contrast, students in advanced courses are required to work with and on that theory. The first stage of our project aims to better understand this transition by building praxeological models of the knowledge to be learned in a succession of two introductory analysis courses.

Keywords: Transition to and across university mathematics, teachers' and students' practices at university level, teaching and learning of analysis and calculus.

INTRODUCTION

Several studies have discussed the specific knowledge taught and learned in pre-calculus, calculus, and analysis courses, from different perspectives: for example, concept image and concept definition (e.g., O'Shea, 2016), APOS theory (e.g., Martínez-Planell, Trigueros Gaisman, & Mcgee, 2016), and the Anthropological Theory of the Didactic (ATD; e.g., Bergé, 2016). Our starting point is the general and relatively vague question of when in an undergraduate degree in mathematics does a student need (*need* in the sense of *to succeed in the course*) to engage in *mathematical activities* that may substantially, or meaningfully, lead to developing *mathematical practices*. We consider and frame this question within the ATD (Chevallard, 1999), which provides theoretical tools for modelling any human activity or practice. The semantic distinction between these two words is essential to us. Our hypothesis is that the kinds of didactic constructs to which professors and students are exposed are decisive in fostering the emergence of practices out of collections of local, particular, and relatively short-lived activities. From the theoretical stance we take, this means the development of mathematical knowledge out of local, particular, and relatively short-lived mathematical activities.

Previous research has found that the activities proposed to students in introductory calculus courses do not necessarily encourage the development of mathematical practices. Lithner's (2004) study of the exercises in undergraduate calculus textbooks used in Sweden led to the conclusion that the majority of tasks students encounter can be solved by mathematically superficial techniques such as finding and copying a similar solution outlined somewhere in the same section of the book. When working in Spanish high school calculus classes, Barbé, Bosch, Espinoza, and

Gascón (2005) observed teachers implementing mathematically incomplete practices: they solved numerous tasks in hopes of guiding students in developing solid mathematical techniques, but struggled to introduce any lasting rational discourse (i.e., theoretical block) that produced or explained the techniques. Hardy (2009), who conducted task-based interviews with students in North American college calculus courses, showed that in the absence of such a theoretical block, students construct non-mathematical reasoning to support the highly routinized practical block they develop (i.e., the techniques and corresponding types of tasks).

To describe the kinds of transitions students are expected to go through as they progress in their university mathematics coursework, Winsløw (2008) introduced the model depicted in Figure 1 below. The conjecture is that students encounter at least two types of transitions in the practices they are supposed to develop. The first requires them to gain some level of awareness of the theoretical block that was once absent from their exclusively practical work; the second occurs when elements of that theoretical block become part of the practical block with which they must engage autonomously. Think, for instance, of how some early university courses spend a significant amount of time in lectures elaborating previously scarce definitions, theorems, and proofs, which students may be expected to understand enough to quote in assignments or reproduce on exams. In contrast, more advanced coursework requires students to develop their own proofs, often involving the more abstract objects that were part of the theoretical block constructed in earlier courses.

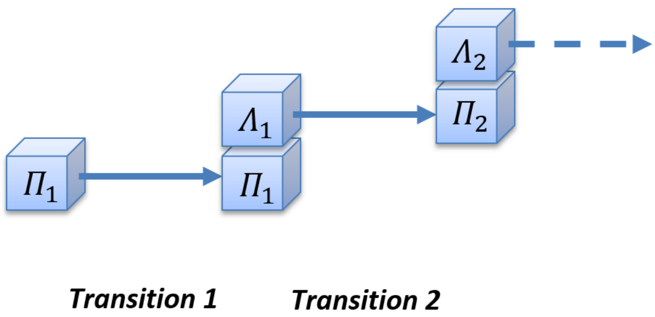

Figure 1: Transitions in university mathematics coursework (from Winsløw, 2008)

A recent study suggests that in the context of undergraduate multivariable calculus courses, students are not yet required to go through Transition 1: the models of the knowledge to be learned in these courses show that students are exposed to a limited practical block (Π), with no need to work with or on the corresponding theoretical block (Λ; Brandes, 2017). This said, students are indeed expected to work with and on mathematical theory when they take advanced courses later on.

A few questions arise from this:

1. What does this "work with and on a theoretical block" look like in comparison to the routinized, principally practical activity in which students seem to be engaging in introductory courses?

2. If the practices students develop in advanced mathematics courses can be modelled by the third stage shown in Figure 1, when, if ever, do students' practices reflect the second stage, and what are the mathematical activities proposed to them in such contexts?

The purpose of our study is to contribute to addressing these questions, and therefore, to the discussion of the transitions students face. To do so, we propose to model the knowledge at different stages in the didactic transposition process in two courses contained in what we will call the "analysis path" in a typical undergraduate mathematics program in North America (US and Canada). Ultimately, the goal is to reflect on the general question mentioned above: Can the activities in which students are obliged to engage lead to the development of mathematical practices (i.e., mathematical knowledge)?

THEORETICAL FRAMEWORK

"Activity and practice"

As mentioned above, we have come to see the semantic difference between *activity* and *practice* as pertinent to our work. The ATD's notion of *praxeology* provides a fundamental model for defining mathematical practice, which, in the context of the theory, is equated to mathematical *knowledge*. According to the model, any practice (or piece of knowledge) can be represented by a quadruplet [T, τ, θ, Θ] involving four interconnected components: a type of *tasks* T, which generates the practice, the corresponding collection of *techniques* τ developed to accomplish T, the discourse used to describe, justify, explain, and produce the techniques (i.e., their *technologies* θ), and the underlying *theories* Θ that serve as a foundation of the technological discourse. As students progress in their studies of mathematics, they engage in numerous activities, which progressively determine the practices they develop.

As a strictly hypothetical example, we could imagine students in an introductory calculus course being asked to engage in the following activities, inspired by a commonly used calculus textbook (Stewart, 2008):

a_1: Estimate the area under the graph of $f(x) = \frac{1}{x^2}$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate? What happens if you repeat the exercise with left endpoints? (Areas and Distances, Section 5.1)

a_2: Evaluate $\int_0^1 (3t-1)^{50} dt$. (The Substitution Rule, Section 5.5)

a_3: Determine if $\int_0^9 \frac{1}{\sqrt[3]{x-1}} dx$ is convergent or divergent. If it is convergent, evaluate it. (Improper Integrals, Section 7.8)

If these were the first activities completed by students in the corresponding sections, we could expect their actions to be localized and particular. In other words, the solutions students produce would likely be the result of their engagement in a relatively isolated act of figuring out how to solve the specific given problem. As the students participate in more activities, however, they may be exposed to tasks of the same type, and may consequently begin to develop a related practice. By the end of a calculus course, for example, students will have typically solved a large number of problems involving the calculation of definite integrals by way of various integration techniques. From this, they may have learned to recognize other activities (e.g., $\int_0^{\pi/6} \frac{\sin x}{\cos^2 x} dx$ or $\int_e^{e^4} \frac{dy}{y\sqrt{\ln y}} dy$) as forming a type of task with a_2, and therefore as requiring the same technique: making a substitution (not forgetting to change the bounds!), determining the anti-derivative of the new function, and calculating the difference of this anti-derivative evaluated at the bounds. In comparison, certain activities may be encountered by students only in insignificant (e.g., unevaluated), rare, and/or disconnected situations. The action of accomplishing those tasks may hence remain isolated and particular, never contributing to the development of practices. Activities like a_1 or a_3, for instance, might never be encountered beyond a few recommended exercises at isolated, unique moments in the course.

Research confirms that the collection of activities given (and not given) to students play a crucial role in determining the kinds of practices they develop (and do not develop). Although students may seem to be learning mathematical practices, they may in fact be engaging in isolated activities or developing practices of a non-mathematical nature. In her research, Hardy (2009) noticed that when first-year calculus students are given activities related to slightly non-routine tasks, they often apply techniques in a mathematically unjust way. For example, when asked to compute $\lim_{x \to 1} \frac{x-1}{x^2+x}$, 20 out of 28 students factored, seven of which did direct substitution first. Her analysis of students' discourse during task-based interviews led her to conclude that the students tended to justify their techniques through perceived norms. She specifies, for example, that "it seems that students were doing substitution not to find the limit or to characterize an indetermination, but because that is 'what you do first'" (p. 351). To explain her observations, Hardy (2009) discusses how the kinds of activities to which the students were exposed led them to develop such practices, composed of a limited practical block and non-mathematical technologies. The activities in which students participated did not only relate to sets of highly routinized tasks, they also required no form of mathematical justification. Engaging in such activities, students observed patterns that led to the construction of techniques based on arbitrary lists of steps that just seemed to work; at least enough to do well on assignments and exams.

In a similar sense, we could imagine a student in our hypothetical example justifying their solution to the activity "integrate $f(x) = 10x^2 + 3x^4 - 1$ over $[-1,1]$" by saying something like: "first, find the antiderivative, then find the difference between the value at 1 and the value at -1, because that's how we always do it!" An activity such as “integrate $f(x) = 1/x^2$ over $[-1,1]$”, might therefore elicit the following erroneous response:

$$\int_{-1}^{1} \frac{1}{x^2} dx = -\frac{1}{x}\Big]_{-1}^{1} = -\frac{1}{(1)} + \frac{1}{(-1)} = -2\,.$$

Unless of course the isolated activities, a_1 and a_3, were eventually, substantially, and meaningfully incorporated into developing the above non-mathematical practice into a practice more mathematical in nature.

“Undergraduate mathematics coursework”

As illustrated in the previous section, an anthropological perspective does not interpret students’ non-mathematical practices (or knowledge) as reflecting a common misconception inspired by difficulties inherent to a given mathematical concept. Rather, it sees such practices as resulting from a concrete situation within which the student finds themselves, under the influence of institutions (Douglas, 1986). In the ATD, the word “institution” is taken in a wide sense. For example, mathematicians work within an overarching institution that we could call Mathematical Research (MR), where their praxeologies are shaped by various shared criteria (concerning consistency, beauty, explanatory power, efficiency, etc.), but survive only if they follow the strict rules of mathematical reasoning. The students of interest to us, in contrast, are subjects of the institution Undergraduate Mathematics Coursework (UMC), which was in large part created to train potential participants of MR. This said, various conditions and constraints within UMC can require and enable a network of praxeologies that is fundamentally different from that built and recognized by MR. The non-mathematical praxeologies described in the previous section provide some examples.

To capture the transposition of knowledge as it moves from MR into UMC, Chevallard, and others (e.g., Bosch, Chevallard, & Gascón, 2005), have introduced a distinction between different types of knowledge (i.e. practice):

- Scholarly Knowledge, produced and used by mathematicians;
- Knowledge to be Taught, as determined by curricula, textbooks, and professors’ teaching plans;
- Knowledge Actually Taught, according to professors’ actual interactions with students, e.g., in lectures;
- Knowledge to be Learned, i.e., the knowledge students are expected to develop, which is often a transposed subset of the knowledge to be taught and actually taught, with the minimal core indicated by assessment tools;
- Knowledge Actually Learned, which can only be predicted, through analyses of student work, in-class observations of students, or other specially-designed interactions with students, such as interviews or problem-solving situations.

Although a lot can happen in university lectures, the minimal knowledge students are obliged to learn to pass their courses is determined by their assignments and exams. It is not surprising that the knowledge actually learned by students is often only a transposed subset of this minimal core. Hence, if we want to know what kind of knowledge students are or could possibly be developing in UMC, then we cannot restrict our exploration to curricula, textbooks, and teachers' lecturing practices: we need to pay careful attention to the way in which students are assessed.

Students' learned knowledge in UMC may also be characterized as a progression through various sub-institutions: from secondary school to early university courses (e.g., in single and multivariable calculus), through to more advanced university courses (e.g., in real analysis, metric spaces, measure theory, and functional analysis), which may eventually lead to graduate studies and beyond. Programs can vary from school to school and from country to country. However, a common phenomenon in secondary schools seems to be that assessments focus solely on the practical block of mathematical knowledge. The teacher may be expected to know the theoretical block for explaining the material to students; but the students are typically not obliged or even invited to develop an awareness of the technology or theory, let alone how it is linked to the practical block (Barbé et al., 2005; Winsløw, Barquero, De Vleeschouwer, & Hardy, 2014). One observed result is that many students interpret mathematical knowledge (practice) as equivalent to identifying a type of task and applying the corresponding technique (Bergqvist, Lithner, & Sumpter, 2008). Several studies confirm that this same kind of situation can arise in early university coursework (e.g., Lithner, 2003; Hardy, 2009; Brandes, 2017).

Indeed, over multiple years of coursework, students not only gain a particular view of what mathematical knowledge is, but they also develop knowledge that, when judged against the scholarly knowledge produced and used by mathematicians, is evidently non-mathematical – from the strategies they develop to identify tasks, to the discourses they use to justify these strategies and the techniques they choose. Nevertheless, as conjectured in the schema shown in Figure 1, a transition is expected to occur at some point: students are eventually required to develop knowledge that is completely and coherently mathematical. These circumstances lead Winsløw et al. (2014) to wonder about how teachers could help students accomplish such transitions. In parallel, we are inspired to validate, specify, and extend these researchers' claims by constructing praxeological models of how the different kinds of knowledge produced in the didactic transposition process progress throughout an entire undergraduate degree. In other words, we are inspired to investigate more closely the nature of the mathematical training being received by future mathematicians in the progression of their undergraduate coursework.

Of course, developing praxeological models to represent the knowledge (to be) taught and (to be) learned throughout an entire undergraduate degree is a hefty task. Within the context of our PhD project, we propose to accomplish a first stage, based on a subset of courses in one coursework path. Like in the Mathematical Research institution, Undergraduate Mathematics Coursework is divided into several sub-institutions according to domain – e.g., algebra, geometry and topology, analysis, statistics, mathematical physics, or probability – each of which contain a grouping of courses, which can be placed in some chronological order according to their prerequisites. Having already carried out research in the early courses of an "analysis path", this is the context that seemed most appropriate for our work.

METHODOLOGY

Although our project aims at modelling different levels of knowledge that can be identified in the didactic transposition process, in this paper we discuss only the modelling of *the knowledge to be learned* (KTL).

Our research is conducted at a large, urban, Canadian university. The mandatory courses in the analysis path of an Honours Bachelor of Science in Mathematics include multivariable calculus (MVC) I and II, and mathematical analysis (MA) I, II, and III. Since previous work (Brandes, 2017) suggests that the KTL in MVC I and II is similar in nature to the KTL in calculus and pre-calculus courses, we decided to start by focussing our attention on the two courses that come next and are likely candidates for housing the transitions of interest to us: MA I and II.

As mentioned above, the KTL represents the knowledge that students are expected to develop, which can be gleaned from the various activities in which they are invited to engage (lectures, assignments, and exams), as well as the materials that frame and support the activities (course outlines and textbooks). Since the minimal core of the KTL is represented in the assessment activities students must complete on their own, we have decided to ignore what happens in lectures and focus on the activities that comprise assignments and (practice) exams.

MA I and II are institutions in themselves in that they enjoy some sort of stability. For various reasons, course outlines and assigned textbooks tend to remain the same from year to year. The courses also maintain the same assessment structure: students complete assignments on a regular basis during the term, a midterm exam halfway through, and a final exam, with most of their mark (90% or more) concentrated in the examinations. This said, the actual activities proposed on assignments, midterms, and finals have less stability in that they can reflect personal choices of the professors assigned to teach the course in a given term. On top of this, our approach to modelling the knowledge actually learned will involve task-based interviews with students after they have passed MA I or II. Hence, we have collected the assignments and (practice) exams proposed only by the professor(s) who would be teaching those students. For instance, from the two MA I professors teaching in Fall 2017, we collected eleven weekly assignments, seven practice midterms, six practice final exams, and the actual examinations they gave to their students (these professors worked together in that they gave the same set of activities to their students).

To analyse such a collection of activities, we think about whether each activity is “isolated” or part of a “path to a practice”. An “isolated” activity may occur only once in the sense that no other activities engage students in accomplishing the same type of task. Since such activities are unlikely to contribute directly to the development of a practice, we reflect on why they are proposed. In comparison, the activities that belong to a “path to a practice” typically combine with other activities to expose students to a type of task. Our goal in studying these activities is to extract a theoretical model of the praxeologies that the *ideal* student (i.e., the student that receives a good passing grade) is expected to develop in the course. We start by constructing punctual praxeologies related to groups of non-isolated activities. Looking at the problem statements, we can establish the types of tasks (T) that generate the praxeologies. Determining the technologico-theoretical blocks ([τ, θ, Θ]), however, requires more data. We rely on the solutions a professor makes available to students to uncover the intended techniques, as well as portions of the expected theoretical blocks; and we complete the latter by checking the course outline and reading the relevant textbook chapters. The resulting collection of punctual praxeologies then becomes part of our data, which we use to construct more generalized praxeologies, think about how they are related to one another, and reflect on the nature of the KTL.

Eventually, we plan to put the models of the KTL for MA I and II together and compare our results with what previous researchers have found in calculus and pre-calculus courses. This, we hope, will allow us to discuss how the ideal student is expected to progress in the early stages of the analysis path. At the time of the INDRUM 2018 conference, we will have completed this initial theoretical stage of our project and will thus be able to share our results.

SUMMARY AND EXPECTATIONS

For a long time, mathematics students *survive* their courses based on developing a transformed version of a practical block, where they learn to recognize routine tasks and apply techniques to solve them in a sort of mechanical, naturalized, or normalized way, void of a mathematical theoretical block. At some point throughout an undergraduate degree in mathematics, however, the conditions for students’ survivability change dramatically and possibly abruptly: they are faced with activities that require them not only to fill the void of a mathematical theoretical block, but also to develop techniques for accomplishing tasks (e.g., proofs) that involve the abstract theoretical objects that have come out of hiding. Through modelling the knowledge the ideal student is expected to develop, as well as, eventually, the knowledge students actually develop in early analysis courses, we expect our project to bring about a more detailed and concrete understanding of praxeological “transitions” that have been theorized to occur, and give us some insight into how (or if) students adapt to them.

Returning to the vague and general question that originally inspired our project, we ultimately hope to learn more about what students are actually learning throughout an undergraduate degree in mathematics. The empirical data we collect will be a contribution to largely anecdotal discussions about when (if at all) students' knowledge is invited to become, and actually becomes, coherently, completely, and complexly mathematical, just like the scholarly knowledge produced and used in the institution of Mathematical Research. The significant difference between elementary and advanced courses, as professors gain more freedom and teach topics more closely related to their field of study, leads us to predict that students are eventually required to develop mathematical practices. After all, in spite of the apparent disconnection that is often observed between university mathematics courses and mathematical research (cf. Broley, Caron, & Saint-Aubin, 2017), the field of mathematics continues to live on, with new mathematicians emerging from the coursework that made up their mandatory professional education. In any case, through studying the principal conditions that currently shape the activities in which undergraduate mathematics students engage, we feel that we will be in a better position to discuss realistic and meaningful ways of encouraging these students to develop practices that are truly "mathematical", within the confines of educational institutions. This, we hope, will serve as complementary to the recent surge of studies (cf. Barquero, Serrano, & Ruiz-Munzon, 2016) aiming to explore innovative teaching approaches that question not the nature of the knowledge developed, but the dynamics of the knowledge development.

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Bridging probability and calculus: the case of continuous distributions and integrals at the secondary-tertiary transition

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This paper focuses on two mathematical topics, namely continuous probability distributions (CPD) and integral calculus (IC). These two sectors that are linked by the formula $P(a \leq X \leq b) = \int_a^b f(x)dx$ *are quite compartmented in teaching classes in France. The main objective is to study whether French students can mobilize the sector of IC to solve tasks in CPD and vice versa at the transition from high school to higher education. Applying the theoretical framework of the Anthropological Theory of the Didactic (ATD), we describe a reference epistemological model (REM) and use it to elaborate a questionnaire in order to test the capacity of students to bridge CPD and IC at the onset of university. The analysis of the data essentially confirms the compartmentalisation of CPD and IC.*

Keywords: Transition to and across university mathematics; Teaching and learning of analysis and calculus; Teaching and learning of probability; Anthropological Theory of the Didactic.

INTRODUCTION

Continuous probability distributions (CPD) and integral calculus (IC) are two topics that are taught in France during the last year of high school (grade 12 of the scientific track). They constitute two sectors (in the sense of the Anthropological Theory of the Didactics, ATD) that belong to the two different but closely related mathematical domains of probability theory and calculus respectively. Indeed, the continuous probability of an event and the definite integral with respect to a non-negative function are both defined as areas of suitable two-dimensional domains in the syllabus (in France), and the formula $P(a \leq X \leq b) = \int_a^b f(x)dx$ is the key for solving several standard tasks in CPD, where X represents a random variable and f its associated density function.

IC is the focus of extended studies in mathematics education: for instance, research (Schneider, 1992; Tran Luong, Bessot & Dorier, 2010; Haddad, 2013) was conducted in the context of Belgian, French, Vietnamese and Tunisian secondary education, including the secondary-tertiary transition (Haddad, 2013). By contrast, there is hardly any literature on CPD and, to make things worse, the available studies mostly put the emphasis on normal distributions (Batanero, Tauber & Meyer, 1999; Wilensky, 1997; Batanero, Tauber & Sánchez, 2004; Pfannkuch & Reading, 2006). Therefore, the teaching-learning phenomena generated by the interrelationship between CPD and IC are still to be investigated.

A first stone was laid by Derouet and Parzysz (2016; see also Derouet, 2016), who studied possible ways to introduce the density function at grade 12 so that students may construct this concept starting from considerations regarding histograms and therefore might relate continuous probability to the integral. By an analysis of textbooks, Derouet could show that the two sectors CPD and IC are very much compartmentalised in the French curriculum (Derouet, 2016, pp. 127-190). For instance, the above formula is seldom justified by a thorough discussion of the definitions involving areas, which certainly hinders the bridging of the two sectors by students.

In this paper, we regard this "compartmentalisation" of knowledge as an institutional phenomenon and therefore use ATD as the theoretical framework (see below). Our goal is to study the impact of this compartmentalisation on the learning of mathematics: are French students able to mobilize the sector of IC to solve tasks in the sector of CPD and vice versa at the transition from high school to higher education?

After the presentation of theoretical constructs from ATD used in this research, we will describe the reference epistemological model (REM) that we elaborated for the types of tasks in CPD and IC with regard to studying interrelations of the two sectors. We will then describe our methodology that builds on the elaboration of a questionnaire, based on the REM, that has been submitted to students at the entrance of university. We finally present results of a primary analysis of the data from the questionnaire and draw some conclusions and perspectives opened up through this study.

THEORETICAL CONSTRUCTS

ATD "postulates that any activity related to the production, diffusion or acquisition of knowledge should be interpreted as an ordinary human activity, and thus proposes a general model of human activity built on the key notion of praxeology" (Bosch & Gascon, 2014). The praxeology Π is represented by a quadruple $[T/\tau\ /\theta/\Theta]$: its praxis part (or know-how) consists of a type of tasks T together with a corresponding technique τ (useful to carry out the tasks $t \in T$ in the scope of τ). The logos part (or know-why) includes two levels of description and justification: the technology θ, i.e. a discourse on the technique, and the theory Θ, which often unifies several technologies.

The elaboration of a reference epistemological model (Florensa, Bosch, & Gascon, 2015) as sequences of praxeologies, for a given body of knowledge, is an important step in any research carried out in the ATD framework. It is the tool that will be used by the researcher to describe, analyse, put in question or design the specific contents that are at the core of a teaching and learning process. In order to build such a model, “mathematical praxeologies are described using data from the different institutions participating in the didactic transposition process, thus including historical, semiotic and sociological research, assuming the institutionalized and socially articulated nature of praxeologies” (loc. cit. p. 2637).

Our study relies on an overview of standard textbooks used at grade 12 in France, as well as the official syllabus, in order to identify the standard praxeologies in CPD and IC that may be related and the nature of this relationship at the praxeological level. An epistemological investigation of the historical development and the interrelation of both domains have previously been carried out in (Derouet, 2016, pp. 67-85). In order to test the effect of the institutional compartmentalisation of knowledge on the learning of mathematics, we need to check the availability of the identified praxeologies in the *praxeological equipment* of students, and then submit tasks to students which need to bridge CPD and IT as mutually interdependent sectors that share techniques or technologies, borrow them from or lend them to each other. Special care must be taken in the phrasing of these bridging tasks, taking into account the effect of *ostensives* (Bosch & Chevallard, 1999), that is to say the role of signs. Indeed, ostensives contribute to the activation of the specific sectors to which they belong and therefore direct students toward specific techniques.

REFERENCE EPISTEMOLOGICAL MODEL

Even though problems of quadratures arose in ancient Greece, IC finds its roots as a systematic method in the 17th century. The emergence of continuous probability may be situated in the 18th century with the theory of errors in physical measurements. Various functions were introduced to model the distribution of errors and the area under the curve permitted to evaluate the “theoretical frequency” (so the probability) of the deviation from the “true” value. CPD was thus naturally connected to IC in its historical roots. The gaussian distribution was proposed later by Gauss in 1809.

To identify the different praxeologies, we have analysed 12 textbooks of the grade 12 of the scientific track (edition 2012). We focused on the exercises with a given solution in the textbooks to have access to the usual techniques for the different tasks.

In our study, we will focus on two main types of mathematical tasks T_I and T_P, which are related to the mathematical domains of integral calculus and the continuous probability respectively:

- T_I: compute a value for an integral $\int_a^b f(x)dx$ for a positive continuous function f;
- T_P: determine the probability $P(a \leq X \leq b)$ for a random variable X endowed with a density function f.

The type of tasks T_I may further be split into two subtypes of tasks, depending on the expected result: an exact value ($T_{I,exact}$) or an approximation ($T_{I,approx}$).

The most useful technique $\tau_{I,exact}$ to solve $T_{I,exact}$ is to compute a primitive of the function and apply the *fundamental theorem of calculus.* The corresponding technology $\theta_{I,exact}$ is given by the fundamental theorem of calculus that relates integrals and primitives: $\int_a^b f(x)dx = F(b) - F(a)$ with $F' = f$. The theory $\Theta_{I,exact}$ includes the definition of the definite integral for a continuous positive function as an area and properties of areas that may be formalised into a local axiomatic theory[1].

The technique $\tau_{I,exact}$ thus resorts to praxeologies dedicated to the computation of primitives. The standard technique at high school level is to use the "tabular of primitives" (deduced from the tabular of derivatives). The technology comprises the properties of the derivative and the theory is that of differential calculus. In the case of piecewise affine functions, an alternative technique to $\tau_{I,exact}$ is to interpret the integral as the area of an elementary surface (or a union of these).

The type of tasks $T_{I,approx}$ may be solved using two main techniques: using a calculator (or software), more or less a blackbox, or applying the "rectangle method". The latter technique $\tau_{I,approx}$ consists in considering the integral as an area, taking a subdivision of the interval of integration and computing the sum of rectangular areas. The technology $\theta_{I,approx}$ comprises the definition of the integral and properties of areas. A further theoretical level $\Theta_{I,approx}$ is mainly non-existent at high school level (cf. endnote 1).

Regarding the type of tasks T_P, two cases need to be distinguished, depending on whether a primitive of the density function f is known ($T_{P,prim}$) or not. The latter case is reduced to that of the normal distribution ($T_{P,norm}$), which is dealt with using the implementation of such a distribution in a calculator or software. Computer scientific tools are mainly used as a blackbox by students, which hinders the possibility for students to make connections with IC.

The generic technique $\tau_{P,prim}$ for $T_{P,prim}$ is to compute $\int_a^b f(x)dx$, in other words to resort to the praxeology $\Pi_{I,exact}$. The technology $\theta_{P,prim}$ is given by the formula $P(a \leq X \leq b) = \int_a^b f(x)dx$, and the theory $\Theta_{P,prim}$ comprises the definition of a continuous probability (as the area of the corresponding domain) and the definition of a probability density function. At high school, two particular cases are emphasised and lead to local techniques, as concrete formulas are available for $P(a \leq X \leq b)$ in the case of the exponential and uniform distributions. For instance, the technique $\tau_{P,exp}$ in the case of the exponential may be reduced to computing $e^{\lambda a} - e^{\lambda b}$ with the technological argument $P(a \leq X \leq b) = e^{\lambda a} - e^{\lambda b}$.

Let us recall that our model is based on the study of standard textbooks used in grade 12 classes in France and is dedicated to the description of the teaching-learning of CPD and IC as it actually is (we are not planning task-design at this stage of the research).

In this model, we note the following links between CPD and IC: at the level of the theoretical blocks, praxeologies in both sectors are anchored on the empirical notion of area. At the level of the technique, $\Pi_{P,prim}$ uses $\Pi_{I,exact}$, so that, from an ecological point of view (Bosch & Gascon, 2014, p. 72), CPD contributes to the thriving of such IC praxeologies. By contrast, $\Pi_{I,approx}$ does not seem to be reinvested in CPD (whereas the computation of a probability of the Gaussian distribution could be an opportunity to mobilize $\Pi_{I,approx}$). Conversely, we did not detect elements of the praxis of CPD in the IC sector. This isn't a surprise: IC is regarded as a prerequisite to CPD and precedes the teaching of CPD in all textbooks. Nevertheless, the normal distribution is a prototypical example of a function whose primitive cannot be expressed in terms of available elementary functions. This fact explains the choice of techniques in $\Pi_{P,norm}$ and contributes also to the logos of $\Pi_{I,exact}$ (by complementing the statement that every continuous function admits a primitive). What about the type of tasks $T_{I,norm}$: compute $\int_a^b \frac{e^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}dx$? It could appear in the IC sector through $\Pi_{I,approx}$ only but we didn't find it in any textbook and it is never stated as such in the CPD sector. The most efficient technique requires to use the formula $P(a \leq X \leq b) = \int_a^b f(x)dx$ from right to left: although the equality is symmetrical as an equivalence relation, it isn't symmetrical as a sign which denotes a succession of operations in performing a computation. How would students react to such a task that asks to bridge CPD and IC in an unusual way? This question came to us as a starting point for the elaboration of our questionnaire dedicated to the investigation of the educational effects of the institutional phenomenon of compartmentalisation of knowledge, in the case of CPD and IC.

THE QUESTIONNAIRE

Our main goal is to test whether students are able or not to connect CPD and IC, and especially mobilize the CPD sector to solve an IC task when ostensives do not indicate explicitly the probability domain. To do so, we have elaborated a questionnaire both to check the availability of standard praxeologies of CPD and IC in the praxeological equipment of students and the capacity of students to complete such bridging tasks.

Bridging tasks appear at the very end of the questionnaire and are stated as follows:

Question 6: Expliquer toutes les méthodes que vous pouvez utiliser pour déterminer une valeur exacte et/ou approchée de l'intégrale suivante : $I = \int_{-0,5}^{1} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$. On pourra se limiter à donner une idée de la méthode si sa mise en oeuvre est trop compliquée.

Question 7: Soit A la fonction définie par $A(\lambda) = \int_0^\lambda f(x)dx$ avec $f(x) = xe^{-x}$, pour tout $\lambda \in [0; +\infty[$. On peut démontrer que $\lim_{\lambda \to +\infty} \int_0^\lambda xe^{-x}dx = 1$. D'après ce résultat, expliciter tout ce que vous pouvez dire sur la fonction A et la fonction f.

Translation:

Question 6: Explain all the methods that can be used to determine an exact and/or approximate value of the following integral: $I = \int_{-0,5}^{1} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$. It suffices to give an idea of the method if its implementation is too complicated.

Question 7: Let A be the function defined by $A(\lambda) = \int_0^{\lambda} f(x)dx$ with $f(x) = xe^{-x}$, for all $\lambda \in [0; +\infty[$. It can be proved that $\lim_{\lambda \to +\infty} \int_0^{\lambda} xe^{-x}dx = 1$. According to this result, note everything that you can say about the function A and the function f.

Figure 1: bridging tasks submitted to students

Question 6 contains an instance of the type of tasks $T_{I,norm}$ discussed in the REM. The task is stated in an opened way, asking for every method that students may know to compute an exact or approximate value for the Gaussian integral. Praxeologies $\Pi_{I,exact}$ and $\Pi_{I,approx}$ should therefore also show up. In question 7, we intend to check whether students can say that A is both a primitive for f and a probability associated with the density function f, or restrict to the IC sector with an interpretation in terms of areas.

Previous questions intend to "activate" both sectors CPD and IC equally. In this respect, question 1 offers a routine task of type $T_{I,exact}$ in the case of a straightforward exponential function. Analogously, the first part of question 4 is a routine task of type $T_{P,exp}$ (compute $P(1 \leq X \leq 5)$ when X has an exponential distribution of parameter 3). In its second part, students are asked for a graphical interpretation of the probability $P(1 \leq X \leq 5)$ that is in fact defined as an area in the high-school syllabus, as well as the integral: this interpretation is therefore essential in order to link the logos of Π_P and that of Π_I.

Question 2 activates the CPD sector by soliciting an element of the logos of Π_P, namely the properties that define a density function. This logos is crucial in the bridging question 7, which is stated in the IC sector without any reference to CPD. Question 3 tests if students are able to retrieve the definitions of both the exponential and normal distributions by specifically asking for those in the case of simple parameters (the reduced centered gaussian law). The latter is an element of the logos of $\Pi_{P,norm}$: we wish to check if students are able to recognize the normal distribution in the statement of the bridging question 6, while taking care not to direct them towards a specific technique (hence the order of questions).

In question 5, we rather activate the IC sector, more precisely elements of the logos of $\Pi_{I,exact}$ (primitives), but we have in mind the praxeology $\Pi_{P,norm}$ in relation to the bridging question 6: students are asked to provide an example of a continuous function, if it exists (or justify the impossibility), that a) doesn't possess primitives b) admits primitives but expressions for these are not "explicitly known" (expected answer: the density function of the normal law).

Summarizing, by the questions 1 to 4 we want to investigate whether the students master techniques $\tau_{I,exact}$ in a calculus context and $\tau_{P,prim}$ in a probability context and whether they know technologies related to IC and CPD. Then, by questions 6 and 7, we focus on relationships between CPD and IC and the previous questions: we analyse links between the questions 3, 4 (second part), 5 and 6, on the one hand, and links between the questions 2 and 7, on the other hand.

DATA ANALYSIS AND RESULTS

The questionnaire was used at the beginning of September 2017 (the first week of classes) in two classes of first year CPGE (French engineers school preparatory classes) students, which is in fact at the transition between secondary and tertiary levels. The first class (called class N) is a class of MPSI (Mathematics, Physics and Engineering Science) and the second class (called class R) is a class of PCSI (Physics, Chemistry and Engineering Science) of a rather prestigious establishment. The students working on the questionnaire are in selective classes, so we can assume that they are "good" scientific students, and in particular, if they meet difficulties then these are shared by the other students. We only analysed answers from students who studied in French high school during the past year because we constructed the questionnaire taking into account the context of the French high school institution. We retrieved 82 questionnaires (40 of the class N and 42 of the class R). Except for a few students (less than 5), the students didn't use a calculator during the test.

From the 82 students, only one does not mobilize the technique $\tau_{I,exact}$ to resolve the routine task concerning IC (question 1). 85% find a correct expression of the primitive and 78% obtain the correct result, which means that this technique is mastered quite well by students. Regarding the computation of the probability for an exponential distribution (question 4), 63% of the students get a correct result. 82% of the students use the technique $\tau_{P,prim}$ and the other students directly apply a formula. More than 70% of the students could identify the probability as an integral. Summarizing, except for some errors regarding the primitive or the computation, the majority of students are able to pass from a probability to an integral and, moreover, know the fundamental theorem of calculus. In praxeological terms, they are able to mobilize the technique $\tau_{P,prim}$ that implies the use of the technique $\tau_{I,exact}$ in the case of an exponential distribution.

Regarding CPD and neglecting formulation and formalization issues, only 39% of the students know the definition of the density function (question 2) and only 27% recollect the density function of the normal distribution (question 3b). In view of question 5, only 32% mobilize the theorem claiming that all continuous functions on an interval admit a primitive. 52 % give an example of a function for which they don't explicitly know a primitive (although it might exist and be expressed in terms of standard functions[2], for instance $ln(x)$). Among these, 15% mention the density function of the Gaussian distribution (or a function of the type e^{-x^2}). Of the 22 students who know the density function of the normal distribution, half propose it as an example in question 5 (13% of all the students).

9% of the students do not answer question 6. The method most often proposed is technique $\tau_{I,exact}$ (46%). Less than 20% (16 students) mention the normal distribution and 16% propose the rectangle method ($\tau_{I,approx}$). A few students propose the technique "integration by parts", which is beyond the curriculum in grade 12. Moreover, only 23% of the students who propose the rectangle method are able to illustrate the method by drawing the graph of the Gaussian curve (the other students draw a wrong curve or do not consider any graph). Only 11 of the 16 students (69%) who mention the normal distribution in this question write that the integral I is equal to the probability $P(-0{,}5 \leq X \leq 1)$ with X a random variable of a reduced normal distribution and 7 of them state more precisely that they have to use the calculator to evaluate this probability ($\tau_{P,norm}$). So, our results indicate that the ostensive $\int_{-0,5}^{1} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$ without indication invite students to stay in the IC sector and even more particularly in the praxeology $\Pi_{I,exact}$ (even if it is not possible here), which reflects that $\Pi_{I,exact}$ is the praxeology most developed in grade 12. To pass from an integral to a probability and to change the sector in this direction does not seem to be natural for students. Moreover, of the 22 students who know the expression for the density function of the reduced normal distribution (question 3b), only 10 (45%) recognize the density function of the normal distribution is this context. Of the 15 students who say that the Gaussian function does not admit primitives expressed by using standard elementary functions (question 5b), around 27% proposes to use the technique $\tau_{I,exact}$ nevertheless and only 53% recognize the normal distribution in question 6. This means that most of the students were not able to mobilize $\Pi_{P,norm}$ in the IC-context of question 6, that is $\Pi_{I,norm}$. In particular the application of $\tau_{I,exact}$, although question 5b is answered correctly, demonstrates the strong compartmentalisation between CPD and IC.

Regarding the answers to question 7, we notice that 38% of the students (31) state that the function f is a density function. 42% of them (13 students) could justify that it is a density function including 4 who forget to mention the positivity of the function (because they don't write this condition in their definition in question 2). Moreover, 12 of the 31 students as well as one additional one identify $A(\lambda)$ as $P(0 \leq X \leq \lambda)$. 8 students think that the distribution in question 7 is an exponential distribution of parameter x and 7 students state that A is a density function. Finally, more than 46% of the students mention "probability", which means that they manage to identify at least some link between the IC embedding of question 7 and CPD. Probably, the questionnaire itself influenced students and the percentage would be lower otherwise, i.e. if question 7 was asked independently. Overall correct results with justification are rare and of the 32 students who master the definition of a probability density function (one element of the technology $\theta_{P,prim}$ tested in question 2), 31% (10 students) do not identify f as a density function. 16% of the students do not at all answer question 7.

Again, we observe also with respect to question 7 that CPD praxeologies, although they are in principle available, often cannot be mobilized in the IC contextualization.

Summarizing, the data analysis shows that techniques related to one sector are available for the majority of students only when ostensives related to this sector are provided. Additionally, related technologies are much less mastered by students. Perhaps, this could be an explanation why it is not natural for students to mobilize praxeologies of the CPD sector for a task in the IC sector, in addition to the fact that these tasks are not taught in the classroom. The data analysis by all means shows a strong compartmentalisation between CPD and IC.

CONCLUSION AND PERSPECTIVES

The results of our primary data analysis clearly demonstrate a strong compartmentalisation between CPD and IC. In particular, techniques from CPD, although available in a CPD task, could not be mobilized in an IC-contextualized task. A next step in our research will be more detailed data analyses looking for correlations and interdependencies between techniques and technologies of CPD and IC. We further observed that available techniques were not accompanied by related technologies. One could claim that more elaborated technologies might support the transfer of techniques. More generally, we think about studies investigating the impact of changes in the institutional setting, i.e. establishing innovative teaching sequences with less compartmentalisation. A teaching sequence articulating CPD and IC is proposed in Derouet (2016). The effect of this teaching on the answers of students to the questionnaire could be analysed by comparing the latter with the present results.

NOTES

1. This axiomatic remains implicit at the secondary level; it may be related to Measure Theory at university level.

2. We realised *a posteriori* that our question was not phrased properly: "a function that admits a primitive whose expression is not explicitly known to you" may be interpreted as a lack of techniques to actually compute the primitive and not the impossibility to provide an expression (in terms of elementary functions).

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Affective variables in the transition from school to university mathematics

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The transition from school to university mathematics is a difficult step for students, which many of them do not succeed to manage immediately. In this contribution we use questionnaires, which measure mathematics-related affective variables as well as subject-unspecific affective variables and students' achievement during the semester to predict the outcome of the exam at the end of the first semester (as a first indicator for success in their studies) and the students' attendance in this exam (as an indicator for early dropout). We are interested in whether the mathematics-related or the "general" affective variables are more suitable to predict the students' exam attendance and the exam outcome. The students' achievements during the semester turned out to be the best predictor for the exam outcome, whereas the students' attendance was best predicted by their interest in mathematics.

Keywords: transition to and across university mathematics, teaching and learning of analysis and calculus, dropout, study success, affect

INTRODUCTION

Dropout is a big problem for German universities, especially in mathematics. Nearly 80% of all mathematics students drop out or change their subjects – most of them during their first year at university (Dieter & Törner, 2012). However, this is not a typical German phenomenon: Chen (2013) reports similar figures of dropout and subject change for the United States. In this paper, we do not distinguish between students who drop out and leave the university system without examination and those who "just" leave mathematics and change to another subject.

At Ruhr-University Bochum, where our study takes place, students begin their studies of mathematics with two lecture courses in the first semester – calculus I and linear algebra I. Two so called "mini-tests" are written during the semester to prepare the students for their first "real" exam at university. These "mini-tests", designed by the lecturer, cover conceptual and procedural knowledge about definitions and proof which have been discussed in the lecture before. 73% of the students at Ruhr-University (who attended the exams) failed their calculus exam in 2017. Furthermore, 25% of the students did not even attend the exam. Following the goal to improve the support of students who are at risk to fail their exam and/or drop out, we are interested in which way students at risk differ from those who succeed.

Due to data protection regulations, it was not possible for us to identify which students really dropped out from mathematics at Ruhr-University in 2017. Instead, we could match the results from our questionnaires with the results of the exam at the end of the first semester. We therefore could identify which students were successful in their exam, which failed and which did not attend the exam. Students who did not attend the exams might have dropped out before or may be at risk to drop out soon. Baars and Arnold (2014) found that students who do not attend their exams in the first semester have a high risk to drop out.

THEORETICAL BACKGROUND

Both, dropout and study success, are considered to be influenced by multiple factors, which are often called predictors, such as the socio-economic and school background, personal psychological prerequisites, learning behaviour and study conditions (Tinto, 1975; Heublein et al., 2009; Thiel et al., 2008).

The predictors, which are listed by the students for their decision to drop out, are called dropout reasons. In Germany, most dropped out mathematics students name the course requirements (e.g. failed exams, work-overload) (33%) and low motivation (25%) as their main reasons to quit their studies of mathematics. Other reasons such as study-conditions (13%) and reasons related to health- or financial problems (12%) and personal reasons like family problems (10%), were less important – specially for early dropout (Heublein et al., 2009).

Given the fact, that most dropped out mathematics students in Germany name the requirements at university and their lack of motivation as crucial for their decision to drop out, we want to shed light on the following affective variables, which are considered to influence the students' motivation and academic achievements: mathematical self-concept, interest in mathematics, beliefs concerning the nature of mathematics, basic needs and general self-efficacy. These variables are briefly discussed in the following.

Mathematical self-concept

The self-concept can be seen as the mental model of one's personal competences, abilities and properties, or "in very broad terms, self-concept is a person's perception of himself" (Shavelson, Hubner, and Stanton, 1976 as cited in Bong and Skaalvik, 2002). The self-concept is influenced by the students' former experiences and achievements and can itself influence students' motivation (Bong and Skaalvik, 2002). The self-concept is considered to be domain specific. Rach and Heinze (2016) found that the mathematical self-concept is a significant predictor for dropout but not for students' success in the first semester.

Interest in mathematics

The interest in mathematics is considered to have a positive impact on the learning of mathematics. Schiefele et al. (1993) define interest as a specific relation between a person and an object. The interest in the subject that one is studying is rather stable, since it has been developed over a longer time through different experiences.

Due to contradictory results in various studies the impact of interest in mathematics on students' performance and success in their studies of mathematics remains uncertain. Rach and Heinze (2016) found no significant influence of the interest in mathematics on students' success during the first semester or on their risk to drop out. However, Blömeke (2009) found significant correlations between the interest in mathematics and the students' intention to drop out.

Beliefs concerning the nature of mathematics

It has been widely discussed that the nature of mathematics changes with the transition from school to university (e.g. Rach and Heinze, 2016). Mathematics in German schools is often focused on applying mathematical techniques to solve real world problems (modelling, problem solving). New mathematical contents are regularly presented more intuitively with examples and illustrations and yield on an intuitive or practical understanding of the concepts. Mathematics at university is more theoretically and proof oriented. New concepts are presented in a rather formal and abstract way and therefore less illustrated than in school. The focus often lies on encouraging logical and abstract thinking. The students have to develop understanding for deductive argumentations and proof – applying the theory is less important than at school. This change from a practical to a theoretical approach is not easy for most students. Many of them feel a big gap between mathematics at school and university (Geisler, 2017). This feeling might be a result of unfulfilled expectations and incongruences between the mathematical "reality" at university and their established beliefs concerning the nature of mathematics, which are based on their school experiences. Daskalogianni and Simpson (2001) call this phenomenon "belief overhang". Andrà, Magnano and Morselli (2011) found hints that students' beliefs concerning the nature of mathematics can influence their decision to drop out or to stay. Traditionally we distinguish between a *static* view, where mathematics is viewed as a summary of (unconnected) rules, facts and techniques, and a *dynamic* view, where mathematics is considered as a process and a creative field of research (Grigutsch and Törner, 1998). However, it is yet unclear which beliefs are beneficial for a successful transition from school to university.

Basic Needs

Following the framework of self-determination theory (Ryan & Deci, 2000), there are three basic psychological needs that are important for the well-being of humans and to generate motivation: social relatedness, competence and autonomy. In the special situation of the transition from school to university mathematics, many students do not experience autonomy and competence (Liebendörfer and Hochmuth, 2013). This is problematic since Faye and Sharp (2008) found that especially the feeling of competence is strongly associated with motivation in university. In an explorative case-study, we found hints for the impact of social relatedness and competence on the decision to drop out (Geisler, 2017).

General self-efficacy

The general self-efficacy is the strength of a persons' belief to be able to reach certain goals and to solve problems by his or her own competences and abilities (Luszczynska et al., 2005). This general belief is not limited to a special domain like mathematics or special academic settings. In contrast to the self-concept, self-efficacy is more focused on the consequences of one's own competences and abilities than on the competences and abilities themselves. That's why self-concept is rather past oriented whereas the self-efficacy focuses on the future (Bong and Skaalvik, 2002). Besides, self-concept is considered to be the more stable variable. Self-efficacy can influence the students' motivation in the sense that students who believe that they are able to succeed in their studies of mathematics are more motivated to put effort in their learning than those who believe that they have no chance in the exams. Self-efficacy is therefore associated with academic achievement (Luszczynska et al., 2005; Bong and Skaalvik, 2002). Students with lower self-efficacy have a higher risk to drop out than those with higher self-efficacy (Krieger, 2011).

Achievement during the semester

The students' achievement is an important factor for success and dropout. In Tinto's (1975) framework, achievement, as a part of the academic integration, is important for the decision to drop out or to stay. In an explorative case-study, Geisler (2017) found hints that students who are not satisfied with their achievement during the first semester sometimes drop out, even if they are successful in their exams at the end of the semester. Though achievement is closely connected with the perceived feeling of competence.

RESEARCH QUESTIONS

In order to support students who are at risk to fail their exam or even to drop out we want to know in which way these students differ from those who succeed. Following the theoretical background described above, we decided to focus on the students' achievements during the first semester and on affective variables which are likely to influence students' motivation. Since the dropout rate in mathematics is high compared to other subjects, it seems plausible that mathematics related variables have an important impact on dropout and success. We therefore distinguish between mathematics-related affective variables (mathematical self-concept, interest in mathematics, beliefs concerning the nature of mathematics) and more "general" affective variables (basic needs, self-efficacy). We are interested in whether the mathematics-related or the "general" affective variables are more suitable to predict students' exam attendance and their exam outcome. This leads to the following research questions:

Differences between the three groups of students

1) Which differences in the affective variables and the achievements can be found between students who do not attend the exam, students who fail in the exam and those who succeed?

Prediction of the students' exam attendance

2.1) In which way can the mathematics-related affective variables predict the students' attendance for the exam at the end of the first semester?

2.2) In which way can the "general" affective variables and the students' achievements approve this prediction?

Prediction of the exam outcome

3.1) In which way can the mathematics-related affective variables predict the outcome of the exam at the end of the first semester?

3.2) In which way can the "general" affective variables and the students' achievements approve this prediction?

METHODOLOGY

209 students in the calculus lecture in wintersemester 2016/17 voluntarily participated in our study. Undergraduate mathematics students as well as pre-service teachers in mathematics usually attend this lecture during their first year at university. The questionnaires were filled out during the lecture in the mid of the first semester, taking into account that students cannot rate their satisfaction of the basic needs at the begin of the semester. Due to incomplete datasets, only N=193 cases could be included in our analysis. The instruments used in our questionnaire can be found in Table 1.

construct	**source**	**No. of items / Crobach's α**	**Item-example**
Interest	Schiefele et al. 2007	12 / 0.82	"It is personally important for me that I can study mathematics."
Self-Concept	Kauper et al. 2012	4 / 0.82	"I am very good in mathematics."
Beliefs: static	Laschke & Blömeke 2013	6 / 0.67	„Mathematics means learning, remembering and applying."
Beliefs: dynamic	Laschke & Blömeke 2013	6 / 0.73	"Mathematics involves creativity and new ideas."
Social Relatedness	Kauper et al. 2012	6 / 0.78	"I feel comfortable with the other students."
Competence	Kauper et al. 2012	3 / 0.66	"I get clear and detailed feedback on my achievements."
Autonomy	Kauper et al. 2012	3 / 0.58	"I can do tasks in my way."
Self-Efficacy	Beierlein et al. 2012	3 / 0.88	"I can solve most problems on my own"

Table 1: Instruments with numbers of items, reliability and item-example

All reliabilities (Cronbachs α) were at least sufficient – except for the reliability of the autonomy-subscale. All items were answered on a five-point Likert scale (1=totally disagree; 5=totally agree). To measure the students' achievement during the semester, we used their results in the first "mini-test" (1 to 12 points).

The results of the questionnaires were analysed using a MANOVA (to prevent the accumulation of the α-error compared with t-tests) to answer research question 1. To answer the other research questions, we used linear and binary logistic regressions.

RESULTS

Students who do not attend the exam at the end of the first semester differ significantly from those who fail the exam and those who succeed in almost all affective variables (research question 1), except for the static beliefs (Table 2). Focussing on the mathematics-related affective variables, the biggest difference between the three groups of students can be found in the interest in mathematics, which can explain 13% of the variance (η^2=0.13***). Regarding the "general" affective variables, the self-efficacy turned out to explain the most variance between the three groups of students (η^2=0.1***). Taking into account all measured variables, the biggest difference between the three groups of students can be found in their achievements in the "mini-test" (η^2=0.19***).

	No Attendance N = 54		Failed N = 101		Succeeded N = 38			
	M	*SD*	*M*	*SD*	*M*	*SD*	F	η^2
Interest	**2.99**	**0.66**	**3.43**	**0.7**	**3.7**	**0.56**	**13.89**	**0.13*** **
Self-Concept	**2.64**	**0.83**	**2.93**	**0.67**	**3.11**	**0.58**	**5.45**	**0.05** **
Beliefs: static	*3.77*	*0.66*	*3.79*	*0.58*	*3.68*	*0.5*	*0.46*	
Beliefs: dynamic	**3.07**	**0.68**	**3.46**	**0.65**	**3.56**	**0.62**	**8.22**	**0.08*** **
Self-Efficacy	**2.46**	**0.96**	**2.78**	**0.78**	**3.25**	**0.69**	**10.18**	**0.1*** **
Social Related.	**3.77**	**0.68**	**4.07**	**0.55**	**4.23**	**0.57**	**7.69**	**0.08** **
Competence	**2.94**	**0.87**	**3.23**	**0.86**	**3.43**	**0.81**	**3.91**	**0.04* **
Autonomy	**2.89**	**0.72**	**3.28**	**0.79**	**3.42**	**0.66**	**6.83**	**0.07** **
Achievement	**6.19**	**3.9**	**7.81**	**3.16**	**10.89**	**2.75**	**22.74**	**0.19*** **

Table 2: Means, standard deviations and results of the variance analysis
***p<0.05 **p<0.01 ***p<0.001**

To answer research questions 2.1 and 2.2, three different logistic regression models were tested (Table 3). Model 1 only contains the mathematics-related affective variables. The only significant predictor for the students' attendance in this model was interest in mathematics. Model 1 can explain 19% of the variance and is able to classify 71% of the students correctly as attending or not attending. Model 2 additionally contains the affective variables basic needs and self-efficacy. None of these variables has a significant influence on the students' attendance and they do not improve the students' classification. In contrast, the students' achievements are a (weak) significant predictor for the students' attendance (Model 3). The inclusion of the students' achievement can improve the classification of the students (75.1%

correct) and increases the explained variance (Nagelkerke's R^2=0.26). Note that the interest in mathematics is still the most significant predictor in Model 3.

	Model 1	Model 2	Model 3
Interest	**2.43****	**2.5****	**2.42****
Self-Concept	*1.22*	*0.89*	*0.8*
Beliefs: static	*1.14*	*1.3*	*1.29*
Beliefs: dynamic	*1.66*	*1.3*	*1.24*
Self-Efficacy		*1.33*	*1.18*
Social Relatedness		*1.3*	*1.14*
Competence		*0.93*	*1.04*
Autonomy		*1.4*	*1.39*
Achievement			**1.13***
Nagelkerke's R^2	**0.19**	**0.22**	**0.26**
Correct classification	**71 %**	**71 %**	**75.1 %**

Table 3: Results (coefficients Exp(B)) of the logistic regression to predict the exam attendance - *p<0.05 **p<0.01

We used three linear regression models to answer research questions 3.1 and 3.2 (Table 4). In Model 1 only the mathematics-related affective variables were included.

	Model 1	Model 2	Model 3
Interest	*0.08*	*0.06*	*0.01*
Self-Concept	*0.17*	*-0.03*	*0.04*
Beliefs: static	*-0.09*	*-0.01*	*0.02*
Beliefs: dynamic	*0.05*	*-0.02*	*-0.05*
Self-Efficacy		**0.41*****	**0.2***
Social Relatedness		*0.11*	*0.07*
Competence		*-0.08*	*0.02*
Autonomy		*-0.03*	*-0.07*
Achievement			**0.54*****
R^2	**0.05**	**0.13**	**0.37**

Table 4: Results (standardized beta coefficients) of the linear regression to predict the exam outcome - *p<0.05 ***p<0.001

None of them can predict the exam outcome significantly. All these variables together only explain 5% of the variance in the exam outcome. The "general" affective variables, specially the self-efficacy, seem to have a bigger impact on the exam outcome (Model 2). The inclusion of these variables improves the variance that can be explained (R^2=0.13). The self-efficacy is a highly significant predictor. The most important (highly significant) predictor for the exam outcome is the students' achievement in the "mini-test" (Model 3). The inclusion of the achievement increases the explained variance to 37%. Multicollinearity of the variables in our models is at least tolerable (tolerance>0.48, VIF<2) and should not have a big impact on results.

CONCLUSION

The interest in mathematics is the most important predictor for exam attendance in our study, whereas the mathematical self-concept has no significant influence. This is contradictory to Rach's and Heinze's (2016) findings, where only the mathematical self-concept was able to predict the attendance. However, both studies have in common that a mathematics-related affective variable is the most important predictor for the attendance, whereas the "general" affective variables have no influence.

The mathematics-related affective variables do not predict the exam outcome. The only affective variable that significantly predicts the exam outcome is the general self-efficacy. This is rather surprising, taking into account that self-efficacy and self-concept are (at least theoretically) closely connected variables and it seemed plausible that the mathematics-related variable provides more insights. However, in contrast to the mathematical self-concept, the general self-efficacy is not only focussed on one's competences and achievements in mathematics but also takes into account subject unspecific competences which could be beneficial at university, too.

All in all, it turned out that the students' achievements can predict both, exam outcome and exam attendance. Interest in mathematics is suitable to predict the exam attendance, whereas self-efficacy can predict the exam outcome. Firstly, this finding shows that, mathematics related as well as general affective variables play an important role in the transition from school to university. Secondly, it suggests that success and dropout should not necessarily be viewed as two sides of a coin.

Our study has some limitations. We conducted data from only one university, which could lead to cohort specialities. Furthermore, questionnaires were filled out in the mid of the semester during the lecture. Students who do not (regularly) attend lectures or have dropped out before have not been captured by our study. Some results might be different if we could capture those students, too.

Our on-going research will now focus on a more detailed characterisation of dropped out students, taking into account cognitive and metacognitive variables (e.g. students learning behaviour) as well. In addition, it seems to be useful to identify different types of dropped out students in mathematics. This might help to design and evaluate more individualized supporting programs for students in the transition.

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Discussing Mathematical Learning and Mathematical Praxeologies from a Subject Scientific Perspective

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This programmatic contribution discusses the link between concepts from Anthropological Theory of Didactics (ATD) and the "subject-scientific point of view" according to Holzkamp (1985, 1993). The main common concern of ATD and the subject-scientific approach is to conceptualize and analyse "objects" like "institutionalized mathematical knowledge" and "university" not as conditions that cause reactions but essentially as meanings in the sense of generalized societal reified action possibilities. The link of both approaches is illustrated by the issue of "real numbers" in the transition from school to university: Hypotheses are derived for further actual-empirical research, which intrinsically incorporate content- and subject related perspectives as well as societal and school-related findings.

Keywords: Curricular and institutional issues concerning the teaching of mathematics at university level, transition to and across university mathematics, subject scientific approach, mathematical praxeologies, real numbers.

INTRODUCTION

This paper contributes to an ongoing major research project that describes and analyses form and content of advanced mathematics and its teaching and learning from a subject scientific point of view. This approach is grounded in "Critical Psychology", framed by Holzkamp (1985) (see Tolman (1991) and Schraube & Højholt (2015) for English written introductions). Recently this theory becomes internationally more known in the mathematics education community due to Roth & Radford (2011), who assessed "German Critical Psychology" as a further development of the culture-historical activity approaches by Leontjev (1978) and Vygotsky (1978). It's beyond the scope of this paper to describe and analyse in which respects "German Critical Psychology" differs and goes beyond culture-historical activity theory. Instead the paper intends to point out its compatibility and (partial) complementarity with ATD as well as to illustrate its potential relevance for further research concerning university mathematics education.

Main features of "Critical Psychology" and its subject-scientific point of view are well elaborated psychological categories (roughly: basic notions, see for details (Holzkamp, 1985, pp. 28)) for describing and analysing cognitive and emotional-motivational dimensions of subject [individual] related experiences, in particular thoughts, actions and learning, in a way that major societal aspects are inherently be incorporated. It aims (besides others) to provide individuals with analytic tools for their self-reflection of problematic experiences and situations to reveal their inherent dependencies and circumstances, thus allowing individuals to achieve a more reflective learning. Within this framework, there is so far a lack of research that relates to mathematical learning in general and in university in particular. “Critical Psychology” provides points of contact for incorporating research results concerning the societal and historical genesis of knowledge and reference structures as well as institutionally framed (e.g. school, university, study courses) external and internal transposition processes (Chevallard, 1991).

ATD’s praxeological analyses could principally inform any psychological or sociological theory considering teaching and learning. Already in Castela (2015) a link between ATD and cultural historical activity theory is discussed, in particular between Roth’s concept of crossing boundaries between different socio-cultural contexts and the issue of inter-institutional transitions. This basic idea is in the following taken up in a broader sense.

In view of the ongoing major research project, this contribution has the status of an intermediate step presenting programmatic ideas about linking ATD with the subject scientific approach. This combination might in particular be fruitful for research connecting detailed analyses of mathematical practices with a complex vision of learners and teachers in a way that (with respect to both sides) their intrinsic societal mediatedness is systematically incorporated. Though this rather pretentious goal can easily be formulated (at least by using abstract notions, which would require a lot of pages to be embedded in a coherent theory and to be explained in detail (Holzkamp, 1985, 1993)), the following lines also indicate that there is still a way to go combining both approaches in actual-empirical research (Holzkamp, 1985, pp. 509).

The structure of the paper is organized as follows: In the first two sections we introduce some notions from ATD and the subject scientific approach. Then we discuss the link of both approaches. Finally we illustrate the link and some of its aspects and opportunities considering the issue of “real numbers” in the transition from school to university: After an ATD-orientated overview about the nowadays typical treatment of real numbers in German secondary schools and considering various options of extending this discourse in the transition from school to university , we discuss subject scientific related aspects taking into account societal and school-related findings and how they might contribute to validate mathematical and didactical practices. This theoretical analysis exemplarily demonstrates how the link intrinsically integrates content- and subject-related perspectives and leads to hypotheses for actual-empirical research projects.

SOME NOTIONS FROM ATD

ATD (Chevallard, 1992; Winslow, Barquero, Vleeschouwer & Hardy 2014) aims at a precise description of knowledge and its epistemic constitution. Its concepts make possible to explicate institutional specificities of knowledge and related practices. An underlying conviction of this approach is that cognitive-oriented accesses tend to misinterpret contextual or institutional aspects of practices as personal dispositions. A basic concept of ATD are praxeologies represented in so called “4T-models (T,τ,θ,Θ)” consisting of a practical and a theoretical or logos block. The practical block (know-how, “doing math“) includes the type of task (T) and the relevant solving techniques (τ). The logos block (knowledge block, discourse necessary for interpreting and justifying the practical block, “spoken surround“) covers the technology (θ) explaining and justifying the used technique and the theory (Θ) justifying the underlying technology. Praxeologies give descriptions of mathematics by reference models that are activity oriented (techniques, technologies). The interconnectedness of knowledge is modelled in ATD by means of local and regional mathematical organizations that allow contrasting and integrating practical and epistemological aspects in relation to different institutional contexts. Therefore ATD is in particular helpful for analysing institutional realizations of mathematical knowledge within different learning contexts, e.g. the use of mathematics in signal theory (Hochmuth & Schreiber, 2015).

More than 15 years ago Chevallard has introduced the additional notion of “scale of levels of codeterminations” that in the meanwhile has become rather important in ATD analyses (Bosch & Gascón, 2006). The hierarchical sequence of levels covers civilisation, society, school, pedagogy, discipline, domain, sector, theme and subject (in the sense of topic). Each level provides some kind of framework, within among others actions on lower levels are possible, supported or hindered and praxeologies are in a certain sense embedded. In Barbé et al. (2005) is shown, for example, how general didactic restrictions for teaching mathematical topics in school can affect teachers’ practices and their established praxeologies, in particular the shaping of the practical and the logos block and the relations between them. To mention one further example, Job & Schneider (2014) argue that smoothing the transition-gap from calculus to analysis shows at least the tendency to blur the distinction between the different discourses in school and university, which tends to reinforce an empirical positivist attitude by students as an epistemological obstacle to learning (ditto; p. 641). Generalizing their arguments, one might say that there are issues relating to general world views (society) that affect institutionally settled praxeologies.

Moreover, Chevallard (1991) introduced the notion $R_I(x, o)$ indicating the relation of a position x (a typical position of an individual) within an institution I to a praxeology o. The “scale of levels of codeterminations” underlines that the institution and, with the institution, the position x and the praxeology o have to be considered as being societal situated, i.e., that in their analyses specific emphasize also has to be put to societal assignments that are related to societal mediation processes. The subject scientific perspective, which is introduced and discussed in the next both sections, allows to further specify positions x keeping the significance of societal and institutional mediatedness (in a materialistic sense, see for example (Arndt, 2013)).

SOME NOTIONS CONCERNING THE SUBJECT SCIENTIFIC APPROACH (“CRITICAL PSYCHOLOGY”)

“Critical Psychology” claims to present a scientific discussable and criticisable elaboration of basic psychological concepts (categories). The starting point is a historical-empirical investigation of general historical-specific characteristics of relations between societal and individual reproduction as well as its dialectic mediatedness. One of the central subject related categories is “action potence”, which is the potence to ensure the disposal about the subject’s individual living conditions together with others (Holzkamp, 1985, pp. 239).

Within the context of this paper there are three important points to notice: First, the actual historical-specific form of subjectivity is characterized by the “possibility relation” regarding the societal reality, which includes in particular the basic experience of intentionality. Second and connected to the first, the specific modality of subjective experiences comprises the discourse form “reasoning discourse”: “I” speak about my “own” actions in terms of subjective reasonable (not necessarily “rational”) activities and of premises in the light of “my” life interests. A third crucial point is that the “human’s relationship to the environment is almost always mediated. […] Categories of psychology like learning, emotion, motivation and cognition cannot fail to be significantly altered by the fact of our existence’s social mediatedness. The most important mediation category is meaning.” (Tolman, 1991, pp. 14-15)

These three interrelated issues are combined in the assertion that conditions are given to “me” in terms of meanings in the sense of generalized societal action possibilities and that reality aspects, which are relevant for “me”, denoting again the generalized subject standpoint, become premises for “me” in the light of “my” life interests. Therefore, subject scientific considerations are essentially given by meaning-premises-reasons-relations, which a priori situate experience and activities of the (individual) subject “within the world” Accordingly, Holzkamp (1985, pp. 342) figured out the level of subjective action reasoning as the main subject specific level: It represents the level with respect to which individual experiences and activities (e.g. learning) has to be reconstructed and analysed.

Via the specific notion of meaning, human activities, like teaching and learning, are intrinsically thought as societal mediated. This implies that any analysis of subjective actions requires the reconstruction of subjectively relevant conditions in the sense of generalized action possibilities and the consideration of their societal mediatedness. Since meanings appear (via objective-subjective premises) as the medium within which subjects’ reasoning discourses are grounded, their study is a prerequisite for describing and analysing related cognitive, motivational and emotional processes as aspects of subjects’ activities. But, although meanings in the indicated sense are rather relevant for acting and thinking, they do not determine them. Instead, they represent action possibilities that might become premises in the light of subjectively perceived “life interests”.

THE LINK BETWEEN ATD AND THE SUBJECT SCIENTIFIC APPROACH

The position x within an institution can be (re-)interpreted as the “position” and/or “situation of life” from the subject point of view, which includes intentionality, the modus of reasoning discourse and societal mediatedness. As an “element” (a “position x”) of an institution a subject is typically not confronted with the whole world but only with a local “situated” section represented by those meanings typically produced and reproduced within the institutional context. Hence, institutional contexts provide specific frameworks for premise-reasoning-patterns. In this sense ATD’s praxeological analyses contribute to concretisations of meaning-premise-reasoning-patterns that are typical within an institution at the position x. In particular the concept of praxeologies allows capturing substantial aspects of mathematical practices in such a way, that they can be injected as facets of action related meanings, i.e., they can be (re-)interpreted as generalized societal action possibilities, which were potentially reflected in subject related reasoning schemes as premises and/or reasons. In this sense praxeological analyses can be seen as one non-trivial first step within subject scientific research projects: They might inform microanalyses of task solution processes by exploring institutional established practices. They are relevant for describing and analysing related activities, since they appear as institutionalized medium, within subjective action reasoning grounds. With respect to premises-reasoning-patterns the technological dimension of praxeologies, i.e. the justification and validation of techniques, is of specific importance. But, see above, praxeologies do not determine subjects’ activities, since there is an unconscious-conscious step by subjects of selecting, neglecting or highlighting facets of praxeologies in view of their evaluation of “life interests” and how they are perceived by them at “position x” in the “institution I” in view of all prospects addressed by “the scale of level of codeterminations”. Thus, the latter is rather relevant for both, the analysis of meanings (essentially by ATD) and the analyses of premises-reasons-relations (essentially by the subject scientific approach). In fact, both strands can’t be seen as totally separated but as dialectically interrelated, since institutionalized practices live through subjects’ [individual] activities.

THE ISSUE OF “REAL NUMBERS”

In this section we give first a short overview of the nowadays typical treatment of real numbers in German secondary schools in grade eight or nine. Because of the space limitation a detailed praxeological analysis can’t be presented.

The Treatment of Real Numbers in German Schools

The treatment of real numbers in German schools presumes that rational numbers are known and can be represented by ratios, decimal fractions and points on the number line. Moreover it is presumed that students are able to switch between those representations. In particular basic calculations should be understood and can be executed with respect to the different representations. The typical starting point for the introduction of real numbers is the observation or proof (sometimes!) that there are quadratic equations like $x^2 = 2$ without rational solutions. Next it is observed (but typically not proven) that one can find approximations by proper decimal fractions that fulfil those equations up to an arbitrary chosen error. On the other hand it is (geometrically) argued that there is a magnitude x, the length of the square diagonal, satisfying $x^2 = 2$ that corresponds to a certain point on the number line. The intuitive conviction about the existence of those points on the number line supports the idea, that (somehow converging) infinite sequences of approximating proper decimal fractions give a (unique!) final finite result, a number, that can be represented by a non-terminating decimal fraction. This type of discourse justifying the existence of infinite-finite objects (i.e., infinite processes giving in a certain sense a finite result) has in particular been considered in Lakoff & Nūñes (2002) as basic for the whole analysis and denoted as "basic metaphor of infinity". The new objects of non-terminating and non-periodic decimal fractions are called irrational numbers and build, together with the already known rational numbers, the set of real numbers. Moreover, the calculation rules that are known for rational numbers are assumed to be also true for all real numbers. Whereas in former years one can at least find Descartes' geometrical arguments for explaining multiplication and division for general real numbers such arguments are nowadays missing. Corresponding to the sketched treatment of real numbers there are nearly no tasks that are related to structural aspects of real numbers or that enforce to reflect arguments of the discourse concerning limits or the existence of points or numbers respectively. Instead the tasks focus on various isolated techniques that are locally established, for example approximation techniques like interval bisections and the Heron algorithm or the use of calculation rules.

In terms of the 4T-model the established mathematical praxeologies can be characterized as essentially punctual (or at most local) with isolated types of tasks and corresponding isolated techniques, where the tasks can be solved without referring to superordinated technological aspects, i.e., there are praxeologies $(T_i, \tau_i, \theta_i^{weak},\ )$ with technologies θ_i^{weak} having in particular weak connections to T_j, τ_j for $j \neq i$.The technological and theoretical discourse remains (so far it is represented at all) mostly implicit and essentially incomplete. These observations blend with those presented by González's et al. (2013) institutional analysis and with results from a qualitative study by Bauer, Rolka & Törner (2005). For corresponding results considering prospective secondary mathematics teachers we refer to (Sirotic & Zazkis, 2007) and for an actual rather detailed study, which problematizes in particular the use of the number line and investigates the knowledge of fresh French university students see (Durand-Guerrier, 2016).

Real Numbers in University Education and Potential Foci for Transitions

First of all it is interesting to notice that mostly the treatment of real numbers in university is either done axiomatically (courses for math majors) or (more or less) skipped (courses for engineers and natural scientists), but does generally not intend to connect or complete school praxeologies, i.e. for example: showing the one-to-one correspondence of number line and decimal fraction views (Kirsch, 1966); discussing geometrically the completeness of the number line (Artmann, 1983); showing how to add and multiply non-terminating and non-periodic decimal fractions.

For improving this situation one might think of transition measures that are adjusted to study courses for, e.g., math majors, prospective grammar school teachers or engineering students. A general scheme suitable for describing and analysing desirable transitions is given as follows (Hochmuth, 2018):

$$R_S(s, o) \rightarrow R_U(\sigma, \omega[X(o)])\ with\ X \in \wp\{\tau, \theta, \Theta\},$$

where o represents a praxeology and s a student within the institution S (school), ω a praxeology and σ a student within the institution U (university) in relation to one or several blocks of the praxeology $o = [T, \tau, \theta, \Theta]$ and $\wp$ is the power set symbol. The scheme works as a heuristic tool and allows to express that techniques, technologies or theories of o might be differently relevant for the relation of S within U to a (perhaps new) praxeology ω (see (Biehler & Hochmuth, 2017) for a slightly more restricted scheme).

Applying the scheme we illustrate next various transition foci: At first the focus might be on techniques τ_i and technologies θ_i^{weak} of o such that related skills are improved, but tasks, techniques and technologies are only slightly extended, for example: ordering of square roots and decimal fractions; applying calculation rules for simplifying terms. This might be important for all above mentioned study courses. Secondly the focus might be on technologies θ_i^{weak} and their further development (possibly also their theoretical embedding), for example: knowing, that square roots like $\sqrt{2}$ or $\sqrt{3}$ can only be approximated by finite decimal fractions; justifying real exponentials and powers (Winsløw & Grønbæk, 2014). This might in particular be important for prospective grammar school teachers. Thirdly the focus might be on techniques τ_i and the replacement of technologies θ_i^{weak} by technologies θ_i that are strongly and systematically embedded in real analysis, for example: constructing the set of real numbers by Dedekind cuts or Cauchy sequences; starting with axioms for $\mathbb{R}$ and identifying natural and rational numbers within this new set of objects. This might in particular be important for math majors. Within the ATD-framework each case could be analysed, explored and specified in greater details and in view of the scale of level of codeterminations. We skip further details in this paper.

Links to the Subject Scientific Approach

From the subject scientific point of view the analysed praxeologies represent meanings in the sense of institutionalized action possibilities. Students' reasoning and activities ground in those praxeologies but select, neglect or highlight them in view of an evaluation of their "life interests" and how they are perceived in view of the institutions *S* and *U*, corresponding to positions s and σ, and in particular aspects related to levels of codetermination. It is well-known that in the transition a lot of issues play a role, see e.g. Gueudet (2008). In the following we will discuss two different but complementing issues that are specifically linked to "Critical Psychology".

The first issue refers to the level “society” and in particular to Dowling’s sociological analysis of myths, exemplarily the myth of “reference” (Dowling, 2002). Within our frameworks Dowling’s myths can be (re-)interpreted as technological aspects of mathematical praxeologies in school that are related to the societal significance of educational processes. With respect to the latter Dowling differentiates between assignments concerning general societal aspects and those concerning their historic specificity. Considering the historic specificity the exchange-value aspect comes into play, which somehow undermines the use-value aspect and establishes a problematic mixture of both. This fits to the observation that actual school introductions of real numbers typically refer to “real world” problems like doubling the area of a quadratic piece of chocolate. Such references dominate the justification of introducing real numbers although treating the “real world” problems does not require “exact” solutions and, moreover, algebraic extensions would be sufficient to resolve this issue. Following Dowling, the myth of “reference” is not only a didactical issue that relates to illustrative introductions, but possibly leads to problematic technological and theoretical ideas, which do not disappear by establishing new, for example, axiomatic praxeologies, instead they possibly survive and constitute a strong epistemological (and motivational) obstacle (similar to the epistemological obstacle considered in (Job et al., 2014)) for students’ learning of university mathematics, in particular for future grammar school teachers. This could in particular happen, since Dowling’s myths might dominate students’ general view of their “situation of life” and therefore their accentuating of meanings.

The second issue relates to the organization of learning in school and to the “school-and-exam system” mentioned for example in Chevallard (2013). Partly because of this issue Holzkamp (1993) introduced the notion of defensive learning, a learning which primarily intends to prevent negative consequences. An important aspect of this notion is the opposition between ostensive and conceptual thinking that represents, according to (Holzkamp, 1985), the historic-specific societal concretization of the cognitive aspect of human activities: Ostensive thinking is essentially characterized by taking things as they appear to be and, in terms of the 4T-model, by strongly focusing on locally situated technical and technological issues, which blend with the above described praxeological organisation of “real numbers” in school.

Again, corresponding “ostensive” students’ views on their “situation of life” and related meanings-premises-reasoning-patterns might let transition measures’ intentions fail and in particular the incorporation of technological school-blocks within technological university-blocks, which results in new isolated praxelogies with new but still weak technologies. It is an empirical open question how this tendency is amplified by actual initiatives aiming to reduce transition problems by establishing a “university-and-exam system”.

FINAL REMARK

The hypotheses derived in the last section illustrate the necessity that an analysis of measures supporting students in the transition from school to university have systematically to take into account both approaches, the praxeological and the subject scientific as well as, with respect to both, the scale of level of codeterminations. The established link between ATD and the subject scientific approach facilitates theoretical and actual-empirical studies factoring in systematically aspects, which are intrinsically connected to the institutional and societal level and have impact both on institutionalized praxelogies and subjects' meaning-premises-reasoning-patterns.

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Evaluating Innovative Measures in University Mathematics – The Case of Affective Outcomes in a Lecture focused on Problem-Solving

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The transition from high school to university mathematics has proven to be difficult for many students but especially for pre-service secondary teachers. To support these students at mastering this transition, various universities have introduced support measures of various kinds. The WiGeMath project developed a taxonomy that makes it possible to describe and compare these measures concerning their goals as well as their frame characteristics. We exemplify the use of the taxonomy in the description of one specific innovative measure that was part of the WiGeMath evaluations. Moreover, we present first results concerning the goal-fulfilment of this measure concerning affective characteristics of the student cohort and their predominant beliefs.

Keywords: Transition to and across university mathematics, Novel approaches to teaching, Teacher education, Motivational developments, Beliefs.

BACKGROUND

In German mathematics teacher education, pre-service teachers first study at university before they enter a practical training. In this first phase, there is a strong focus on mathematical content, in particular in higher secondary teacher education where students mostly attend the same courses as mathematics major students. In these shared lectures, many internationally known problems of the secondary-tertiary transition arise (Gueudet, 2008), in particular, motivational problems and drop-out are often reported. There is a substantial decline in students' mathematical interest in the first semester with Cohen's d around 0.4 (Rach und Heinze, 2013, 2016), a decline in their mathematical self-concept with Cohen's d ranging from 0.5 to 0.7 (Rach und Heinze, 2013, 2016), and a strong dominance of controlled motivation over autonomous motivation (Liebendörfer, in press) in the terminology of Ryan and Deci's (2017) self-determination theory. Consequently, many pre-service students experience their university courses as a necessary evil rather than a helpful qualification towards their aspired job (Kalesse, 1997; Liebendörfer, in press).

THE WIGEMATH PROJECT

To counteract the negative effects which for many students seem to occur at the transition between school and university mathematics, many universities have introduced support measures of various kinds. Even though university internal evaluations of these measures mostly exist, a framework that helps to facilitate the comparison of design and outcomes of different measures has until recently been lacking. The WiGeMath project (Wirkung und Gelingensbedingungen von Unterstützungsmaßnahmen für mathematikbezogenes Lernen in der Studieneingangsphase; Effects and success conditions of mathematics learning support in the introductory study phase) [1], which is a joint research project of the Universities of Hannover and Paderborn (Colberg et al., 2016) led by Biehler, Hochmuth and Schaper, has developed a framework for goal dimensions and frame conditions of mathematics learning support in universities (Liebendörfer et al., in press) that aims at such a comparison. Moreover, the project has used the framework in first evaluations of various support measures at different universities in Germany. Some exemplary results for one representative of the category of redesigned lectures, which is one type of support measure that was evaluated in the project, is presented below.

THE WIGEMATH TAXONOMY

The aim of the WiGeMath project is to develop and exemplify in use a taxonomy that categorizes features and goals of Projects of Mathematics Learning Support (PMLS) and to use this taxonomy to evaluate different support measures at German universities. All projects that fall under the category of PMLS have in common that they try to support students at the beginning of their university studies in mastering the critical transition to university mathematics. They are innovative insofar as they deviate from the standard format of lectures and tutorials that is encountered in university mathematics even though the way how they do this differs. In the WiGeMath project, different PMLS are subsumed under one of four categories, namely bridging courses, mathematics support centres, support measures that parallel courses and redesigned lectures. Due to space limits, this text focuses only on redesigned lectures; a description of the other types of PMLS is given in (Liebendörfer et al., in press).

Redesigned lectures are lectures that offer particular support to students that have been shown to have higher risks at failing mathematics courses or focus on very specific learning goals in a non-traditional way. We examined both redesigned lectures that address preservice secondary teachers, who often show the greatest problems with the transition from school mathematics to more abstract mathematical content, and redesigned lectures which address engineering students who had already failed a compulsory mathematics test of some kind. All redesigned lectures have in common that new mathematical content is not the focus of teaching.

Different PMLS have different aims some of which are explicit but some of which stay hidden even to the teaching staff until they are inquired about them by an outsider. WiGeMath aimed at evaluating different PMLS based on their own assumptions following a program evaluation approach (Chen, 1990) as well as comparing them on grounds of an encompassing taxonomy that should categorize descriptive (non-normative) goals in the sense of criterions that the PMLSs set out to meet in their conception, features and conditions of PMLSs.

The taxonomy was constructed in a two-fold process. A first draft was developed by the project members by means of a document analysis, taking into account documents provided by project partners. The WiGeMath project collaborates with 11 partner universities in Germany at which PMLS have been implemented. The draft for the taxonomy was then used as a guiding thread for guided interviews with teaching staff of eight PMLS, two of each category. The interviews were taped and transcribed and afterwards the draft for the taxonomy was tested by trying to fit mentioned goals into the draft's categories. This led to minor refinements and reformulations of categories and yielded the final WiGeMath taxonomy.

This final WiGeMath taxonomy consists of three main categories, namely frame conditions, measure categories/ characteristics and goal categories. The frame conditions include various sub-categories, which help to characterize the student cohort addressed by a PMLS, the way it came about and developed, its embeddedness in the university course system, organisational characteristics that may pertain to it, characteristics of the room where it is held, financial and staff conditions and lastly characteristics of the learning culture. Measure categories/ characteristics serve to describe certain elements that characterize the PMLS in its structure, its didactical elements and its teaching staff. Finally, goal categories encompass various sub-categories of goals that either regard the individual learner or goals that the university may have in implementing the PMLS as a broader organization as well as a sub-category that allows to describe the quality of the goals. Not every sub-category has to be relevant in the description of every PMLS and some aspects of a PMLS may pertain to more than one category but the use of these categories aims to give an all-encompassing description of a PMLS's characteristics.

In the following, the use of the WiGeMath taxonomy shall be exemplified by applying it to one of the redesigned lectures.

CHARACTERIZATION OF REDESIGNED LECTURES

All in all, six redesigned lectures at five German universities were evaluated in the WiGeMath project. Out of these, four addressed preservice secondary teachers and two addressed engineering students. To reach a characterization of each lecture, an interview guided by the WiGeMath taxonomy was held with the teacher before the start of the semester. It was audiotaped and transcribed and the transcript was then used to name the measure's characteristics and sort them into the right categories of the WiGeMath taxonomy. What follows is the exemplary description of one of the evaluated lectures along the lines of the taxonomy's categories and sub-categories.

Frame conditions

Concerning the **characteristics of the student cohort**, those students that attended the given redesigned lecture were preservice secondary teachers in their first semester meaning that most of them were less than 20 years old and had just graduated from high school. The lecture has had round about 200 participants in each of its turns.

The **development of the lecture** officially started in 2011. Since then, there has not been a strict script, which is followed each year, but the different lecturers who have been responsible focused on different aspects. Nonetheless, the basis of the lecture always is the book by Grieser (2013) which deals with problem-solving strategies and proofs.

As to the **embeddedness of the lecture** in the wider system of university lectures, it is compulsory for preservice secondary teachers and voluntary for mathematics majors in their third semester. For preservice secondary teachers the lecture has substituted Linear Algebra as a first semester lecture though they still have to attend Linear Algebra in their second semester.

Staff conditions have been marked recently by problems to find qualified tutors to give the tutorials that support the lecture. As noted before, though there is only one lecturer per semester, it is not always the same one.

Finally, the **learning culture** is characterized by a strong focus on the students being active in their learning. They are supposed to try new methods and solve tasks during the lectures as well as during the tutorials. The concept of a “thinking pause” is very much enforced in the problem solving process. The lecturer gives some new input at the beginning of each class and collects and discusses results after the students have worked on problems or proofs.

Measure categories/characteristics

As to the **structural characteristics**, the measure consists of a lecture of two times 90 minutes per week with a tutorial of 90 minutes per week. One cycle of the measure starts at the beginning of each winter semester and finishes at its end (October through January).

The **didactical elements** include weekly homework and tutorial work of three to four exercises. All exercises may be worked on in groups. The solutions are discussed in the tutorials but no exemplary solutions are handed out. During the lecture there are phases of teacher talk, partner work and individual work. The book by Grieser (2013) is named as a reference text and can be accessed online on campus. At the end of the semester, a written exam concludes the course.

Concerning the **characteristics of the teaching staff**, the lecturer has his PhD in mathematics and is responsible for the contents of the lecture as well as the tutorials, the exercises and the final exam. The tutorials are given by six tutors who are students in higher semesters. These same tutors also have to correct the exercises which are handed in by students. As mentioned before, the selection of tutors proved difficult due to a small number of qualified applicants.

Goal categories

In the category of **individual learning goals** the measure focuses on activity-oriented rather than on knowledge-related goals: Both mathematical working strategies, like problem-solving strategies or use of examples and counter-examples, and learning strategies shall be improved. Attitudinal goals play a major role, as well: The measure aims at strengthening process beliefs and weakening toolbox beliefs in the sense of Grigutsch and Törner (1998) and wants to introduce the students into the mathematical professional community. Affective characteristics are to be influenced insofar as anxieties shall be lessened, interest and motivation shall be strengthened and the students shall gain a higher mathematical self-efficacy. Moreover, the measure wishes to let students recognize the relevance of its contents for further university studies.

The **system-related goals** include the preparation of the participants for their further university studies and the decrease of the number of dropouts. Besides, the measure wants to increase the quality of the feedback that students receive during their studies.

The **quality of the goals** as understood in the WiGeMath taxonomy is not to be understood in a normative sense but rather as a description of their substantiality. With this aim in mind, goals are examined concerning how specific, measurable, accepted, realistic and time-phased they are. Such a description of the quality of a goal would be done for every goal individually in a thorough analysis but due to space restrictions we will only focus on one specific goal in this paper to illustrate the point: One of the measure's goals is to improve affective characteristics of the participants, i.e. to lessen maths anxiety, increase motivation and interest and improve the participants' mathematical self-efficacy, in the course of the semester. This goal is specific to the point that it explicates what shall be achieved when by whom. It remains unspecific in naming why the goal is important, who holds the responsibility to reach it and which preconditions or limitations possibly exist. The goal is indirectly measurable through a survey directed at the students and it is accepted as it was named by the lecturer as a goal he wants to achieve rather than a goal he has to achieve due to orders given from above. The goal seems to be realistic to the point that the affective characteristics of the students seem to change for worse quite fast at the beginning of their university studies so it seems plausible that they may be changed for better even within a single semester. Still, this has to be checked as will be shown below. Finally, the goal is time-phased as it shall be reached within a limited time, namely the duration of the measure.

In this paper we will evaluate the following research question: To what extent was the described redesigned lecture successful in achieving the last-mentioned goal of influencing affective characteristics of the student cohort and in how far was the lecture successful in changing beliefs away from toolbox beliefs towards process beliefs?

METHODS

To measure the extent to which the above mentioned goals were reached, two questionnaire surveys were conducted with the participants of the described lecture. The first survey (t1) took place in the second week of the winter semester 2016 and the second one (t2) was conducted in the second to last week of the same semester. For each survey a questionnaire was developed, each one laid out to take about thirty minutes to complete. These questionnaires were handed out at the end of a lecture so that only those who were present that day could participate and participation was voluntary which the students were informed about. Moreover participation was anonymous but students used an individually constructed code so that the results of each participant in the first survey could be compared and contrasted to the results in the second survey. 163 students participated in the first survey and 103 in the second. We analyze the data of the 76 participants who answered both questionnaires.

We used adopted versions of the scale of Schiefele, Krapp, Wild and Winteler (1993) to measure interest, the scale of Schöne, Dickhäuser, Spinath, & Stiensmeier-Pelster, (2002) to measure mathematical self-concept, a translation of the instrument of Longo, Gunz, Curtis and Farsides (2014) to measure the experience of competence, autonomy and social relatedness, an adopted version of the PISA 2000 instrument for self-efficacy (Kunter et al., 2002), a shortened version of the scales by Grigutsch and Törner (1998) to measure beliefs (application, process, system and toolbox) and a translated version of the academic motivation scale (Vallerand, Pelletier, Blais, Briere, Senecal and Vallieres, 1992) to measure different types of motivational regulation (intrinsic, identified, introjected and extrinsic). We used Likert scales ranging from 1 to 4 for self-concept, self-efficacy and beliefs, from 1 to 5 for motivational regulation, from 1 to 6 for interest and from 1 to 7 for the experience of competence, autonomy and social relatedness. All reliability coefficients (Cronbach's alpha) are acceptable or better, compare Table 1.

For each scale and each survey, a descriptive data analysis was conducted in order to get an overview of the results. Though the entirety of scales included more than the ones mentioned above, we concentrate on these only as our focus is to check to what extent affective characteristics of the student cohort, their experience of competence and their attitude towards different beliefs was changed in the course of the semester.

RESULTS

Table 1 shows the changes in mean values during the semester and effect sizes (Cohen's d) as well as p-values of paired t-tests.

Scale	Number of items	Cronbach's α		Mean value		Cohen's d	p-value
		t1	t2	t1	t2		
Interest for mathematics	9	.83	.83	4.23	3.96	0.32	**.001**
Mathematical self-concept	3	.81	.81	3.03	2.96	0.12	.149

Experience of competence	6	.80	.81	4.63	4.25	0.40	< **.001**
Experience of social relatedness	6	.85	.89	5.39	5.40	0.01	.900
Experience of autonomy	6	.73	.77	4.81	4.61	0.20	.132
Mathematical self-efficacy	4	.83	.87	2.72	2.66	0.10	.353
Application beliefs	4	.80	.88	3.01	3.02	0.02	.889
Process beliefs	4	.67	.85	3.26	3.18	0.12	.306
System beliefs	7	.79	.84	2.97	2.93	0.07	.534
Toolbox beliefs	5	.66	.74	2.75	2.56	0.34	**.002**
Intrinsic regulation	5	.88	.88	3.82	3.55	0.33	**.001**
Identified regulation	4	.72	.78	4.01	3.81	0.24	**.026**
Introjected regulation	4	.73	.78	2.04	2.19	0.18	.097
Extrinsic regulation	4	.64	.72	1.78	1.88	0.12	.278

Table 1: Scales and their Cronbach's alphas, means, effect sizes of changes between the two surveys and p-values for a significant change.

We see a substantial decline in interest and in the experience of competence, whereas students' mathematical self-concept and self-efficacy did not change significantly. We can also see that the objective of reducing toolbox beliefs was clearly achieved, but not the objective of strengthening process beliefs. The mean values of motivational regulation show that intrinsic and identified regulation are dominating although they are decreasing in the course of the semester.

DISCUSSION

The observation of a decline in interest is similar to the results showing a decline in traditional courses; however, student's mathematical self-concept does not change significantly, which is a major difference (Rach und Heinze, 2013, 2016). Although in our tests we were not able to show that the course could raise student's self-efficacy, it did not reduce it significantly either, which may still be an achievement. The dominance of intrinsic and identified motivation is a positive result as a study in traditional courses found extrinsic and introjected regulations to be dominant (Liebendörfer, in press). Thus, although students' interest in university mathematics and their intrinsic motivation may reduce, they do not seem to develop a stronger feeling of being inadequate for studying mathematics in the newly designed lecture. The decline in interest as well as intrinsic and identified regulation may be explained by a decline in the experience of competence. The change in students' toolbox beliefs is remarkable as beliefs are rather stable by definition and toolbox beliefs did not change in other studies in the first year of lower secondary or primary mathematics teacher education (Kolter, Liebendörfer & Schukajlow, 2016; Liebendörfer & Schukajlow, 2017).

These results show that a specifically designed lecture may reduce problems of the secondary-tertiary transition in mathematics. Nonetheless, the question remains whether such lectures prepare the students for their further studies just as well as traditional teaching does, considering that the course covered fewer mathematical topics.

Moreover, the analysis that has been done to this point cannot ensure that the results obtained were produced by the innovative measure alone. First of all, the lecturer's personality has an influence on the measure's outcomes that could not be separated from the outcomes of the measure itself in our study. A possible further effect may be caused by the change in order of other lectures, in explicit the postponement of Linear Algebra to a later semester: Whereas students usually experience their low competency in both Analysis and Linear Algebra in the first semester, in this case it is only one lecture.

In order to test this hypothesis, a next step in the WiGeMath project will be to distribute the same questionnaire that was used in the investigation described above to a different innovative measure at a university where different courses are attended simultaneously, as well as to a traditional lecture. This will make comparisons more explicit.

As to the taxonomy that was developed by the WiGeMath project, this in part resembles the objectives of other taxonomies (Krathwohl, 2002) though with a different focus. Whereas other taxonomies are mostly concerned with individual learning outcomes, the WiGeMath taxonomy targets a description and ensuing comparison of innovative measures as a whole. Though other taxonomies exist which classify systems of higher education institutions (for example the Carnegie Classification of Institutions of Higher Education, described in Bartelse & Vught, 2009), the perspective taken by WiGeMath to interpret such characteristics as goals is a new one.

So far, the taxonomy is only laid out to serve innovative measures and even in this area will have to be adapted as measures develop and improve. Traditional lectures have not been taken into consideration so far but we propose that these would also benefit from a similar taxonomy in terms of communicating frame conditions and learning goals. In our interviews with lecturers, we found that often even to them goals remained implicit until they were asked about them specifically. This might even more be the case in traditional lectures that have "worked" for a long time.

As mentioned above, in many cases goals stay hidden until a framework like the one developed by the WiGeMath project provides a common language to talk about them. Even though lecturers have specific intentions when they design a course with specific learning goals that a student cohort with certain characteristics shall achieve in a setting framed by staff conditions, learning culture, etc., they often lack guidelines to arrange these in a way that is comprehensible for others. Yet, only if they can explicate their ideas, can an evaluation be successful and show strengths as well as possible weaknesses of the designed course. In our example, the lecturer had his PhD in mathematics and had hardly been in contact with didactical theories and frameworks until the point of the WiGeMath evaluation. Hence, he had would not speak in terms of mathematical beliefs, for example. When the concept was explained to him, though, he clearly saw that one intention of the lecture was to change students' beliefs but to that point he simply lacked the vocabulary to explain this intention.

Our taxonomy will help to communicate goals between universities, staff and students as it provides a frame of reference and a common language as has been shown for one example in this text.

NOTES

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Defining as discursive practice in transition – Upper secondary students reinvent the formal definition of convergent sequences

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The investigation of limits is at the heart of analysis at university. Accordingly, it is a worthwhile topic for transition courses. The present study engages upper secondary students in reinventing the definition of convergent sequences. Using a commognitive framework, the central development stages of the definition from experiential to abstract are empirically investigated in terms of activated secondary school discourses. The students' familiarity with secondary school discourses is critical, as it allows them to transition from grasping processes with metaphors towards grasping them as formal and abstract objects. For this, school objects act as intermediate steps. Further studies of transition courses should explicitly address the role of students' secondary notions as resources for reifying processes into abstract objects.

Keywords: transition; limits; convergence; practice of defining; design research.

INTRODUCTION

The investigation of limits is at the heart of analysis on the university level (Cornu, 1991). Accordingly, limits are a worthwhile object of investigation in a transition course. In the German context, students also have previous knowledge about limits in the domain of derivatives from upper secondary school. At school, the students' notions are usually not developed into a formal understanding of limits, as it is not expected in the German curriculum. Students transitioning to university have to develop their formal notion of limits in the regular lecture, which can be difficult. Thus, transition courses located at school have a huge potential for supporting students in developing a more compatible formal and abstract understanding of limits.

Objects in school have an experiential basis, while objects in tertiary mathematics "are specified by formal definitions and their properties reconstructed through logical deductions" (de Guzmán, Hodgson, Robert, & Villani, 1998, p. 753). For different reasons, students in transition often do not use definitions as starting points for their reasoning about objects, as would be expected in tertiary mathematics (Edwards & Ward, 2008, Vinner, 1991). Engaging students in activities of defining objects on a trajectory from experiential to formal approaches might help alleviate these issues.

In this paper, five students in a transition course from secondary to tertiary education in Germany are asked to create a definition of convergent sequences, after having studied them with the model of epsilon strips the week before. The paper illustrates how students' progress from their experiential understanding of convergence towards a formal definition based on deductions. The qualitative analysis also unfolds how the students use their previous school knowledge for this.

DEFINING AS A MATHEMATICAL PRACTICE IN TRANSITION

Students' intuitions about limits

The transition from secondary to tertiary mathematics has been extensively addressed in terms of changes and obstacles (Thomas et al., 2015). A specific issue of transition in regard to the understanding of limits are the students' intuitions (for typical metaphors: Oehrtmann, 2009). The students' intuitions about limits are not surprising: Historically, mathematicians used their intuitions to think about infinity based on infinitesimals, and this kind of reasoning still permeates modern analysis despite not being accepted as adequate (Cornu, 1991). Students' intuitions are a fruitful starting point for reasoning about limits. Activities of using an epsilon-strip help students to understand the convergence of sequence in terms of neighborhoods, in which are "almost all terms of the sequence" (Przenioslo, 2005, p. 88). They can also help students to understand the logical relations of ε and N (Roh, 2010).

Another issue is the role of pre-formal notions about limits. In upper secondary classrooms in Germany, pre-formal notions of limit are encouraged by teachers and textbooks, as for example $n \to \infty$ is commonly referred to as "tends to". By using and extending metaphors of "how many terms make a party" (p. 335) into "an infinite amount of terms will be in that epsilon neighborhood and a finite amount of terms will be outside", this understanding can be developed into a formal understanding of convergence (Dawkins, 2012). It has been illustrated that students can reinvent the formal definition (Swinyard, 2011), but not how students' intuitions and previous knowledge from secondary schools can systematically be activated in this process. Hence, carefully guided activities of investigating sequences with epsilon strips might help students make experiences that connect to their intuitions, but might at the same time be rich enough for students to develop a formal and abstract understanding of limits. While the first aspect has been investigated (see above), the latter aspect of transitioning to the formal and abstract while connecting to previous knowledge and intuitions has not yet been investigated.

The practice of defining

Definitions are the central means for grasping objects on the tertiary level, and the starting point for mathematical reasoning about these objects (Vinner, 1991; Alcock & Simpson, 2002). However, in nearly all mathematical domains, students in transition often rely on their informal understanding of objects, instead of definitions (overview in Thomas et al., 2015). There are several reasons why students have difficulties with defining and definitions. Definitions describe objects in arbitrary ways (Vinner, 1991), i.e. students do not have an experiential basis with the defined objects. Furthermore, students are usually not engaged in practices of defining. Instead, students encounter definitions in processes of proofing or validating (Swinyard & Larsen, 2012). When students can encounter definitions as product of their own making, it might help them to become aware of the nature of definitions at the university level (Swinyard, 2011).

In sum, students can and should be engaged in practices of defining limits. From a transitionary standpoint, practices of defining should be rooted in familiar secondary activities with empirical investigations to create an experiential basis (de Guzmán et al., 1998). Then, practices of defining in a transition course should start with an experiential basis, and let students reinvent an own 'arbitrary' but viable definition, and later progress to a more formal and abstract definition.

THEORETICAL BASIS: COMMOGNITION AND GUIDED REINVENTION

Commognitive Perspective on Learning and tertiary mathematics

Within the framework of commognition, students' participation in discourses is investigated in terms of changes in the ways students discursively realize mathematical objects in their utterances, where such changes constitute learning (Sfard, 2008).

Mathematical discourses can be distinguished from each other by their use of keywords (e.g. "tends to"), of visual mediators (e.g. graphs and symbols), of practices ("routines", patterned activities like defining or proving) and of narratives (like definitions) (Sfard, 2008, p. 134). Of special interest in the present study is the students' use of visual mediators and of narratives. Visual mediators are central means for grasping convergent sequences with epsilon strips and with mathematical symbols. Narratives are the means for students to discursively realize relations between objects, or between the facets of the object convergent sequence, like ε or N.

The development of a definition (as a narrative) and its associated mathematical can be characterized through the ways it is discursively realized over time. This development of the definition can be visualized as a "realization tree". The branches of the realization tree represent the distinct ways in which a definition, in this case about the object convergent sequence, is discursively realized by the students (Sfard, 2008, p. 153), up to a certain point in time in the discourse.

Research program of design research and framework of guided reinvention

The present study follows a guided reinvention approach based on realistic mathematics education. It engages students in context problems in which a mathematical object is the result of own experiences and activities (Gravemeijer & Doorman, 1999). This is a fruitful approach for introducing tertiary mathematical objects (e.g. Dawkins, 2012; and many others). It is located in the research program of design research (Prediger, Gravemeijer, & Confrey, 2015).

The underlying hypothetical learning trajectory builds on the research on the learning of limits. Building on the epsilon-strip activity in the previous session, students are asked to document all facets which are relevant for convergence, for example the height of the strip. Afterwards, the students have to bring these previously found facets into a logical relationship in the form of a narrative. At the same time, the students have to formalize the activity, including the process of finding a limit candidate. A critical step in this trajectory is the need to progress from an x-first perspective, in which students focus on inputs (x-values) and their respective outputs (y-values), to a y-first perspective. The x-first perspective emphasizes finding a candidate for a limit (Swinyard & Larsen, 2012). In regard to continuity, the y-first perspective and finding a limit candidate is implied in the epsilon-strip activity, as students first choose a strip with a certain height and afterwards arrange it on the sequence. Nevertheless, previous experiences with functions might still guide students to consider an x-first-perspective, so that the y-first perspective needs to be stabilized.

Research questions

In the present study, the students' progression from their experiential notion of convergence with epsilon-strips towards a more the formal, abstract notion of convergence along the outlined trajectory is investigated. The students are engaged in practices of defining, in line with the previous considerations. This encompasses, among others, the following activities: 1. Identifying central elements that should constitute a definition, and baptizing them, and 2. exploring or deducing relations of these elements. Of special interest are the students' ways of building on and connecting with their previous secondary school discourses. The study focuses on the emergent practice of defining, as students' preceding exploration of sequences with epsilon strips has already been studied in other studies (Przenioslo, 2005).

The following research questions will be investigated in the present study:

Q1. What are the crucial steps in the students' learning trajectory from experiential investigations towards a formal definition of convergence?

Q2. How do students activate their previous knowledge from school, and what role does this knowledge play to proceed to a formal, abstract definition?

METHODOLOGY

Participants and implementation

The design research project encompasses three design experiment cycles, the data analyzed in this paper stem from the third cycle in which five students participated (Ludwig, Lawrence, Dominic, Leif and Tanja). These students are highly proficient eleventh-graders, in their penultimate year of upper secondary education. They participate voluntarily in a one-year long transition course, designed by the author for preparing for university STEM-studies. From their regular mathematics classrooms, the students are familiar with an informal understanding of continuity ("Drawing without lifting the pen") and with limits as "tends to". The teaching unit comprised 5 sessions of 90 minutes each, three sessions on convergent sequences, two sessions on continuity. The teaching unit was taught by a Master student with tutoring experiences at university level. All sessions were videotaped and transcribed.

The analysis focuses on Task 2 of Session 2, in which the students attempt to find a formal definition for convergence by drawing upon their activities with the epsilon-strips (Session 1). Before Task 2, the students recapitulated the activity of using the epsilon-strips with a given convergent sequence. Furthermore, the students have generated a 'knowledge storage' in the first session, where they documented the relevant mathematical facets of convergence, and visualized them. Session 2 finishes with the completion of the here investigated task.

Analysis of data

The transcripts of Task 2 in Session 2 were analyzed qualitatively in the framework of commognition (Sfard, 2008). The central steps of the analysis are:

To answer Q1, the collective ways to realize the object convergent sequences over time are analyzed and depicted in form of a realization tree. For that, the transcript is segmented according to the qualitatively different ways in which the students realized the object convergent sequence. As illustrated, changes in the ways of realizing objects with keywords, visual mediators and their relations with narratives in the discourse are indicative of learning. Each branch in the tree hence represents one distinct way of grasping convergence discursively. The progression from one branch of the realization tree to the next one marks a crucial step in students' learning trajectory, as the students change the discourse about convergence in some substantial regard.

To answer Q2, the transcript is analyzed in regard to episodes in which the students' discourse builds on school objects, which is indicated by keywords like "function", "y-value", and narratives, e.g. the metaphor of "tends to". These episodes are analyzed in terms of how the students proceed from secondary utterances about school objects towards more tertiary abstract and formal utterances. An object becomes abstract, when its narratives do not refer to empirical phenomena, but to other (previous) mathematical narratives. An object is considered formal, if it is typically realized with a symbol as visual mediator (Sfard, 2008).

RESULTS

Different Realizations of convergent sequences and their progression

After having investigated a convergent sequence for 20 min, the students engage in finding a definition for convergent sequences in Task 2 for the remainder of Session 2. For that, they can build on their "knowledge storage", in which they documented the central facets of convergence together with their respective graphical representations. The realization tree in Figure 1 illustrates how students progressively realize convergent sequences, from left to right, and achieve an abstract, formal definition (fourth branch). The small rectangles denote the predominant discursive means used. The turn numbers "Tx" under each oval localize the realization branches in the classroom conversation. The conversation lasts 393 Turns.

In regard to the question of proceeding from an experiential notion of convergence towards an abstract definition (Q1), this realization tree reveals several interesting features of the developing practice of defining in transition. At two points in the process students specifically go back to the experiential basis of working with the epsilon strip, in the first and third branch of the realization tree. In these instances, the students rely on metaphors to make sense of their experiences. In interaction with the teacher, who explicitly states the rule to not use the metaphor "tends to" after #237, the students adopt the phrase "it exists" from the teacher to form a new narrative about *A* in terms of functional relationships, leading to the notion of $A(m)/\ N_\varepsilon$.

In the second and fourth branch of the tree, the students activate their previous school knowledge in order to grasp their experiences in a more formal and objectified way. This will be investigated in more detail in the next section.

Activation of secondary discourses

The starting point for the students to define convergence are narratives about limits grounded in "tends to", resulting in a narrative about two interconnected limit processes: "*m* tends to *0* if *A* tends to infinity. And we need a target value" (#171-172). In this first branch of the realization tree (Fig. 1), the students rely heavily on the familiar school narratives of "tends to". Now, the teacher establishes the discursive rule that the students should avoid using "tends to" (#213-230). The students attempt to follow this rule, which leads to a change in the discourse (second branch in Fig. 1). The following conversation occurs right in the beginning of these attempts:

237	Lawrence	Yes, A is getting smaller and smaller, I mean, bigger, if m is getting smaller and smaller. But we should avoid the tends to.
238	Leif	One could try somehow proportional, like proportional-technique to plot it.
239	Ludwig	Dependent from A. Hence, writing in the index.
240	Lawrence	A in dependence of m.
241	Leif	I would have said, like, somehow m is proportional to 1 divided by A.

In this episode, Lawrence summarizes the result of the previous discussions in terms of "tends to". Leif proposes to think about the relations in terms of proportionality, and Ludwig and Lawrence pick this up in terms of functional relations.

The students can give up narratives with "tends to" by first replacing the "if…then" relationship with a functional relationship, as indicated by keywords of 'dependence'. These keywords show that the students' ideas are rooted in familiar secondary discourses about functional relationships. It seems that these secondary notions are brought into the discourse associatively. Accordingly, the viability of these notions is up to debate, and competing narratives are uttered (*#240, 241*). This guides Leif to specify his proposal into "*m* is proportional to *1/A*". In this episode, school notions allow the students to collaboratively develop new narratives. They help students to engage in a new discourse in which the discursive rules have changed.

In the following third episode, the students try to decide about the nature of the hypothesized functional relationship.

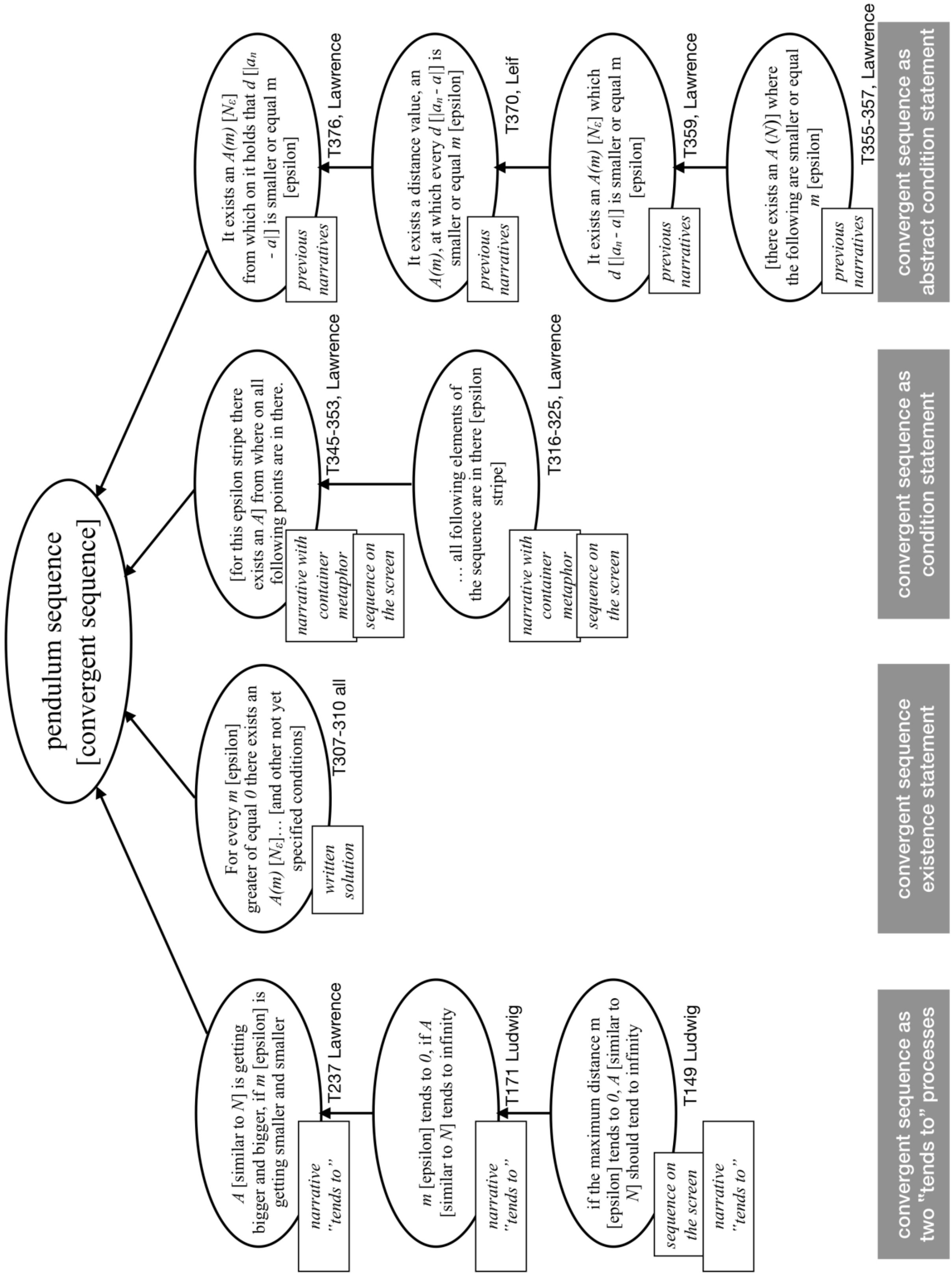

Figure 1. Realizations of the object convergent sequence

279	teacher	Yes. You already said, what else is dependent on m?
280	Lawrence	Yes, A.
281	Leif	We have the dependence of m and A.
282	Lawrence	I write this down now like in short as tends to. [Lawrence writes $m \to 0$, $A \to \infty$]
283	teacher	What is dependent on what? […]
287	Ludwig	m is the y-value, thus it is always dependent from the x-value.
288	Lawrence	You take, like, the strip and look then, from which value on all of them are in it.
289	Leif	Yes, but you, you #
290	Lawrence	We have never taken a value and then looked at how high the strip needs to be. […]
293	Leif	Yes, OK, then A is dependent from m.
294	Lawrence	If m gets smaller, A gets bigger.

While the students still document their ideas in terms of the narrative of "tends to" (#282), the teacher engages them in thinking again about the functional relationship.

Ludwig treats sequences in terms of properties of functions, namely in terms of y- and x-values (#287). Based on this, he concludes that, as with functions, *m* has to be dependent from *A*, taking a x-first perspective. Lawrence goes back to activities of using the epsilon strips, and how to choose a strip (#288), proposing an y-first perspective as implied by the epsilon strip activity. Hence, Ludwig treats sequences as a variation of the familiar secondary objects of functions, but his functional perspective is contested by Lawrence, who enforces his narratives about the epsilon strips and their height and placement (#288, 290). In the end, both narratives are merged as "A is dependent from m" (#293, 294), resulting in an endorsed narrative about *A(m)*.

Here, the students' previous school knowledge about functions and functional relationships is the link between the students' experiences with epsilon strips and the formal notion of *A(m)* as an object. It provides the narrative that A and m can be brought together as a combined object *A(m)* under a y-first perspective, and the visual mediator/symbol of *A(m)*. However, as Ludwig's utterance illustrates, treating sequences as functions is at the same time misleading, as it reinforces an x-first perspective. Above that, there is still an echo of the metaphor "tends to" in *A(m)*, as the students use this narrative to summarize their ideas about the functional relationships (#282, #294), suggesting that *A(m)* (analogue to N_ε) is an evolution of this narrative.

Synthesis and Summary

In answer to Q2, secondary discourses seem to have at least two functions in the students' progression to more abstract discourses (branches of the realization tree in Fig. 1), where abstract means that students more and more endorse narratives *about narratives* instead of narratives about experiences. First, they are the source for the students to develop new discourses with a new set of narratives and visual mediators – in this case about *A(m)* [N_ε] – but similar keywords, after the teacher establishes the discursive rules that "tends to" is not to be used (episode 1). Second, secondary discourses unlock objectified narratives. They allow to transform the narratives about experiences with "tends to" or with "being in the strip" into narratives about abstract objects of $A(m)$ [n_ε] (episode 2) and about d [$|a_n - a|$] and "$d \leq m$" (Fig. 1). This is an example of *saming*, where "proportionality" (#238) is imported into narratives about experiences with the epsilon strips (Sfard, 2008, p. 170). This step is critical in progressing from an experiential process-notion of relations, which is suggested by the activities of using the epsilon strips, towards understanding these relations as abstract objects. Nevertheless, the teacher is needed to provide the phrases and scaffolds by which the students can engage in new discourses with new narratives, e.g. about logical relations. This is expected, as the students neither have previous knowledge about the typical phrases and narratives from the tertiary level, nor about the practice of defining.

DISCUSSION

The here presented study contributes to the ongoing investigation of teaching interventions that help students to understand the abstract definition of limits. It especially highlights the role of the students' previous experiences with secondary mathematical discourses in practices of defining: Students from an upper secondary classroom connect to secondary narratives about limits and other objects (functions) in order to make sense of their experiences with the epsilon strips, and to formulate a definition. The students' secondary school notions are critical for transposing secondary narratives with metaphors ("tends to") into more abstract narratives *about narratives* that grasp logical relations. As relatively formal objects, school objects mediate between informal experiences of relationships and a more tertiary, abstract understanding of these relationships as objects, by allowing *saming* (Sfard, 2008).

Investigating students' practices in terms of familiar secondary discourses and of distinct steps of realizations has proven highly insightful to understand students' resources in transitioning to tertiary mathematical discourses. The results of the present study call for investigating students' transition to tertiary mathematics not only in terms of difficulties, but also in terms of students' (secondary school) resources and how these resources are situationally activated when the discursive rules change.

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Fostering Heuristic Strategies in Mathematics Teacher Education

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The "double discontinuity" stated by Felix Klein 1908 is still relevant in the mathematics teacher education at German universities. We are developing a course approach, which is intended to bridge the double discontinuity in a didactic dimension. We offer additional learning opportunities for teacher students, which are characterized by clarifications of mathematical methods that are fundamental to the regular lecture contents. We focus especially on reflections on heuristic strategies, which are the core part of mathematical methods according to George Pólya. Feedback shows that students consider our approach as helpful in the transition from school to university.

Keywords: Novel approaches to teaching, teaching and learning of analysis and linear algebra, mathematical methods, heuristic strategies, teacher education, Double discontinuity.

INTRODUCTION

In Germany, the first year at university is often associated with great problems for mathematics teacher students. The dropout rates in the study entrance phase are extremely high in mathematics compared to other study programs. (Dieter et al., 2008). Klein's "double discontinuity" is still an issue in the mathematics teacher education at German universities (Hefendehl-Hebeker, 2013). On the one hand, the double discontinuity can be understood as a discontinuity of content, but Klein also assigns a didactic dimension to it (Allmendiger, 2016). While mathematics teaching at school can be described as intuitive and problem-based, mathematics at the university is characterized by a deductive structure (Klein, 1905). The ideas and strategies behind the findings of mathematics are rarely presented at the university. The mathematics teacher students take part in the same mathematics lectures as regular students who study mathematics as a major subject. Therefore, bridging the double discontinuity is only possible by offering additional learning opportunities for teacher students. A few years ago, we implemented additional seminars for teacher students where we showed connections between school mathematics and university mathematics. Feedback of such seminars showed that students did not consider our approach as helpful for bridging the discontinuity. They said that the regular lectures were challenging and time consuming enough so they did not want to think about additional issues in our seminars. For this reason, we have developed an approach for additional learning opportunities for teacher students, which is characterized by reflections on intuitions and strategies that are fundamental to the regular lecture contents. Our approach is intended to bridge the didactic discontinuity. On the one hand, we want to support teacher students in the transition from school to university, and on the other hand, knowledge about strategies and intuitions is important in the future teaching activity at school. In the following report, we describe first experiences we have made with our seminars.

THEORETICAL FRAMEWORK

Subjects and methods in mathematics

In the philosophy of science, it is generally accepted that each science has its specific subject and its specific method[i]. As mathematical subjects in university mathematics, we deem the mathematical definitions, theorems and their proofs including mathematical algorithms. The mathematical methods describe how the subjects of mathematics are created or investigated by a mathematician[ii]. Which methods does a mathematician use, when formulating a new definition or a theorem? Furthermore: how does the mathematician come to the ideas when developing a proof? Sometimes the steps of a proof are easy in the sense that a beginner could do them but the decision to do these steps needs a lot of experience. In fact, producing a proof that does not already exist is an act of problem solving according to the definition of Schoenfeld (1985, p.74) or the idea Pólya (1973, p. 3) describes by mentioning "problems to proof". This point of view corresponds also with the definition of the concept "problem" the German cognitive psychologist Dörner gave in 1976 (p. 10[iii], own translation):

> What a problem is, is easy to define: an individual faces a problem if he or she is in an inner state that is unwanted by any reason, but the individual has not the means in the moment to transform the unwanted situation into the desired one. Three aspects characterize a problem:
>
> 1. An unwanted start situation s_α.
> 2. A desired final situation s_ω.
> 3. A *barrier* that prevents the transformation from s_α to s_ω at the moment.
>
> "We distinguish problems from tasks. Tasks are intellectual challenges for which the methods[iv] are known. […]. Tasks only need reproductive thinking while in problem solving you have to produce something new." (1976, p. 10, own translation)

Dörner (1976, S. 10, own translation) emphasised: "Whether a question is a problem or a task depends on the experience of the individual." If a student just learned induction "Prove the Bernoulli Inequality by induction!" as the first exercise might still be a problem but having done a series of similar proofs, it should be a task.

Based on this definition of a problem Dörner (1976) defines heuristic strategies as „the structure that organises and controls a problem-solving process" (Dörner, 1976, p. 273, own translation), which covers a very wide range of methods and is not bound to mathematics. Following this concept, heuristic strategies cover much of what is meant by the concept of "method" in the philosophy of science once applied to mathematics. Pólya (1973) already underlined this as the subtitle of "How to solve it" is "A new aspect of mathematical methods".

The following list of heuristic strategies was formulated based on the work of Pólya and others relying on the broad definition by Dörner. These strategies are arranged in groups for a better overview:

- organise your material / understand the problem: change the representation of the situation if useful, trial and error, use simulations with or without computers, discretize situations,
- use the working memory effectively: combine complex items to supersigns, which represent the concept of 'chunks', use symmetry, break down your problem into sub-problems,
- think big: do not think inside dispensable borders, generalise the situation,
- use what you know: use analogies from other problems, trace back new problems to familiar ones, combine particular cases to solve the general case, use algorithms where possible,
- functional aspects: analyse special cases or extreme cases, in order to optimise you have to vary the input quantity, discretize the situation,
- organise the work: work backwards and forwards, keep your approach – change your approach – both at the right moment.

Examples and detailed descriptions for each strategy are shown in Stender (2017). These strategies do not cover all the methods in the work of mathematicians but a broad range due to our literature analysis. Many of these strategies are also applicable beyond the field of mathematics[v] and this way of importance as general problem solving strategies. We would call a list of heuristic strategies that are used only in mathematics as they rely on a formal language "proof strategies". This list contains for example mathematical induction, proof by contradiction, proof by exhaustion, the Invariance Principle (Engel, 1998, p. 1), Cantor's diagonal argument, the box principle (Engel, 1998, p. 59) but also very simple ideas like "use an adequate substitution" or "add zero in an appropriate way." We do not claim that any of these lists is complete. A third aspect is the use of mathematical language like reading a mathematical text appropriately (for example see Hodds, 2014). We visualised these aspects in figure 1.

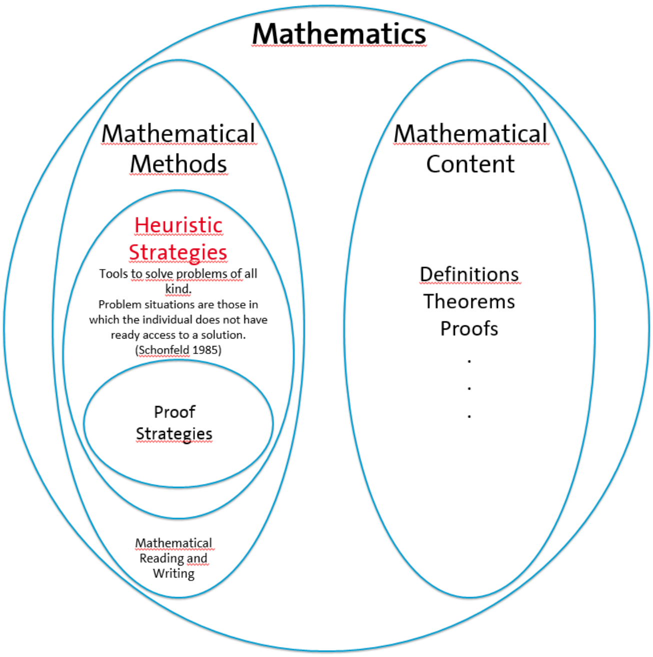

Fig. 1: Mathematical Content and Methods

The importance of the methods in mathematics

The discussion on problem solving since Pólya (1973) shows the importance of heuristic strategies in mathematics and thus (if one follows the argument above) the importance of mathematical methods in the process of learning mathematics. But mathematical methods are not emphasized often in (German) lectures in the mathematics departments. Therefore, we formulate an additional argument here.

We look at a special case of mathematics competence. Imagine a person that knows many mathematical definitions, theorems and even proofs by heart, but has hardly any mathematical methods available. This person would not be able to solve any mathematical problem.

The other extreme would be a person with little declarative knowledge of mathematical facts but competent in applying mathematical methods. He or she just has forgotten most of the mathematical content he or she learned. This person would be able to understand new mathematical content very fast and could fill gaps in the mathematical knowledge quickly if necessary.

This thought experiment shows that a lack of mathematical methods is a much more severe problem while doing mathematics than a lack of knowledge about mathematical content.

A mathematician often uses mathematical methods unconsciously. This is appropriate, as this is faster in the most cases than channelling everything through the restricted working space (see Miller, 1956) in the prefrontal cortex. For a future teacher it is much more important to be aware of his own methods as he or she shall later explain his actions to children. An explicit reference to his own methods is essential in these explanations.

Furthermore, the methods mentioned are the same in university mathematics and in school mathematics while the content shows little similarities as Klein (1945/1908) pointed out. This way the methods are an important link between school mathematics and university mathematics and give studying abstract mathematics a meaning for students who want to become a teacher.

These considerations lead us to the following goal for our research: we want to implement and evaluate teaching and learning of heuristic strategies in the mathematics teacher education in a close connection to the mathematics content trying to show that learning abstract mathematics means learning mathematical methods which is crucial for teaching mathematics in school.

IMPLEMENTED INTERVENTIONS

The approach we have developed for teacher students includes different additional learning opportunities:

- We offer special tutorials for mathematics teacher students connected to the Linear Algebra and mathematical Analysis lectures.
- Within the scope of Linear Algebra and Analysis lectures students must do weekly exercises. We have implemented a weekly workshop where students get support in creating and formulating proofs.
- All students may choose a preparation lecture before beginning their studies. Part of this lecture is a tutorial. We offer special tutorials for mathematics teacher students too.

These learning opportunities have in common that they do not intervene in the regular lectures and are not obligatory. This way we are free to realise our teaching approach, but this also leads to smaller groups so a standard pre-test-post-test design to evaluate the tutorials is not appropriate. We offered the tutorial over four semesters with 20 participants. This number was quite stable while participants changed: some gave up studying mathematics, some new students joined during the semesters when they heard about the tutorial. Due to the individual timetable of students, some of them couldn’t participate over all four semesters. The tutorial was open to students with a major in math too (parallel traditional tutorials exists that only focus on mathematical content) so overall five teacher students participated over the whole time.

As the tutorials fulfil different functions in the learning process, we offer the following aspects for the students:

Among other aspects as answering questions related to the lectures, we point out the appearance of heuristic strategies in the lecture content. Besides students solve small exercises so we can exemplify the application of heuristic strategies. We discuss the application of heuristic strategies in school by means of examples. Furthermore, we give a structured overview over the lecture content several times per semester to help students to network their knowledge. The overview is closely connected to some heuristic strategies as *using analogies* and *generalise the situation*.

An example for the use of heuristic strategies is the use of equivalence classes in linear algebra for example using quotient rings. Quotient rings were presented in the lecture using a formal definition. The lecturer gave $\mathbb{Z}/m\mathbb{Z}$ as an example but this structure was not familiar to the students. In the tutorial we showed the following ideas to the students:

- Using a special example, we presented the ring $\mathbb{Z}/4\mathbb{Z}$.
- For this structure, we gave several different representations – a number line where the numbers of the same equivalence classes were coloured in the same way, the four classes as sets, a representation on a circle and a table that showed the calculations in this structure.
- The importance of the concept of equivalence classes was emphasised: a number of objects is chunked together to one new object according to Miller (1965) so an element of a quotient ring is what we call a supersign according to Kießwetter (1983).
- We pointed out, that similar structures occur in school when dealing with rational numbers, which are classes of pairs of natural numbers (supersigns) and where several representations are necessary while working with fractions. Even if fractions are not implemented in this abstract description in school, students must deal with this mathematical structure.

This way similarities in school mathematics and university mathematics are brought into the focus of the students and showed that students in school dealing with fractions might experience the same difficulties as students in university dealing with quotient rings.

A second example from the tutorial is a short exercise we presented to the students: In the tutorial matrices were used in vector spaces with bases. In the tutorial, we showed how the matrix describing a rotation ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$, angle α) could be build. Then we asked the students to develop the matrix for a reflection through an axis that has the angel α with the x-axis. As this group was already used to heuristic strategies, we gave the following help:

- Change the representation (figure).
- Look for special cases (basis).

- Trace back new problems to familiar ones (use the knowledge from the rotation).
- Break down your problem into sub-problems (work on one element of the basis at a time).
- Use symmetry (to work on sin(2α-180°).

This example could also be situated in high school, so it breaks down university mathematics to school mathematics and shows in addition how heuristic strategies can be used supporting students work.

In the weekly workshops, students do exercises on their own. Mostly students have to prove small theorems related to the lecture content. They work in groups of three or four and a tutor supports them in developing and formulating proofs. In case students are not able to overcome barriers in proving processes by themselves, the tutor promotes the application of an appropriate heuristic strategy. Following a proving process, students reflect on the strategies they have applied in the process.

EVALUATION RESULTS

As said before the number of students participating in the additional learning opportunities is too small for quantitative designs. We are in the process of design-based research as our interventions are an early approach for the use of heuristic strategies in mathematics education. This means results of efficiency do not make sense in this phase of the research.

We have conducted interviews with five participants of the tutorials and the workshops at the end of each semester. We have analysed these interviews using qualitative content analysis according to Mayring (2010). In the interviews, the students were asked what they remembered from the tutorial and what they think was helpful (not discriminating helpful for studying mathematics or helpful for the later profession). If whole aspects were not mentioned the interviewer stimulated an answer using a key word. The two kinds of answers (with or without stimulation) were separated in the analysis.

Over the four semesters of this process, a slightly increasing awareness of the heuristic strategies in mathematics can be observed in the responses of the students. Over the time, the students gave more examples of the use of heuristic strategies during the interviews. The students gave a positive feedback according to the heuristic strategies as a link between school and university. They said that it delivered a meaning to the fact that the content of the mathematical studies had hardly any connect to school mathematics. One student mentioned that he used heuristic strategies to explain mathematics when he was working in school as a substitute teacher. The students regard the workshops as very helpful for their learning progress. They emphasize the importance of developing proofs by their own to understand the lecture content. Therefore, they consider it beneficial that the tutor gave only a few strategic hints in case of barriers in the proving process rather than presenting solutions. Especially the heuristic strategy of changing the representation helps students to develop new ideas.

We think that it can be expected that the heuristic strategies integrated in the lecture as metacognition, added to the mathematical content would be far more effective for learning mathematical methods.

CONCLUSIONS AND LOOKING AHEAD

Overall, we can observe that there are effects of the metacognitive input according the heuristic strategies as mathematical methods, but the learning process is very slow. It needs many different examples for each strategy to be able to realise the importance of these strategies and even more to identify them in your own mathematical work or use them explicitly to solve problems. Pólya (1973, p. 208) quotes a fictional traditional professor “A method is a device which you use twice.” Obviously “twice” is far not enough in the learning process – and not enough to make a device so important that it should be taught.

The teaching concept to support students by stressing the mathematical methods particularly by stressing the heuristic strategies seems to have a positive impact but is not realised easily. This corresponds with the difficulty of teaching problem solving and heuristic strategies (Schoenfeld1985, p. 95). The examples we gave to the students during our tutorials and workshops had an impact on realising heuristic strategies in mathematics on the long run. However, we do not know if they were not enough to let the student’s use these strategies in their mathematical work explicitly.

We redesign the tutorial if necessary and expand the number of learning situations where we implement metacognition on the use heuristic strategies. A long-time goal is to include metacognition in the lectures itself but that means to convince mathematics lecturer to change their lectures.

Our next project is developing a tutor training for master students tutoring beginners. In the tutor training, we will use former homework tasks and analyse the solutions according to the use of heuristic strategies. This will be presented to the students in few examples and then done on further tasks by the students working in small groups. In the next step, we will demonstrate how these strategies can be used explaining mathematics and supporting students to overcome barriers during problem solving. The students shall use their own results from the first step to practise this.

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[i] The concept of “method” is used in the philosophy of science in a much broader way than used by e.g. Schoenfeld (1985) when he mentions “This way a method becomes a strategy”. In this paper method is mostly used in this broader meaning of the philosophy of science.

[ii] There is a broad discussion whether mathematics is produced by humans or discovered and for both standpoints and for the idea, that mathematics is a mixture of both there are good arguments. For this paper it makes no difference if one changes the verb “producing” by “discovering”.

[iii] The original is in German language, this is an own translation.

[iv] Dörner uses the word “method” not in the same way “scientific method” is used in this paper. The word method has more the meaning of an algorithm here.

[v] Pólya (1973) gave examples from mathematics for the heuristic strategies he mentioned but also examples from outside mathematics for most of the heuristic strategies shown.

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