

First conference of the
International Network for
Didactic Research in University Mathematics

INDRUM 2016
March 31
April 1-2

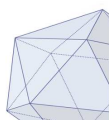
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PROCEEDINGS of INDRUM 2016

First conference of the International Network for Didactic Research in University Mathematics

March 31- April 2, 2016
Montpellier, France

Editors: Elena Nardi, Carl Winsløw and Thomas Hausberger
Publisher: University of Montpellier and INDRUM
Place: Montpellier, France
Year: 2016
ISSN: 2496-1027 (online)

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Editorial

Elena Nardi¹, Carl Winsløw², Thomas Hausberger³

¹University of East Anglia, E.Nardi@uea.ac.uk

²University of Copenhagen, Winslow@ind.ku.dk

³University of Montpellier, thomas.hausberger@umontpellier.fr

The International Network for Didactic Research on University Mathematics (INDRUM) was founded at CERME9 (9th European Congress on Research in Mathematics Education) in Prague, February 2015. A scientific committee of 18 scholars from 12 different countries was set up to plan a first INDRUM conference in the spring of 2016; it was decided that this congress was to be held in Montpellier, France, and an organising committee with 12 members from Montpellier, Barcelona and Paris was established shortly after. At CERME9, the board of ERME (European society for Research in Mathematics Education) launched a call for proposals on ERME Topic Conferences, aiming to be “organised on a specific research theme or themes related to the work of ERME as presented in associated working groups at CERME conferences”. Indeed, the proposal of INDRUM grew out of the Thematic Working Group on University Mathematics Education, in which most of the founding members had participated in CERME9 or at the two preceding ERME conferences (CERME8, CERME7) where this TWG had convened. The proposal for the first INDRUM conference, drawn up in the months following CERME9, was accepted by the ERME Board as the second in a series of (by now, three) ERME topic conferences.

The first announcement, published in March 2015, followed up on the decisions made at the founding meeting, calling for papers on two broad themes:

- teachers’ and students’ practices at university level; and,
- teaching and learning of specific topics in university mathematics.

Papers were to be worked on during the conference according to the CERME principles of organisation: short ‘reminder’ presentations of the papers that are made available to delegates a few weeks prior to the start of the conference and with a large part of meetings dedicated to discussions in thematically split working groups. It was expected that the two broad themes would result in a finer distribution of papers to a larger number of working group themes – five, as it transpired. As the conference was to be held in France, which hosts a large community of didactics researchers in the field of INDRUM, the papers could be submitted not only in English (as in CERME) but also in French – thus, making INDRUM a flexible, bilingual space for academic exchange.

The second announcement was published in May 2015, with further details for submission. We also had the immense pleasure to announce that Michèle Artigue (University of Paris-Diderot, France) had accepted to deliver the plenary address that would open the congress.

Following a slight extension of the deadline for paper submission initially set on November 1 to November 20, 2015 (an adjustment necessitated by the proximity of the ICME13 deadlines in that period), we were delighted to observe a first, and in fact overwhelming, success of the INDRUM enterprise: a total of 81 submissions (63 full papers, 18 poster proposals), almost equally distributed on the two broad themes. The review process was organised by the Chair and co-Chair, and involved invitations for two reviews per full paper, one by a member of the Scientific Committee and one by an author of another submission. Decision letters – that announced acceptance or recommendation for converting a full paper to a poster proposal or rejection – were sent out in January 2016. A total of 50 full papers and 18 poster proposals were accepted. The final number of papers and posters presented at the conference and included in these proceedings (46 full papers and 15 posters, with the latter represented in the Proceedings as two-page short papers) varied slightly as a small number of delegates retracted submissions or cancelled attendance for personal reasons.

Discussion of the accepted papers and posters was organised in five thematic working groups (TWG1-TWG5), based on a rough classification of contents. Two members of the Scientific Committee were invited to lead¹ each of the five TWG:

TWG1 (Calculus and Analysis): Stephanie Bridoux, Fabrice Vandebrouck.

TWG2 (Modelling and Mathematics in other disciplines): Alejandro S. González-Martín, Reinhard Hochmuth²

TWG3 (Number, Algebra and Logic): Faiza Chellougui, Maria Trigueros

TWG4 (Teachers' practices and institutions): Simon Goodchild, Nicolas Grenier-Boley

TWG5 (Students' practices): Ghislaine Gueudet, Chris Rasmussen.

The following table summarises the 46 papers and 15 posters finally presented at the congress, and published in these Proceedings:

	TWG 1: Calculus and Analysis	TWG 2: Modelling and Mathematics in Other Disciplines	TWG 3: Number, Algebra and Logic	TWG4: Teachers' practices and institutions	TWG5: Students' practices
Papers	13	7	8	8	10
Posters	3	2	1	6	3

¹ The success of the congress owes plenty to the leaders' thorough, sensitive and creative approach to organising the work of TWG1-TWG5.

² Due to personal reasons Reinhard did not attend the conference but worked with Alejandro towards the preparation of the TWG2 sessions which were then led single-handedly, and adroitly, by Alejandro. Reinhard's contribution to the panel was also incorporated in the panel discussion by the Chair.

The congress programme comprised: the plenary lecture of Michèle Artigue, entitled “Mathematics education research at university level: achievements and challenges”, seven (2x60m, 3x90m, 2x120m) sessions per TWG, poster presentation sessions and a plenary panel discussion led by Marianna Bosch, with Caroline Bardini, Reinhard Hochmuth², Chris Rasmussen and Maria Trigueros as panel members, on the theme “Current interactions between mathematicians and researchers in mathematics education”. A text reflecting the work of the panel³ can be found at the beginning of this volume of proceedings, after the text of the plenary lecture.

The conference was held between March 31 and April 2, 2016 and was attended by a total of 100 registered participants. In the light of the volume and quality of submissions, and substance of exchanges during the sessions, we are happy to conclude that the first INDRUM conference turned out as an eminent success.

Papers appear in these Proceedings in a version chosen by the authors following the (optional) request by the editors (congress Chairs, Elena Nardi and Carl Winsløw; member of the Organising Committee, Thomas Hausberger) to upload a final version of their paper soon after the congress.

A very special thanks is due to the organising committee, chaired by Viviane Durand-Guerrier and co-chaired by Marianna Bosch, for their tireless work of many months towards this large event. Thomas Hausberger was responsible for the website, with Simon Modeste and Pedro Nicolás. Administrative support was offered by Eric Hugounenq and Bernadette Lacan. These colleagues worked unstintingly before, during and after the congress to ensure that every participant had a smooth, productive and enjoyable INDRUM experience. They have set the bar high for the congresses to follow and we are indebted to them all.

What next?

A measure of success for an event like INDRUM is legacy and longevity. We are delighted to be able to announce the following three developments.

An International Journal for Research in Undergraduate Mathematics (IJRUME) Special Issue will be guest-edited by Elena Nardi and Carl Winsløw, with support from IJRUME Editor Chris Rasmussen and reviewers including members of the INDRUM2016 Scientific Committee. The Call for Papers is due soon after the publication of these Proceedings and the deadline for paper submission is November 1, 2016. We will invite papers of 6,000-8,000 words in length, written in English (and, to honour the bilingual approach taken during the congress, their abstracts in both English and French). We will aim to send decision letters to authors by January 31, 2017. Revised versions of the papers will be submitted by March 31, 2017. Final decisions for the papers that have been accepted for publication will follow by May 31,

³ We thank the panel Chair and its members for a vibrant, current and forward-looking session during the congress, skilfully reflected in the text prepared for these proceedings.

2017 with final versions of the papers, and production of the *Special Issue*, planned for late 2017 / early 2018. We aim that the *Special Issue* will be ready in time to celebrate its publication at INDRUM2018.

An ERME Topic Conference book is planned for publication after INDRUM2018. We envisage that INDRUM2016 TWG discussions and presentations will inform the planning of this volume which will continue at a meeting at CERME10 in February 2017. We envisage that substantial part of the activity in INDRUM2018 will involve the shaping of the book's chapters. Viviane Durand-Guerrier and Carl Winsløw will lead the editorial team.

Finally, we are absolutely delighted to be able to announce that the Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC, University of Agder, Norway) will host INDRUM2018 between Wednesday 4 and Friday 6 April 2018⁴. The conference will be chaired by Viviane Durand-Guerrier and Reinhard Hochmuth, with MatRIC director, and INDRUM2016 *alumni*, Simon Goodchild as Chair of the Organising Committee.

We now invite you to carry on reading this volume and we hope that the promise of its contents will encourage you to consider joining, or continuing to be part of, the ambitious, bold enterprise that is INDRUM!

⁴ These dates are provisional and reflect the intention to hold the next INDRUM congress during the week after the 2018 Catholic Easter (April 1).

Plenary talk

Mathematics Education Research at University Level: Achievements and Challenges

Michèle Artigue

LDAR, Université Paris Diderot – Paris 7, France, michele.artigue@univ-paris-diderot.fr

Abstract: In this text, associated with my opening lecture at the first INDRUM Conference, I will reflect on the achievements and challenges of mathematics education research at university level, in the light of my experience both as a researcher and a university teacher. First, I will come back to the historical evolution of this field of research through the retrospective analysis of some selected episodes of my personal trajectory. I will then consider the current state of the field, presenting my vision of its strengths and weaknesses, of the challenges it faces and the resources existing to take them up.

Keywords: mathematics education, didactics, university mathematics, university teaching, advanced mathematical thinking, Calculus, anthropological theory of the didactic

INTRODUCTION

Having been invited to give the opening lecture at this first conference of the new International Network for Didactic Research in University Mathematics is an immense honour. For several reasons, this is also a big challenge. The first reason is that this field, seen as an academic field of research, has a rather long history which can be traced back up to the seventies at least. To make sense of research achievements and of the limitation of these, one needs to be aware of some elements of this history. The second reason is that the field is currently a burgeoning field of research, addressing a huge diversity of questions and mobilizing an increasing diversity of theoretical approaches for their study. Making sense of recent research achievements, through the multiplicity of existing discourses, through the multiplicity of changing contexts where they are produced, is a very challenging task. The third reason is that the vision that each one of us has of the field is necessarily today a limited vision. This is also a biased vision, shaped by each individual's personal experience. I am not an exception in this.

Reflecting on how I could take up the challenge and organize this opening lecture, I decided to structure it into two main sections. In the first one, perhaps more directed towards researchers in the field who are not necessarily aware of the details of its historical development, I will try to provide an idea of the evolution of the field, through a reflective look at some important episodes of my personal trajectory. Then, in a second part, I will discuss the current state of the field, presenting my vision of its strengths and weaknesses, of the challenges it faces and the resources existing to take these challenges up.

As already stressed, this is certainly a partial and subjective vision of the field, with which the reader is not obliged to agree. I hope, however, that it will provide some useful elements to understand how this field of research progressively constituted and developed its identity, what it was able to achieve, and to outline perspectives for its future.

ENTERING THE FIELD OF UNIVERSITY RESEARCH: A MATHEMATICS-PHYSICS ADVENTURE

I entered research in university mathematics education at the turn of the eighties, when a group of mathematicians and physicists in my university, including mathematics and physics didacticians, decided to create an experimental course for students entering the mathematics and science programme. Our main goal was to challenge the compartmentalization of the two disciplines. The design of this course was very innovative with regular lectures on topics of common interest jointly prepared and given by one mathematician and one physicist, common tests, interdisciplinary projects (Artigue 1981). The whole team met once a week and everyone attended the common lectures.

Everything worked fine with the exception of the planned common lecture on the notion of differential for which the mathematicians and physicists could not reach an agreement. This « differential clash » became a research issue that I first addressed jointly with the two didacticians of physics in the team, Laurence Viennot and Edith Saltiel, then with Marc Legrand and his colleagues in Grenoble, who were working on the teaching of the Riemann integral.

If I consider retrospectively the research we developed (Artigue, Menigaux & Viennot 1990; Alibert et al. 1988), it is rather representative of some predominant characteristics of university research in mathematics and physics education at that time. The research has indeed a clear cognitive orientation. It aims at understanding the conceptions of the differential developed by our students, the difficulties they meet with this notion and the associated processes in mathematics and in physics, and it does so through a series of questionnaires and interviews.

This research also demonstrates a strong epistemological sensitivity, perhaps more characteristic of the French didactic community. With the help of historians, we made specific efforts to understand the source of the « differential clash » observed, why mathematicians seemed so rigidly attached to their vision of the differential as a function, in fact a differential form, and why physicists seemed so rigidly attached to their pragmatic vision of differentials as small, if not infinitesimal, increments. For that purpose, we worked on primary and secondary historical sources, and also systematically studied the traces of the differential/derivative educational debate.

This research also shows the strong desire of researchers for concretizing their research findings into educational action. The results of both the cognitive and epistemological analyses were used in order to make visible the negative effects of

the current situation to mathematicians and physicists. Students declared that it was better for them not to try to understand what a differential is, and to work mechanically, both in mathematics and in physics; they were not able to distinguish between situations requiring or not the use of differential processes, and they only succeeded in solving the tasks proposed to them because they learned to detect the linguistic hints in their texts calling for the use of such processes and to mobilize the rituals within each of the two disciplines. These results were also used to develop a compromise acceptable for the two teams. Moreover, math-physics tasks in line with this compromise were designed and implemented in specific workshops in the following years (Artigue, Menigaux & Viennot 1988).

However, this research, and more globally the work carried out in this experimental course, shows that ecology and sustainability issues were not really part of our agenda. In fact, when the teams moved to other projects and teaching activities, this experimental course disappeared.

EXPERIENCING THE ‘SCHIZOPHRENIA’ OF UNDERGRADUATE UNIVERSITY TEACHERS: THE TEACHING OF DIFFERENTIAL EQUATIONS

The second research I would like to briefly evoke is the research I developed regarding the teaching of differential equations some years later. At that time, I was working on issues related to dynamical systems with some colleagues, with the support of a specialist of this domain, the mathematician Adrien Douady. I experienced the type of ‘schizophrenia’ which is rather common to those who teach undergraduate courses: the complete disconnection between what is their lived experience as mathematicians and their lived experience as university teachers. The programme of the course on ordinary differential equations for second year students I had to teach was indeed focused on the algebraic solving in finite terms of some specific forms of equations, making the students think that the goal of research in this domain was to progressively complete the book of recipes they were shown through some exemplars. Only graduate students could have access to other views.

I decided to investigate the possibility of developing a course for beginners more in line with the epistemology of this domain. Once again, the epistemological work was an essential dimension of the research, leading to the identification of three main historical strands, each one of them having its own problématique and development: the algebraic, the numerical and the geometrical-topological strand initiated much later than the two first ones by Henri Poincaré at the end of the 19th century. Only the first strand, in its most elementary forms, was part of the course (the theory of exact resolution as initiated by Condorcet and Liouville was not considered).

To address the research question I chose a methodology of didactical engineering, and the use of didactic constructs familiar to French didacticians – such as the notions of setting and tool-object dialectics due to Regine Douady (1986) – together with

fundamental constructs of the Theory of Didactical Situations (TDS) (Brousseau 1997). This methodology was of course adapted to the specific context of university education. The didactical engineering was collaboratively designed with Marc Rogalski and his colleagues from the Université de Lille 1, and successfully experimented with for several years at this university (Artigue & Rogalski 1990). However, the experimentation showed that the viability of the design required a different institutional status for graphical representations to the limited heuristic status given to these in university courses; graphical representations had to be credited as a legitimate tool for reasoning and proof, of course in appropriated forms such as those developed in the engineering design (Artigue 1992). Moreover, we discovered that such a change could not be limited solely to the teaching of differential equations; for evident reasons of coherence, this change was to impact the whole approach of the Analysis course. This certainly contributes to explaining why, despite its repeated successful implementation, only the first situations of this design were more widely used. Today, conceptual tools such as the hierarchy of didactic codetermination proposed by the Anthropological Theory of the Didactic (ATD) (Chevallard 2002) help us systematically consider the different conditions and constraints governing the possible ecology of the didactical engineerings we design, beyond those situated at the level of the mathematical theme or sector directly addressed – and better anticipate their possible effects. We were less equipped to address these ecological issues thirty years ago.

These are just two examples among many others. They reflect the cognitive and epistemological focus of the research carried out at that time, and also the form that this focus was likely to take in the French didactic culture where TDS, with its underlying systemic approach was the predominant theoretical approach. It also reflects the engagement of researchers in action, but without the conceptualizations that would have allowed them to seriously address dissemination and sustainability issues.

THE INTERNATIONAL SCENE: THE AMT WORKING GROUP OF PME AND THE ICMI STUDY ON THE TEACHING AND LEARNING OF MATHEMATICS AT UNIVERSITY LEVEL

On the international scene, the state of research at that time is well reflected by the work of the Advanced Mathematic Thinking (AMT) working group of PME which I entered in the late eighties, and the synonymous book resulting from this work whose production was coordinated by David Tall (1991). This book confirms the cognitive orientation of research I already mentioned towards the study of students' learning processes, thinking modes, conceptions and difficulties, and also the strong influence of constructivist perspectives. One of the main aims of the working group was to elucidate the specific nature, if any, of what its participants called advanced mathematical thinking, and thanks to this elucidation, to better understand what differentiated learning processes at university from those experienced before by students. Even if a definitive answer is not provided in the book, some criteria are

proposed in terms of relationship to abstraction, symbolism and generalization, role of definitions, formal reasoning and proof. One can also observe the important role played in research by constructions that take the form of distinctions such as that between concept definition and concept image due to Schlomo Vinner and David Tall (1981); or, emerging theories such as the construction developed by Dubinsky based on Piaget's idea of reflective abstraction that would become APOS theory (Arnon et al. 2014); or, that developed by Tall (2013) that would lead to his theory of cognitive development along three different worlds (the embodied, the symbolic and the formal). One can also note the predominance of Calculus/elementary Analysis as a mathematical theme and that many research projects were motivated by the high level of failure in the corresponding undergraduate courses, which form a gateway to any kind of scientific orientation in most universities. The concept of limit, considered as its foundational core concept, was especially addressed.

Another characteristic, well representative of educational research at that time, is the emphasis put on cognitive discontinuities and their epistemological sources in learning processes, and on the persistent difficulties generated by these discontinuities. Different theoretical constructions contributed to the conceptualization of these discontinuities – I just mention three of these: the notion of epistemological obstacle borrowed from Bachelard's epistemology (Bachelard 1938; Brousseau 1983) that shows the role played in students' resilient difficulties by forms of knowledge which have proved to be effective in other contexts (as shown by researchers such as Bernard Cornu (1991) and Anna Sierpiska (1985) for the concept of limit); the discontinuity between proceptual (Gray & Tall 1994) and formal thinking; the discontinuity between concepts that emerge as necessary ingredients of the solution of specific problems, such as the concept of derivative or integral, and concepts that respond to unifying and formalization needs, such as the concept of abstract vector space (what Aline Robert (1998), Jean-Luc Dorier (2000) and their colleagues named FUG (formalizing, unifying, generalizing) concepts).

Discontinuities were also identified between domains, for instance between algebra and analysis. I just mention below some of those which have been proved especially challenging for students:

- the change needed in the perception of equality, which, in order to understand the mechanism of analytic proofs, must be perceived as a sign expressing arbitrary level of closeness;
- the predominant role taken by inequalities over equalities, and, more than that, the transition from global perspectives regarding the solving of inequalities to a subtle combination of local and global perspectives;
- the change induced from reasoning modes based on equivalence to reasoning modes based on the use of sufficient conditions, whose effectiveness requires the ability to lose information in a controlled way, taking into account both the

respective orders of magnitude of terms in symbolic expressions and the local character of analytic reasoning.

Let me stress that what is more globally addressed here is the change and reconstructions needed in mathematical practices when moving from one domain to another – what today I would call a change in mathematical praxeologies.

As stressed earlier, research first focused on discontinuities, but progressively became more sensitive to the essential role played by connections and flexibility in teaching and learning processes. Such an evolution has been supported by the increasing attention paid to the semiotic dimension of mathematical activity in educational research, and also by the technological evolution and its specific semiotic affordances. In university research, this evolution is visible for instance in the presentation of the state of the art of research about the teaching and learning of linear algebra co-authored by Jean-Luc Dorier and Anna Sierpinska in the ICMI Study devoted to the teaching and learning of mathematics at university level (Dorier & Sierpinska 2001). This chapter makes the complexity of connections at stake in linear algebra clear: connections between different languages (geometrical, algebraic, abstract), between different registers of representations (graphical, algebraic, symbolic representations, tables), between Cartesian and parametric points of view, and synthetic-geometric, analytic-arithmetic and arithmetic-structural modes of reasoning. Moreover, analyzing teaching practices, researchers show that university teachers, most often, jump without any precaution between these different systems, underestimating the difficulties that these jumps provoke for their students. Of course, connections and flexibility are not a specificity of university mathematics, but what changes is their intensity, and the autonomy given to the students regarding their management.

I was involved in the ICMI Study just mentioned co-ordinated by Derek Holton as a member of its International Programme Committee and this was a very interesting experience. As is the case for any ICMI Study, our collective work intertwined general reflection on the themes identified in the discussion document, syntheses of research advances, and the presentation and discussion of many innovative realizations carried out in different contexts. Published ten years after the AMT book, this ICMI Study (Holton 2001) shows the diversity of issues addressed by those interested in teaching and learning at university level, not just researchers in the field: curricular and assessment issues, teaching practices, relationships between mathematics and other disciplines, affordances of technology, teacher education including that of university teachers still in an emerging state at that time. Issues related to the secondary and university transition are addressed in several chapters, but they are more widely approached than in the AMT book, considering the diversity of social and psychological moves that this transition entails for students. The research section, however, tends to show that the “socio-cultural turn”, as denoted by Steve Lerman, has not yet substantially impacted research at that level.

THE SOCIO-CULTURAL TURN THROUGH THE LENS OF THE ATD

For me, in fact this socio-cultural turn was tightly linked to the incorporation of the ATD in my research perspectives. It first occurred with the supervision of Brigitte Grugeon's doctoral thesis on the transition between vocational high school and technological high school in France (Grugeon 1995), and it turned out to be so productive that I engaged Frederick Praslon, another doctoral student of mine, with adopting ATD as a macro-theoretical framework to study the secondary/university transition on the concept of derivative and its mathematical environment (Praslon 2000).

As expressed very well by Marianna Bosch and her colleagues (Bosch, Fonseca & Gascón 2004), adopting such a perspective on the secondary/university transition represents a radical move. The lens is no longer directed towards the student and her cognitive functioning or development, but towards the institutional practices that condition and constrain, both explicitly and implicitly, what she has the possibility to learn or not, and the associated systems of norms and values which remain partly tacit. In his pioneering work, Frederick Praslon in fact used the ATD to question the common at the time perspective on transition as a transition from the proceptual to the formal world, from intuitive and pragmatic reasoning modes to rigorous mathematical ones. Carefully analyzing mathematical praxeologies in scientific high school and first university year, through a diversity of institutional sources, he showed that contrary to what was often claimed by university teachers, a substantial universe around the notion of derivative was already established at the end of high school in France at least for students in the scientific stream, but that a dramatic extension of the landscape was taking place in the first six months at university, which he visualized using concept maps. He also showed that the transition between secondary and mathematics-sciences programmes at university was not a radical move from the proceptual to the formal world, from an intuitive and algorithmic Calculus to the approximation world of Analysis; it was rather an accumulation of micro-breaches, thus less visible and not appropriately addressed by the institution. The main breaches he identified are the following:

- an increasing speed in the introduction of new objects;
- a greater diversity of tasks making routinization much more difficult;
- much more autonomy given in the solving process for similar tasks;
- a new balance between the particular and the general, the *tool* and *object* dimensions of mathematical concepts;
- objects more controlled by definitions, results more systematically proved, and proofs which are no longer “the cherry on the cake” but take the status of mathematical methods.

As he evidenced, the conjunction of these breaches created a substantial gap but university teachers were not aware of it in their great majority and tended thus to under-estimate the cognitive charge induced for their students. To make university

teachers and students sensitive to these changes, Praslon designed a set of tasks that could be considered in the gap between the two cultures: a priori compatible with high school knowledge but fully exotic in high schools, and at the same time not really university tasks.

I will not enter into more details. Since that time, the anthropological perspective has been used for the study of institutional transitions, with the incorporation of conceptual tools such as the hierarchy of didactic codetermination which were not available at the time of Praslon's doctoral thesis, the development of specific notions such as the notion of completeness of praxeologies (Bosch, Fonseca & Gascón 2004). These have made the identification of new characteristics of the secondary-university transition possible: incompleteness and isolation of high school praxeologies, changes in the respective importance attached to the praxis and theoretical blocks of praxeologies, and in the topogenetic distribution of roles between teacher and students. However, this is only one part of the changes potentially induced by the adoption of the ATD lens, and does not take into account more recent developments of the theory such as its design dimension based on the paradigm of "Questioning the world" and the idea of Study and Research Path (SRP) (cf., for instance, the pioneering doctorate thesis by Barquero (2009), the recent one by Cristina Oliveira (2015) and several contributions to this conference), or the refinements of the notions of technology and theory introduced by Castela and Romo Vazquez (2011) in order to better take in charge the circulation of knowledge between institutions and the reality of practices in engineering courses.

Mentioning these recent developments helps me make the transition to the next section of this text in which I discuss more broadly the strengths and weaknesses of this field of research, as I see them in the light of its historical evolution, and the current challenges that the field faces.

STRENGTHS, WEAKNESSES AND CHALLENGES

Strengths

There is no doubt that the field of research in university mathematics education presents evident strengths. It has developed over more than four decades, with regular efforts of syntheses. I have already evoked two of these, the AMT book and the ICMI Study volume, but in the recent years, new syntheses have been produced, such as the chapter I co-authored with Carmen Batanero and Philip Kent for the second NCTM Handbook (Artigue, Batanero & Kent 2007), the survey led by by Mike Thomas for ICME-12 (Thomas et al. 2014), or the book *Amongst Mathematicians* (Nardi 2008). All these syntheses show that a substantial amount of knowledge has been accumulated, and also that important efforts have been made to build structured, connected and coherent accounts of this knowledge. This is certainly a strength in itself.

I have stressed the importance of epistemological reflection in the emergence of this field and evoked some forms it has taken. This epistemological work is going on accompanying the development of the field, more and more benefitting from productive interactions with other communities and from the progression of their research problématiques and results. Being attached to a doctoral school structured around philosophy, history and epistemology of sciences and didactics of sciences, I have a regular experience of such productive interactions. I can also measure the fascinating evolution of epistemological perspectives since the time of the AMT working group, more and more open to the diversity of forms of life that mathematics has according to the contexts and cultures where it is practised and developed.

The emergence of the field was also characterized by the domination of cognitive and constructivist perspectives. I consider as a strength of our field the fact that we have succeeded in emancipating ourselves from these perspectives, whose limitations are evident, but also the fact evidenced by the consideration of most research publications, that this emancipation went along a reconstruction of their main outcomes, thus making possible some form of incorporation of these outcomes in the new paradigms. I personally experienced such reconstructions in the diverse doctoral theses I supervised on institutional transitions, and I see also a sign of it in the current enterprise of networking between APOS and the ATD, two theoretical constructions I tend to position at the extreme opposites of our field.

Another strength of the field is certainly its move from investigation focusing on the student to a more balanced interest in both the student and the teacher. This move is not proper to the field of research at university level as is well known, but it seems to have been more difficult to achieve at this level of schooling. Today, however, this obstacle seems finally overcome, and university teacher practices are becoming an object of study in their own. Research also investigates more and more possible strategies for the didactic acculturation of university teachers, extending the pioneering work of Barbara Jaworski and Elena Nardi at Oxford University years ago (Nardi, Jaworski & Hegedus 2005), and benefiting from the potential of new theoretical perspectives, such as those offered by the theories of community of practice (Biza, Jaworski & Hemmi 2014) and community of inquiry (Jaworski 2008).

More generally, theoretical evolution in the field is both promising and challenging. I have already evoked the increasing use in research of the ATD, the potential of which for university research has been especially analysed by Carl Winsløw in his regular lecture at ICME12 (Winsløw 2014). Furthermore the *Research in Mathematics Education* Special Issue (Nardi, Biza, González-Martín, Gueudet & Winsløw 2014) is especially insightful from this perspective, considering the affordances of a range of socio-cultural, institutional and discursive theories: ATD, TDS, instrumental and documentational approaches (Gueudet, Pepin & Trouche 2012), the theories of Communities of Practice and Communities of Inquiry, and the theory of Commognition (Sfard 2008). There is no doubt for me that these theories offer

evident potential for research at university level and at the transition between secondary and university education. Without entering into more details about this potential, I would say that the most challenging perspective for me is that offered by commognition. I have personally some reservation at accepting all the implications of adopting such a radical discursive approach, but the ways it engages us to analyze communicative acts involving students and teachers (Nardi, Ryve, Stadler & Viirman 2014), to think about the teacher role and the resilience of university practices such as lecturing (Sfard 2014) is for me really insightful.

Weaknesses

This being said, there is no doubt that the field also presents some weaknesses, and here I would like to mention some of these. In the ICMI Study already mentioned, it was pointed out that research concentrated too much its efforts on the classical formation of future mathematicians despite the fact that these represented only a very small percentage of university students being taught some mathematics. This was the reason why, when I was asked to lead the authorship of the chapter on learning mathematics at post-secondary level of the second NCTM Handbook (Artigue et al 2007), I proposed as co-authors Carmen Batanero and Philip Kent, who could help us realize a more balanced perspective that paid due attention to stochastic and engineering education. However, there is no doubt that research is still biased both in terms of domains and in terms of population. I still have the impression that: fields of increasing importance in mathematics – such as the stochastic field including probability and statistics, and more generally applied and computational mathematics – are still under-investigated, and that still the practice of the mathematician researcher, and even the pure mathematician researcher, is the implicit reference in most research studies; and, that the diversity of forms of professional relationship with mathematics for which university courses may prepare graduates is still not sufficiently investigated and taken into account. Perspectives are moving as attested by the contributions at this INDRUM conference, and we are much better equipped for tackling such issues; however, interests in the field remain too unbalanced.

Another weakness in my opinion is the excessive predominance of very small-scale qualitative studies, involving a very limited number of students or teachers. Moreover, reviewing submissions or reading articles, I am also often disappointed to read that the authors have collected a huge amount of data, but that quite often the evidence they provide for supporting their claims is reduced to the micro-analysis of some very limited episodes; and, that the triangulation that is a priori possible between different levels and dimensions of analysis that would make the results more convincing is hardly present.

I have to confess also that I often have the impression that what I am reading has been already said years ago – admittedly with some variation in the discourse – but with variations that do not show evident progression of knowledge. This is for me especially the case in Calculus and Analysis, but this may be just because I have been

involved much longer in that area. I do not deny the necessity of going on working on foundational concepts such as the concept of limit, incorporating new perspectives, taking into account the evolution of contexts, of populations, educational means and resources. After attending this conference where many contributions have dealt with Calculus and Analysis, in the first thematic working group as well as in the other four, I am, however, more optimistic.

The last weakness I would like to mention is the insufficient dissemination of research results towards the relevant communities or practitioners, and the very limited influence of our research on university teaching practices. Reading recent publications, for instance the three first issues of the new *International Journal of Research in Undergraduate Mathematics Education*, I find nearly the same description of standard university practices at undergraduate level as decades ago. Of course, such difficulties are not specific to the field of university mathematics education, but one could expect that, being themselves researchers, university teachers would be more open to considering research advances and what these can offer them to better understand their students and to improve their teaching practices. Unfortunately, this does not seem to be the case in general, for many reasons which range from the low institutional value attached to teaching activities, in comparison to research activities at university, to the image of the didactic discipline itself in the mathematics community, in most countries.

However, we have to acknowledge also that making sense of research results in mathematics education, converting them into something useful in practice, is not an easy task. The activities of networking between theoretical frameworks I have been involved through different projects in the last decade (Bikner-Ahsbahr & Prediger 2014), (Lagrange & Kynigos 2014) have evidenced that, even for a didactician, to make sense of other research approaches and results, just by reading the associated literature and by attending seminars or conferences, is difficult. In these projects, appropriation resulted in fact from the collaborative building of networking praxeologies on top of our own research praxeologies (Artigue & Bosch 2014), and it was very progressive. The communication between mathematicians and didacticians does not face exactly the same problems, as extensively discussed in (Fried & Dreyfus 2013), but this experience reinforced my conviction that to overcome the current limitations, we must not think in terms of dissemination of research results, but in terms of collaborative projects, building and negotiating, jointly with mathematicians and other university teachers, problématiques that make sense for all those involved, and meet their respective interests and needs. And then we must combine our respective knowledge and expertise in these projects through appropriate praxeologies.

Of course, collaborative projects have existed for decades. The two experiences I mentioned at the beginning of this text were clearly collaborative projects; the ICMI Study volume (Holton 2001) presents a range of examples; however, each of them

more or less appears as a particular and isolated case. We can go further today; we have more powerful conceptual tools in order to approach ecological and institutional issues, to approach collaborative work and relationships between communities, to build, analyse and compare projects, and, last but not least, to consider the long term dynamics necessarily at stake.

Challenges and resources

There are strengths and weaknesses, but there are also many challenges, old and new challenges. I will focus here on some of those raised by the fact that we live in a fast moving world.

How can we maintain some connection between the living field of mathematics, so dynamic and diverse, and undergraduate mathematics education, both in terms of content and practice?

There is no doubt that, contrary to graduate mathematics education, undergraduate mathematics education is poorly connected to the mathematics of today in most universities. It is generally argued that the limited mathematical background of undergraduate students makes the connection with the sophisticated world of current mathematics impossible. This may be the case if we consider that this connection must necessarily be expressed in terms of operational knowledge. However, if we consider mathematics as a part of human culture, we must admit that, as any cultural form, our mathematical culture is not reducible to its operational part. The distinction between different forms of relationships with mathematical objects and practices opens the landscape towards alternative didactic strategies and practices. Those developed and used by the very active community engaged in the popularization of mathematics could be a source of inspiration. However, as was evidenced by ICMI Study 16 which addressed this topic (Barbeau & Taylor 2009), up to recently at least, didactic research has paid limited attention to popularization practices, and more generally informal mathematics education. Moreover, as far as I know, the didactic community has still limited contact with the community of research in science communication with which the collaboration could certainly be helpful. There are thus resources that could be more systematically explored for addressing this challenge.

How can we make our students really experience the subtle and original combination mathematics currently offers of experimental and deductive games, thanks to the evolution of technology?

Technological evolution has substantially impacted professional mathematical practices, in particular by providing much more powerful tools for supporting an experimental dimension of mathematical practices that has always existed, and by making this experimental work more visible and sharable (cf. for instance the journal *Experimental Mathematics*). However, in many places, undergraduate mathematics education seems still blind to this evolution, even when those in charge make

extensive use of technology in their professional activity. Making visible the experimental dimension of mathematics tends to be perceived as an obstacle to the entrance into the deductive game of mathematics aimed at, leading in some first year programmes to the banishment of any technological tool in algebra or analysis courses. Overcoming such a limited epistemological view and its negative consequences is a challenge that researchers in mathematics education have faced for decades, but there is no doubt that the situation remains critical today at university level in many places, contributing to the rupture with mathematical practices in secondary education.

How can we address the dramatic changes that the technological evolution more generally induces in the ways we and our students access information and resources, learn, communicate, interact, work and produce with others?

In fact, the changes induced by the technological evolution do not limit to those just evoked. The digital era in which we have entered induces dramatic changes in the way we access information and build knowledge, in the way we communicate, interact and work. New pedagogical strategies develop, such as reversed pedagogy, MOOCs and diverse forms of hybrid pedagogy, which need to be studied. The number of on line resources increases exponentially as well as the diversity of learning sources, and modalities of use. Once again, didactic research offers promising tools to take up this challenge, the documental approach initiated by Ghislaine Gueudet and Luc Trouche (Gueudet et al. 2014; Gueudet, Pepin & Trouche 2012) being one of the most recent ones.

And, finally, how can we make our students consider mathematics as a resource for thinking about this fast moving world, questioning it, and trying to make it a bit better?

CONCLUSION

Opening the first INDRUM Conference, I have tried to share with the participants the experience of a researcher who has been active in the field of university mathematics education for more than three decades. I organized my reflection around the historical evolution of this field of research convinced that this could help understand its current state and better appreciate its achievements, identify its strengths and weaknesses, as well as perspectives for future research. I have tried to make clear that substantial advances have been made, both from a theoretical and empirical point of view, that knowledge has progressively accumulated, and that, even if weaknesses still exist, we are today reasonably equipped to take up the many challenges that we have to face. As pointed out in the introduction, the vision I have given of the field is certainly both partial and subjective, and shaped by my own experience and by the research and educational cultures in which it has mainly developed. I hope, however, that it has found resonance with the perspectives and experiences of many

participants, and that it has been a stimulus for the discussions and work carried out during the three days of the INDRUM conference.

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Panel

Current interactions between mathematicians and researchers in mathematics education

Caroline Bardini¹, Marianna Bosch², Reinhard Hochmuth³,
Chris Rasmussen⁴ and María Trigueros⁵

¹Melbourne Graduate School of Education, Australia; ²Univ. Ramon Llull, Barcelona, Spain; ³Leibniz Universität Hannover, Germany; ⁴San Diego State University, USA; ⁵Instituto Tecnológico Autónomo de México, Mexico

The interaction between researchers in mathematics and in mathematics education has always been diverse, depending not only on the country and historical period, but also on the educational level considered. In the case of university education the roles become even more varied: producers and “organisers” of knowledge; textbooks authors; designers of university syllabus and programmes; teachers; disseminators; educational research interlocutors; etc. The aim of the panel is to discuss the nature and extent of the collaboration between mathematicians and researchers in mathematics education, the benefits and pitfalls of the relationships and the directions that seem important to strengthen.

Keywords: university research; university education; research communities; interactions between research and teaching.

A DIVERSITY OF ROLES AND INTERACTIONS

One of INDRUM's main objectives is to contribute to the development of research in didactics of mathematics at all levels of tertiary education, with a particular focus on strengthening the dialogue between mathematicians involved in multifaceted university practices such as research in mathematics, research in mathematics education, university teaching, etc. The interactions between the community of researchers in mathematics and the community of researchers in mathematics education have always been diverse, depending not only on the country and historical period, but also on the educational level considered. In the case of university education, the roles of mathematicians and didacticians become closer. Mathematicians are at the same time producers and “organisers” of knowledge, textbooks authors, designers of university educational programmes, teachers of future mathematicians and mathematics teachers, innovators, etc. They not only assume various levels of interlocution with didacticians, but also become part of the object of study of didactics research. Many questions can be asked concerning the relationship between researchers in university mathematics education (UME) and mathematicians depending on the different roles assumed by the latter. When mathematicians act as teachers, the main questioning refers to the ways of collaboration between educational researchers and “practitioners”, from the design, experimentation and data gathering of teaching practices, to the dissemination and

impact of research results. To what extent do teaching questions nourish UME lines of research and what are the conditions needed for the reception and implementation of research results? Environmental factors related to the possible different positions of mathematicians and didacticians in mathematics or education departments, as well as the interactions between research societies, could also be crucial to facilitating or hindering fruitful relationships between both disciplines. And, finally, how do the institutional positions assigned to both communities in the university policy making (curriculum design, university entrance requirements, teacher selection, research project assignment, etc.) shape these relationships?

THREE EXPERIENCES

Three experiences from different countries (Australia, USA and Mexico) can illustrate different ways of collaboration between mathematicians and didacticians. In the Australian case (Bardini & Pierce 2015) the starting point is a divide between mathematicians and mathematics educators, who are located in different faculties (Science and Education) and do not need to share many teaching or research projects. Some collaboration is emerging slowly, especially through national funded projects about the transition from school to university mathematics and the low retention rate in university mathematical sciences. In the case of the USA (Bressoud, Carlson, Mesa & Rasmussen 2013), the initial situation seems to be different, since mathematics departments often offer mathematics education positions and one third of the doctoral programmes in mathematics education are offered by mathematics departments. The study “Characteristics of Successful Programs in College Calculus” launched by the Mathematical Association of America in 2009 and supported by the National Science Foundation is an example of collaboration: amongst the outputs of the project we note the participation of mathematics education researchers in the design and implementation of teacher assistants’ training to help them implement active learning problems. The Mexican case (Possani, Trigueros, Preciado & Lozano, 2010) shows an example of collaboration between 3 research mathematicians, 3 university mathematics teachers and 2 didacticians who worked in the same university institution during a 4-year project about teaching linear algebra using models. The results show some conditions for the collaboration to be effective, such as the need for mathematicians to be open to innovative methods, and some difficulties resulting from the range of approaches in mathematics education. In the three examples we see a growing importance of undergraduate mathematics education within mathematics, especially through its formal inclusion in the respective Mathematical Societies and some joint projects. Difficulties still remain, due to the lack of recognition of educational research, which still appears to be undervalued, in comparison to mathematics research, and the consequent reluctance of mathematicians to become involved in projects that might not bring them enough returns in terms of prestige or publications.

FINAL QUESTIONS FROM THE AUDIENCE

The panel raised several questions related to the current interactions between mathematicians and researchers in mathematics education:

- What are the reasons for the distance between the two communities? What are the motives to invest in bringing them closer?
- How can we communicate the main results in mathematics education in order to foster their dissemination?
- Can the fact that some mathematicians gravitate towards mathematics education help bring the two communities closer?
- Can we learn something from an international comparison, for example: how much is the evaluation of teaching connected with the interest of mathematicians for mathematics education?
- What are the conditions for the collaboration with university teachers of mathematics? Do they need to understand mathematics education theories?
- When mixed teams, involving mathematicians and didacticians, are formed to deal with a teaching problem at university level, which are the epistemological, didactic or pedagogical obstacles that may appear?
- How are new ideas in mathematics education received by university teachers, by mathematicians and, more generally, by the public or the media? Is there resistance to or support for these ideas?
- Much collaboration between mathematicians and mathematics educators has emerged in the last 10-20 years. There is a need to establish collaboration also within other disciplines in which mathematics is taught (engineers, physicists, biologists, economists, etc.). What forms might this collaboration take?

These are questions which open new lines of research in the field of university mathematics education and should be approached as such.

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TWG1: Calculus and Analysis

Le rôle de la borne supérieure (ou supremum) dans l'apprentissage du système des nombres réels

Analía Bergé

Université du Québec à Rimouski, Canada, analia.berge@uqar.ca

Dans cet article nous montrons que des étudiants universitaires, à qui on a présenté la complétude de l'ensemble des nombres réels à travers l'existence de la borne supérieure (ou supremum) de sous-ensembles non vides et majorés, perçoivent généralement le supremum comme une notion qui ne possède pas d'autre utilité que d'être un majorant. Nous faisons l'hypothèse que la notion de supremum (ainsi que celle de complétude) sera difficilement comprise par les étudiants universitaires à moins qu'elle ne soit introduite par des situations où elle prend son caractère essentiel, soit dans des situations de preuve ou de détermination de valeurs particulières liées au travail en analyse telles que les distances, les limites, les zéros de fonctions ou autres.

Mots clefs: ensemble de nombres réels, complétude, supremum, borne supérieure.

LA COMPLÉTUDE DE L'ENSEMBLE DES RÉELS

L'ensemble \mathbf{R} des nombres réels est le domaine naturel des fonctions étudiées dans les cours d'analyse mathématique. Dans les livres d'analyse contemporains il est défini comme étant un corps, ordonné et complet. Le fait que \mathbf{R} soit un ensemble complet est souvent énoncé par un axiome qui détermine l'existence d'un plus petit majorant, nommé borne supérieure ou supremum, pour chaque sous-ensemble non vide et majoré. Il y a, pourtant, d'autres façons équivalentes d'exprimer la complétude. L'inclusion d'un axiome de complétude pour \mathbf{R} , au-delà de la forme choisie pour l'exprimer, se justifie par le besoin de compter sur un système numérique adéquat afin d'être capable de développer l'analyse : on veut pouvoir déterminer l'existence d'éléments par l'entremise de l'intersection d'intervalles emboîtés dont la longueur tend vers zéro ou par l'existence de la limite des suites numériques de celles qui « doivent » converger (monotones et bornées, par exemple, ainsi que les suites fondamentales ou de Cauchy); on veut aussi pouvoir affirmer l'existence d'un zéro d'une fonction continue sur un intervalle où elle prend des valeurs de signes différents, etc. Ces résultats et d'autres qui s'en déduisent sont démontrables seulement dans un domaine qui soit un corps ordonné et complet. Les mathématiciens s'en sont pourtant servis afin de faire avancer l'analyse bien avant la définition d'un système numérique possédant ces caractéristiques. Durant les 17e et 18e siècles, l'analyse s'est développée sans compter sur un énoncé arithmétique définissant ce que nous appelons, à partir du 20e siècle, la complétude. Dans les problèmes étudiés alors, l'existence des nombres cherchés n'était pas un problème à analyser; elle était déjà donnée par la nature des phénomènes abordés, par exemple, le calcul d'une distance ou d'une hauteur (Bergé et Sessa, 2003). Les notions de

courbe et de droite en jeu étaient celles de lieu géométrique d'un point mobile, qui héritaient la continuité de ce mouvement. Ainsi, Galilée considérait la parabole comme le lieu géométrique décrit par un point suivant la trajectoire d'un projectile (De Gandt, 1988); pour Newton les quantités mathématiques n'étaient pas constituées par les parties les plus petites possibles mais décrites par un mouvement continu. Les lignes étaient engendrées, selon Newton, non pas par l'adjonction de parties mais par le mouvement continu des points (De Gandt, 1990). Remarquons que considérer les lignes comme la somme de ses « parties les plus petites possibles » implique de faire face à une somme ne pouvant être autre chose qu'une somme non dénombrable.

La naturelle continuité de la droite comme support aux développements de l'analyse a été remise en question au début du 19^e siècle. La création des géométries non euclidiennes a motivé plusieurs mathématiciens à se questionner sur la portée de la géométrie comme modèle de l'espace physique. Les arguments basés sur des représentations graphiques ont été rejetés, ce qui a amené des mathématiciens à vouloir reconstruire l'analyse en s'appuyant seulement sur des concepts arithmétiques. Nous rencontrons les premières tentatives dans les travaux de certains mathématiciens, dont B. Bolzano et J-A Cauchy, de la première moitié du 19^e siècle (Jarnik, 1981, Van Roostelar, 1962, Cauchy, 1994); ils ont explicité, et utilisé dans l'écriture de preuves, quelques-unes des propriétés avec lesquelles le système numérique devait compter, telles que (dites dans notre langage contemporain) la convergence de suites fondamentales, celle de suites croissantes et majorées ainsi que l'existence d'une borne inférieure pour un ensemble minoré. Pourtant, ces propriétés ne pouvaient pas être démontrées, faute d'un système numérique adéquat; elles étaient plutôt considérées comme des propriétés que le système possédait sans discussion sur leur validité. Dans la deuxième moitié du 19^e siècle, des mathématiciens, dont Dedekind et Cantor, conscients que ces propriétés n'étaient pas fondées sur une base arithmétique, construisent, chacun de leur côté, un système numérique s'appuyant sur les nombres rationnels dont les propriétés mentionnées plus haut, et d'autres équivalentes, pouvaient être démontrées (Bergé et Sessa, 2003, Durand-Guerrier, 2012). Vers la fin du 19^e siècle, la complétude prend la forme d'axiome dans la définition de \mathbf{R} dans les travaux de Hilbert et lesdites constructions prennent à leur tour le rôle de modèles satisfaisant l'ensemble d'axiomes de cette définition.

La raison d'être de la définition de la complétude de \mathbf{R} , telle que nous la connaissons aujourd'hui, est donc celle de faire partie de la définition d'un domaine numérique dont on puisse démontrer des énoncés nécessaires pour l'analyse. Les étudiants universitaires suivant des cours en analyse, la reconnaissent-ils? Que comprennent-ils à propos de la complétude? C'est une question large, qui admet plusieurs sous-questions et à laquelle nous avons partiellement répondu en Bergé (2010), nous y reviendrons dans les sections suivantes. Dans ce travail, nous nous penchons sur une

de ces sous-questions. Nous nous intéressons à des étudiants à qui on a introduit la complétude via l'existence du supremum de sous-ensembles de \mathbf{R} non vides et majorés : dans quelle mesure reconnaissent-ils une utilité au supremum liée à la complétude? C'est la question à laquelle nous tenterons de répondre ici.

CADRE DE RÉFÉRENCE

Ce travail s'inscrit dans un projet en cours dont l'objectif général est double :

- 1) connaître comment évoluent les connaissances des étudiants universitaires sur la notion de complétude de l'ensemble des nombres réels dans le cadre de l'enseignement universitaire reçu et
- 2) élaborer des hypothèses sur la façon dont cette notion peut s'acquérir dans les études universitaires.

Pour préciser ce que nous interprétons comme connaissances sur la notion de complétude, nous prenons comme référence la formulation de Douady (1986) :

Nous disons qu'*un élève a des connaissances en mathématiques* s'il est capable d'en provoquer le fonctionnement comme outils explicites dans des problèmes qu'il doit résoudre, qu'il y ait ou non des indicateurs dans la formulation, s'il est capable de les adapter lorsque les conditions habituelles d'emploi ne sont pas exactement satisfaites, pour interpréter les problèmes ou poser des questions à leurs propos. (Douady, 1986, 11-12).

Afin de penser au *fonctionnement de la complétude comme outil explicite*, la réflexion sur la nature de cette notion mathématique et sur son développement dans l'histoire des mathématiques nous semble indispensable. Cela explique l'ampleur des analyses historiques et épistémologiques dans ce projet, dont les principales lignes se trouvent sommairement énoncées dans la première section de ce travail. L'intérêt et l'importance de l'analyse épistémologique pour le travail en didactique des mathématiques ont été signalés par de nombreux chercheurs en didactique des mathématiques, en particulier par Artigue (1991, 1995). Un tel projet ne peut pas se faire séparément de l'étude des occasions que les étudiants ont de travailler avec cette notion. Nous prenons le cadre qu'offre la Théorie Anthropologique du Didactique (Chevallard 1998, 1999) pour examiner l'ensemble des pratiques mathématiques réalisées par des étudiants au sein d'une institution. Nous utilisons, dans le projet général, notamment la notion de *praxeologie* à travers laquelle Chevallard modélise les pratiques institutionnelles, composée des descriptions de quatre éléments fondamentaux de chaque pratique (tâches, techniques, technologies et théories). Nous utilisons également la notion d'*organisation mathématique* comme un ensemble organisé, structuré de praxeologies (Chevallard 1998).

TRAVAUX PORTANT SUR LA COMPLÉTUDE ET LE SUPREMUM

Nous avons analysé l'organisation mathématique mise en place par une institution donnée à propos de la complétude pour quatre cours corrélatifs d'analyse à

l'université que nous nommons cours I, II, III et IV (Bergé, 2008). Nous avons utilisé comme données l'ensemble des fiches de tâches mathématiques adressées aux étudiants, des copies de notes de cours d'étudiants et des entrevues avec des professeurs. Nous nous limitons ici à mentionner que la complétude fait partie des quatre cours, prenant des formes moins explicites ou plus explicites selon le cours. Les deux premiers cours correspondent à ce qu'est appelé en anglais *Calculus*. Nous identifions alors l'explicitation de la complétude comme une marque de la transition entre le *Calculus* et l'Analyse. Nous nous interrogeons sur la raison d'inclure explicitement la complétude dans des cours qui ne demandent généralement pas aux étudiants de produire des justifications théoriques comparables à celles qui demandent de déployer la complétude.

Dans une autre étude (Bergé, 2010) nous montrions que des étudiants de trois de ces cours (cours II, III et IV) à qui \mathbf{R} avait été présenté par les axiomes de corps, d'ordre et de complétude, interrogés par écrit sur la convergence de suites monotones et bornées et sur la complétude, répondaient majoritairement en termes de continuité naturelle (par l'entremise de représentations) et par des énoncés non-opérationnels, c'est-à-dire, qui ne sont pas utilisables pour argumenter formellement. Ce portrait changeait une fois réalisée l'étude d'espaces métriques, au cours IV.

Concernant l'apprentissage de la notion de supremum, nous avons trouvé deux travaux qui soulèvent les difficultés des étudiants. Chellougui (2006) a analysé l'utilisation des quantificateurs universel et existentiel en première année de l'université, prenant comme étude de cas la notion de supremum; elle repère chez les étudiants plusieurs difficultés concernant la formulation du supremum en termes de deux quantificateurs et leur ordre dans la définition. Bills et Tall (1998) ont de leur côté réalisé des entretiens avec cinq étudiants d'un cours d'analyse sur la définition du supremum; ils concluent que l'effort de rendre la définition de supremum opérable peut signifier pour quelques étudiants une demande cognitive trop élevée; à leur avis d'autres étudiants ne disposent que de *concept images* inopérables.

Dans la section suivante nous présentons la méthodologie et les données utilisées dans ce travail.

DONNÉES UTILISÉES ET METHODOLOGIE

Tel que mentionné plus haut, nous voulons connaître dans quelle mesure les étudiants reconnaissent une utilité au supremum liée à la complétude. Le contexte est celui des cours II, III et IV mentionnés en Bergé (2008 et 2010). Voici quelques caractéristiques des cours en relation à l'apprentissage du supremum. Brièvement, le cours II est un cours de *Calculus* à plusieurs variables. Les étudiants de ce cours appartiennent aux programmes de mathématiques, de physique et de chimie. Durant les premières semaines de ce cours, une révision a été faite, comprenant entre autres l'ensemble des réels, connu déjà des étudiants, qui avaient réussi un cours de *Calculus* en une variable. Le premier cours a démarré avec la définition de majorant

d'un sous-ensemble de \mathbf{R} et de supremum d'un sous-ensemble majoré. Le professeur a montré que si le supremum existe, alors il est unique et il a énoncé l'axiome de complétude : tout sous-ensemble de \mathbf{R} non vide et majoré admet un supremum. Le professeur a déduit l'existence de l'infimum de tout sous-ensemble de \mathbf{R} non vide et minoré. Il a présenté la propriété d'Archimède comme une conséquence de l'axiome de complétude et il a démontré que toute suite non décroissante et majorée converge au supremum. Dans les travaux dirigés, en lien avec la complétude, les étudiants ont eu à déterminer et justifier les supremums, les infimums, les maximums et minimums de certains ensembles (s'ils existaient); montrer que certains sous-ensembles de \mathbf{R} ne sont pas bornés; montrer que l'ensemble des rationnels dont le carré est inférieur à $2n$ n'a ni maximum ni supremum en \mathbf{Q} ; déterminer si certaines suites convergent (par exemple en analysant si elles sont monotones et bornées) et, plus tard, se servir du théorème de Bolzano pour démontrer des résultats tels que l'existence de zéros de fonctions sous certaines conditions. Les étudiants des cours III et IV appartiennent exclusivement au programme de mathématiques. Au cours III le supremum et l'infimum prennent un rôle d'outil (dans la définition de distances, ou de l'intégrale de Riemann par exemple); ils sont vus aussi comme des objets possédant plus d'une définition (le plus petit majorant d'un sous-ensemble ou le majorant étant la limite d'une suite contenue dans l'ensemble) dont les étudiants doivent prouver leur équivalence. Durant la première semaine du cours IV les étudiants ont à prouver l'équivalence de cinq énoncés exprimant la complétude de \mathbf{R} , dont l'existence du supremum de sous-ensembles non vides et majorés, l'existence de la limite de suites de Cauchy, celle de suites non décroissantes et majorées, celle d'un élément dans l'intersection d'intervalles emboîtés et celle d'une sous-suite convergente pour chaque suite bornée.

Un questionnaire de cinq questions a été rempli par les 145 étudiants qui se trouvaient présents le jour de sa passation : 124 sur 192 du cours II, 11 sur 24 du cours III et 10 sur 16 du cours IV. La traduction au français du questionnaire est :

1. Si tu voulais expliquer à un étudiant plus jeune qu'une suite non-décroissante et majorée a une limite, comment le ferais-tu?
2. D'après toi, quelle est l'utilité de la notion de supremum? [Cette question, pour le cours II, était précédée de la suivante : Si tu te souviens de la définition de supremum, peux-tu donner des exemples de a) ensembles qui possèdent un supremum? b) ensembles qui ne possèdent pas de supremum?]
3. Que veut dire pour toi « \mathbf{R} est un ensemble complet ? » [Seulement cours III et IV]
4. Considérant ce que tu as appris jusqu'à présent en analyse, quelles « parties » demandent l'utilisation de l'axiome de complétude de \mathbf{R} ? Autrement dit, quels concepts et propriétés des nombres et des fonctions on n'aurait pas pu connaître sans cet axiome? [Seulement cours III et IV]

5. Pour chaque $c \in \mathbf{Q}$ considère la fonction $f_c: \mathbf{Q} \rightarrow \mathbf{Q}$, $f_c(x) = x^2 - c$. Malgré le fait qu'elle n'est pas définie sur \mathbf{R} , elle est continue. a) Y a-t-il une valeur $c \in \mathbf{Q}$ telle que l'affirmation « si f_c prend deux valeurs $f_c(a)$ et $f_c(b)$ alors elle prend toutes les valeurs entre les deux? » b) Si f_c était définie sur \mathbf{R} à la place de \mathbf{Q} , est-ce que cela ferait une différence?

Nous avons présenté en Bergé (2010) l'analyse des réponses aux questions 1 et 3. Nonobstant les limites de ce questionnaire, nous concluons alors que la majorité des étudiants considéraient la limite de la suite mentionnée dans la question 1 comme étant évidente, comme allant de soi. Quant à la perception de la complétude de \square (question 3), tel que mentionné plus haut, elle était exprimée par des expressions non-opérationnelles (une vision naturelle, une mention à la représentation sur la droite numérique, une utilisation du sens courant du mot complet, absence de « trous ») par la plupart des étudiants du cours III. Au cours IV, la majorité des étudiants a utilisé des énoncés mathématiques pour l'exprimer. Nous présentons ici l'analyse correspondant à la question 2, qui n'avait pas été analysée auparavant.

La question supplémentaire à la question 2, adressée aux étudiants du cours II, demandant des ensembles possédant et ne possédant pas un supremum a été posée essentiellement pour nous assurer de la pertinence de considérer les réponses à la question 2. Les réponses à la question 2 peuvent montrer dans quelle mesure les étudiants ont de connaissances du supremum : quels usages du supremum les étudiants ont « à leur portée » s'ils l'identifient dans les théorèmes qu'il permet de démontrer, s'ils l'identifient comme outil dans la détermination de nombres, s'ils l'identifient comme faisant partie de la définition de la complétude.

VISION DES ÉTUDIANTS SUR LE SUPREMUM

La totalité des étudiants du cours II a donné des réponses correctes à la question supplémentaire. Après la lecture de toutes les réponses à la question 2, nous avons trouvé une certaine régularité qui nous a permis de les classer dans les cinq catégories suivantes :

Majoration

Il s'agit de réponses dont on manifeste que l'utilité du supremum est celle de borner. Le supremum est donc vu comme une borne supérieure ordinaire. Des exemples de réponse sont :

« C'est très utile, car cela m'indique si un ensemble est borné ou pas »;

« L'idée de supremum, je la rapporte à celle de majorant. Le fait de savoir que s est le supremum me dit que n'importe quel $x \in$ à l'ensemble, il est plus petit que s et par conséquent je sais que l'ensemble est majoré par s »;

« Si une fonction ou un ensemble ont un supremum, on peut dire qu'ils sont majorés »;

« Si je sais qu'un ensemble a un supremum, alors je sais que les éléments de l'ensemble ne seront pas plus grands que le supremum. Cela peut être utile pour majorer ».

« Le supremum est utile pour pouvoir affirmer que tout élément appartenant à un ensemble A est plus petit ou égal au supremum de A »

Ces étudiants ne voient pas dans le supremum une autre fonction que celle d'être un majorant.

Absence de réponse

Nous avons groupé dans cette catégorie l'absence de réponse ainsi que les réponses manifestant que le supremum n'a aucune utilité ou manifestant ne pas connaître d'utilité. Nous avons inclus dans cette catégorie aussi des réponses se limitant à réécrire la définition de supremum. Des exemples :

« La notion de supremum ne m'a pas été utile, sauf pour les exercices dont on me demandait de le trouver »;

« Je ne vois pas d'utilité. C'est peut-être un concept qu'il faut connaître. Il a peut-être une utilité que je ne connais pas »;

« La notion de limite me semble très utile. Celle de supremum, je ne sais pas »

« L'utilité que je vois c'est d'être le plus petit majorant ».

Présence dans une démonstration non spécifiée

Ce sont des réponses affirmant que le supremum est utilisé dans plusieurs démonstrations de théorèmes sans les préciser :

« Les supremums, on les utilise pour démontrer des théorèmes »;

« Je me souviens que nous l'avons utilisé à plusieurs reprises dans des démonstrations »;

« Il est très utile pour certaines démonstrations mathématiques et il peut être utile pour prédire un majorant en équations de la physique ».

« Partant de l'axiome de supremum on a déduit plusieurs propriétés, spécialement de suites, qui ont été la base du cours. C'est très utile »

Ces étudiants ont le souvenir d'avoir « vu passer le supremum » dans une démonstration sans pouvoir l'identifier.

Réponses erronées

« Ça sert à définir des ensembles ouverts et fermés ».

« Ça sert à introduire les limites ».

Utilisation dans des démonstrations spécifiques, définition des réels

Ce sont des réponses qui reconnaissent une utilité spécifique du supremum, soit afin de définir la complétude ou de déterminer un nombre, soit dans la démonstration d'un résultat qui est nommé. Des exemples de ce groupe de réponses sont :

« Pour démontrer qu'une suite majorée non décroissante a une limite » ;

« Afin de définir la complétude de \mathbf{R} » ;

« Il est important pour démontrer le théorème de Bolzano » ;

« Avec la notion de supremum nous pouvons formaliser des idées intuitives comme celle que nous venons de voir [il fait référence à la question 1]. Elle permet aussi de donner une première présentation des réels avec ses 14 axiomes. L'axiome de supremum permet de distinguer entre \mathbf{R} et \mathbf{Q} . D'autre part, elle est nécessaire pour démontrer la plupart (selon ce que je crois) des théorèmes d'analyse » ;

« Il sert à prouver la propriété des intervalles emboîtés et qu'une suite de Cauchy est convergente » ;

« Il sert à démontrer des résultats comme celui de la question 1 ou comme le théorème de Bolzano ».

Ces étudiants reconnaissent une utilité du supremum liée à la complétude afin de définir l'ensemble \mathbf{R} ou afin de l'utiliser dans la justification de propriétés requérant de la complétude pour être démontrés. Nous pourrions dire qu'ils identifient le supremum dans son rôle d'*outil explicite*, dans le sens de Douady (1986).

Tableau 1 : réponses des étudiants à la question « D'après toi, quelle est l'utilité ou l'intérêt de la notion de supremum? »

Catégories	Cours II, sur 124	Cours III, sur 11	Cours IV, sur 10
1) Majoration	72 (58 %)	2 (18%)	1 (10%)
2) Absence de réponse	19 (15 %)	1 (9%)	0
3) Présence dans une démonstration non spécifiée	16 (13%)	1 (9%)	0
4) Réponses erronées	10 (8 %)	1 (9%)	0
5) Utilisation dans des démonstrations spécifiques, définition des réels	7 (6 %)	6 (55%)	9 (90%)

DISCUSSION ET CONCLUSIONS

Nous voulions savoir dans quelle mesure des étudiants, à qui on a présenté la complétude à travers l'existence du supremum de sous-ensembles non vides et majorés, reconnaissent une utilité au supremum en l'identifiant comme outil explicite. La question analysée, malgré ses limites (car en fait nous n'avons pas posé aux étudiants des situations à résoudre) nous permet d'obtenir quelques conclusions.

La notion de supremum, pour la majorité des étudiants du cours II ne semble pas avoir d'utilité reconnue ou l'utilité d'un majorant ordinaire. La principale tâche que les étudiants ont accomplie concernant le supremum dans ce cours a été de le trouver

pour quelques ensembles (notamment des intervalles, des unions de quelques intervalles ou des ensembles définis par les valeurs qu'une suite prend). Il n'est donc pas surprenant que dans ce cours le supremum soit vu par la majorité des étudiants comme un majorant tout simplement. Les étudiants ont vu le professeur utiliser le supremum pour déterminer l'existence de la limite d'une suite non décroissante et majorée, mais étant appelés à expliquer ce fait, le résultat semble pour eux aller de soi (Bergé 2010) et seulement 6% des étudiants de ce cours peuvent identifier le supremum dans l'écriture de preuves spécifiques ou dans la définition de \mathbf{R} . Pendant que le supremum fait partie indispensable du texte du professeur pour s'assurer de pouvoir écrire des démonstrations, les étudiants n'ont qu'à accomplir des tâches où le caractère d'outil du supremum n'est pas déployé. Un changement majeur se produit quand les étudiants rencontrent le supremum dans la définition ou la détermination de nombres comme c'est le cas aux cours III et IV. En effet, la majorité des étudiants du cours III et la quasi-totalité de ceux du cours IV peuvent nommer différentes situations de fonctionnement du supremum dans son rôle d'outil et dans son rôle d'objet. Douady affirmait que *du point de vue mathématique, les problèmes mettent rarement en œuvre les caractères essentiels des notions, ceux qui en justifient scientifiquement l'emploi* (Douady, 1986, p.12). Peut-on penser à une introduction du supremum mettant en œuvre le caractère essentiel de cette notion? Pour cela il faut, nous semble-t-il, confectionner des situations qui sortent du numérique et intègrent les fonctions et les suites; des situations, en somme, reprenant des conditions similaires ou équivalentes à celles qui ont motivé l'émergence de la définition de la complétude.

Nous faisons l'hypothèse que si le caractère d'outil de la complétude (qu'elle soit définie par l'existence du supremum ou par d'autres voies) n'est pas intégré dans les organisations mathématiques prévues pour l'apprentissage des nombres réels qui l'incluent, les étudiants continueront à ne pas la reconnaître, au-delà de la présence du supremum dans les tâches à accomplir.

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A model to analyse the complexity of calculus knowledge at the beginning of University course

Two examples: parametric curves and differential equations

Isabelle Bloch¹ and Patrick Gibel²

¹University of Bordeaux, France isabelle.bloch@u-bordeaux.fr ²University of Bordeaux, patrick.gibel@u-bordeaux.fr

Our research focuses on the difficulties students encounter with the learning of calculus, considering that they have to cope with much more mathematical objects but also with new ways of reasoning – not only algebraic calculation, but also the practice of approximation, and a scaffolding way of using functions, limits, derivative, integrals, etc. for proving. The semiotic facet of new objects, and the way to manage it, is also a source of great difficulties. We use a model (Bloch & Gibel, 2011) to describe students' work when they have to deal with the resolution of exercises about parametric curves and differential equations.

Keywords: Calculus, students' understanding of mathematical signs and objects, reasoning processes, parametric curves, differential equations.

INTRODUCTION

Every researcher knows that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the University. We have studied the transition between the secondary mathematical organisation in teaching (pre)calculus, and the University one; in this perspective we aim at classifying the different 'things' students have to cope with when they are practicing Calculus.

The organisation at Secondary school takes into account some mathematical objects, as functions, derivatives, integrals: but a number of researchers underline the fact that the way these objects are introduced leads to algebraic calculation and not to analytic work. For instance, students are supposed to calculate an integral but not to justify why it exists; to study the variations of functions with derivative, but not to have a knowledge about which functions get derivatives at which points, or not. So we can see that the *raison d'être* of a mathematical concept is not highlighted.

We notice that, even if teachers think of the structural level, in most cases they confront students only with the operational one. For instance, Ghedamsi (2015) analyses a first year regular course at University and she concludes that:

the teacher does not intervene to enrich (the work) by emphasizing relationships among notions, by changing the setting of semiotic representations, by leaving openings to organize knowledge, by making assessments of knowledge, etc. (So) students employ methods used at the secondary school and do not succeed to shift to the university ones.

In our case, the problem seems not to be the way the notions have been taught; we got an access to the students' course notes, and they show relevant justifications and

explanations. The didactical repertoire of the class has been elaborated by suitable exercises and situations, leading to highlight the operating mode of these concepts. But to better explain the students' work, it is necessary to classify the objects, signs and reasoning processes they have to cope with during resolution of calculus problems.

I. THE LEARNING OF CALCULUS: OBJECTS, SIGNS AND REASONING

Mathematical objects and signs: complexification

At the beginning of University studies, students meet functions as in Secondary school like rational ones as polynomials, or sinus or cosine; they have to solve problems with exponentials, logarithms, but the derivatives can also generate new functions, and integrals too, or series: so objects may have different status, and signs become polysemic. With respect to these signs, we notice that at Secondary school students operate frequently by implementing isolated techniques: they can calculate on a rather straightforward way. But at University, they face complex signs and they have to associate different kinds of symbols, sometimes through a long proving process, for example to calculate an rather complex integral or to prove that a theorem is valid, which is not their responsibility at Secondary school. At University too, signs are multiform: for instance derivative can be written f' but also df/dx ; or x can be the function, so it will appear as dx/dt ; a letter can nominate a variable, a function, or a parameter, which status is sometimes difficult to understand.

Moreover, rules about the use of signs are imbricated, so if you try to calculate $\int \cos^2 x dx$ you have to linearize $\cos^2 x$ because you cannot apply the rule of the primitive of x^2 , just 'mixed' with the primitive of $\cos x$, to $\cos^2 x \dots$ and find $1/3 \cdot \sin^3 x$, as we saw once a student. This evolution of signs is even more evident considering the procedures for proving within the calculus work: students have to understand and use new analytic methods, as it is well known, for limits with ε and α , and to master quantifiers, which reveals to be rather hard (see Chellougui & Kouki, 2013).

Reasoning processes

This complexity requires that students adapt themselves to improve and perfect their reasoning processes: they have to become able to deal with all the facets of knowledge and to adapt their "way of doing", taking into account all the aspects of a question and the requirements of the proofs.

We can say that throughout the reasoning processes, signs (and then objects) work in a strong interaction, as seen above: integrals with the primitive of sine and squares, but also techniques and technologies to prove. Among these technologies it is very important that students learn how to manage the new tools, as quantifiers and the way to perform a valid reasoning up to its end.

II. A MODEL TO ANALYZE STUDENTS' PRODUCTIONS

We need then a tool for modelling students' reasoning processes and try to seize how they manage with this complexity at each level of a situation. This tool takes its origin in the TDS (Theory of Didactical Situations, see Gonzalez-Martin, Bloch, Durand-Guerrier, and Maschietto 2014). Let us recall that TDS is trying to implement situations with an adidactical component, that is, situations that allow students to live a heuristic phase of research. Then they can validate their conjectures through a confrontation to the elements of an adequate milieu. Eventually, a phase of institutionalization is managed by the teacher. The whole model can be found in Bloch & Gibel, 2011.

The theoretical tools used in the elaboration of the model

We want to take into account the semantic dimension – the *meaning* of the aimed knowledge – to analyse reasoning processes: this contributes to justify our choice of the TDS as a basis of our model. TDS organizes adidactical situations with three phases (corresponding to levels of the milieu): a heuristic one (students' action) grounded with a question; a formulation and validation one; and a last one, institutionalization by the teacher. In this configuration reasoning processes we take into account are as well valid or erroneous ones. This theoretical frame allows developing also an analysis of the *functions* of the reasoning processes within the situation (Gibel, 2004; 2015). So in our model we consider signs, functions of reasoning, and levels of argumentation.

The semiotic dimension of the analysis

In order to complete and enhance this theoretical framework we add a semiotic content to TDS. In a previous research (Bloch & Gibel 2011; Gibel 2015) we highlighted the fact that reasoning processes elaborated by the pupils and the teacher during a lesson can occur in various ways: linguistic, calculative, scriptural, and graphic elements (see also Bloch 2003). Consequently the semiotic analysis constitutes one of the dimensions of our model, completing those previously presented: on the one hand the function of the reasoning processes and on the other hand the corresponding level of the didactical milieu. Let us notice that signs can be either formal or linguistic: both will be taken into account. What is significant are the arguments embodied in those signs. This is why Peirce's semiotics seems particularly appropriate for our research and will enable us to study more precisely the evolution and the transformations in the signs used by the different actors within the situation.

In our application of Peirce's semiotics we use the three usual designations: icon, index-sign and symbol-argument. Yet we do not consider the whole intricacy of Peirce's theory: it would be too complex to take into account and not necessary to correctly interpret students' actions in the situation. So we just correlate icons with students' intuitions, drawings, examples, resolution attempts; indexical signs with local proofs, first tools for validation, more accurate reasoning, formulations of

mathematical objects; and symbols-arguments with the concluding validation and mathematical formulation of the rules, and of the aimed knowledge.

The didactical repertoire and the repertoire of representations

The work in the students group leans first on the existing *repertoire*: all the semiotic means used by a teacher, and those he expects from his pupils through his teaching, establish the didactical repertoire of the class – as defined by Gibel (2004). The didactical repertoire of the class can be identified as being part of the mathematical knowledge the teacher has chosen to explain, namely during validation and institutionalization phases of previous situations or previous lessons. The repertoire of representations is a constituent part of the didactical repertoire. It is made up of signs, diagrams, symbols and shapes and also linguistic elements (oral and/or written sentences), which make it possible to name the objects encountered and to formulate properties and results.

A model to analyse reasoning processes

The model of structuration of the didactical milieu used in this construction is that of Bloch (2006). The chart below (Table 1) sums up the levels of milieu – from M1 to M-3 – corresponding to the *experimental* situation.

M1 Didactical milieu	E1: reflexive subject	P1: P. planner	S1: sit. of project
M0 Learning milieu : institutionalization	E0: generic student	P0: professor teaching	S0: Didactical situation
M-1 Reference milieu : Formulation and validation	E-1: The subject as learner	P-1: Professor Regulator	S-1: Learning situation
M-2 Heuristic milieu : action, research	E-2: The subject as an actor	P-2: P devolves and observes	S-2: Situation of reference
M-3 Material milieu	E-3: epistemological subject		S-3: Objective situation

Table 1 – Structuration of the didactical milieu

The negative levels are of particular interest in the sequences we frequently study since they allow describing the emergence of a proof process in the setting up of an adidactical situation. The place where we hope to see the expected reasoning processes appear and develop is located at the articulation between the heuristic milieu and the reference milieu.

In our previous research (Bloch & Gibel, *ibid.*), we decided to focus our didactical analysis on three main axes to study the reasoning processes. The first axis is linked to the nature of the situation: in a situation involving a research dimension, students produce reasoning processes which depend to a great extent on the involved phase of the situation, that is, the level of milieu (heuristic milieu, milieu of formulation or validation) (Table 1).

The second axis of our study is the analysis of the functions of reasoning. We aim at linking these two axes, showing how the reasoning functions are linked specifically to the levels of milieu and how these functions also *manifest* these levels of milieu.

The third axis concerns noticeable signs and representations. These elements can be observed through different forms which affect the way the situation unfolds.

The application of this model to a situation will then include an analysis of the milieu and semiotic analysis of the students and teacher's productions. We will interpret the conjectures, intuitions, signs and reasoning processes as an evolution of the didactical repertoire of the class, knowing that the situation aims at developing a mathematical knowledge in the field of calculus. This is summarized in Table 2:

	Milieu M-2 Heuristic level	Milieu M-1 Formulation, validation	Milieu M0 Institutionalization
Nature and functions of reasoning	R1.1 SEM - Intuitions on a drawing - Decision of calculation - Heuristic tools; errors - Exhibition of an example /a counter ex.	R1.2 SYNT/SEM - Generic calculations and conjectures (right or wrong) - Decision on a mathematical objet	R1.3 SYNT - Formalization of proofs within the mathematics involved theory
Level of use of symbols	R2.1 SEM Icons or indices depending on the context (schemas, intuitions...)	R2.2 SYNT/SEM Local or more generic arguments: indices, calculations	R2.3 SYNT Formal and specific arguments: symbols <i>hypoicons</i>
Actualisation of the repertoire	R3.1 SYNT/SEM - Ancient knowledge - Enrichment at the heuristic level: calculations, conjectures	R3.2 SYNT/SEM Enrichment at the argumental level: - statements, reasoning	R3.3 SYNT - Formalized proofs - Signs within the relevant theory - theoretical elements

Table 2 – A model to analyse situations

Table 2 then includes levels of milieu, nature of signs, functions of reasoning, level of the repertoire. We have also pointed out that some formulations are made on a semantic mode (SEM), as more evolved (in a mathematical sense) ones can also be formulated on a syntactic mode (SYNT). Let us notice that this model allows not only the study of didactical situations, but also to analyse students' productions while solving 'ordinary' problems: in this text we choose to develop this feature of our work (see examples in III.). The matrix notation R1.1 etc. allows to quickly situating the level of arguments where students are located.

We want to underline the fact that an *a priori* analysis is necessary for each situation we choose to study: the model we built is also useful and efficient to perform this *a priori* analysis, as it allows anticipating resolution processes and difficulties. In this perspective, we classify reasoning, calculations, formulas, the nature of signs

produced, and knowledge(s) expressed by students in the different phases, reflecting the situation in which they are located.

The use of the model to analyse 'ordinary' teaching

Our model can also be used to analyse 'ordinary' secondary or university teaching, as it allows detecting students' reasoning processes, use of symbols, and understanding of mathematical objects involved. We can analyse the students' templates while they try to solve a problem: they are first in a heuristic milieu M-2, trying to find a resolution process. Then they decide to undertake calculations, use of theorems, and a final issue. They are then in a milieu M-1. This can be seen also in the context of an evaluation which we present in part III.

III. TWO EXAMPLES: HOW STUDENTS COPE WITH PARAMETRIC CURVES AND DIFFERENTIAL EQUATIONS

We consider the productions of fourteen students in the context of a first year terminal exam at the University of Pau, in May 2014. The teaching unit involved is named: "Mathematics of the movement", which is interesting because a link is made between mathematical knowledge and physics problems. The exam includes three exercises, the first one on polar and parametric functions, the second and the third ones on differential equations. Parametric curves and differential equations are especially interesting to study as they involve complex new signs, unusual processes for secondary students, and new kinds of reasoning. These reasoning encompass also mathematical objects, as functions, limits, derivatives, but in a new way of thinking.

1. Parametric curves

A parametric curve is of the type: $x = f(t)$, $y = g(t)$. There are two functions x and y to study; students must understand that what is required finally is to describe the variations of y with respect to x , in the case of a movement for instance; so the study of the two functions f and g (including the calculation of their derivatives) is just a step (of R1.2 type) to interpret what happens with the curve of y while x being the final variable. Sketching the graph needs to give values to t , being sure that we got the 'whole' curve; or eliminating the parameter t , which may reveal to be complex.

Another difficulty comes from the existence of tangents: in contrast to what happens with algebraic curves, parametric ones can have two tangents at the same point: this is a singular point that students did not meet before. They are expected to identify the nature of this singular point, for instance a cusp. It needs to first apply a formula ($x'(t)=0$, $y'(t)=0$) and then try to find the tangents at this point to be able to identify the nature of the singular point (a calculation and reasoning of successive derivatives that takes place at R2.3 or R3.2 level at least and involves specific interpretation about the objects at stake).

We classified students' productions from S1 to S14. In May 2014 students were confronted to the following question:

Let us study the parametric curve defined by $x(t) = a t^2/(1+t^2)$, $y = a t^3/(1+t^2)$ with $t \in \mathbb{R}$. Show that it is sufficient to study for $t \geq 0$. Determine the variations and confirm that the curve gets symmetry, an asymptote and a singularity.

Students have to calculate $x(-t)$ and $y(-t)$ and conclude about the kind of symmetry; calculate the derivatives, build the variation table and do not forget the limits; and they must undertake pertinent interpretations of these results. The curve has a singularity, a cusp: they must find its coordinates and its nature. We expect that a difficulty can occur in the interpretation of derivatives: students are accustomed to calculate such derivatives but for algebraic functions one derivative is enough to find the variation of f . The asymptote can be a problem too, as $t \rightarrow +\infty$ when $x \rightarrow a$ and $y \rightarrow +\infty$. So the asymptote is vertical, but nevertheless when $t \rightarrow +\infty$, which can be a source of misunderstanding: for algebraic curves a limit where the variable tends to infinite corresponds to a horizontal asymptote.

Analysis of students' productions

Student S1 does perfectly all what is expected: she calculates the derivatives, the behaviour of the function, draws the graph with the asymptote, and determines the cusp with its tangent, which needed to calculate $x^{(3)}(t)$ and $y^{(3)}(t)$ for $t=0$. S1 reaches the level R1.3, she makes a formalization of proofs within the required theory. Student S14 cannot do anything; five other students encounter difficulties to calculate derivatives, to interpret the symmetry, and to find the singular point. One student says that a should be the parameter. One other writes that the equation of the curve is $x(t)+y(t)$... So we can see that even in M-2, some students do not reveal to be able to undertake local adequate calculations, as they do not understand that they are no more in the case of a Cartesian function. There are errors about the nature of the asymptote, for instance: only six students calculate the limits and conclude about the asymptote, reaching the R2.2 level, but among these six two of them write a wrong equation: $y=a$ instead of $x=a$. Students' productions also show calculation mistakes, especially in derivatives and primitives. The handling of singular points is not properly integrated: students are *unsettled* with the conditions for being a singularity, with the ways of finding the tangent... For instance S6 tries to find the point by calculating $x=0$ and $y=0$ instead of their derivatives; S2, who succeeds in the exam, writes that: "every non collinear vector to the curve is tangent to the curve"...

Some students who calculate without mistakes encounter problems with the interpretation of their calculations: their use and interpretation of signs do not exceed the R2.1 or R2.2 level. Those who succeed very well (four from the twelve) write sentences to explain that a singular point is given by $x'(t)=0$, $y'(t)=0$, applying a R2.2 or even R2.3 knowledge; one student says that it means that the speed is equal to zero; but only the first one S1 is able to calculate the tangent and identify the nature of the singularity, being clearly in the position R2.3 for all needed symbols.

2. Differential equations

In the exam students had to cope with the solving of these two differential equations:

Exercise 3: Given the first order differential equation: $e^x yy' - x^2(y^2 - 9) = 0$

After separating the variable, solve the equation. Then solve the Bernoulli differential equation:

$$y' - \frac{4}{x}y - x\sqrt{y} = 0$$

A priori analysis

First, we consider the first order differential equation. Separating the variables implies preserving the initial shape, that is, not to develop the term $x^2(y^2 - 9)$, to obtain the following shape: $\frac{yy'}{y^2 - 9} = \frac{x^2}{e^x}$. This requires analysing preliminarily the features, the characteristics of the different mathematical signs appearing in the equation to anticipate the expected form. To solve this equation, students have then to transform y' as $y' = \frac{dy}{dx}$; then they can produce an algebraic form allowing to integrate the terms.

Dealing with the term $\int \frac{x^2}{e^x} dx$ requires necessarily applying *two times* integration by parts. Considering the second part of the exercise, solve the Bernoulli equation gives rise to a number of difficulties: the first one consists in being able to make the substitution leading to the equation $2zz' - \frac{4}{x}z^2 = xz$. After simplification it can then be written: $2z' - \frac{4}{x}z = x$.

Students must solve first the homogeneous differential equation associated, and then they have to solve the inhomogeneous differential equation by variation of the constant, which can be source of new difficulties. The technique of variation of the constant is a part of the new technical and technological tools of first year University course, so it is of Level R3.3 in our model.

Analysis of students productions

First we analyze main difficulties encountered by students to solve the differential equation $e^x yy' - x^2(y^2 - 9) = 0$. The first one is to separate the variable to obtain $\frac{yy'}{y^2 - 9} = \frac{x^2}{e^x}$: among fourteen students only eight of them accomplished this task; for two of them this task was difficult and required several attempts as we expected. The next step of the resolution needs to compute $\int \frac{y}{y^2 - 9} dy$. Seven students out of eight were able to fulfil this task, but two of them represented the quotient as a sum of rational functions, because they did not acknowledge the derivative of the function

$\ln(y^2 - 9)$. Recognize this primitive is of Level R2.2 because students have to identify a schema – a hypoicon according to Peirce – of different 'models' of derivatives/primitives, which variable is not always 'x'. It supposes that the students' repertoire encompasses a lot of 'forms' that at this level they did not meet often enough.

We notice that only five students were able to deal with the term $\int \frac{x^2}{e^x} dx$ applying two times integration by parts. Then, only four students resolved this equation and obtain the whole solution.

As regards the Bernoulli equation, half of the students recognized an equation such as $y' + a(x)y = b(x)y^n$, with $n = \frac{1}{2}$ and $a(x) = -\frac{4}{x}$, $b(x) = x$. They have been able to make the substitution $z = y^{\frac{1}{2}}$. But only five of them succeeded in obtaining $2z' - \frac{4}{x}z = x$; two students did not allow themselves to reduce the equation, they could not admit the possibility of dividing each term by z. Among these five students, the three other students implemented successful method of solving.

CONCLUSION

We can conclude that it is really difficult for students to access to Level 3 of our model, although this level being the 'expert' one required: they frequently keep blocked at Level 1 with old non-adapted knowledge or false calculations, or they try to work at Level 2 but do not succeed in more complex calculations, especially when schemas are involved; or they make the expected calculation but are no more able to interpret it within the problem.

We can notice that the involved activities, at this level, imply a very rich assortment of technics, procedures, and a variety of occasions to apply formulas. Yet the familiarity with this knew field of knowledge is not established for a majority of students, and their algebraic skills are undersupplied. Then the students' productions highlight their numerous attempts to try to calculate and recognize well-known shapes within the heuristic milieu. We think that the difficulties highlighted in this study are not linked with the teacher's didactical choices, but they are common within the population of mathematical students, due to the reasons we evoked in the first part of this paper.

We want to point out the missing knowledge also in the (French) secondary curriculum: students study no more the composition of functions. Yet recognize the kind of schemas we see in a differential equation as above implies to detect which functions are at stake and how they appear in the formula. As students have no familiarity with 'the whole formula' they try to interpret each element separately, which has no meaning. So, most of them do not achieve the level R3.

We could also formulate these obstacles by saying that students fail in doing a pertinent association between syntactic and semantic methods: they are stressed with calculations and cannot control the meaning of the operations they have done. The next step of our work should be finding relevant situations for the teaching of Calculus, both in an introductory way at Secondary school, and at University.

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Introduire la notion de convergence avec une ingénierie des années 1980 : rêve ou réalité didactique pour l'enseignant?

Stéphanie Bridoux¹

¹Université de Mons, Belgique et LDAR, EA 4434, stephanie.bridoux@umons.ac.be

Nous présentons une expérimentation visant à introduire la notion de convergence d'une suite numérique à des étudiants belges en première année universitaire en reprenant une ingénierie de 1983. Nous nous centrons ici sur le rôle que l'enseignant peut jouer à chaque étape de l'ingénierie pour faire émerger la définition formelle de la notion. L'expérience témoigne de la robustesse de cette ingénierie « d'un autre temps » pour le public actuel moyennant une gestion adaptée du travail des étudiants par l'enseignant.

Mots clefs: ingénierie didactique, convergence d'une suite, formalisation, conceptualisation, pratiques enseignantes.

INTRODUCTION

La convergence d'une suite de nombres réels est une notion clé dans la plupart des cours d'Analyse en première année universitaire donnés en Belgique et en France notamment. Cette notion est déjà abordée dans l'enseignement secondaire belge. La définition en $\varepsilon - N$ est donnée mais n'est finalement pas ou peu travaillée avec les élèves. L'accent est rapidement mis sur l'étude des suites arithmétiques et géométriques pour lesquelles la convergence est abordée de manière intuitive. Les résultats liés à la convergence des suites (unicité de la limite, théorème des gendarmes,...) sont admis sans démonstration et les exercices proposés sont souvent réduits à des calculs de limites utilisant les règles de calculs qui, elles aussi, ne sont pas démontrées.

Les étudiants qui entrent à l'université ont donc des conceptions, souvent intuitives et/ou erronées, sur la notion de limite. Le rôle de l'enseignant universitaire nous semble alors important au moment de l'introduction de la notion puisqu'il s'agit de montrer aux étudiants la nécessité de la définition formelle de la convergence et de leur faire comprendre en quoi les conceptions acquises dans l'enseignement secondaire ne sont pas toujours suffisantes pour appréhender la notion. Cependant, lorsque la notion de convergence est étudiée à l'université, elle est introduite plus ou moins rapidement avec la définition formelle. D'une certaine manière, tout se passe comme si la notion était présentée indépendamment des connaissances, même intuitives ou erronées, que les élèves ont acquises dans l'enseignement secondaire.

En 1983, Aline Robert a élaboré et expérimenté à de nombreuses reprises une ingénierie pour introduire la définition formelle de la notion de convergence tout en s'appuyant sur les connaissances déjà présentes chez les étudiants. Après avoir décrit notre problématique de recherche, nous présentons l'ingénierie et son expérimentation auprès d'étudiants belges en première année d'université en nous

centrant sur le rôle que peut jouer l'enseignant pour faire émerger la définition formelle.

PROBLÉMATIQUE

De nombreuses recherches ont mis en évidence les difficultés des étudiants universitaires avec la notion de limite d'une suite numérique, que ce soit du côté des représentations développées par les étudiants (Robert, 1983; Roh, 2008), ou encore du côté de la complexité de la définition formelle (Mamona-Downs, 2001; Durand-Guerrier & Arsac, 2005).

Concernant le rôle de l'enseignant universitaire, qui nous intéresse ici, Durand-Guerrier (2005) a par exemple montré que les pratiques des enseignants dans la manipulation des écritures quantifiées pouvaient être mises en relation avec les difficultés des étudiants. Bridoux, Grenier-Boley et Hache (projet de texte pour le colloque INDRUM 2016) étudient l'impact possible du langage utilisé par l'enseignant au moment de l'introduction de la définition formelle de la notion de limite (suite et fonction) sur les apprentissages des étudiants, notamment par le biais des reformulations de la définition. Mais quoi qu'il en soit, le rôle de l'enseignant universitaire dans l'acquisition de la notion de convergence fait l'objet de peu de recherche.

Nous nous intéressons ici au rôle joué par l'enseignant pour introduire la notion de convergence avec une ingénierie didactique. Au fil du temps, force est de constater que l'ingénierie de Robert a été peu reprise par les enseignants universitaires, alors que la notion est toujours enseignée aujourd'hui et que les difficultés rencontrées par les étudiants dans son processus de formalisation sont toujours d'actualité. Une explication possible est la difficulté, pour un enseignant, à reproduire une ingénierie dans sa classe en l'adaptant éventuellement à son public. Artigue (1988) explique que les ingénieries didactiques posent effectivement la question de leur reproductibilité, menant parfois à un phénomène d'obsolescence.

Dans le cadre des travaux menés par la CIIU (Commission Inter-IREM Université, en France), nous avons fait le pari que l'ingénierie de Robert pouvait être adaptée au public actuel, en y apportant quelques modifications mineures (CIIU, 2015). Cependant, Robert a peu développé le rôle de l'enseignant, notamment en ce qui concerne son discours au moment de l'introduction de la définition formelle. Nous abordons donc ici la question suivante: quel est le rôle précis que l'enseignant peut jouer à chaque étape de l'ingénierie pour faire émerger la définition de la convergence d'une suite chez les étudiants?

INGÉNIERIE ET DÉROULEMENT PRÉVU

L'élaboration d'ingénieries pose naturellement la question de caractériser les mathématiques à enseigner. Compte tenu des différentes fonctions que les notions jouent dans les programmes scolaires et dans les cours élaborés par les enseignants, Robert (2011) distingue différents types de notions en s'appuyant notamment sur la

distance entre les connaissances anciennes ou déjà là chez les élèves et les nouvelles connaissances visées. La prise en compte des spécificités des notions à enseigner mène alors à des choix variés pour introduire les nouvelles notions. En ce sens, la notion de convergence d'une suite numérique est une notion formalisatrice, unificatrice et généralisatrice (notion FUG). En effet, elle unifie et généralise des notions (même intuitives) rencontrées par les élèves au lycée à partir d'un nouveau formalisme. Les premières notions de topologie (Bridoux, 2011) et celles d'algèbre linéaire (Dorier, 1997) possèdent des caractéristiques semblables. Pour les notions FUG, l'écart entre l'ancien et le nouveau est très grand. Les caractères F, U et G de la notion de convergence font que celle-ci est difficile à introduire tout en lui donnant du sens. Cela amène le didacticien à concevoir pour les notions FUG des situations plus partielles que les situations fondamentales (au sens de Brousseau, 1998), n'ayant pas que des phases adidactiques et où on alterne des phases de recherche individuelle chez les étudiants avec des phases d'institutionnalisation de l'enseignant, en s'appuyant sur des leviers tels que des commentaires explicatifs, des changements de cadres ou de registres (au sens de Douady, 1986).

Nous avons expérimenté l'ingénierie de Robert (1983) avec un groupe de 45 étudiants d'une filière informatique en première année universitaire en Belgique. La notion n'avait pas encore été abordée avec eux à l'université, leurs seules connaissances sur la notion étaient donc celles construites dans l'enseignement secondaire. Par ailleurs, durant la semaine qui a précédé l'expérimentation, trois séances (environ 5 heures) ont été consacrées à la définition d'une suite, à la représentation graphique et à l'étude des caractères majorés, minorés ou bornés des suites. Ces notions ont été travaillées avec les étudiants sur de nombreux exemples. Voici la séquence telle qu'elle a été proposée à ce public.

1) *Considérons les suites de terme général suivant :*

1. $u_n = \frac{n^2-25}{2n^2+1}$ (échelle sur l'axe des ordonnées : une unité = 2cm).
2. $u_n = \frac{(-1)^n}{20}$ (échelle sur l'axe des ordonnées : une unité = 10cm).
3. $u_n = \frac{1}{n} \cos n$ (échelle sur l'axe des ordonnées : une unité = 2cm).
4. $u_n = \cos n$ (échelle sur l'axe des ordonnées : une unité = 5cm).
5. $u_1 = 1, u_2 = 2, u_3 = 3, u_4 = -1, u_n = 2$ pour tout $n \geq 5$.
6. $u_n = \frac{(-1)^n}{n^2+1}$ (échelle sur l'axe des ordonnées : une unité = 10cm).
7. $u_n = \cos n \frac{\pi}{6}$ (échelle sur l'axe des ordonnées : une unité = 2cm).
8. $u_n = \sin \frac{1}{\sqrt{n}}$ (échelle sur l'axe des ordonnées : une unité = 10cm).
9. $u_n = n^2 + 1$ (échelle sur l'axe des ordonnées : une unité = 0,5cm).
10. $u_n = \frac{1}{n+(-1)^n \sqrt{n}}$ ($n \geq 2$) (échelle sur l'axe des ordonnées : une unité = 10cm).

Après avoir dressé un tableau de valeurs permettant de calculer les 10 premiers éléments de chaque suite, représentez graphiquement chaque suite sur un dessin différent.

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- 2) *Pouvez-vous classer ces dessins ? Expliquez les critères permettant vos classements.*
- 3) *Dans chaque cas, pouvez-vous ou non trouver un nombre l et un entier n^* à partir duquel $l - \frac{1}{10} \leq x_{n^*} \leq l + \frac{1}{10}$. Expliquez brièvement votre choix. Même question en remplaçant $\frac{1}{10}$ par $\frac{1}{100}$. Mettez en relation ce que vous venez d'obtenir avec vos classements.*
- 4) *Les énoncés suivants sont-ils vrais ou faux ? Justifier vos réponses par écrit.*
- i) *Une suite à termes positifs qui tend vers 0 est décroissante à partir d'un certain rang.*
 - ii) *Si une suite a une limite strictement positive, tous ses termes sont strictement positifs à partir d'un certain rang.*

Le fil conducteur de l'ingénierie est de prendre appui sur une représentation dynamique [1] de la notion de convergence, très présente chez les étudiants (Robert 1983), pour classer les dessins (question 2) et d'enrichir cette première intuition avec une formulation numérique (question 3). Celle-ci s'avère insuffisante pour rédiger les démonstrations de la dernière question. La définition est alors donnée par l'enseignant et est accompagnée de commentaires pour la construire. La formalisation choisie pour la notion de convergence est une représentation en termes de bandes contenant tous les termes à partir d'un certain rang. L'écriture symbolique correspondante émerge au fur et à mesure grâce au jeu de cadres (au sens de Douady, 1986) entre les dessins et l'écriture formelle. Le nouveau formalisme est donc introduit pas à pas; ce qu'il unifie et généralise se fait par la reprise de diverses situations (des bandes de largeur $1/10$, puis de largeur $1/100$ et enfin toutes les bandes de largeur $\varepsilon > 0$ à la dernière question). En ce sens, l'ingénierie de Robert prend effectivement en compte la nature FUG de la notion de convergence.

Nous décrivons maintenant comment nous envisageons le rôle de l'enseignant à chaque étape de l'ingénierie, en dégagant quelques « passages commodes » qui peuvent selon nous favoriser l'émergence de la définition à la question 4 tout en laissant des marges de manœuvre à l'enseignant.

Pour la première question, nous avons conservé l'idée de faire travailler les étudiants en petits groupes de trois ou quatre étudiants, comme le prévoyait Robert, et de répartir la représentation des dix suites dans chaque groupe. La calculatrice est autorisée pour la construction des tableaux de valeurs mais les dessins sont réalisés à la main. L'objectif est que les étudiants discutent entre eux de leurs dessins. Le rôle de l'enseignant est ici de maintenir les conditions de travail.

À la question 2, les étudiants doivent produire des critères dans lesquels ils intègrent les dix suites. Certaines suites peuvent évidemment faire débat. Le comportement de convergence pourrait ne pas émerger dans tous les groupes comme un critère de classement. Nous pensons que pour mener à bien cette question, un certain nombre de connaissances doivent être disponibles chez les étudiants. Il nous semble évident que l'objet « suite » et la représentation graphique d'une suite doivent en faire partie mais

les notions de croissance et de suite majorée/minorée/bornée s'avèrent pertinentes comme critères de classement. En d'autres termes, le rôle de l'enseignant est aussi d'avoir préalablement organisé un milieu (au sens de Brousseau, 1998) pour les étudiants suffisamment riche pour questionner les liens entre les notions durant la séquence. L'enseignant circule donc auprès des étudiants pour prendre connaissance des critères retenus et les amène, si nécessaire, à discuter du comportement de convergence. Il peut par exemple rappeler que le dessin ne montre pas tout et que certains comportements nécessitent un regard plus global sur la suite. L'objectif est de lancer une discussion dans les groupes sur ce qui se passe « loin dans la suite ». L'enseignant collecte ensuite les classements. À cette étape, il peut expliquer aux étudiants que le choix est de se centrer sur le comportement de convergence, qui est le seul critère dont on n'a pas une définition précise. Il peut aussi insister sur le fait que la représentation (intuitive) qui leur a permis de classer les suites par rapport à ce comportement est peut-être correcte mais que les choix de classement restent des conjectures. De plus, certaines suites peuvent faire débat et ne pas avoir été classées de la même manière dans tous les groupes. Nous allons voir que le travail proposé par l'enseignant à la question 3 peut amener des conjectures correctes dans tous les groupes.

À la question 3, les étudiants reprennent le travail sur les dessins pour étudier les inégalités avec $1/10$ et puis avec $1/100$. L'objectif est d'abord de comprendre comment utiliser les dessins pour répondre à la question. On n'attend donc pas ici un travail algébrique pour prouver que les inégalités sont vérifiées. Tous les dessins ne permettent pas de répondre à la question avec $1/100$ puisqu'on n'a pas assez de points. L'enseignant peut le rappeler si nécessaire. Nous prévoyons alors une phase d'institutionnalisation. L'enseignant interprète avec les étudiants cette première formulation numérique de la convergence avec le vocabulaire suivant : « on cherche s'il est possible de construire une bande autour de l dans laquelle les éléments de la suite rentrent à partir d'un certain rang [2] ». L'enseignant peut également montrer des dessins sur ordinateur où, pour quelques suites, le nombre d'éléments représentés est plus important et des bandes de largeurs différentes sont tracées. L'enseignant peut aussi choisir l'une ou l'autre suite pour réaliser le travail algébrique permettant de justifier le choix du réel l et du naturel n^* et le fait que les inégalités sont vérifiées. Il y a également lieu d'amorcer une discussion sur la largeur des bandes en relation avec la convergence; le fait que la suite 2 vérifie les inégalités pour $1/10$ mais pas pour $1/100$ doit être pointé. À cette étape également, l'enseignant peut représenter une suite qui suscitait un débat à la question précédente et montrer que si on construit une bande autour d'un nombre qui n'est pas le bon candidat limite, la propriété n'est pas satisfaite.

À la question 4, les étudiants travaillent de nouveau en petits groupes. L'enseignant peut rappeler que les suites de la question 1 sont des exemples pour guider l'intuition, voire pour trouver un contre-exemple. Pour la deuxième affirmation, l'enseignant doit poursuivre la discussion sur la largeur des bandes, qui dépend ici de la limite,

pour pouvoir définir la convergence et amener l'idée que la formulation algébrique de la question précédente doit être vérifiée quelle que soit la largeur de la bande. L'enseignant présente alors la définition en $\varepsilon - N$.

L'EXPERIMENTATION

Quinze groupes de trois ou quatre étudiants ont été formés, la constitution des groupes étant laissée à la charge des étudiants. Deux de nos collègues ont participé à l'expérience pour encadrer le travail des étudiants dans les phases de recherche individuelle. L'expérience a duré 3h30, réparties sur deux séances de cours.

Il n'est évidemment pas possible de retranscrire tous les échanges entre l'enseignant et les étudiants dans ce texte. Nous étudions plus précisément le rôle de l'enseignant durant la correction de la question 4, où la définition de la notion devient nécessaire pour justifier la deuxième affirmation. Nous nous centrons sur le vocabulaire introduit par l'enseignant pour donner petit à petit du sens à la notion de convergence.

Pour la seconde affirmation, la majorité des étudiants pense qu'elle est vraie mais aucun groupe ne peut la justifier. L'enseignant demande quelle intuition a guidé les étudiants pour dire que c'est vrai. Un étudiant prend la parole :

Étudiant 7 : On a vu que si une suite converge vers l , toutes les bandes centrées en l ... il y a toujours un moment où les éléments sont dans la bande.

Le fait d'évoquer toutes les bandes centrées en l n'était évidemment pas prévu par l'enseignant. Il revient donc sur les constatations qui avaient émergé à la question 3 de la présentation des dessins montrés sur ordinateur:

Enseignant: Ce n'est pas exactement ça qu'on a vu, on n'a pas parlé de toutes les bandes mais de toutes les bandes entre $1/10$ et $1/100$ et on était resté avec l'idée que la bande de largeur $1/10$, ça ne fonctionne pas pour toutes les suites, $1/100$ ça fonctionnait, qu'effectivement entre $1/10$ et $1/100$ ça avait l'air de fonctionner aussi. Est-ce que cette idée peut être reprise ici pour expliquer ?

L'enseignant réalise en même temps un dessin au tableau. Il s'agit du premier graphique (à gauche sur le tableau de la figure 1). Le dialogue se poursuit :

Enseignant: Ça veut dire que les éléments sont compris entre $l-1/100$ et $l+1/100$. Vous voyez que si les éléments sont dans cet espace, conformément à ce que dit l'affirmation, à partir de n^* , les éléments seront positifs. Maintenant est-ce que ce dessin fait office de preuve ?

Étudiant 7 : Non parce que vous avez choisi un exemple qui allait bien pour nous montrer.

Enseignant : Tu peux me donner un exemple qui va moins bien ?

Étudiant 8 : Il faut prendre quelque chose de petit pour l , par exemple l qui vaut 0. Du coup, on a le $l-1/100$ en-dessous et les éléments ne sont plus positifs.

- Enseignant : Sauf que je ne peux pas prendre $l = 0$, par rapport à l'énoncé.
- Étudiant 8: On prend $l = 1/100$ par exemple.
- Enseignant : Oui. Vous voyez, mon dessin, je l'ai fait avec une valeur de l qui est suffisamment grande pour qu'en faisant $l-1/100$ et $l+1/100$, mes éléments soient dans l'espace et soient donc positifs.

L'enseignant fait un autre dessin au tableau pour mettre en défaut la situation (à droite sur le tableau de la figure 1).

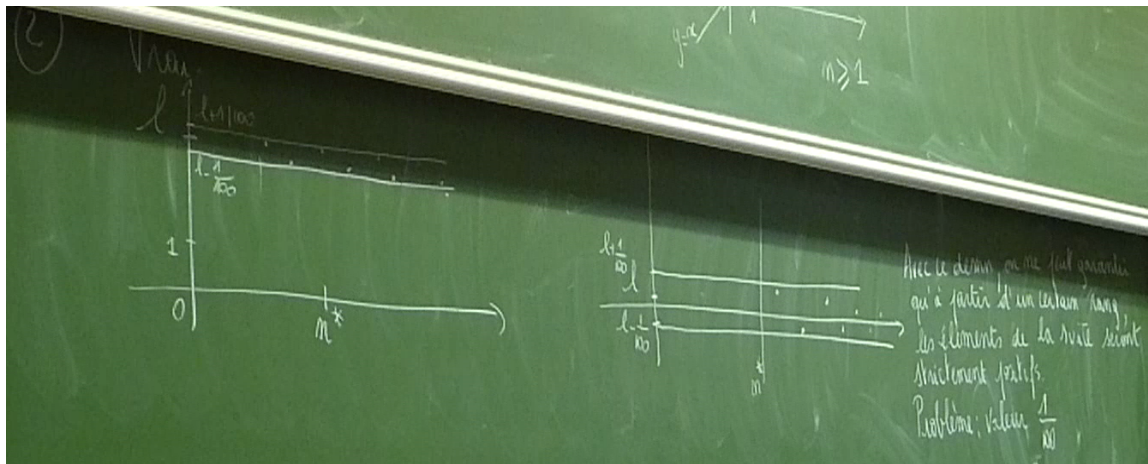


Figure 1: Question 4, largeur des bandes

- Enseignant : Bon, comment pensez-vous pouvoir régler le problème ? Qu'est-ce-que ça met en défaut ça (en montrant le dessin qui est à droite sur la figure 1) ?
- Étudiant 8 : C'est la valeur $1/100$ qui pose problème.

L'enseignant amène alors lui même l'idée de choisir un espace qui dépend de l en reprenant toujours le vocabulaire introduit à la question 3.

- Enseignant : Exact, ce qui pose problème ici, c'est l'espace qu'on laisse autour de l . Donc ici, avec ce dessin (il écrit au tableau ce qu'il dit), on ne peut pas garantir qu'à partir d'un certain rang, les éléments de la suite seront strictement positifs. Le problème est la valeur $1/100$ (l'enseignant reprend oralement). Ce qu'on commence à sentir, c'est que cette possibilité de faire rentrer les éléments dans la bande qu'on délimite autour de l , ça va dépendre de l . Est-ce que vous auriez une valeur à me proposer comme espace autour de l qui garantit que les éléments vont rentrer à partir d'un certain rang dans l'espace que je délimite ?

Pas de réponse.

- Enseignant : Peut-être pas une valeur précise mais me dire comment vous pouvez construire cet espace ?
- Étudiant 9: Inférieur à l .

Enseignant : Donc compris entre 0 et l , c'est ça.

Le professeur fait un dessin qu'il complète au fur et à mesure de l'échange suivant et écrit au tableau (voir figure 2) :

Enseignant : Pour que ça fonctionne, il faudrait laisser autour de l un espace ε tel que $0 < \varepsilon < l$. Allez, il y a beaucoup de nombres entre 0 et l , vous pouvez m'en donner un exemple ?

Plusieurs étudiants : On divise en deux.

Étudiant 10 : $\varepsilon = \frac{l}{2}$ fonctionne.

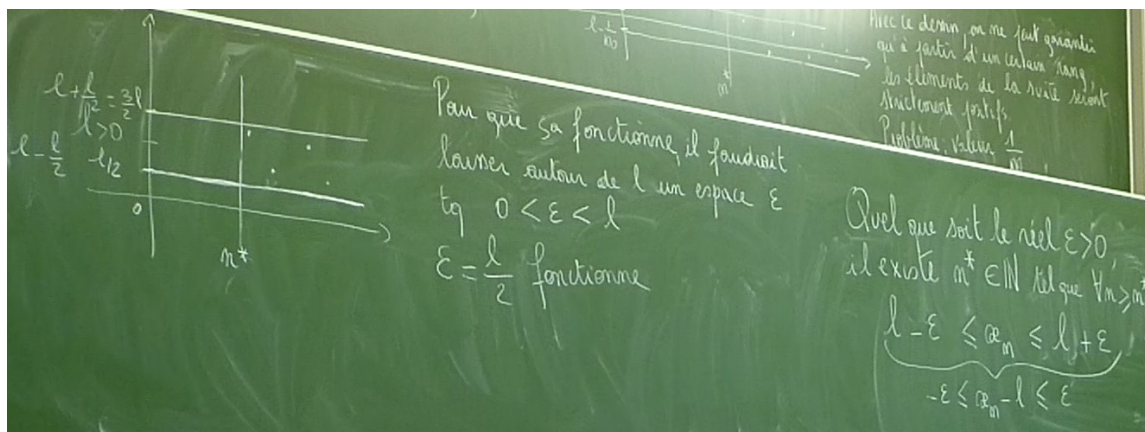


Figure 2: Émergence de la définition

Le professeur dresse un bilan oral et fait émerger la définition, comme le montre la figure 2:

Enseignant : Donc on est arrivé à la question 3, $1/10$ ne fonctionnait pas, $1/100$ fonctionnait pour toutes les suites de la liste mais cette valeur est remise en défaut ici où la largeur doit dépendre du candidat limite. Donc une manière simple choisie par les mathématiciens pour regrouper tout ça, c'est de se dire, bon, avec $1/100$ ça marche, plus on réduit l'espace, ça a l'air de marcher aussi, mais de temps en temps, l'espace doit aussi dépendre de la limite. Au final, ce qu'on va demander pour caractériser la convergence, c'est que cet espace qu'on délimite autour de l , ce soit valable tout le temps. On va demander que quel que soit l'espace qu'on délimite autour de l , à partir d'un certain moment, les éléments doivent rentrer dans cet espace, sachant et on l'a vu sur les dessins, que si vous laissez beaucoup d'espace, ce n'est pas intéressant. Les éléments vont rentrer facilement dans cet espace, ce n'est pas ça l'idée de la convergence. L'idée c'est d'autoriser des espaces tellement petits, et que là aussi ça garantit que les éléments rentrent dans l'espace. Donc une définition de la convergence c'est d'avoir :

Quel que soit le réel $\varepsilon > 0$, il existe $n^* \in \mathbb{N}$ tel que $\forall n \geq n^*, l - \varepsilon \leq x_n \leq l + \varepsilon$.

L'enseignant s'appuie ici sur le vocabulaire introduit tout au long de la séquence pour introduire la définition formelle avec des mots tout d'abord puis avec des symboles.

Bilan

Durant la correction de cette question, on peut voir l'enseignant prendre constamment appui sur les questions précédentes ainsi que sur l'image en termes de bandes autour du candidat limite pour faire émerger la définition. La réalisation des trois premières questions est donc essentielle pour le bon déroulement de la question 4.

Par ailleurs, pour l'enseignant universitaire, cette expérimentation nécessite probablement des changements de pratiques. Tout d'abord, il y a lieu d'accepter de consacrer un temps long à l'introduction d'une nouvelle définition. Comme nous l'avons expliqué, l'expérience a duré 3h30. L'enseignant doit également mettre en place un vocabulaire peut-être plus informel que d'ordinaire ainsi que des images au lieu de commencer par les définitions formelles, comme souvent dans les cours magistraux. La réalisation de dessins est elle aussi un élément essentiel dans le déroulement de l'ingénierie. Un post-test réalisé après l'enseignement complet (cours et exercices) de la notion de convergence a révélé qu'environ 90% des étudiants avaient une représentation statique [1] (au sens de Robert, 1983) de la notion.

CONCLUSION

Il nous semble tout d'abord important de souligner la robustesse de cette ingénierie « d'un autre temps ». Pour tenter d'en faciliter sa reproductibilité auprès des étudiants d'aujourd'hui et surmonter le phénomène d'obsolescence évoqué au début de ce texte, c'est le rôle de l'enseignant que nous avons choisi ici de préciser et d'étudier. Dans cette ingénierie, le rôle de l'enseignant est de montrer aux étudiants la nécessité d'une définition formalisée de la notion de convergence en commençant par lui donner du sens et en la faisant apparaître comme un outil de démonstration à la dernière question. Cet enjeu prend selon nous en compte les caractères F, U et G de la notion : on s'appuie sur une « pré-notion » de convergence liée à un caractère dynamique de rapprochement, généralisant à toute suite la définition, ce qui précède ne suffisant pas, avec un formalisme adéquat. Nous avons présenté un déroulement mais d'autres possibilités existent. L'enseignant peut en effet choisir une autre manière de répartir les étudiants, en groupes ou non, de répartir le temps de travail de manière différente (notre découpage tient avant tout à la durée d'une séance), voire d'autoriser la calculatrice graphique pour gagner du temps, même si pour ce dernier point, nous restons d'avis que le travail papier-crayon engendre des discussions dont on ne peut pas affirmer qu'elles se transposeraient naturellement en regardant un dessin « déjà fait » par la calculatrice. De nouvelles expérimentations sont prévues et seront complétées par des analyses des notes prises par les étudiants et des analyses de questions portant sur la convergence issues d'évaluations.

NOTES

1. Les représentations dynamiques sont des représentations en termes d'action où le mot « converger » est exprimé en termes de « se rapprocher de ». Les représentations statiques sont des représentations en langue naturelle de la définition

formalisée. Robert (1983) a montré que les étudiants chez qui le modèle statique était présent réussissaient mieux les exercices portant sur la notion de convergence.

2. L'expression « à partir d'un certain rang » a été déjà rencontrée par les étudiants dans le chapitre précédent sur les suites numériques.

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Entrée des étudiants dans l'Analyse formelle de début d'université : Potentialité des méthodes numériques d'approximation

Imène Ghedamsi

Université de Tunis, IPEIT, Tunisie, ighedamsi@yahoo.fr

Dans cet article, nous nous posons la question des moyens que peut se donner l'enseignement des mathématiques, à l'entrée à l'université, pour permettre aux étudiants d'accéder aux objets de base de l'analyse réelle. En se plaçant dans le cadre de la TSD, nous étudions les potentialités d'un milieu théorique fondé sur les méthodes numériques d'approximation, pour engager les étudiants dans une phase de travail expérimental leur permettant de retourner efficacement sur un système cohérent d'objets de l'analyse. Nous décrivons succinctement comment l'entrée dans un processus de preuves mixtes – pragmatiques vs formelles – a été rendu obligatoire dans le travail des étudiants, à travers l'émergence du problème général de l'existence et de l'accessibilité des nombres, des limites et des suites.

Mots-clés : existence, approximation, pragmatique, formel, milieu.

INTRODUCTION

De nombreux travaux en didactique de l'analyse soulignent les modifications majeures qui devraient accompagner le travail des étudiants à l'entrée à l'université. Ces modifications ont été modélisées dans la littérature en utilisant plusieurs angles de vues. Les travaux qui étudient le développement cognitif mettent en exergue les exigences de flexibilité requises dans le travail des étudiants : mise en œuvre de plusieurs registres de représentations sémiotiques ; mise en fonctionnement de plusieurs niveaux des connaissances (la disponibilité de ces connaissances étant le niveau habituellement exigible à l'entrée à l'université) ; transiter "efficacement" entre les dimensions intuitive (incarnée sur la base d'expérimentations nouvelles et plus anciennes), proceptuelle (dualité processus-objet) et formelle des notions de l'analyse, etc. (Praslon, 2000 ; Bloch & Ghedamsi, 2005 ; Tall & Mejia-Ramos, 2006 ; Chellougui, 2009 ; Bergé, 2010 ; Vandebrouk, 2011 ; González-Martín et al., 2011, etc.). Plusieurs de ces travaux tablent leurs analyses sur la modélisation à priori de l'organisation mathématique en jeu, en praxéologies mathématiques. L'usage de la modélisation praxéologique dans certaines études, a de plus permis de pointer essentiellement la prédominance du bloc théorique dans les praxéologies de début d'université (Bosch et al., 2004 ; Winslow, 2008 ; Diaz et al., 2008, etc.). Dans tous les cas, l'existence formellement établie des objets de l'analyse réelle est souvent invoqué pour investiguer les difficultés que pourraient engendrer, dans le travail des étudiants, ces nouvelles attentes. Le lien entre nombre réels et limite cristallise à son tour d'une manière fondamentale cette existence, et ceci qu'il s'agisse de l'existence d'une borne supérieur, d'une limite (convergence d'une suite), d'une suite (densité de Q dans R ou théorème de Bolzano-Weierstrass par exemple), d'une valeur intermédiaire, d'un point fixe, d'un nombre dérivé (théorème des accroissements

finis, théorème de Rolle ou formule de Taylor par exemple) ou encore d'un infinitésimal (reste intégral ou un équivalent d'une fonction par exemple).

Par conséquent, dans une problématique de type recherche d'une progression de situations de l'analyse réelle à l'entrée à l'université, l'un des enjeux est celui de convaincre les étudiants qu'il est possible d'explicitier et de donner plus de visibilité à ces objets. L'approfondissement et la reprise du travail sur les nombres réels, ainsi que l'élargissement du champ d'expérience des étudiants relativement à la nature des nombres réels et à leur manifestation constituent donc un second enjeu incontournable d'une telle progression. Le questionnement que nous posons globalement est le suivant : Dans quelle(s) mesure(s), un travail d'investigation numérique sur nombres réels et limite permettrait-il de réintroduire d'une manière consistante du point de vue de l'apprentissage, un ensemble d'objets constitutifs de l'analyse réelle à l'entrée à l'université ? Nous plaçons d'emblée l'étude investiguant un tel questionnement dans le cadre de référence de la Théorie des Situations Didactiques (Bloch, 2002 ; González-Martín et al., 2014). La question de la recherche d'une dimension d'expérience dans le travail des étudiants est de ce fait posée d'un double point de vue : celui des composantes souhaitables pour la construction d'un milieu théorique de la progression de situations envisagée (d'un point de vue essentiellement épistémologique), ainsi que celui du rôle de la classe - y compris du professeur, conçu sur la base des différents niveaux d'un milieu expérimental a priori. Dans le cadre de ce travail, notre réflexion sera essentiellement portée sur les conditions d'élaboration des situations plutôt que sur les modalités d'expérimentation.

LES METHODES NUMERIQUES : UNE PLACE DE CHOIX DICTEE PAR LE CADRE DE REFERENCE DE LA TSD

La construction du milieu théorique de la progression envisagée stipule à un premier niveau de réintroduire des questionnements d'ordre épistémologique et/ou mathématique. Ce niveau d'étude devra permettre de définir un schéma théorique modélisant les principales caractéristiques d'une organisation de l'analyse réelle, qui prenne appui sur le lien entre nombres réels et limite afin de revisiter le réseau des objets de base l'analyse standard (Convergence de suite, densité de Q dans R , segments emboîtés, borne supérieure, valeur intermédiaire, accroissements finis/Rolle, point fixe, formule de Taylor).

Il est largement reconnu que l'organisation mathématique en analyse réelle telle qu'introduite à l'entrée à l'université a complètement supplanté les préoccupations épistémologiques liées à la construction des nombres réels. Or, il est aussi amplement connu que la théorie de l'analyse standard s'est stabilisée à l'issue de la construction rigoureuse de l'ensemble des réels et de la mise en place des propriétés topologiques des réels, qui constituent un fondement théorique des savoirs sur suite, limite et fonction. Par conséquent, avoir un niveau de signification suffisant des objets de base de l'analyse réelle suppose que l'on a acquit le sens de l'adage "*approcher, majorer, minorer*" (Dieudonné, 1980), et on en a établi le lien avec le formalisme que requiert

l'existence des nombres réels. Finalement, la préoccupation ultime est celle de se donner les moyens d'investiguer la question ultime de Cauchy de ce qu'est un nombre réel et de ce qu'il entretient comme relations avec les objets qui fondent la base de l'analyse standard. Toutes ces considérations nous portent à choisir de ne pas évacuer le problème du numérique (et/ou de l'approximation) et de la structure de \mathbb{R} , mais de le mettre au cœur des problèmes mathématiques et des situations qui doivent être travaillées par les étudiants.

L'évolution historique de la conception des nombres est accompagnée par la genèse et le développement des méthodes numériques d'approximation successive. A travers l'histoire, l'usage de ces méthodes a porté en germe un travail avec les objets de base de l'analyse énoncés ci-dessus ; les questions relatives aux notions de suite et fonction dérivent de l'idée d'approximation et de la nature des nombres réels/limite. Ces méthodes ont très longtemps manqué de bases solides pour la validation qui demeurait imprécise, et aucun progrès significatif ne s'est fait avant le XIX^e siècle, un siècle qui marque l'entrée dans l'ère de la formalisation de la notion de nombre (Dieudonné, 1978 ; Hairer & Wanner, 2000). Finalement, le recours aux méthodes numériques d'approximation permet de mettre en œuvre un système satisfaisant des objets de l'analyse, de faire apparaître sa cohérence et de rendre accessible ses objets, dont l'existence théorique n'est pas forcément "constructible". L'étude de nature épistémologique et mathématique concernant le développement de différentes méthodes d'approximations dans leur lien avec l'évolution de la conception de nombre (Ghedamsi, 2008), nous a permis de préciser un modèle théorique de situations engendrant, dans la mesure du possible, un ensemble satisfaisant de problèmes spécifiques des nombres réels dans leurs relations avec les notions de limite, suite et fonction. Parmi les caractéristiques de ce modèle, nous citons : 1) concevoir la nature des nombres (distinguer entre rationnel, irrationnel algébrique, irrationnel transcendant) ; 2) faire une hypothèse et la valider sur la possibilité de construire une suite approchant un nombre ; 3) calculer des valeurs approchées d'irrationnels (par exemple solution d'une équation non résoluble par l'usage des techniques algébriques à disposition) avec une approximation arbitrairement fixée ; 4) construire une suite convergeant vers un réel et contrôler la rapidité de la convergence (exhiber une approximation d'un réel, contrôler l'erreur dans les termes de la suite) ; 5) contrôler la validité des résultats numériques (les relier aux définitions et théorèmes déjà introduits).

La question se pose des moyens qui favoriseraient l'engagement des étudiants dans de telles situations à travers une phase expérimentale permettant : d'une part, un travail heuristique sur des objets spécifiques, avec des preuves pragmatiques et des opportunités de conjectures, de calculs, etc., et d'autre part, un retour au formalisme dans un milieu de référence (de validation) informé par le milieu objectif (et/ou d'action) précédemment visité. Dans ce qui suit, nous explicitons succinctement comment le recours aux méthodes numériques permet d'organiser de telles situations dans les conditions susmentionnées. Dans le cadre de ce travail, les méthodes qui

nous sont apparues comme les plus riches en possibilités didactiques sont celles des fractions continues, introduites historiquement par Théon de Smyrne et la méthode de Newton, qui du fait de leur histoire complexe, tant au niveau des motivations intrinsèques qu'au niveau des questions futures qu'a provoqué l'usage implicite de savoirs, s'énoncent à l'origine comme pragmatiques et imprécises mais suscitant de plus en plus de rigueur. C'est en grande partie sur la densité de Q dans R , la convergence des suites, les segments emboîtés, la valeur intermédiaire, le point fixe, les accroissements finis, Rolle et la formule de Taylor, ayant historiquement manqué longtemps de base théorique précise, que se fondent ces méthodes. Nous anticipons de la part des étudiants des va et vient entre des manques théoriques restés longtemps inaperçus et la rigueur de l'édifice final tel qu'il est enseigné actuellement.

Au moins deux paramètres sont à prendre en considération afin d'introduire un milieu d'expérience des objets de l'analyse tels que nous le visons : 1) le milieu disponible pour la dévolution doit permettre la manipulation d'une variété d'ostensifs (graphiques, géométriques, numériques) de nombres, suites et fonctions. De plus, on peut s'attendre à ce qu'un registre donne lieu à une manipulation d'ostensifs d'un autre registre ; ceci sous-entend que les réponses ne devraient pas être données dans un contrat d'ostension. Les tâches d'action et de formulation sollicitées devraient amener les étudiants à questionner les savoirs institutionnels de l'analyse réelle et les exemplifier. La richesse des questions que peut susciter le problème d'approximation et les objets auxquels elles réfèrent, peut mettre les étudiants sur la voie de validation pragmatique/sémantique, déclencher leurs intuitions et les confronter (dans un milieu de référence conçu à cette fin) à des savoirs qu'ils ont reçu sans dimension d'expérience. Dans ces conditions, le milieu objectif pourrait a priori, aussi bien être un milieu graphique, géométrique ou numérique. Bien entendu, ceci n'empêche pas le fait que les étudiants pourraient recourir à des ostensifs algébriques ou analytique au cours de leurs investigations ; 2) à la suite d'investigations sur les nombres, le milieu de référence (particulièrement algébrique et analytique) est supposé mettre à défaut le recours à une application "automatique" des définitions, théorèmes et formalisme du cours. Nous attendons que les étudiants, au cours de leur contact direct avec les nombres, développent des évidences sous forme d'intuitions fondées graphiquement, géométriquement, numériquement ou autre. Nous attendons aussi qu'ils soient en mesure de conforter ces évidences en les confrontant aux objets de l'analyse visés dont l'existence a été formellement introduite, et réciproquement. Encore faut-il que le milieu de référence, censé permettre de formuler des preuves à l'aide des règles du calcul formel, puisse favoriser ce phénomène, et ce d'autant plus qu'il ne fait pas explicitement référence aux nombres et aux approximations.

La question du contrôle par les étudiants de la validité des résultats empiriques (numériques ou autres) suppose d'exhiber un objet spécifique (générique ou non) et d'attester de son existence en le reliant à l'objet général qui lui est associé. Réciproquement, l'idée est aussi de conforter et d'interroger cette existence tout en gardant le contrôle pragmatique. Par exemple, dans le cas de la notion de valeur

intermédiaire, l'idée intuitive étant que toute fonction continue sur un intervalle ne peut prendre deux valeurs distinctes $f(a)$ et $f(b)$ sans prendre les valeurs comprises entre $f(a)$ et $f(b)$ (le tracé d'une fonction continue sur un intervalle ne présente pas de trous). Mais alors, cela nous aura ouvert à d'autres affirmations (ou considérations intuitives) disant, par exemple : lorsqu'une fonction continue sur un intervalle prend deux valeurs opposées, elle prend nécessairement la valeur 0 en, au moins, un point de cet intervalle. De même, si on se donne à réfléchir sur la question (appuyée par les graphiques ci-dessous), de ce qui se passe dans le cas où f est discontinue (existe-t-il t compris entre $f(a)$ et $f(b)$ et qui n'est image d'aucun c compris entre a et b ?).

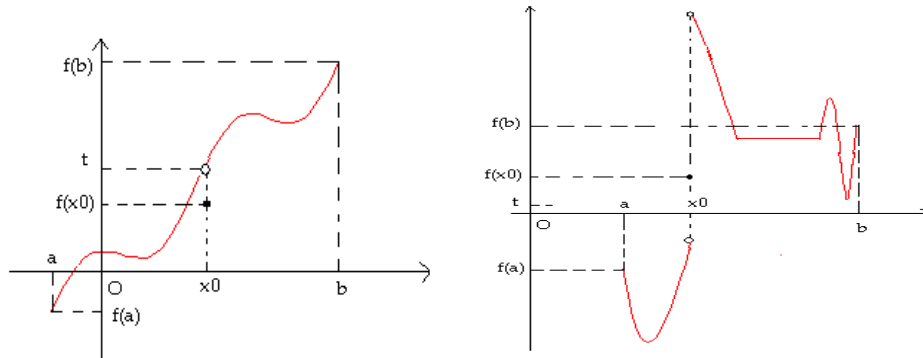


Table 1: Exemples de graphiques de fonctions

Posons- nous maintenant la question de ce qui se passe si on veut expliciter c . Cette question est légitime si on s'intéresse à la résolution de l'équation $f(c) = t$. Puis posons- nous la question plus générale de savoir traiter l'équation dans le cas où l'outil algébrique ne le permet plus. Ceci nous amène à la preuve du théorème des valeurs intermédiaires qui se fait à l'entrée à l'université en fonction de l'axiomatique choisie (borne supérieure, segments emboîtés, etc.). Se pose maintenant la question du rôle conceptuel de cette démonstration, en d'autres termes à quel recours supplémentaire faudrait-il faire appel pour développer d'autres idées, affiner et/ou renforcer l'intuition ? Ceci du fait par exemple, que dans le cas où t admet plus qu'un antécédent, la preuve d'existence concernera "un seul terme c ", comment retrouver les autres ? Une source possible d'intuitions appropriées serait les approximations successives en l'occurrence au moyen de support graphique. Ce raisonnement s'applique aussi aux objets introduits dans le théorème des accroissements finis et le théorème de Rolle. Dans tous les cas, l'appui sur le contenu conceptuel de la densité de \mathbb{Q} dans \mathbb{R} permettrait d'exhiber une suite de rationnels d'approximation des objets spécifiques en jeu en traduisant la convergence de la suite vers un nombre par : quel que soit l'erreur que l'on considère, on peut toujours trouver un ordre à partir duquel tous les termes de la suite sont des valeurs approchées de ce nombre à cette erreur près.

Dans la suite, nous décrivons succinctement un exemple d'une progression de deux situations que nous avons construite et expérimentée en se fixant trois enjeux : 1) reprendre le travail sur les nombres réels, et élargir le champ d'expérience des

étudiants relativement à la nature des nombres réels et à leurs manifestations de façon à ce que le savoir de base sur la structure de \mathbb{R} devienne disponible et opérationnel ; 2) conduire les étudiants dans une phase expérimentale à un travail heuristique, permettant de développer un processus de preuves pragmatiques ; 3) amener les étudiants à saisir le lien entre expérience pragmatique et existence des objets de l'analyse réelle par le biais d'un aller/retour preuves pragmatiques/preuves formelles (pour plus de détails concernant les situations expérimentales, voir Bloch & Ghedamsi, 2010 ; Ghedamsi & Chellougui, 2013 ; González-Martín et al., 2014).

CONSTRUCTION/EXPERIMENTATION DES SITUATIONS

Les variables didactiques qui caractérisent le modèle théorique des situations sont au nombre de sept : nature du nombre (rationnel/irrationnel, algébrique, transcendant), nature de la suite, type de suite (explicite/récurrente), comportement de la suite (croissante, décroissante, etc.), types d'approximation et d'erreur, nature de l'équation en jeu, méthode d'approximation. Les valeurs données à ces variables sont conditionnées par l'objectif ultime qui concerne la problématisation par les étudiants de la nature des nombres et de leur lien avec limite et les objets de base de l'analyse formellement introduits. Par exemple concernant la variable "nature du nombre", on écarte de prime abord le cas d'un rationnel ; le choix d'un transcendant défini implicitement comme solution d'équation permet de soulever les questionnements requis. Pareillement en ce qui concerne la variable "type de suite" si l'on veut bénéficier de l'avantage que représente le fait de pouvoir visualiser le processus d'approximation ; le choix des suites récurrentes se justifie par l'accessibilité et la clarté du procédé qui permet de les représenter, et afin de ne pas négliger les conditions sur le continu, il faudrait avoir le moyen de les représenter sur la droite réelle. Il est clair que pour la variable "comportement de la suite", le prototype de suites bien faites croissantes ou décroissantes est à éliminer.

En regard de toutes les conditions auxquelles les valeurs des variables doivent être soumises, deux situations ont été retenues : 1) Antiphérèse de racine de 2, qui table sur la construction de la meilleure approximation rationnelle de $\sqrt{2}$ et, dans la mesure du possible, sa généralisation à certains irrationnels. La méthode utilisée est basée sur les fractions continues et a été introduite historiquement par Théon de Smyrne au temps d'Euclide ; 2) Point fixe de cosinus, qui prend appui sur une discussion de la qualité de l'approximation du point fixe de cosinus donnée par la méthode de dichotomie par comparaison à celle donnée par la méthode de Newton. Dans tous les cas, l'ingénierie prévue doit amener les étudiants à travailler sur la nature des nombres (rationnels, irrationnels, algébriques, transcendants), les sous-ensembles de \mathbb{R} et l'approximation des réels non rationnels ; ceci dans la perspective plus large de revenir sur la réintroduction de la densité de \mathbb{Q} dans \mathbb{R} , la convergence d'une suite, la valeur intermédiaire, les segments emboîtés, les accroissements finis, le point fixe et les fonctions contractantes, la formule de Taylor.

Antiphérèse de racine de 2

La situation de l'antiphérèse vise à permettre aux étudiants de prendre appui sur le contenu conceptuel du théorème de densité de \mathbb{Q} dans \mathbb{R} par le biais d'instanciations appropriées, ainsi que de mettre en œuvre et revenir sur un réseau de savoirs de l'analyse réelle dont les segments emboîtés et les accroissements finis. L'institutionnalisation prévue concerne des procédures et la méthode de construction d'une suite de rationnels qui converge vers un irrationnel de la forme \sqrt{d} , d un entier supérieur à 2, avec $d - 1$ un carré parfait. L'organisation générale de la situation part de l'idée d'une instanciation de ce théorème d'existence via la recherche d'une suite rationnelle convergeant vers un irrationnel algébrique, de sorte que la construction d'une telle suite : 1) problématise la nature des nombres (différence entre rationnels/irrationnels, représentations numériques des nombres et lien avec les approximations successives, les limites, etc.) ; 2) soit une méthode généralisable à d'autres irrationnels algébriques de la même forme. Nous avons guidé l'introduction par un travail sur l'irrationalité inspiré du principe d'exhaustion s'appuyant sur la figure d'un triangle rectangle isocèle, utilisé par Euclide pour montrer l'incommensurabilité de la diagonale d'un carré. A condition de disposer de notations convenables, le procédé itératif appelé antiphérèse conduit à ce que l'on appelle aujourd'hui, le développement de $\sqrt{2}$ en fraction continue illimitée périodique. Plus précisément, dans le cas d'un entier $d \geq 2$ vérifiant les conditions requises, l'égalité $\sqrt{d} - \alpha = \frac{1}{\alpha + \sqrt{d}}$, où $\sqrt{d - 1} = \alpha$ permet de donner le développement en fraction continue de $\sqrt{d} = \alpha + \frac{1}{2\alpha + \frac{1}{2\alpha + \frac{1}{2\alpha + \dots}}}$ et la suite des réduites de \sqrt{d} , obtenue en arrêtant

à chaque fois le processus itératif à un ordre précis, n'est autre que la suite définie par $u_0 = \alpha, u_{n+1} = \alpha + \frac{1}{u_n + \alpha}$. Cette situation met en jeu des questions sur irrationalité, approximations, construction, etc. et oblige les étudiants à confronter le fonctionnement du formalisme sur des objets spécifiques.

Le point fixe de cosinus

A côté des objectifs fixés à la situation de l'antiphérèse, celle du point fixe de cosinus vise de plus à amener les étudiants à un travail plus approfondi sur les nombres : différence entre rationnels /irrationnels (irrationnels algébriques, irrationnels transcendants), approximation des réels/limite, précision d'approximation, rapidité de convergence, etc. La méthode d'approximation que nous avons choisie est celle de Newton, issue d'une idée de ce dernier pour la résolution d'une équation algébrique du troisième degré. Elle est fondée sur l'idée de l'interprétation graphique d'une courbe approchée par sa tangente. Rappelons qu'à cette époque, l'énoncé du procédé était empirique. Aujourd'hui, on généralise le procédé de Newton à toute équation de la forme $g(x) = 0$ où g est une fonction de classe C^2 sur un intervalle fermé de \mathbb{R} , et telle que l'équation possède une unique solution simple α ($g(\alpha) = 0$ et $g'(\alpha) \neq 0$) à l'intérieur de cet intervalle. La convergence vers α de la suite construite par la

méthode de Newton résulte d'un choix conséquent de son premier terme. Nous avons orienté le travail de sorte que les étudiants : 1) procèdent à un premier contact graphique et numérique avec les idées qui fondent la méthode de Newton celles du rôle de la tangente, du rôle du premier terme, des conditions sur la fonction ; 2) fassent le lien entre les conditions de convergence et les outils de preuve de la convergence (en particulier l'usage de la formule de Taylor et ce que cet usage nécessite comme hypothèses aussi bien explicites qu'implicites et propres aux conditions de convergence de la suite). Cette situation a confronté les étudiants à la question de l'existence de la solution d'une équation non résoluble par les outils algébriques à disposition, et les a obligé à s'interroger sur la nature des nombres transcendants et le lien qu'ils entreprennent avec le formalisme de l'analyse.

EFFETS DE L'INGENIERIE : SYNTHESE ET DISCUSSION

Au terme des résultats des expérimentations, nous avons pu souligner la portée de travail des étudiants sur les situations expérimentées via trois points essentiels : 1) Saut conceptuel sur les nombres : le travail dans les situations a permis aux étudiants d'envisager les nombres réels avec leur statut d'objet mathématique. La focalisation sur les méthodes numériques d'approximations successives a favorisé un accès indirect à cet objet. Au final, les questions qui portaient au départ sur les nombres, concourent à l'émergence du problème général de l'accessibilité, et donc d'une réflexion métamathématique. Cette accessibilité est ce qui donne la clé d'un travail sur un réseau de savoirs de l'analyse réelle : densité de \mathbb{Q} dans \mathbb{R} , limite d'une suite, point fixe, segments emboîtés, valeurs intermédiaires, accroissements finis, formule de Taylor ; 2) Problématique existence/accessibilité : la question de l'existence des objets spécifiques permet, non seulement de dévoluer les situations, mais aussi de tirer le travail des étudiants vers la dimension pragmatique, ce qui les conduit à problématiser la question d'existence des objets mathématiques. De ce fait, un travail de va et vient entre preuves pragmatiques et preuves formelles s'installe, favorisant l'émergence du lien entre la procédure de recherche (mise en œuvre dans le milieu objectif) et la preuve finalement établie ; 3) Lien entre variables didactiques et les modifications requises dans le travail des étudiants en analyse à l'entrée à l'université : les valeurs des variables didactiques dont le modèle théorique est issu d'une réflexion épistémologique, contribuent, par l'intermédiaire d'un milieu mixte, à la prise en compte d'une évolution progressive dans travail des étudiants. Cette progression est ce qui permet d'articuler les éléments du processus d'entrée dans les connaissances de l'analyse réelle : la situation de l'antiphérèse donne un statut à des nombres vus jusque là comme des "notations" ; celle du point fixe de cosinus permet d'aller plus loin dans la conceptualisation. Elle donne accès à des nombres réels qu'on ne sait pas expliciter, et donc oblige à mettre en œuvre des procédures formelles de traitement de ces nombres. Bien entendu ces procédures ne peuvent être que des énoncés analytiques.

Nous avons donc clairement identifié deux étapes irréductibles l'une à l'autre dans l'enseignement/apprentissage de l'analyse réelle : la première mène d'un travail

pragmatique de recherche d'approximation d'un nombre tel que $\sqrt{2}$ à la conception de nombres irrationnels ; la deuxième amène à concevoir que, pour contrôler des objets non descriptibles de façon simple, il est nécessaire de mettre en œuvre des énoncés formels d'une certaine nature. En dehors des savoirs académiques ainsi construits, il apparaît de façon incontournable que les situations sont porteuses de connaissances métamathématiques. Finalement la question de l'existence des objets de base de l'analyse réelle a induit la question de la nature du travail des étudiants dans des situations qui veulent prendre appui sur le contenu conceptuel de cette existence (Comment inciter les étudiants à interroger ces objets et susciter leur intuition ?). Cette question d'existence génère aussi un autre type de questionnements ; celui lié à l'unicité (dans le cas où ce qui existe est unique, par exemple la notion de limite). Or, traiter de l'unicité de l'objet existant dans l'analyse standard se déroule en général suivant une démarche classique qui conduit à supposer la non unicité et à en conclure une absurdité. Cette question n'a pas eu un statut particulier dans ce travail.

Hasardons-nous par exemple sur la question de l'unicité de la limite d'une suite convergente. Si l'on part de la définition formelle, rien ne nous permet de dire que la limite est unique. Une fois la preuve établie – et qui consiste à prouver que $\forall \varepsilon > 0, |l - l'| < 2\varepsilon$, on aura réglé la question sur le plan syntaxique. On pourrait penser que la contraposée de la définition est plus convaincante $l \neq l', si \exists \varepsilon > 0, |l - l'| \geq \varepsilon$. Implicitement fondée sur le continu archimédien de l'ensemble des réels, les étudiants de début de première année d'université ont-ils jamais mis en doute la forme de cette définition ? Certains travaux investiguant le statut de l'égalité en analyse, ont montré les difficultés des étudiants à concevoir la forme de la définition. Face à des questions du type que peut-on dire de a et b sachant que $\forall \varepsilon > 0, |a - b| < \varepsilon$, les étudiants sont dans l'incapacité de conclure sur l'égalité. Dans le cadre de cette étude, nous pouvons attester que la problématique syntaxe/situations se retrouve dans cette question. Ce que nous pourrions investiguer concerne la consistance et/ou pertinence du recours aux méthodes numériques d'approximation pour traiter des questions liées au statut de l'égalité en analyse standard.

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Understanding irrational numbers by means of their representation as non-repeating decimals

Ivy Kidron

Jerusalem college of Technology

ivy@jct.ac.il

Research study on students' conceptions of irrational numbers upon entering university is of importance towards the transition to university. In this paper, we analyze students' conceptions of irrational numbers using their representation as non-repeating infinite decimals. The majority of students in the study identify the set of all decimals (finite and infinite) with the set of rational numbers. In spite of the fact that around 80% of the students claimed that they had learned about irrational numbers, only a small percentage of students (19%) showed awareness of the existence of non-repeating infinite decimals.

Keywords: extension of rational numbers to irrational numbers; irrational numbers, intuition, non-repeating infinite decimal, transition to university.

INTRODUCTION

The paper deals with students' conceptions of irrational numbers. The importance of real numbers towards the learning of analysis is well known. The understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers. Therefore, research study on students' conceptions of irrational numbers upon entering university is of importance towards the transition to university. Artigue (2001) wrote about the necessarily reconstructions which deal with mathematical objects already familiar to students before the official teaching of calculus:

Real numbers are a typical example...Many pieces of research show that, even upon entering university, students' conceptions remain fuzzy, incoherent, and poorly adapted to the needs of the calculus world...the constructions of the real number field introduced at the university level have little effect if students are not faced with the incoherence of their conceptions and the resulting cognitive conflicts (Artigue, 2001, p.212).

This study is a part of a broader study which aims to investigate students' conceptions of rational and irrational numbers upon entering university. Using epistemological considerations, three different representations of the irrational numbers were considered in the broader study. The first one relates to the decimal representation of an irrational number. The second representation relates to the fitting of the irrational numbers on the real number line. The third representation considers the relationship between incommensurability and the irrational numbers. In this paper, we consider the first representation and analyze students' understanding

of irrational numbers by means of their representation as non-repeating infinite decimals.

THEORETICAL BACKGROUND

Monaghan (1986) observed that students' mental images of both repeating and non-repeating decimals often represent "*improper numbers which go on for ever*". Because of their infinite decimal expansions, these numbers are often considered as *infinite* numbers. Tall (2013) relates to students' difficulties with irrational numbers:

The shift from rational numbers to real numbers proves to be a major watershed for many students. In school, students meet irrational numbers such as $\sqrt{2}$, π and e , and begin to realize that the number line has numbers on it that are not rational, though it is not clear precisely what these irrational numbers are (Tall, 2013, p. 265).

Kidron and Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximation or as a dynamic creature which is in an unending process- a potentially infinite process: in each next stage we improve the precision with one more digit after the decimal point. Vinner and Kidron (1985) analyzed the concept of repeating and non-repeating decimals at the senior high level. The present study is a broader study of the part that relates to non-repeating decimals. Fischbein, Jehiam and Cohen (1995) observed that the participants in their study were not able to define correctly the concepts of rational, irrational, and real numbers. Zaskis and Sirotic (2004) analyzed how different representations of irrational numbers influence participants' responses with respect to irrationality. Sirotic and Zaskis (2007) observed inconsistencies between participants' intuitions and their formal and algorithmic knowledge. The authors claim that constructing consistent connections among algorithms, intuitions and concepts is essential for understanding irrationality.

From the epistemological approach, the difficulties that are inherent to the nature of the specific domain should be taken into account (Barbé et al., 2005). Some of the cognitive difficulties in relation to the concept of irrational numbers might be a consequence of the way we conceive the concept of infinity. Fischbein, Tirosh, and Hess (1979) observed that the natural concept of infinity is the concept of potential infinity. Therefore, students' intuition of infinity might become an obstacle in the understanding of irrational numbers as non-repeating infinite decimals. Fischbein's theory which offers a rich insight in the mechanisms of intuition will serve as theoretical framework for the present study. Fischbein considers the intuitive structures as essential components of productive thinking. Fischbein (1987, pp.129-130) distinguishes different types of analogies which may intervene, tacitly or explicitly, in mathematical reasoning. He also refers to some kind of analogies that manipulate the reasoning process from "behind the scenes". Fischbein (2001) analyzes several examples of tacit influences exerted by mental models on the interpretation of mathematical concepts in the domain of actual infinity. He

describes the concept of mental models as mental representations which replace, in the reasoning process, the original entities.

METHODOLOGY

The task

A questionnaire (which served as a research tool) was compiled and administered and the results concerning one of its questions is brought and analysed here.

Question: A teacher asked his students to give him an example of an infinite decimal.

Dan: I will look for two whole numbers such that when I divide them I would not get a finite decimal; for instance: 1 and 3

Ron: I will write down in a sequence digits that occur to me arbitrarily, for instance: 1.236418..

Dan: Such a number does not exist because what you write down is not a result of a division of 2 whole numbers

Ron: Who told you that what you write down must be the result of a division of two whole numbers?

Who is right? Please explain!

The question aimed to examine whether the students are mathematically matured for the idea of irrational numbers as infinite non-repeating decimals.

Participants and data collection

The question was posed to 91 10th graders and 97 11th and 12th graders learning at the same academic high school in Jerusalem, which is academically selective. The 10th graders learned mathematics at the same level. One group of the 11th graders learned mathematics in an advanced level class (5 units). The other 11th and 12th graders learned in an average level class (4 units).

We asked the students in the sample whether they studied the concept of irrational numbers in the past. 77% of the 10th graders (78% of the 11th and 12th graders) claimed that they studied it; 7% of the 10th graders (12% of the 11th and 12th graders) claimed that they do not remember if they studied it or not and 16% of the 10th graders claimed that they did not study it (10% of the 11th and 12th graders). The part of the questionnaire that related to irrational numbers requested more concentration from the students in comparison to the part that related to rational numbers. The 11th and 12th graders were focused in their work and wrote detailed answers. Even after answering the questionnaire they remained in the class and discussed their answers. The situation was different for the 10th graders. It was difficult for them to concentrate on the questions on irrational numbers. In contrary to the first part of the questionnaire which dealt with rational numbers in which all of the 10th graders

participated, around 20% of the 10th graders did not participate in the second part which dealt with irrational numbers.

RESULTS AND ANALYSIS

The distribution of answers to question 1 is given in Table 1

Categories	Percentages of 10 th graders N=91	Percentages of 11 th and 12 th graders N=97
A. Any decimal must be a result of a division of 2 whole numbers	56%	54%
B. An infinite decimal can be obtained not only as a result of a division of 2 whole numbers	23%	43%
C. No answer	21%	3%

Table 1: Distribution of answers

Category A : Any decimal must be a result of a division of 2 whole numbers

For 55% of all students in the sample “any decimal must be a result of a division of 2 whole numbers”. Analysing students’ detailed answers, we observe four sub-categories of answers.

The answers of 14 % of all students belong to the first sub-category:

Every number is a result of a division of two whole numbers

8% of all students refer only to rational numbers. For example, the following answer:

Dan is right because we asserted that an infinite number, namely, an infinite decimal is a certain kind of a rational number and in order to obtain a rational number we should divide two whole numbers.

Some students proposed to check if the number given by Ron is a result of a division of 2 whole numbers. The students wrote:

- If a number is not the result of a division of 2 whole numbers then it is impossible to define it or to express it.
- Ron is wrong. He proposed a number which is not defined and we do not know what will be the next digits. We do not know if the number will be finite or infinite.

The students are reluctant to deal with an irrational number because there is no enough information about this number.

The answers of 17 % of all students belong to the second following sub-category:

We do not create numbers. All numbers are formed by means of division of whole numbers

In a decimal number the digits after the decimal points should be linked to some division which give them.

Students are not ready to accept the irrational number:

- I think that Dan is right contrary to Ron who creates something out of nothing, a meaningless number.
- Ron can add as many digits as he wants it will not be a decimal number since a decimal number is a result of a division of two numbers. It will be something else.

Some students have difficulties to imagine an infinite procedure of writing digits.

Dan is right since his decimal number has infinite digits in contrary of Ron's decimal. When Ron will stop adding digits it will result in a finite decimal.

The belief that we do not create numbers and every number is obtained by means of dividing two whole numbers was expressed in two main groups of answers. In the first group (7% of all students) the reason for this belief is that this is the only way to control the infinite number of digits in the decimal representation.

Dan is right. Ron invented a number and he is not able to know if it is finite or not since we do not have here two whole numbers that he can divide.

2/3 of students' answers in this group are 10th graders' answers.

For the second group of answers (5% of all students), every infinite decimal is "at the end" a repeating decimal. This conception might be a consequence of the fact that it is not easy for the students to give an example of an infinite non repeating decimal with a rule which guaranties the infinite digits with no repetition. A similar percentage of answers of 10th graders and 11th and 12th graders belong to this group. The following answer was given by a student who expressed in other questions his awareness of the existence of irrational numbers. This answer shows the student's erroneous conception regarding randomness.

At the moment you just write digits after the decimal point there will always be a repeating pattern since you only have 10 digits and an infinite number of places. Therefore there is a probability that a periodicity will appear.

The answers of 12 % of all students belong to the third following sub-category:

An "infinite decimal fraction" is identified by mistake with "fraction"

Dan is right. A decimal fraction is another name or another way of writing a fraction a/b .

There is no fraction which might be obtained not by means of dividing two numbers. It might be a consequence of the fact that the questionnaire was in Hebrew and in Hebrew the infinite decimal is called "infinite decimal fraction".

The answers of 12 % of all students belong to the last following sub-category:

The student thinks only in terms of rational numbers

Ron is right since even when we divide $1/3:1/2 = 2/3$, $2/3$ is also an infinite decimal and it is a result of the division of two rational numbers which are not whole numbers.

Category B: An infinite decimal can be obtained not as a result of a division of 2 whole numbers

For 34% of all students in the sample “an infinite decimal can be obtained not only as a result of a division of 2 whole numbers”. We observe two sub-categories of answers.

The answers of 15 % of all students belong to the first sub-category: **There might be such a number with no relation to the question “what is this kind of number?”**

- Finally, such a number as the one given by Ron must exist and not every infinite decimal must be the result of a division of two whole numbers.
- The number exists although the thought process by Dan is safer
- One can obtain a number merely by writing down its digits.

In some answers, we note some reservation:

An infinite decimal has an infinite number of digits after the decimal point and we can write down its digits as we want since theoretically it exists.

The answers of 24% of 10th graders who complete the questionnaire belong to this first sub-category vs 10% of 11th and 12th graders.

The answers of 19 % of all students belong to the second sub-category: **An infinite non repeating decimal is not a result of a division of 2 whole numbers**

The situation is now different: 4% of 10th graders wrote answers that belong to this category vs 33% of 11th and 12th graders (50% of 11th graders that learn in the advanced level class). The following answers belong to this category:

- Ron is right. Dan claims that the number $1.236\dots$ is not a number. I do not agree with Dan because for him the word “number” only means rational numbers and he does not recognize irrational numbers as a number.
- Ron is right. There exist infinite decimals that cannot be obtained as a result of a division of 2 whole numbers. For example, π is an infinite non-repeating decimal. As a result of a division of 2 whole numbers, we always obtain a repeating decimal.
- Ron is right: (i) as a result of a division of 2 whole numbers, you always obtain a rational number. An irrational number cannot be obtained by means of a division of two whole numbers. (ii) it is always possible to define a new group of numbers (for example, if there is no solution to a quadratic equation, you can define a new group of numbers in which there is a solution).

Even if the students are aware of the existence of irrational numbers we note

reservation especially in those answers that emphasize that one should not add verbal explanations to a mathematic object like a number.

- We should require to express the infinite decimal by means of conventional signs in order to assure that it is an infinite decimal.

Categories	Percentages of the entire sample N = 188	Percentages of 10 th graders N=91	Percentages of 11 th and 12 th graders N=97	Percentages of 11 th and 12 th graders at the average level N=64	Percentages of 11 th graders at the advanced level N=33
A. Any decimal must be a result of a division of 2 whole numbers	55%	56%	54%	58%	48%
I Every number is a result of a division of two whole numbers	14%	14%	14%	16%	12%
II We do not create numbers. All numbers are formed by means of division of whole numbers	17%	20%	15%	17%	12%
III An “infinite decimal fraction” is identified by mistake with “fraction”	12%	5%	16%	14%	21%
IV The student thinks only in terms of rational numbers	12%	16%	8%	11%	3%
B. An infinite decimal can be	34%	23%	43%	39%	52%

obtained not as a result of a division of 2 whole numbers					
I There might be such a number-no relation to the kind of number	15%	19%	10%	16%	0%
II An infinite non repeating decimal is not a result of a division of 2 whole numbers	19%	4%	33%	23%	52%
C. No answer	11%	21%	3%	3%	

Table 2: Distribution of answers with sub-categories of perceptions

DISCUSSION OF FINDINGS

55% of the students identify the set of all decimals (finite and infinite) with the set of rational numbers. In spite of the fact that around 80% of the students claimed that they had learned about irrational numbers, only 4% of the 10th graders and 33% of the 11th and 12th graders showed awareness of the existence of non-repeating infinite decimals. In addition, 21% of the 10th graders could not even answer the question. The 11th and 12th graders did not receive any additional learning experience about irrational numbers. Therefore, the difference between the two groups may be explained by maturation. The mental ability to imagine an infinite procedure of writing digits, in an arbitrary way, to the right of the decimal point requests such maturation. A large number of students in both groups did not show awareness of the existence of non-repeating infinite decimals. In the next three subsections, we propose an explanation of students' reluctance to deal with irrational numbers.

The conception that numbers exist and we have no control on it

We can point on (natural) numbers or define rules of operating on two of these numbers (by means of addition, subtraction, multiplication or division). The larger set that the students can obtain by means of this conception is the set of rational numbers. The transition to real numbers is more problematic. The thinking "I will define a larger set of numbers that includes the previous one and keeps its properties" is a thinking which is opposed to the intuition that numbers exist without our intervention. In category A_{II}, we find explicit expression of this intuition. This view might be a consequence of the influence from the outside real world and the

analogy with natural phenomena that exist without our involvement. We can investigate them but their existence does not depend on us. We noticed here a possible conflict between the learners' intuition in the sense of Fischbein and the formal rules of thinking.

The extension from the rational numbers to the real numbers is of a different kind than the previous extensions

The previous extensions from the natural numbers to the whole numbers and from the whole numbers to the rational numbers were done and expressed by means of the previous set. A rational number is simply defined by means of whole numbers. This kind of request to define the irrational number by means of rational numbers is well expressed in the historical development of irrational numbers.

I demand that arithmetic shall be developed out of itself...Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone (Dedekind translated by Woodruff Beman in D.E. Smith).

The need to know the process that leads to the infinite decimal

In the first part of the questionnaire that deals with rational numbers and the way students conceive repeating infinite decimals we observed that the students are unable to differentiate between the result -the infinite repeating decimal and the process that gives this decimal. For example, the students identify the number $0.3333\dots$ with the process: $0.3; 0.33; 0.333; 0.3333; \dots; \dots; \dots$

The process (1:3) promises a single fixed result and this is important because of students' dynamic view of the repeating infinite decimal. The task in the present paper deals with non-repeating infinite decimals. Ron's number with the infinite arbitrary digits reinforces this conception of a number that changes all the time. We are also unable to predict how it changes. This situation reinforces students' dynamic process view of the infinite decimal and, as a consequence, the need for a process that promises a result. This dynamic process view of the infinite decimal corresponds to Fischbein's description of intuition of infinity as a potential infinite. When 55% of the students claim that a decimal (including infinite decimal) is the result of a division of two whole numbers they express their view that this process is a division. Why? It might be by analogy with the extension from whole numbers to rational numbers. This is right for a finite decimal and the student wants to suppose that it also works for any infinite decimal repeating or not-repeating. This need to identify the infinite decimal as a result of a division of two numbers was also observed among students who did express their awareness of the existence of irrational numbers. Even so, we read some answers like the following one

Every number is always obtained by division of two numbers.. We can also obtain an infinite decimal by means of a division of irrational numbers.

We have here an expression of tacit influences exerted by mental models on the interpretation of mathematical concepts in the domain of actual infinity even for students who have constructed formal knowledge (Fischbein, 2001). The findings of this study help towards the effort of facing students entering university with the incoherence of some of their conceptions and the resulting cognitive conflicts.

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Rationality and concept of limit

Thomas Lecorre University Joseph Fourier in Grenoble, Fourier Institute, France,
thomas_lecorre@orange.fr

We present a didactic situation aimed at the formal definition (delta-epsilon) of the limit of a function; the experimentation of this didactic situation has been made many times with French students in the last year of high school and the first year of university using "scientific debates" between students. From an excerpt of the script of an experimentation, we study the evolution of students' reasoning. We specifically study the kinds of rationalities used by students to try to solve the given problem.

Keywords: limit, rationality, definition, calculus, scientific debate.

CONTEXT

Teaching the definition of the limit of a function is known to be difficult and didactic studies related to this topic started more than thirty years ago. The large number of studies on this subject shows the efforts of the didactic community to address this difficulty which corresponds to the foundation of calculus (Robert 1982, Sierpiska 1985, Bloch 2000, Mamona-Downs 2001, Przenioslo 2004, Ghedamsi 2008,...). We will look specifically at the beginning of calculus in the last year of lycée, in Terminale S (last year of high school). The notion of limit of a sequence is introduced the year before (première S) with intuitive approaches where the dynamic conceptions of limit are given as: "the function tends to L", or in terms of a function "approaching" a number. Then, in Terminale S, a more formal definition is given. The official texts specify that "to express that a sequence u_n tends to L when n tends to + infinity, we say that any interval that contains L contains any value of u_n from a certain rank n.". However, in this article we specifically discuss the introduction of the definition of the limit in delta-epsilon of a function in the level of Terminale S (for all $\epsilon > 0$, there exists a delta such that for all x greater than delta, $f(x)$ is between $L - \epsilon$ and $L + \epsilon$).

PROBLEMATIC

Our aim is to add to the first quite dynamic conceptions new conceptions called static ones, corresponding to the common formalism in calculus, which means showing this formalization as determined by two quantified systems of proximity (delta and epsilon). Studies show (Robert 1982, Williams 1991) that the dynamic conception is very resistant at university, which can be easily understood: academic formalism is not only very complex (because of quantifiers, real numbers, linked variables, the notion of infinity,...) but it can also give first the feeling that it has nothing to do with the object it is supposed to define.

We think that a part of the difficulty comes from the kind of rationality really available for students. How can a student understand such a definition if he usually does not really need any definition to use mathematical objects (Job 2011)? How can he understand the subtleties of this definition if he does not feel that this responds to a

problematics of proving calculus results, i.e. if he does not really feel that mathematics are a special building where definitions, bricks, are fit for demonstrations, as a special cement (Lakatos 1984)? How can he accept such a “repulsive” definition whereas he feels he already has a good idea and a very much easier one of what it can be, i.e. how can he accept without adopting a theoretical rationality? We finally reverse all these questions and ask: can the study of the limit be an opportunity for developing theoretical rationality? And how?

We distinguish three kinds of rationalities. Pragmatic rationality consists in closely examining specific cases. There is no attempt to generalize observations. The Empirical rationality is used when one wants to have a general law. One uses facts to deduce generalizations. Theoretical rationality begins with theory (theorems, properties, definitions, axioms...) to establish new properties and theorems (more details will be given later).

We approach this problem of rationality by creating a didactic environment in which students have no indication about the kind of rationality to use to solve the limits problems they are given; and we will follow the evolution of the kinds of rationalities used by them. We suggest a didactic engineering in two steps.

Didactic engineering is a didactic research method which is quite common in French didactic research (González-Martín & al. 2014). A specific subject is first studied in terms of theory, then a didactic experimentation is designed and an a priori analysis formulated. The experimentation is conducted and finally a comparison is made between what was expected and what really happens. The study of any eventual discrepancy between the hypothesis and the result often results in the discovery of something new and unexpected.

The aim of the first step of our didactic engineering about the limit is to destabilize the first conception of the limit (the dynamic one) and to create a need for formalization. The aspect of rationality targeted/singled out in this first step is the way that objects are given and exist in mathematics: by definitions. The aim of the second step is to give a meaning to the kind of formalization used by the definition in delta-epsilon of the limit of a function. The aspect of rationality targeted in this second part is the way to establish the truth in mathematics: by demonstration. In this article, we study this second step. We are using a "scientific debate" (Legrand 2001) where the reasoning of a group of students is not led by the teacher but is the result of interaction within the group. The corpus used in this article comes from experimentations in lycée in TS (2014 for second step). The involved students are accustomed to this type of activity.

DESCRIPTION OF THE DIDACTIC ENGINEERING

We first have a look at the first step (widely described in Lecorre 2015). The teacher gives a conjecture about two functions having no infinite limits in infinity.

C1: "If the limit of f is strictly less than the limit of g in plus infinity then for every real x , $f(x)$ is strictly less than $g(x)$ "

This conjecture is obviously false and students quickly agree that it is false. Then, the teacher asks the students to "repair" the conjecture: the main idea, the order of the limits gives the order of the functions, must be kept but the conclusion must be changed in order to make the conjecture true. The aim, of course, is to make the students confront the need to formalize the concept of neighborhood of infinity. The first suggestion given by a student is:

C2: " If the limit of f is strictly less than the limit of g in plus infinity then $f(x)$ is upper bounded by $g(x)$ in infinity".

The debate between students leads them to say that C2 looks like "if A then A" which they find pointless ("It's true without any demonstration"); then emerges the question of the definition of "f(x) is upper bounded by g(x) in infinity". (the ask for a definition shows that the students are here adopting a theoretical rationality). The teacher, then, introduces the idea of neighborhood of infinity as an answer to this question and its formalization with the conjecture.

C3: "If the limit of f is strictly less than the limit of g in plus infinity then a real A exists so that for every real x more than A , $f(x)$ is strictly less than $g(x)$ ".

The debate among students for C3 leads to a broad agreement about its validity, but some students remain uncertain considering that it has not been proven yet. The teacher then suggests to study what he calls "the monster" (see below), a function f that remains below g (g which finishes to be constant and equal to 3) except in rare but regular peaks (every 10^6) which go over g on a small interval (less than 10^{-6}).

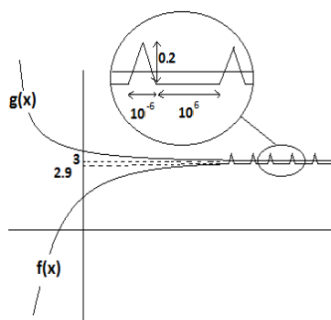


Figure 1 : The 'monster'

Next, the teacher asks if "the monster" is an example, a counter-example or a 'hors-sujet' for the conjecture C3 ('hors-sujet' means that it does not fit the hypothesis). There is a debate and finally a student asks: "But what is the limit of the function f ?". This inquiry is not exactly a request for a definition, but the dynamic conception of the limit is starting to look fragile, so it seems that it would be time to go on to the second step: give a meaning to the formalization of the limit, formalization which will be given by the teacher in the end.

This second step is based on the notion of equality in calculus. Indeed, if in algebra the equality $A=B$ is usually translated by " $A=B$ if $A-B=0$ ", and in the set theory in terms of double inclusion " $A=B$ if $A\subseteq B$ and $B\subseteq A$ ", in calculus the equality between two numbers is usually translated by " $A=B$ if for every $\varepsilon>0$ the distance between A and B is less than ε ". And this equality can be applied to a function like: f verifies $P1$ if "there exists A such that for all x more than A and for all $\varepsilon>0$, $L-\varepsilon<f(x)<L+\varepsilon$ ". In these conditions, if f verifies $P1$ then exists A such that for every x more than A , $f(x)=L$. What is quite interesting is that if we exchange the quantifiers with $P2$ ($P2$: "for every $\varepsilon>0$ there exists A such that for every x more than A , $L-\varepsilon<f(x)<L+\varepsilon$ ") then we get the definition of the limit: f has got the limit L in plus infinity if f verifies $P2$. The didactic engineering is based on the idea that the study of $P1$ may be used to give a meaning to the variables A and ε as proximities of infinity and of the limit, and then, inverting $P1$ into $P2$ the formalization of the limit can be much better understood. Here we will deal with the study of $P1$ by the students. This study of $P1$ is divided into three phases. The first phase is a call for a conjecture. The property $P1$ is given by the teacher under the simplified form (to reduce the number of variables): $P1$: "for every real $x> 50$, for all $\varepsilon>0$, $2-\varepsilon<f(x)<2+\varepsilon$ " and students are asked to say what can be concluded if f verifies $P1$. Three suggestions are made by them:

C3.1 : "If f verifies $P1$ then f is upper bounded"

C3.2 : "If f has got a limit and f verifies $P1$ then f is between $2-\varepsilon$ and $2+\varepsilon$ "

C3.3 : "If f verifies $P1$ then f is bounded by $2-\varepsilon$ and $2+\varepsilon$ since 50 "

The conjecture C3.1 is invalidated and the conjecture C3.3 is validated through the debates, whereas the study of the conjecture C3.2 is postponed.

For the second phase, the teacher asks to give examples of functions that verify $P1$. The first suggestion ($f(x)=2$) is soon validated. The second suggestion ($f(x)= 2+\varepsilon$) is rejected by the teacher who will point out that he wants functions of a single variable x . Then a third and alternative suggestion is given ($f(x)=2+1/x$). This suggestion will be invalidated by a student taking $x=51$ and $\varepsilon=0.001$.

Finally, a fourth conjecture is made under the form of a drawn curve of a function f (see below).

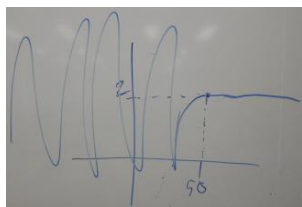


Figure 2 : Suggestion C3.4

The students' debate will soon validate this function f as an example of C3. For the third and last phase of the study of $P1$, the teacher gives the conjecture C4 and asks the students to study it: C4: "If f verifies $P1$ then for every $x>50$, $f(x)=2$ ".

In this article we are going to study the debate about the conjecture C4 in terms of rationality. And we will first give more details about this model of rationalities.

SOME TOOLS TO ANALYZE OUR CORPUS

In Boero(2006) the Habermas' rationality construct is used to analyze the production of a student. This rationality is split into three interrelated components: *epistemic rationality*; *teleological rationality*; *communicative rationality*. The kind of rationality we are using here is an epistemic one: we call rationality the means used by humans to try to understand a domain of reality, i.e. to try to avoid contradiction between what they may think and the reality they are attempting to figure out. Rationality achieves this aim of no contradiction by maintaining the coherence between three aspects: the kind of domain of study (reality or theory), the kind of logic (everyday logic or mathematical logic), and the kind of validation (abduction, induction, deduction, argumentation...: see Balacheff 1988, Pedemonte 2003, Meyer 2008. We will also split this epistemic rationality into three components. The first component, which we will call *pragmatic rationality*, using a kind of reality that does not need to be defined: it is here and it is so. Natural logic is usually used (implication is viewed as causality for example) and validation is linked to efficiency (it works or doesn't) or to obviousness. The second component, which we will call *empirical rationality*, seeks to produce general laws with systematic verification of cases. This empirical rationality can be of two kinds. The first one, inductivism, considers that a statement is true if it is true for many various cases. We can often see students explain that it 'must' be true because the studied property always seems to work or there seems not to be any counter-examples. The kind of truth associated with this rationality is plausibility. The second one, falsificationism, considers that a statement cannot be taken as true with only verified cases. The only certainty given by special cases is that the statement is false when they are counter-examples. Students using such rationality will declare that there is no objection for a statement for which no counter-example has been found. But they will also add that they cannot say that such a statement is true since it has not been proven. Usually, they use that kind of rationality because they are not able to prove it, and they are aware of that, or because they are only trying to find counter examples. The kind of truth is also plausibility. The third component, called theoretical rationality, uses the means of demonstration to prove statements. A statement is true if it can be proven by a demonstration. A student using this rationality tries to provide a proof using other properties, theorems, and definitions. These three components will help to see the different forward and backward movements of students' reasoning and attempts to understand, and will help to highlight the steps of the debate.

STUDY OF THE SCRIPT

The teacher gives little time to the students to consider if they think that the conjecture C4: "if f verifies P1 then for every real $x > 50$ we get $f(x) = 2$ " is true or false (P1: "for every real $x > 50$, for all $\epsilon > 0$, $2 - \epsilon < f(x) < 2 + \epsilon$ "). Then a vote is taken. Twelve students think C4 is true, five think it is false, and nine vote "something else" (which

means that they have good reasons not to vote True or False. For example they do not understand the conjecture, or they think it is true and false, or they think it is true but they have no proof, or...).

Valentin starts to explain that he thinks it is false because he has a good candidate for a counter-example but he wonders if he has the right to take a function with two variables. The teacher says that the rule is that such a function is forbidden.

Then Erika, who thinks that the conjecture is false, comes to the board and draws what she considers as a counter-example.

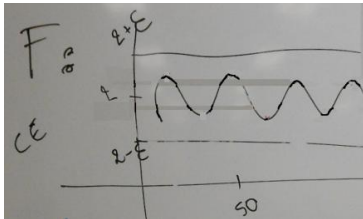


Figure 3 : The false counter example of Erika

Erika's suggestion can be taken as the statement: "the function drawn here verifies P1 because the drawing shows that the function is really between $2-\varepsilon$ and $2+\varepsilon$ ". We can say that this argument comes from a pragmatic rationality as it verifies only one value of ε . Five other students declare that they also think the conjecture is false for the same reason. Baptiste, who voted false, adds that he had another counter-example $f(x)=2 + \varepsilon \sin(x)$ but he gave it up because of the invalidation of Valentin's argument. He also adds that he is not sure about Erika's suggestion, concerning the "little epsilon". This new argument, the "little epsilon" is probably coming from an empirical rationality (falsificationism) which leads Baptist to imagine some other verification. Of course "little epsilon" is playing a special role in the problem and this new information will be helpful. Then Mathieu comes to the board to show that Erika's drawing does not verify P1 as "In P1 it's written for every epsilon, so we have the right to change the epsilon" (see fig. 4). This also corresponds to an empirical rationality, falsificationism, (maybe theoretical rationality but no demonstration of C4 is given) in the sense that the universal quantifier is highlighted: a general law is involved. Mathieu adds to the drawing:

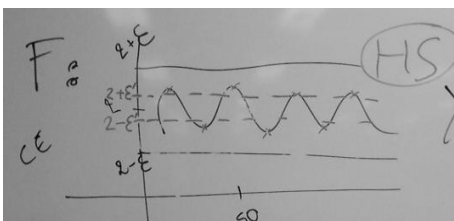


Figure 4 : Invalidation of the counter-example by Mathieu

Then, Louis, who thinks the conjecture is false, expresses his disagreement: we don't have the right to use the same symbol twice on the same drawing (Mathieu, like Erika, had used the symbol ε). So Thomas suggests using a prime (ε') but this option does not please Louis. Indeed, Louis explains that when one takes an epsilon which is

smaller, the function f still remains between the bounds because it decreases at the same time. In fact, Louis is mistaken because of an inversion of quantifiers. This can be linked to a lack of knowledge about the interpretation of a double quantification which is known to be difficult (Dubinsky & Yiparaki 2000) but also to an insufficient definition of the function by Louis, consequently to a pragmatic rationality. At this moment, the teacher explains that if the function decreases with epsilon it means that it is dependent on epsilon just as Valentin said.

Then Maxim wants to remind (wrongly) the class that number 3 and number 2,99999...(infinity of nines) are two different numbers (probably to build a new counter-example, but he does not say it). The teacher says that a proof had been done, supposing that they were different but in fact it had never been concluded that they were so. Maxime seems to get confused between the validity of an inductive reasoning and the truth of the hypothesis. This corresponds to a pragmatic rationality in which no absurd supposition is usually done to prove something. We will see that this question will come up again later with another reductio ad absurdum.

Quentin who thinks C3 is true goes to the board to make a drawing:

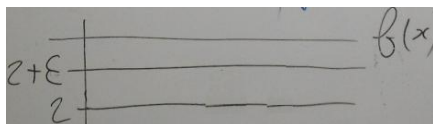


Figure 5 : The Quentin's attempt with a reductio ad absurdum

Quentin: To show that it's true I'm going to show that $f(x)$ can't be different from two. If it's not different from two, it's equal to two, do you agree? So if $f(x)$ is different from two, there are for sure some values between $f(x)$ and two. So P1 won't be verified. So the example, $f(x)$ different from two, is false.

Quentin gives a reasoning which is a reductio ad absurdum while taking a function which never equals two. This is an argument within a theoretical rationality. The desire to prove is clear. We will call this argument a "generic example" as Balacheff defined (Balacheff 1988). This kind of proof is used by students to prove a statement by taking an example, not for itself but as the representative of a class of elements. Here, the example is a constant function: but what is important is that the function is not equal to two. For students, this reductio ad absurdum is not easy to accept. There is the same confusion as with Maxime between the validity of the reasoning and the hypothesis.

Aya: Can you explain another time because I did not quite understand?

Elsa: Quentin, you voted True?

Quentin: Yes, I voted True.

Elsa: There's something wrong!

Nathan: In fact, you're saying it's False. You voted True, but you're saying it's false.

Elsa: Yes, I've got the feeling that it's an argument for False.

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- Margot: Yes, he's saying that False is false.
- Teacher: We're going to listen to him once again because he's pretending that it's an argument of Truth. Go on Quentin, and listen to him.
- Quentin: I'm saying that it's true for $f(x)=2$ by showing that $f(x)$ different from 2 is false. I'm showing that $f(x)$ cannot be different from two. (big silence)

Louis then asks "to develop Quentin's line of reasoning":

- Louis: If we take the first function, we had $f(x)=2$. That's what we're trying to say. In fact, I will start like Quentin: if the function is different from two, it means that somewhere it makes a "jump"; this way it has at least one value different from two. So, for example, here! We can make it little or we can make it big. Here it is less than two minus epsilon. And if we take a value, here h , we will always find a value for epsilon which will be less than h , h which is the highest peak of the function which would be close to two.

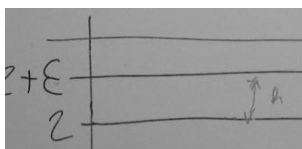


Figure 6 : reductio ad absurdum by Louis

This argument is clearly coming from a theoretical rationality. We will call it "mental experience" as Balacheff called this kind of proof (Balacheff 1988). We can see that in this proof there is some action ("jump", "make it big"), time is playing a role ("always"), and the proof is personalized ("we will"). All these are elements that characterize a proof called "mental experience". It is considered for Balacheff as a proof but still not a demonstration because it has to be depersonalized, decontextualized, and time has to be removed.

However, this proof gains unanimous acceptance: many hands go up to say that they want to change their vote. A last vote is organized: almost everybody votes True. A few remain who are not completely convinced.

The teacher will, then, call for a conjecture: What can we say of a function that verifies P2? (P2 which is the epsilon-delta definition of "f having a limit of 2"). And a new debate will begin...

TWO KEYS: RATIONALITY AND KNOWLEDGE

If we try to understand what underlies the students' queries and what leads them to the solution, we can see that it is the suggestion of arguments and counter arguments (use of rationalities) which slowly becomes deep and robust (knowledge) until a proof is valid and collectively accepted. We can note that the intervention of the teacher - apparently rather light - consists in fact in eliminating a tough problem: the inversion of quantifiers (there exists f such for every $\varepsilon > 0$... or for every $\varepsilon > 0$ there exists f ...) which will have to be faced (in fact it is supposed to be done after the study of P1 with the study of P2). So his intervention is crucial to allow students to

focus on the "right" problem. We will then especially note the decisive role that the counter-example of Erika plays, as it leads to make the *reductio ad absurdum* "concrete". Indeed, those who believe the conjecture to be true are thus forced to show that this counter-example in fact is invalid. So they are led to suggest a reasoning that invalidates this argument: they have to choose a "good" epsilon. This choice is in fact the second part of the *reductio ad absurdum*. Then Quentin suggests the negation of the hypothesis (which is the first part) but, as the conclusion includes a universal quantifier, he gives the negation with a universal quantifier also ($f(x)$ never equals two). Then Louis can "repair" that with a (more) existential quantifier. We can notice that Louis uses both kinds of language in his proof (semantic and syntactic): the "jump", "the "highest peak" but also "there is at least one value different from two" (for communicative rationality of Habermas, see Boero 2006). It is this second demonstration that leads to a collective agreement.

In the end, students can admit the irrefutability of the reasoning when all their reluctances have been taken into account by their peers. What seems decisive, in term of rationalities, is that students can choose on their own the one they consider best adapted to the situation and to their own understanding of it, which could not happen if the teacher intervened. It gives them the opportunity to enter the problem with a pragmatic point of view, taking examples, to use falsificationism to test beliefs or to use a theoretical point of view for proving. It gives them also the opportunity to experiment the robustness of their own beliefs and potentially to modify them. Here, the shared culture of debating and, thus, of moving from a rationality to another, allows students to convince their peers or to be convinced by them. The collective reasoning of students becomes a progressive, rational and guided construction of syntactic and semantic elements that will constitute the bricks of the meaningful proof which is finally realized.

If rationality can lead to a good idea of limit, the reverse can be also observed: it seems clear that the concept of limit, with its unique complexity and proximity, can constitute a real opportunity for rationalities to be developed.

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Student understanding of the relation between tangent plane and the differential

María Trigueros Gaisman¹, Rafael Martínez-Planell², and Daniel McGee³

¹Instituto Tecnológico Autónomo de México, ²Universidad de Puerto Rico en Mayagüez, rafael.martinez13@upr.edu; ³Kentucky Center for Mathematics

Action-Process-Object-Schema (APOS) is used to study students' understanding of the relationship between tangent planes and the differential. An initial conjecture, called a genetic decomposition, of mental constructions students may use in constructing their knowledge of planes, tangent planes, and the differential is proposed. It is tested with semi-structured interviews with 26 students. Results of the study suggest that students tend not to relate these ideas on their own and suggest ways to refine the initial genetic decomposition in order to help students to better understand these concepts.

Keywords: APOS, calculus, tangent plane, differential, function of two variables.

INTRODUCTION

Functions of several variables play a role of great importance in mathematics and the applied sciences. This study focuses on student understanding of the relation between two of the most basic ideas in the differential calculus of functions of two variables. Very little research has been published relating to this topic. An early work by Tall (1992) suggests using a geometric model to visualize differentials in three dimensions. Weber (2012) discussed the rate of change concept in the case of functions of two variables focusing on the use of covariational thinking to help students build a notion of rate of change in space. McGee and Russo (2015) used a model similar to that of Tall (op. cit.) in a study that applied semiotic representation theory to explore the effect of a semiotic chain in student understanding of partial and directional derivatives of functions of two variables. Martínez-Planell, Trigueros, and McGee (2015) applied APOS to study different components of the differential calculus of these functions: partial derivatives, planes, tangent planes, directional derivatives, and the differential. The present report expands their discussion of student understanding of the differential and its relation to the idea of tangent plane in accordance with Tall's model.

THEORETICAL BACKGROUND

Since APOS is a well-known theory we will only give a brief overview. For more information the reader may consult Arnon et al (2013).

In APOS, an Action is a transformation of a mathematical object that is perceived by the individual as external. It may be a step by step implementation of an explicitly

available set of rules or a rigid application of a memorized fact or algorithm. An individual is said to have an action conception of a given mathematical notion when he/she is limited to applying actions in problem solving activities involving the notion. As an individual repeats and reflects on an Action, it may be interiorized into a Process. A process is perceived as internal. An individual with a process conception of a mathematical idea may, without recurring to any external source, reflect on the steps of the process, omit steps, and anticipate the result without having to explicitly perform the process. A process may be coordinated with other processes, and it may also be reversed to the actions it came from as needed in a problem situation. As an individual needs to apply actions on a process he/she may come to see the process as a totality. When the individual is able to perform or imagine performing actions on a process it is said that the process has been encapsulated into an Object. An object may be de-encapsulated into the process it came from as needed in a problem situation. An individual with an object conception of a mathematical concept may recognize the applicability of the concept without any prompt in different problem situations, even in an unfamiliar context, as would be in a different discipline. A Schema for a particular mathematical idea is a coherent collection of actions, processes, objects, and other previously constructed schemas that are related to the mathematical idea. A schema is coherent in the sense that the different components of the schema are inter-related in the individual's mind and the individual can decide when a problem situation falls within the scope of the schema.

Even though one may think there is a linear progression from action, to process, to object, and then to having the different actions, processes, and objects organized in schemas, the progression is dialectical in nature, with partial developments, and passages back and forth between conceptions (Czarnocha, et al 1999). However, the theory is unequivocal in its recognition that a student's tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept is different depending on whether the student understands the concept as an action, a process, or an object.

In APOS, research on student understanding of a particular mathematical concept starts by establishing a conjecture, called a genetic decomposition (GD), of specific mental constructions (in terms of the constructs of the theory) that students may do in order to come to understand the concept. The GD depends on the mathematics itself, the experience of the researcher teaching the concept, and any available data. A GD is not unique, different researchers may propose different genetic decompositions. What is important is that the GD needs to be supported by experimental data from students. What typically happens is that a preliminary genetic decomposition is proposed and the data obtained (usually with semi-structured interviews) shows that students make unexpected mental constructions and have difficulty with some of the mental constructions predicted in the GD. This leads to refining the genetic decomposition in order to reflect the constructions that students actually do and to

the development of activities to help students make the mental constructions with which they had difficulty. This ends a first cycle of research. The second research cycle would start with a classroom implementation of the newly developed student activities and would further refine the GD based on new interviews and classroom observations. These research cycles continue until they stabilize in a GD that serves to both, predict student behaviour and guide instruction. The present work uses APOS theory to study the level of cognitive development of students who completed a course using a traditional lecture/recitation model, as discussed in Arnon et al. (2013, p. 106). Thus, this is a report of a first cycle of APOS research.

GENETIC DECOMPOSITION

We now present a GD for plane and tangent plane. We also present a preliminary GD for the differential concept. This preliminary GD guided the development of the instruments for this study.

Plane

Given a non-vertical plane, the processes of slope of a line and fundamental plane (planes of the form $x=c$, $y=c$, $z=c$, for c constant) are coordinated into new processes of vertical change in the x and y directions, where it is recognized that vertical change in the x direction can be described as a function of the horizontal change in the x direction ($\Delta z_x = m_x \Delta x$), and similarly for vertical change in the y direction ($\Delta z_y = m_y \Delta y$). These processes are coordinated into a process of total vertical change on a plane in three-dimensional space so that total vertical change in any plane is given in terms of the sum of vertical changes in the directions of the coordinate axes: ($\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$ (see Figure 1). The need to perform actions which are treatments and conversions in and between representations (Duval, 2006) on the process of total vertical change promotes its encapsulation into the object conception of plane in three dimensions. In particular, the equation $z - z_0 = m_x(x - x_0) + m_y(y - y_0)$ can be seen as the vertical change on a plane with slopes m_x and m_y from an initial point (x_0, y_0, z_0) to a final generic point (x, y, z) and is also associated with the set of points (x, y, z) on a plane that contains the point (x_0, y_0, z_0) and has slopes m_x and m_y (point-slopes formula for a plane).

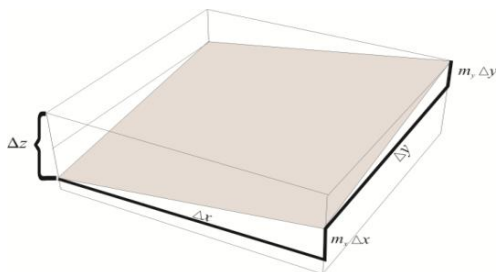


Figure 1: $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$

Tangent Plane

The process of partial derivative is coordinated with that of plane into a new process where tangent planes to any surface at different points can be considered and computed. When there is a need to consider particular tangent planes and perform actions on them to describe the surface in terms of behaviour associated with its tangent plane(s), this process is encapsulated into an object conception of tangent plane.

The Differential (from the preliminary GD)

Treatment and conversion actions (Duval, 2006) are performed on the tangent plane process to recognize it as the differential.

To summarize, the above GD essentially proposes that students first do the mental construction of the process of total vertical change on a plane: $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$. Then they coordinate this process with a process of partial derivative to obtain a process of tangent plane at $(x_0, y_0, f(x_0, y_0))$: $z - f(x_0, y_0) = f_x(x_0, y_0) (x - x_0) + f_y(x_0, y_0) (y - y_0)$. Students then do actions of notational change and of geometric interpretation on the process of tangent plane and interiorize these actions into a process of total differential: $df(a,b) = f_x(a,b) dx + f_y(a,b) dy$.

METHOD

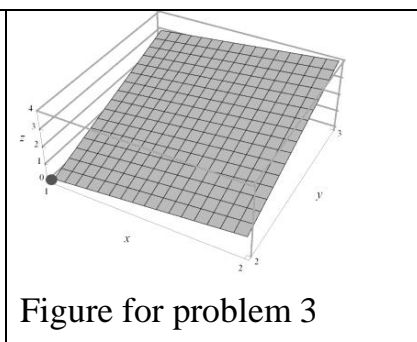
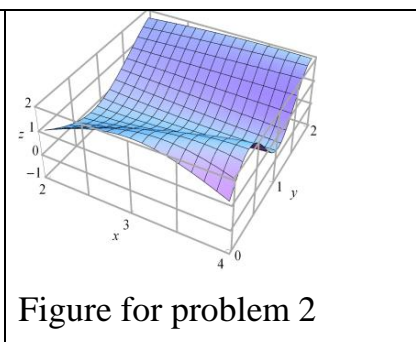
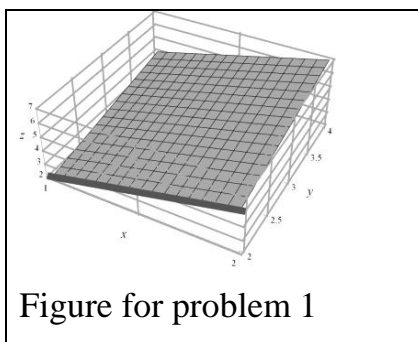
An instrument consisting of six questions was prepared to test student understanding of the different components of the GD. We are reporting mainly on one of these questions (problem 3a). However, when pertinent we will cite (although not always quote) student response to other questions (problems 1 and 2). The instrument was used in semi-structured interviews with 26 students who had just finished taking a multivariable calculus course. The 26 students were chosen from three sections that had different professors. Section T (9 students) used a traditional textbook (Stewart, 2006) and syllabus with all the homework problems chosen from the text. Section E1 was an experimental section (9 students) using the same textbook but with a set of activities designed to help students make the mental constructions in the preliminary GD. Section E3 was another experimental section (8 students) using the same textbook with activities for planes and tangent planes but not for the differential. In this section the differential was defined but was not discussed in class. They had the same set of textbook homework problems for the differential as section T. All three professors were experienced (over 20 years teaching), having taught the course many times, popular with students (as judged by student evaluations), and concerned with student learning. Each of the professors was asked to choose 9 students: 3 above average, 3 average, and 3 below average, providing as balanced a distribution as possible. One student did not show up. The interviews lasted from 40 to 60 minutes. At the same time they were interviewed, students produced written answers and the interviewer took notes of the hand gestures of the students. The interviews were

recorded, transcribed, individually analysed, and then discussed as a group and results were negotiated among the researchers. The questions of interest are reproduced below:

Problem 1. Students were given the plane below and were asked to find the slopes in the x and y directions (m_x and m_y), the total vertical change (Δz) for $\Delta x = 4$ and $\Delta y = 5$, and the equation of the plane. (Observe that $m_x = 3$, $m_y = 1$, and if $\Delta x = 4$ and $\Delta y = 5$ then $\Delta z = m_x \Delta x + m_y \Delta y = 3(4) + 1(5) = 17$. Also, the equation of the plane is $z - 2 = 3(x - 1) + 1(y - 2)$.)

Problem 2. Students were given the graph below and were asked for the sign (positive, negative, or zero) of $\partial f / \partial y (4, 0, 0.7)$ and of $D_{\langle -2, 1 \rangle} f(4, 0)$. (Observe that $\partial f / \partial y (4, 0, 0.7) < 0$ and $D_{\langle -2, 1 \rangle} f(4, 0) > 0$.)

Problem 3. The following plane is tangent to the graph of $z = f(x, y)$ at the point $(1, 2, 0)$. (a) Find, if possible, the differential of f at the point $(1, 2)$, $df(1, 2)$. (b) Find $D_{\langle 1, 1 \rangle} f(1, 2)$. (Observe that since $m_x = 1$ and $m_y = 3$ then $df(1, 2) = 1dx + 3dy$.)



RESULTS

None of the interviewed students clearly exhibited a process conception of the differential. Only one student (from section E1) could be considered to be in transition to a process conception of this concept, 6 showed an action conception (5 of them from section E1), and the other 19 students showed no knowledge or recollection of the concept.

Most students seemed to depend entirely on a symbolic representation, to the extent that seeing the “d” in the symbol $df(a, b)$ they concluded that the differential was some kind of slope or derivative. Indeed 19 of the 26 interviewed students showed this type of response. Tania is one such student. In problem 1 she showed no difficulty finding the slopes in the x and y directions of a given plane (m_x and m_y), finding the total vertical change (Δz) for $\Delta x = 4$ and $\Delta y = 5$, and writing the equation of the plane. Further, in problem 2 she correctly found the sign of the requested partial derivative by identifying it with the slope of a tangent line she drew on the given graph.

Tania: (after reading problem 3) What does it mean by the differential of f ? Is that a slope?

Interviewer: The differential of f at a point. That was defined in class. What is the meaning of the differential of a function?

Tania: That was the slope at this point, isn't it?

Interviewer: No.

Tania: Ok, that would be, the vertical change.

Interviewer: OK

Tania: The point (1, 2) is this point, but to look for a vertical change I need two points, to be able to look for a z .

Note that Tania might be thinking of vertical change along the graph of the function (hence her need for two points). If this was the case, then she was not looking at the information about the function that may be obtained from the given tangent plane. This could indicate that she had not constructed the relation between the differential and the tangent plane. If she was thinking about vertical change along a plane, then she would seem not to have constructed a process of total vertical change on a plane, $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$, suggesting that she succeeded in problem 1 by applying actions (using memorized formulas). In any case, she seemed not to need to perform any treatment actions (Duval, 2006) on the process of tangent plane to obtain a formula for the differential, nor was she doing any conversion action (Duval, 2006) to relate the analytical and graphic representations of the tangent plane in order to obtain the differential from the graph of the tangent plane, as conjectured in the GD. Considering that she might be showing difficulty thinking of dx and dy as independent variables the interviewer asked:

Interviewer: If I tell you that the differential has to do with the variables dx and dy as independent variables, do you remember what is the differential?

Tania: Let's see, df is equal to $m_x dx$ plus $m_y dy$.

Interviewer: Ok... and what is the differential then?

Tania: This would be df (she went on to correctly compute the total vertical change on the plane using $\Delta x = 1$ and $\Delta y = 1$ –rather than leaving dx , dy as independent variables- by calculating m_x and m_y).

Tania had shown that she could identify a partial derivative with the slope of a tangent line in problem 2. Further, in problem 3 she could compute m_x and m_y from the given tangent plane. Hence, it seems she was able to obtain $f_x(1,2)$ and $f_y(1,2)$ from the graph of the tangent plane. However, she needed the interviewer's comment to help her remember a formula and to link it with the given plane, and did not consider dx and dy as independent variables. This, and other similar cases, seemed to show the importance of recognizing the differential at a point as the total vertical change on the tangent plane as a function of the horizontal change (dx , dy) in the mental construction of the differential.

Ramon's performance on problem 1 suggested that he had a process conception of total vertical change on a plane $\Delta z = m_x \Delta x + m_y \Delta y$. He was able to explain this formula in his own words, showing that he could imagine the geometric interpretation of the different components of the formula. Further, when obtaining the equation of the given plane in problem 1, he made clear reference to the notion of total vertical change on a plane. However, when asked about the differential, he did not relate it to the total vertical change.

Ramón: I don't remember the formula for df .

Interviewer: And if I were to tell you that the differential of f gives the change in height along the tangent plane for horizontal changes of dx in the x direction and dy in the y direction?

Ramón: It is something like the formula for change, $\Delta z \dots$ but I don't remember exactly how it was written... It was $df = m_x dx + m_y dy \dots$ I don't know if it was something like this.

Interviewer: What would be m_x in this case?

Ramón: It would be the slope with respect to x and this is the slope with respect to y [referring to m_y].

Interviewer: Can you find them?

Ramón: Yes... in x it is equal to... it would be $(1 - 0)$ divided by the horizontal change, which would be $(2 - 1)$. The slope would be 1...

Interviewer: Now look for the slope in the y direction.

Ramón: I would do it in the same way... it would be $(4 - 1) / (3 - 2) \dots$ which is 3. This is the slope with respect to y . The Δx would be the dx , but at the point $(1, 2)$, that is, at $x = 1$, $y = 2$, around here... so it is this point here... I couldn't calculate it.

Interviewer: And if I were to tell you that dx and dy are independent variables? That is, that stays like that as a function.

Ramón: I don't have the change in x which is a very small number, no... I couldn't look for it there... the product of $m_x dx$ gives a change, vertical, but the dx alone only tells me it is a very small horizontal change in this figure...

This suggests that students need to explore the relationship between the differential and the notion of total vertical change on a plane and that the GD should be revised to make this explicit. It may also be observed that Ramón resists thinking of dx and dy as independent variables.

Some students, like Karla, seemed to lose sight of the function once the tangent plane is given. Karla was able to quickly find and justify the sign of the partial and directional derivatives from the graph of the function given in problem 2.

Interviewer: Will that be positive, negative, or zero? (Referring to the slope of a tangent line to the surface on problem 2 that Karla correctly drew at the given base point and in the requested y direction.)

Karla: Negative... because as y increases, z decreases.

Interviewer: And how about the directional derivative?

Karla: I would say that positive

Interviewer: Why?

Karla: Because... I take this (referring to the direction vector) as Δx and Δy ... and as we move this way (pointing in the right direction) z is increasing.

However, later when working problem 3:

Interviewer: Could you tell me from the drawing of that plane there what is the meaning of directional derivative?

Karla: With this? [Pointing to the given plane in problem 3.]

Interviewer: So you drew a line segment on the plane where $y=2$ [see Figure 2].

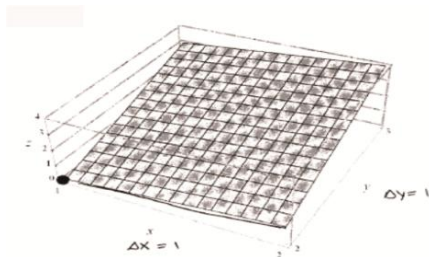


Figure 2: Karla's drawing on problem 3

When working with problem 3, Karla seemed unable to relate the tangent plane to the function when it was not present in the graph, or to say anything about the partial derivatives, showing evidence of not considering the tangent plane as a local approximation of the function.

All results obtained from students' interviews were similar. They suggest making explicit the construction of dx and dy as independent variables when considering the differential of the function and they also suggest that the coordination between processes on the function (like those for partial derivatives, directional derivatives, and vertical change) and the same processes on the tangent plane, is important in the mental construction of the differential. This also suggests the need to make constructions related to treatments and conversions (Duval, 2006) on the tangent plane in order to construct a process conception of the differential.

Results of this study suggest a refinement of the preliminary GD in order to make it a better model of students' constructions. It is also important to incorporate this refinement in activities designed for instruction in order to help students make the mental constructions necessary for a process conception of the differential.

DISCUSSION AND CONCLUSIONS

The notion of the differential appears to be very difficult for students in this study. All the interviewed students had already finished a course on two-variable functions, but still showed they could not even remember what the differential is in this context. Students' performance during the interview clearly evidenced they had not constructed the necessary processes involved in relating the notion of vertical change on a plane, Δz , and that of the differential, $df(a,b)$. Students' difficulties seem to be due to the fact that, in students' minds, the notion of differential remains isolated from that of the tangent plane. Further, the construction of the relation between processes on the graph of a function and those performed on a corresponding tangent plane seems absent in students' constructions. It also seems students need to explicitly construct a process involved in the recognition that the differential at a point $df(a,b)$ is a function of two independent variables, (dx, dy) representing horizontal change, which is associated to the vertical change (on the tangent plane) of the original function.

Students' results show that they conceive the differential as an empty symbolic interpretation as 'some kind of derivative' or, in the best case, a procedure where small values are substituted for dx and dy . The above mentioned constructions need to be taken into account in order to help students give meaning to the differential of a two variable function. As the preliminary GD did not describe all of the students' constructions that were shown to be needed in this study, a refined GD taking explicitly into account those constructions that this study showed to be missing was designed as part of the contribution of this study. This new model needs, of course, to be used in the design of instructional activities and tested with students. The refined GD follows.

REFINED GENETIC DECOMPOSITION FOR THE DIFFERENTIAL OF TWO-VARIABLE FUNCTIONS

Perform the treatment actions of graphically comparing the tangent plane to a two variable function at a given point in order to form a construct of the tangent plane as a local approximation of the function. Do actions to express the point-slopes equation for the tangent plane at a given point $(a, b, f(a,b))$ as the differential $df(a,b)=f_x(a,b)dx+f_y(a,b)dy$ together with treatment and conversion actions that relate processes of partial derivatives and vertical change on a function with the same processes on the tangent plane. These actions are interiorized into processes of total vertical change, $\Delta z=m_x\Delta x+m_y\Delta y$ and the differential. These processes are coordinated into a new process that enables students to relate them. Perform actions needed to evaluate the differential at a fixed point (a,b) for different values of dx and dy . Interiorize these actions into a process that recognizes that given a function f and a point (a,b) , the differential $df(a,b)$ is the total vertical change on the tangent plane expressed as a function of the horizontal changes dx and dy (see Figure 3).

Reflection on the action of computing the differential at different points allows interiorization of the differential into a process where the functional dependence of the differential on the starting point (a,b) is recognized. This process is coordinated with that of function so that the consideration of the differential as a two-variable function is made possible. When actions need to be applied, for example, to find specific properties of the differential, it may be encapsulated into an object.

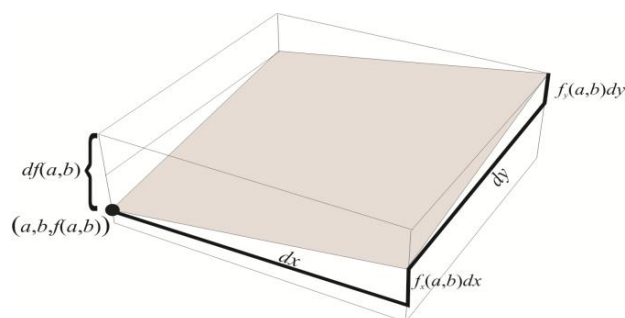


Figure 3: The differential

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Conceptions spontanées et perspectives de la notion de tangente pour des étudiants de début d'université

Rosa Elvira Páez Murillo¹, Elizabeth Montoya Delgadillo² et Laurent Vivier³

¹Universidad Autónoma de la Ciudad de México, Mexique, rosa.paez@uacm.edu.mx;

²Pontificia Universidad Católica de Valparaíso, Chili, elizabeth.montoya@pucv.cl;

³Laboratoire de Didactique André Revuz–Université Paris Diderot, France, laurent.vivier@univ-paris-diderot.fr

Les résultats que nous exposons dans cette étude proviennent d'un questionnaire proposé en début d'université au Chili, en France et au Mexique sur la notion de tangente à une courbe. Nous relevons les conceptions spontanées des étudiants en fonction des registres utilisés. Nous précisons ces conceptions en analysant des productions représentatives avec les perspectives ponctuelle, locale et globale, essentielle en analyse. Nous concluons par une discussion sur les Espaces de Travail Mathématique (ETM) personnels.

Mots clés : Conception, perspectives, tangente, ETM personnels.

INTRODUCTION

Dans cette étude [1], nous nous intéressons aux conceptions spontanées d'étudiants sur la notion de tangente. Nous analysons la première question d'un test écrit de six questions sur la notion de tangente : « **Pour vous, qu'est-ce qu'une droite tangente ?** ». Nous avons recueilli les réponses d'étudiants en première année d'université : 14 étudiants de la Pontificia Universidad Católica de Valparaíso (Chili), 9 de l'Université Paris Diderot (France) et 19 de la Universidad Autónoma de la Ciudad de México (Mexique) [2].

La notion de tangente apparaît dès le premier cycle secondaire dans les trois pays dans le cas spécifique du cercle. La notion réapparaît avec la dérivation pour les tangentes aux courbes représentatives des fonctions : au grade 11 en France ainsi que dans certains lycées chiliens et mexicains, alors que cette définition n'est pas aux programmes d'études officiels de ces deux pays. Finalement, en début d'université, la définition par la dérivation est presque exclusivement utilisée dans les trois pays.

Nous exposons dans un premier temps une typologie des conceptions de la notion de tangente qui ont été identifiées dans des études précédentes (Sierpinska, 1985 ; Castela, 1995 ; Biza y Zachariades, 2010 ; Páez et Vivier, 2013 ; Montoya et Vivier, 2015 ; Vincent et al. 2015) et qui constitue une analyses a priori des réponses possibles. Nous relevons, avec cette classification, les conceptions que les étudiants expriment spontanément. Dans un second temps, nous précisons ces conceptions en

identifiant le ou les registres (Duval, 2006) utilisés, les perspectives (locale, globale et ponctuelle) qui apparaissent dans les traces écrites. Ces éléments permettent de préciser, en partie, l'ETM personnel (Kuzniak, 2011) des étudiants constitué des composants de l'ETM qui sont activés spontanément. Néanmoins, l'ETM personnel ne s'y réduit pas et, souvent, le travail sur des questions plus spécifiques montre que d'autres composants sont alors activés, parfois sans lien.

CONCEPTIONS DE LA NOTION DE TANGENTE

Nous utilisons le terme *conception*, dans le sens de Duroux (1983) pour nous référer aux connaissances disponibles d'un sujet pour un concept mathématique et dans une situation donnée. Pour cette étude, une *conception* est associée avec des images mentales, des caractéristiques, des propriétés et des procédés associés au concept en jeu. Une conception peut apparaître dans une certaine situation, mais pas dans d'autres où peuvent apparaître d'autres conceptions ce qui peut laisser apparaître un manque d'articulation ou de cohérence, voire des contradictions.

Dans les recherches comme celles énoncées en introduction, il a été identifié les idées ou connaissances qu'ont les étudiants ou professeurs pour la notion de tangente. Dans la suite, nous proposons une typologie de conceptions issues des recherches précédentes [3] mais qui nous permet de proposer une analyse des conceptions spontanées des étudiants. Notre idée directrice est de percevoir ces conceptions en deux blocs : un premier bloc issu du domaine de la géométrie, directement lié à la tangente à un cercle, et un deuxième bloc issu du domaine de l'analyse, directement lié à la dérivation.

Conceptions issues de la géométrie

Unique Point d'Intersection (UPI). Cette conception est fondée sur des connaissances de la géométrie en identifiant la droite tangente comme la droite qui n'a qu'un seul point d'intersection au cercle, comme par exemple pour l'étudiant **CIF4** (Figure 1) :

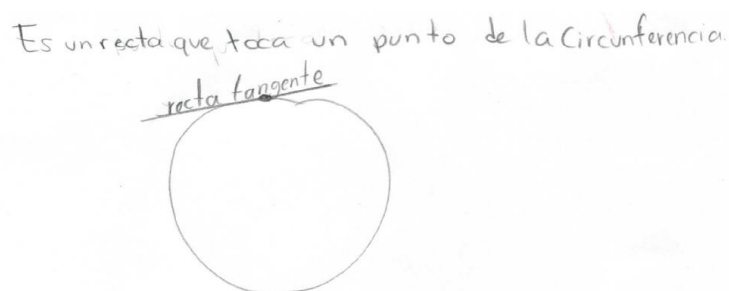


Figure 1. Etudiant CIF4 – C'est une droite qui touche un point du cercle ; droite tangente.

Cette conception s'étend à d'autres courbes, comme une droite qui *touche*, *coupe* ou *frôle* la courbe en un seul point (voir Figure 2). Il y a, en quelque sorte, des cas de *dégénérescence* de la conception UPI, comme lorsqu'un étudiant ne trace qu'une demi-droite ou un segment pour éviter que la trace graphique ne recoupe la courbe

(presque comme dans la figure 2), ou encore une droite qui coupe de manière non tangente la courbe, pour s'adapter à la situation. Toutefois, si ces réponses sont fréquentes lorsque l'on demande de tracer une tangente à des étudiants (Montoya et Vivier, 2015), ces situations sont ici soigneusement évitées par les étudiants puisque ce sont eux qui choisissent la courbe et le point de tangence. Cela dit, certains étudiants ont réduit, en partie tout du moins, le conflit cognitif relatif au fait que la tangente recoupe la courbe comme **C3** (figure 3).

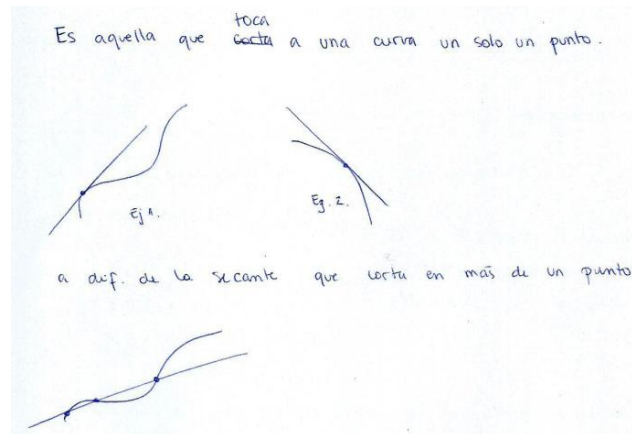


Figure 2. étudiant C3 – C'est celle qui touche une courbe en un seul point ; à la dif. de la sécante qui coupe en plus de un point.

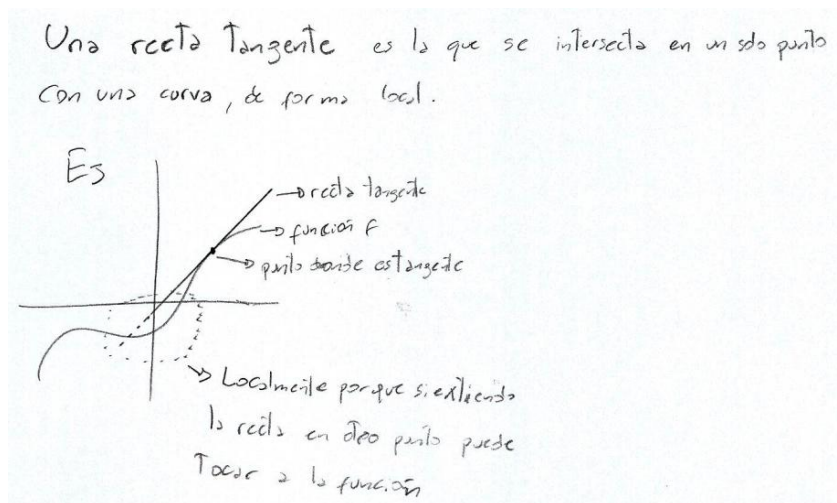


Figure 3. étudiant C11 - Une droite tangente est celle qui s'intersecte en un seul point avec une courbe, de forme locale. C'est ; droite tangente ; fonction f ; point où c'est tangent ; localement parce que s'étendant, la droite peut toucher en un autre point la fonction

La conception UPI s'accompagne du fait que l'unicité du point en commun entre la tangente et la courbe n'est requise que localement autour du point considéré, dans un voisinage. Ici, le jeu local-global (voir section suivante) apparaît comme un élément important pour réduire les conflits cognitifs relatifs à la conception UPI. On note, malgré tout, le tracé de la droite qui passe en ligne non continue, ce qui marque une

incertitude, un changement de statut (nous avons rencontré par ailleurs ce type de réponse en pointillés qui permet d'alléger le conflit cognitif).

Perpendiculaire au rayon (PR). Quand il est explicité avec des mots ou avec des représentations graphiques ou formelles la relation de perpendicularité entre la tangente et le rayon du cercle. L'étudiant **F8** (figure 4) présente cette conception avec également une cohérence, notamment par le codage des angles droits, avec la conception *Tangente Horizontale* (voir ci-dessous) malgré une confusion de langage entre parallèle (à l'axe des abscisses) et perpendiculaire (à la courbe).

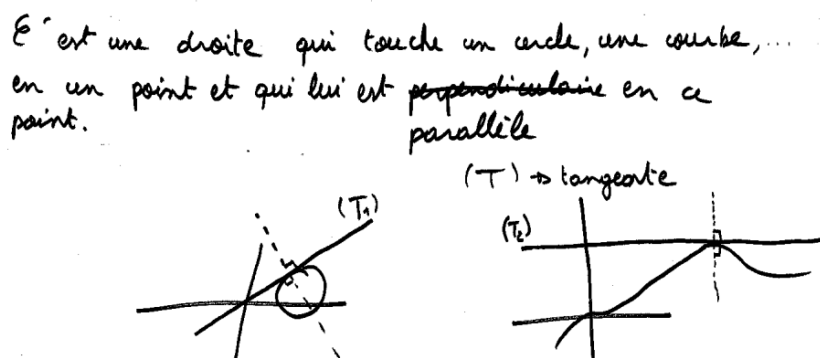


Figure 4 : Etudiant F8, conceptions PR (à gauche) et TH (à droite)

Conceptions issues de l'analyse

Pente (P). Cette conception réfère à l'utilisation du mot « pente » pour décrire la tangente. Des degrés de raffinement existent, comme l'exprime par exemple CDF1, où nous voyons, implicitement, l'expression d'une approximation affine de la courbe.

Etudiant **CDF1** : *Une droite tangente à une courbe en un certain point d'elle, est une droite qui au moment de passer par le dit point, a la même pente que la courbe*

Processus de Dérivation (PD). La droite tangente est la droite qui passe par un point de la courbe et dont la pente est le nombre dérivé obtenu par dérivation (implicitement par un travail formel). De même, des raffinements, dépendant du registre, peuvent apparaître comme l'étudiant **F2** qui utilise le langage naturel à la différence de l'étudiant **F1** qui utilise le registre algébrique. On peut également noter que **F2** présente les conceptions UPI et P.

Etudiant **F2** : *Une droite tangente est une droite qui ne touche une courbe en un seul point et qui a pour pente celle du point quel touche. Elle est représenté par la dérivé.*

Etudiant **F1** : *On définit la droite tangente d'une fonction à un point x_0*

$$\rightarrow y = f(x_0) + f'(x_0) (x - x_0)$$

Il s'agit en fait d'une approximation de degré 1 de la fonction f au voisinage de x_0 .

Tangente Horizontale (TH). Quand il est fait mention de maximum ou de minimum de la courbe pour expliquer la notion de tangente, comme dans le cas de **F8** dans le registre graphique ou l'étudiant **F6** dans le registre de la langue naturelle (LN) :

Etudiant **F6** : *C'est une droite qui passe par un minimum ou un maximum local de la courbe d'une fonction*

En terme général, nous présentons dans la table suivante les conceptions identifiées dans les trois groupes de l'étude. Il est important de mentionner que des étudiants présentent plusieurs conceptions spontanées. Mais il existe aussi des étudiants qui présentent d'autres conceptions que celles mentionnées ci-dessus ou, comme pour l'étudiant suivant, qu'il est difficile de classer par manque d'information (bien qu'elle soit en apparence correcte) :

Etudiant **CDF5** : *C'est une droite qui touche une courbe au point donné*

La table 1 synthétise l'ensemble des réponses ; certains étudiants présentent plusieurs conceptions, souvent dans des registres différents, alors que d'autres n'ont pas répondu à la question. Cela dit, les deux blocs apparaissent équilibrés : UPI et PR d'un côté, des conceptions de la géométrie, et P, PD et TH de l'autre, des conceptions de l'analyse (nous associons la conception P à PD du fait qu'il y a l'idée de définir un nombre au point considéré). Les différences entre les trois populations peuvent s'interpréter comme des différences d'enseignement voire de contexte. Mais il faudrait une étude avec plus d'étudiants pour pouvoir conclure sur ce point.

Étudiants	UPI	PR	P	PD	TH	Réponses non identifiées	Sans réponse
France (9)	4	2	2	5	2	0	0
Chili (14)	9	0	3	1	0	0	3
Mexique (19)	2	1	7	0	0	1	8
Total (42)	15	3	12	6	2	1	11

Table 1 : Typologie des conceptions, ensemble des réponses par population

PERSPECTIVES ASSOCIEES A LA NOTION DE TANGENTE

Ces conceptions de tangentes s'affinent ou se *déconstruisent* avec la notion de *perspectives*, terme que nous reprenons de Vandebrouck (2011) pour les fonctions (voir aussi Maschietto, 2003 ; Rogalski, 2008) : perspectives locale, ponctuelle, globale (voir également Kuzniak, Montoya, Vandebrouck et Vivier, 2015 ; Montoya et Vivier, 2015). Ces perspectives s'expriment de prime abord dans différents registres de représentation, mais ce n'est pas la seule source d'influence.

Pour le cas de la tangente, nous identifions : la *perspective ponctuelle* par la nécessité de considérer le point de la courbe où on veut la tangente ; la *perspective locale* par une référence aux liens qu'entretiennent la courbe et une tangente en un voisinage, ou

autour, du point en question (nous différencions l'utilisation des mots « couper » et « toucher ») ; et la *perspective globale* par l'identification de la tangente comme une droite ou une ligne. On a ainsi, dans l'ordre des trois perspectives : un point, une direction et une droite (ou portion de droite).

Ces trois perspectives s'expriment dans chaque registre de représentation. C'est le cas par exemple de **CDF5** où les trois perspectives apparaissent (apparemment correctement) dans le registre LN. Des perspectives peuvent être influencées par les conceptions ce qui entraîne un contrôle ou une adaptation de certaines perspectives. Par exemple, la conception UPI peut entraîner une réponse dans le registre graphique d'une demi-droite pour une tangente afin que la ligne tracée ne recoupe pas la courbe : la perspective ponctuelle guide le travail et entraîne une adaptation de la perspective globale. On trouve ainsi ce type de réponse, même par des enseignants, lorsqu'il est demandé de tracer une tangente à une courbe (Figures 5).

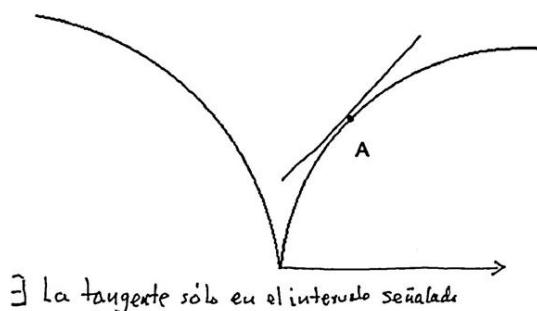


Figure 5a. Réponse d'un professeur mexicain (Páez et Vivier, 2013)

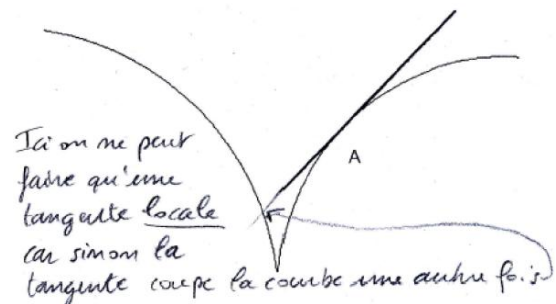


Figure 5b. Réponse d'un professeur français

A la suite, nous décrivons les perspectives qui peuvent apparaître dans chaque conception mentionnée en nous appuyant sur les exemples de réponses donnés. Nous précisons les registres utilisés et les jeux entre les perspectives qui peuvent entraîner des adaptations par une *pression* d'une perspective sur une autre.

Conception UPI

On note que la conception UPI n'a pas été exprimée dans le registre algébrique (Ralg). Nous reprenons les productions des figures 1, 2 et 3. **CIF4**, avec le registre LN, présente les trois perspectives : c'est une droite (globale) qui touche (locale) en un point (ponctuelle) la courbe. Les trois perspectives semblent s'enchaîner et s'articuler sans s'influencer. Il en est de même dans le registre graphique (Rg), même si l'on peut penser que, dans d'autres situations, des problèmes surgiraient.

On note pour **C3** une différence dans LN avec le mot « solo » qui semble indiquer une sorte d'injectivité entre droite et point de la courbe où la droite est tangente, mais cela est précisé à la suite dans LN, avec le mot « corta » et non « toca », et dans Rg où l'on note que ce ne sont pas les points de tangence qui comptent. Mais les points en communs, d'intersection. La perspective ponctuelle est prépondérante.

Pour **C11**, la perspective locale semble disparaître de LN, mais il ne fait aucun doute qu'elle est présente dans Rg, la droite coïncidant localement avec la courbe, ainsi que par le caractère local de la propriété de tangence (que deviendrait cette conception avec une fonction affine ou du type $x^2 \sin(1/x)$ en 0 ?).

Nous avons déjà remarqué (Montoya et Vivier, 2015) la fréquence d'une réponse issue d'une conception UPI où la perspective ponctuelle influence la perspective globale dans Rg avec une réponse constituée d'une partie d'une droite (demi-droite par exemple). On remarque ici, les étudiants étant maître de la courbe proposée, que **CIF4** et **C3** donnent des cas de tangence où la tangente ne recoupe pas la courbe et si **C11** propose un tel cas, il n'en reste pas moins que le prolongement est en pointillés. Nous voyons ici, dans la production même dans Rg (courbe et tangente) une influence de la perspective ponctuelle sur la perspective globale.

Conceptions PR et TH

La conception PR fait souvent référence à un cercle, comme dans le premier graphique de la figure 4 pour **F8**, mais on relève ici une cohérence avec la conception TH. La perspective locale n'est a priori pas présente car la direction est donnée par la perpendiculaire, même si l'on peut penser qu'il y a vraisemblablement un contrôle visuel par la perspective locale. Néanmoins, dans le cas de **F8** on relève de manière claire la perspective locale avec le mot « touche » ainsi que dans le graphique de droite avec le point de maximum. De plus, il est vraisemblable que le tracé ait été produit avec la perspective locale, l'angle droit ayant été marqué ensuite. On peut ainsi penser que les trois perspectives sont présentes et sont articulées, même si la notion de tangente reste étroite.

Pour **F6**, il semble y avoir beaucoup d'implicites, surtout pour la perspective locale. En effet, elle semble absente (on peut faire passer une infinité de droites par un point, fut-il un point de minimum) alors que les deux autres perspectives sont explicitement présentes. Mais on peut penser que le travail sur les fonctions favorise largement l'association "tangente horizontale – dérivée s'annulant – extremum" avec les représentations graphiques associées. Il est fort possible que pour cet étudiant la droite tangente est horizontale en un point d'extremum. Il n'en reste pas moins que la perspective locale n'est pas explicite.

Conceptions P et PD

Dans le cas de **CDF1**, on relève explicitement les perspectives ponctuelle et globale, mais il est difficile de se prononcer sur la perspective locale, sans autre information. Il se peut que le mot *pente* soit pensé en terme de « courbe localement rectiligne » (la propriété de *micro-linéarité* de Maschietto (2003)), auquel cas la perspective locale est bien présente, ou bien en terme de dérivation ou d'association d'une pente à chaque point de la courbe, auquel cas il n'y aurait pas de perspective locale, la perspective ponctuelle donnant la *pente*.

Tout comme la conception P, dans la conception PD la perspective locale peut être occultée, la direction de la tangente étant donnée par le nombre dérivé qui peut provenir uniquement d'une perspective ponctuelle : la perspective ponctuelle semble prépondérante dans le processus de dérivation qui masque la perspective locale. Néanmoins, la perspective locale peut réapparaître, avec un complément. Pour **F2** (voir ci-dessus), la conception UPI dans LN permet d'explicitier une perspective locale (« touche ») qui n'apparaît pas avec le simple mot « dérivé ». Pour **F1** (voir ci-dessus), la première partie montre les perspectives ponctuelles et globales dans Ralg, mais cela est complété par une précision sur l'approximation d'ordre 1 de la fonction au voisinage du point considéré qui marque une perspective locale.

CONCLUSION : DES CONCEPTIONS AUX ETM PERSONNELS

Dans cette étude, nous avons présenté cinq conceptions illustrées par des productions d'étudiants de première année d'université. Il s'agit de conceptions spontanées à une unique question demandant d'expliquer ce qu'est une tangente, elles proviennent essentiellement de deux domaines : la géométrie et l'analyse. L'algèbre est absent des productions des étudiants alors que ce domaine pourrait faire une transition intéressante entre la géométrie et l'analyse (Vivier, 2013). Ces deux domaines mettent en jeu des connaissances différentes. Les conceptions UPI et PR reposent pour l'essentiel sur une connaissance géométrique, courbe et tangente ont un seul point en commun, et les conceptions PD et TH, auxquelles on ajoute la conception P, reposent essentiellement sur la dérivation. Néanmoins, les objets de notre étude sont communs, les courbes et les tangentes. Ainsi, au lieu de considérer deux domaines différents, on peut également considérer que les deux blocs de conceptions sont relatifs à l'Espace de Travail Mathématique (Kuzniak, 2011) de l'analyse relatifs à deux paradigmes différents (Kuzniak et al., à paraître) identifiés comme le paradigme de l'analyse géométrico-arithmétique et le paradigme de l'analyse calculatoire.

Dans l'identification des conceptions spontanées, les étudiants ont le choix des registres et des exemples pour les exprimer. On remarque en particulier le soin apporté aux exemples dans Rg pour illustrer une conception UPI : pour le cas choisi, la tangente ne recoupe pas la courbe alors qu'elle le pourrait pour un autre point (figure 2). La notion de conception nous paraît limitée pour analyser ce phénomène et c'est pour cela que nous avons introduit les perspectives car elles permettent de rentrer plus finement dans les conceptions. De plus, elles permettent de comprendre le travail d'un étudiant comme **C3** (figure 2) : le choix du point (perspective ponctuelle) a été effectué afin que la tangente (perspectives locale puis globale) ne recoupe pas la courbe pour s'accorder avec la conception ou avec une connaissance géométrique en jeu. Le modèle des ETM est un système qui prend notamment en compte une genèse sémiotique liant les signes et le processus de visualisation et les connaissances théoriques. Ainsi, il peut permettre de mieux comprendre le jeu entre les perspectives, leurs articulations, ici dans le registre graphique.

Avec une conception UPI on relève souvent une articulation des trois perspectives (bien que pas toujours correcte), il n'en est néanmoins pas de même pour les conceptions P et PD. En effet, la perspective locale est en générale absente au profit de la perspective ponctuelle puisqu'au point considéré on associe un nombre (voir **CDF1**). Néanmoins, la perspective locale peut apparaître avec un complément comme pour **F2**, avec le mot « touche », ou **F1**, avec l'approximation d'ordre 1 (voir ci-dessus).

On remarque que les conceptions UPI et PR s'accompagnent fréquemment d'un graphique (avec ou sans repère) mais rarement d'une expression algébrique alors que c'est l'inverse pour P et PD qui s'accompagnent rarement d'un graphique mais où on peut voir des expressions algébriques, comme pour **F1** avec l'équation de la tangente, ou simplement la mention d'une fonction f . Cela conforte l'idée qu'il s'agit d'un travail dans deux paradigmes distincts mettant en avant des signes différents.

La juxtaposition de conceptions peut-être mieux comprise dans le modèle des ETM. Par exemple, **F2** présente trois conceptions : UPI, P et PD. Le lien entre P et PD est explicite et constitue une connaissance du référentiel théorique de l'ETM personnel de **F2** : la pente de la tangente est égale au nombre dérivé. En revanche, il ne semble pas y avoir de lien avec UPI qui semble juxtaposée aux deux autres et que nous interprétons comme un défaut dans l'articulation des deux paradigmes en jeu.

Il est significatif que la conception *Limites des Droites Sécantes* n'apparaisse pas explicitement dans les réponses des étudiants malgré son utilisation fréquente pour introduire la dérivation. Les conceptions liées à la dérivation apparaissent souvent, ce qui n'est pas étonnant car le travail porte beaucoup sur celles-ci, mais on remarque que les conceptions UPI persistent dans les réponses spontanées des étudiants.

Nous n'avons pas analysé ici la suite du questionnaire qui proposait systématiquement une expression algébrique d'une fonction, parfois accompagné d'un graphique. Il est à noter qu'alors le travail s'uniformise beaucoup avec une quasi-exclusivité du processus de dérivation, même si d'autres procédures sont plus efficaces et sans possibilité de contrôle par d'autres conceptions ou par un changement de paradigme (notamment avec un graphique). Ainsi, avec l'utilisation de la tangente pour introduire la dérivation, il semble que le travail proposé aux étudiants réduit la notion de tangente à la dérivation. Pourtant, il s'agit d'une notion mathématique très riche qui ne peut se limiter au seul point de vue proposé par la dérivation. Il nous semble que non seulement l'enseignement universitaire perd ainsi une richesse, des occasions de travail mathématique riche et intéressant, mais qu'en plus on ne permet pas aux étudiants de raffiner leurs conceptions, leurs ETM personnels, comme nous l'avons vu avec l'expression des réponses spontanées.

NOTES

1. Cette étude s'inscrit dans le projet PI 2014-39 de la Universidad Autónoma de la Ciudad de México et la Secretaria de Ciencia, Tecnología e Innovación (SECITI) au Mexique et dans le projet ECOS-Sud C13H03 au Chili et en France.

2. Les étudiants chiliens sont C1, C2, etc., les français F1, F2, etc. et les mexicains CIF1, CIF2 etc. ou CDF1, CDF2 etc.

3. La liste n'est pas exhaustive, on ne peut jamais l'être totalement, chaque sujet ayant sa propre conception.

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Concept Images of Open Sets in Metric Spaces

Safia Hamza and Ann O'Shea

Department of Mathematics and Statistics, Maynooth University, Ireland,
ann.oshea@nuim.ie.

We consider the concept images of open sets in a metric space setting held by some Pure Mathematics students in the penultimate year of their undergraduate degree. Ten students were interviewed and asked to define the concept of an open set, as well as to work on some specially designed mathematical tasks on this topic. The analysis of the interview data revealed five main categories of concept image of open sets based on: the formal definition; the idea of boundary of sets; open sets in Euclidean space; the union of open balls; visualisation.

Keywords: concept image, concept definition, open sets, topology.

INTRODUCTION

This paper concerns a study of the conceptions held by students taking a module on metric space topology. The main topic of interest is the notion of an open set in a metric space. This concept is fundamental in the study of topology but (personal) experience has shown that it can pose problems for students and hinder the development of their understanding of the subject. Our goal was to explore the students' concept definitions and concept images of the notion of open set in order to provide information to lecturers which would help them when planning and delivering courses in this area.

Courses on Metric Spaces often involve significant transitions for students. These students usually have taken a course in analysis on the real line but may not be comfortable with the level of abstraction required to work in general metric spaces. Part of this transition involves coming to an appreciation of the role of definitions in abstract mathematics; Edwards and Ward (2004) investigated students' understanding and use of definitions in an introductory abstract algebra course and found that students seem to place less emphasis on definitions than mathematicians would, and even when they are able to correctly state a definition of a concept they may not always use this when working on problems. Very little research has been carried out on students understanding of topics in introductory topology and we wanted to gain information about how students define concepts in this area and also to explore the concept images related to these notions. Our research question was:

- What elements of students' concept image of open sets in metric spaces can we identify?

THEORETICAL FRAMEWORK

We will use Tall and Vinner's (1981) description of the notions of concept definition and concept image. They used the term *concept definition* to indicate a mathematical definition:

a form of words used to specify that concept (Tall and Vinner 1981, p. 152).

They used the term *concept image* to mean all that an individual has in his/her mind about a concept, and this would include mental pictures, experiences and impressions that are associated with it. They defined the concept image as:

the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. (Tall and Vinner 1981, p. 152).

They explained also that a concept image is not a static item in memory; it builds and is reconstructed over time as individuals meet new stimuli. Tall and Vinner (1981, p. 152) also used the term *evoked concept image* to describe the part of a concept image which is evoked by the concept name at a specific time.

Since the introduction of these notions, they have been used in many research studies (see Alcock and Simpson (2009) for an overview) to understand the development of understanding of various concepts in for example calculus (Bingolbali and Monaghan, 2008), linear algebra (Wawro, Sweeney, & Rabin, 2011), and introductory real analysis (Przenioslo, 2004). To the best of our knowledge there has not yet been a study of the concept images of concepts in general topology.

Przenioslo (2004) studied students' concept images of limits of functions and amongst other results she found that aspects of concept images can be formed very early in a student's development. McGowen and Tall (2010) also addressed the role of early experience (met-before) on the learning of mathematics. They described that

The term met-before applies to all current knowledge that arises through previous experience, both positive and negative. It can be given a working definition as 'a mental structure that we have *now* as a result of experiences we have met-before'. (McGowen & Tall 2010, p. 171).

They explained that previous experience could be supportive (in which the old ideas can make sense in the new context) but could also be problematic.

Fischbein (1989) also referred to the positive and negative effects of previous experience on mathematical reasoning and understanding. He spoke about 'tacit models' as models of abstract concepts which are developed early in the learning process and which continue to influence reasoning and interpretation without the learner being explicitly aware of this influence. Problems occur when a tacit model, or possibly a specific example, becomes a substitute in the learner's mind for the concept in question. If the learner is not aware of the influence of these models and examples on their own thinking, then they can do little to change them. Fischbein (1989) suggests that researchers should therefore investigate the likely tacit models related to a concept, and that teachers should make students aware of the existence of these models and of the problems they may cause; the aim of both should be to provide students with opportunities to recognise and control their own tacit models.

Bingolbali and Monaghan (2008) observed that many of the learning theories that developed using the construct of Tall and Vinner (1981) of the concept image and

concept definition were cognitive theories of learning. They argued that the construct could be also used to study social theories of learning. They studied first year Mechanical Engineering and Mathematics students' concept images of the derivative, in particular the rate of change and tangent notions. They showed that students' development of their concept image is affected by the teaching practices and by their departmental affiliations.

METHODOLOGY

This study involved students taking a course on Metric Spaces at Maynooth University in Ireland. This was a one semester module which was delivered by an experienced member of staff. The authors had access to the course notes and assignments. The course ran for twelve weeks and there were two lectures each week. The syllabus for this course is: Metric spaces: definitions and examples, convergence and continuity in metric spaces; uniform continuity; pointwise and uniform convergence, open and closed sets; basic properties; continuity in terms of open sets; limit points; closure; interior and boundary, completeness and compactness.

All 17 students in the module were asked to participate in this study and 10 volunteered to be interviewed. The interviews took place in the final two weeks of the semester. The interviews were conducted by the first author and were task-based (Goldin, 1997) and semi-structured. They were audio-recorded and fully transcribed; the data was anonymised immediately. We will refer to the 10 students using the letters Q - Z. After some initial introductory questions, the students were asked to define an open set in a metric space and how they would explain the concept to a friend. They were also asked to work on some tasks. The tasks used were designed for the study taking care to use the same language and notation as that employed by the module lecturer; they were piloted in written form by two recent graduates. Four tasks were designed for this part of the study but we will only report on two of them here. These tasks were:

- A. Consider the metric space $(\mathbf{Z}, d_{\mathbf{Z}})$, where $d_{\mathbf{Z}}$ is the standard metric inherited from \mathbf{R} , and let $B = \{m - 1, m, m + 1\}$. Is B an open ball in $(\mathbf{Z}, d_{\mathbf{Z}})$? If your answer is yes, please specify the centre and radius of the ball. If your answer is no, please explain. Can you find an open ball C which is a subset of B ?
- B. Let X be the set of all real sequences. Define:

$$d(\{a_k\}, \{b_k\}) = \begin{cases} 0 & \text{if } a_k = b_k \text{ for all } k \in \mathbf{N} \\ 1/k & \text{if } k = \min\{n \in \mathbf{N} : a_n \neq b_n\} \end{cases}$$

- (i) Can you describe this metric in words?
- (ii) What do you think this metric measures?

-
- (iii) Let $\{0\} = \{0,0,0, \dots\}$. If $d(\{a_n\}, \{0\}) = 1$, what can you say about $\{a_n\}$?
 - (iv) What is $B(\{0\}, 1)$? What is $B(\{0\}, 1/2)$?
 - (v) Is the set of sequences $\{\{a_n\}: a_1 = 0\}$ open? Is the set of sequences $\{\{a_n\}: a_1 = 0 \text{ or } 1\}$ open?

(Note that the lecturer had defined $B(a, r) = \{x \in X \mid d(a, x) < r\}$ in the metric space (X, d) .) The transcripts were analysed using a grounded theory approach (Strauss and Corbin, 1990) by both authors independently, the codes and categories created were then compared and a final coding was agreed.

RESULTS

Students' Definitions

The students were asked:

- (i) To define the term *open set* in a metric space,
- (ii) How they would explain this term to a friend.

We analysed the answers to these questions and classified them into three categories. These were: answers based on the formal definition of an open set; answers based on the notion of an open set as a union of open balls; answers related to the boundary of a set. The formal definition given by the lecturer in this course was:

A subset U of a metric space (X, d) is an open set if for all $x \in U$ there exists $\varepsilon(x) > 0$ and an open ball $B(x, \varepsilon(x))$ in (X, d) such that $B(x, \varepsilon(x))$ is a subset of U .

None of the students gave exactly this definition but some gave something very close to it. For example Student Q said

The set is open if for any point in the set you can draw an open ball around it which is contained in the set.

Students X and Z gave similar definitions. Student Y said

The official definition is you can take any open ball around any point and it's still completely contained in the set.

We can see that this is not correct as it is too strong; we do not need every open ball centered at every point to be a subset of U , we just need at least one for every point. Notice that none of the students spoke about the ball being open in (X, d) , we assume that they mean this implicitly.

When asked to define an open set, Students S, T, U, V and Z all said that open sets were unions of open balls. Note that Student T was the only student who seemed to realise that this is a theorem and not a definition and she used the formal definition in her explanation to a friend.

The last category of definition is made up of answers to Question (i) which mention boundaries when trying to define the term open set. Student R said

Open set – something which doesn't have a clear boundary, you can get as close as you like but never get to the actual end of the set.

Student W gave a similar definition after first admitting that he had forgotten the formal definition. He said

We can say what the general idea, the open set is basically, it isn't like say straight edges, is kinda fuzz out, because it doesn't contain border elements

and in his explanation to a friend he also said

so it kind of fades off infinitesimally close to boundary, but it never quite gets out, fuzzy at the edges

<i>Student</i>	<i>Answer to Question (i)</i>	<i>Answer to Question (ii)</i>
Q	Formal Definition	Formal Definition
R	Boundary	Boundary
S	Union of Open Balls	Union of Open Balls
T	Union of Open Balls	Formal Definition
U	Union of Open Balls	Boundary
V	Union of Open Balls	Union of Open Balls
W	Boundary	Boundary
X	Formal Definition	Formal Definition and Union
Y	Formal Definition	Boundary
Z	Union and Formal Definition	Boundary

Table 1: Students answers to the definition questions

Table 1 shows the category of answer given by each of the students to both questions. It shows that in answering question (i) three students gave a definition close to the formal definition, four spoke about unions of open balls, one student mentioned both these ideas, but only two students mentioned anything to do with boundaries. This is in contrast to the answers given in part (ii) where five students spoke about boundaries, two gave answers based on the formal definition and two explained using unions of open balls, with one student using both the formal definition and the idea of unions. It may be that when seeking an explanation suitable for a friend, students looked for examples or non-mathematical terms to illustrate the idea and that this led

them to concentrate on boundaries or the lack of them. In answer to Question (ii) Student Y said

We got a definition for it last year which is just a set that doesn't contain its boundary. So it's kinda easiest to think of it in that way, I probably explain it kinda like that, that you know, if you go shorter and shorter distance so you know, no matter how close you get, you'll never quite get there.

These students had taken an introductory course on analysis on the real line and it may be that their experience with open intervals there has influenced their definition of openness in a metric space; that is they may have a tacit model of the concept of open sets based on examples familiar from the earlier course.

Students' Concept Images

The answers to Questions (i) and (ii) above show us that the students' concept image of an open set in a metric space includes more than the formal definition and in particular includes results proved about open sets (i.e. that open sets can be expressed as unions of open balls) and previous experience of sets without boundary. We analysed the students' answers to the mathematical tasks in the interviews in order to see if other aspects of the evoked concept image of open sets would emerge. We found that some students used the formal definition when working on these problems and also frequently referred to boundaries but no one used the notion of unions of open balls (which would have been very useful in Task B). The other aspects of concept image that we observed included visualisation of open sets using generic pictures (like discs) and notions related to open sets (particularly open intervals) on the real line. We will give some examples of these two aspects of evoked concept image here (to save space we will not revisit the other aspects of concept image encountered in the last section).

Many of the students spoke about visualising the open sets in the tasks. For most students their picture of an open ball seems to be based on pictures from Euclidean space. For example Student Q when considering Task A commented that:

Student Q B is open if we can draw an open ball around 1 which is inside the set, if its centred at 1 then the open ball would include 0 and 2.

Interviewer you said here, draw an open ball, so that means you have a picture of open ball in your mind?

Student Q yeah!

Interviewer which kind?

Student Q just a ball, a circle, um it has to be inside the set.

Similarly many of the other students drew circular regions when thinking about open sets, for example Students Q and S did this on Task B (see Figure 1). Indeed, some students expressed frustration with that problem because they had difficulty in visualising the open sets concerned (Students R, T, and Y).

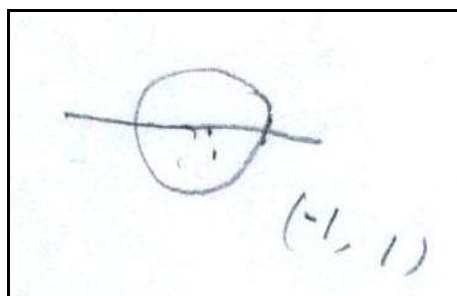


Figure 1: Student S's picture of $B(\{0\}, 1)$ in Task B

Some of the students also spoke about having difficulty with other questions when they couldn't visualise the sets in question. For example Student X, when considering Task A said:

I don't think I have seen a set like that and been asked if it is an open ball, so I can't really picture it.

Some students seemed to realise that their intuitive pictures may not help them in all situations. For example Student Y initially tried to use diagrams to answer Part (iv) of Task B, however later she returned to the definition of the metric d and worked with that analytically. When asked why she did that she answered:

Because I just looked at it to, and it looked too confusing to try and think of a picture of sequences, to try to think of how far they're apart.

It seems that students' concept image of open sets contains visual elements and some of these are based on open sets in familiar metric spaces especially Euclidean space. This space, and the features of open sets in it, appears to have other influences on the students' concept images of open sets. In Task A, some students were reluctant to see B as $B(m, 1+\varepsilon)$ because it consisted of (apparently) isolated points. Student T said

I think I'm going that, r is 1 and x is centre m . But, no, that is not open because it doesn't contain all the points. It's only contains these three points, it's limited, meets these three points, I don't think it's open.

Recall that Student W referred to 'fuzziness' when defining an open set. He used this idea here again

We'd only got three elements, but these elements all have space of the exactly one. So you either have a gap of 0 or 1 between them. There is no kind of fuzziness in between, so you can't make it open. Like, it'll either contain them or not.

These students may be referring to properties of open intervals in \mathbf{R} or open discs in \mathbf{R}^2 such as connectedness and completeness. We saw this idea in the students' answers to other tasks too.

SUMMARY AND DISCUSSION

We found that the students in this study had three main ways of defining open sets in metric spaces: the formal definition; using the notion of boundary; and using the fact that open sets can be expressed as unions of open balls. Our analysis has also showed that these students had varied concept images related to the open set concept. These concept images were based on: the formal definition, the boundary idea, unions of open balls, openness in Euclidean space, and visualisation. When working on mathematical tasks students used both the definition and other aspects of their concept image. We noted that the students who routinely based their reasoning on the definition were more successful when working on problems; this was especially true in Problem B where students' unfamiliarity with the context meant that some components of their concept image did not help them. Most students showed a richness in their concept image and an ability to view open sets in a variety of ways.

We noticed that the students' confusion about boundary points and endpoints of an open set could cause difficulties. Moreover we noticed that the previous experience of open sets in \mathbf{R}^n has an effect on students' understanding of openness in general metric spaces. We also observed that some students used their visualisation of an open ball as a circle or a disc and they appeared to base their reasoning on this when thinking of open balls. This echoes the findings of Przenioslo (2004) and McGowen and Tall (2010) that students' previous experience can influence their thinking in significant ways. It seems that some of the students may have tacit models (Fischbein 1989) of open sets based on their previous experience which influences their reasoning without their explicit knowledge. This influence can be very positive and can help students build intuition and develop understanding however it may also cause difficulties. It is important for lecturers to realise this point when introducing new concepts. Indeed McGowen and Tall (2010) suggested that mathematicians should not only consider the positive influence of students' prior learning on their understanding of a new concept but also should address the possible ways in which it could hinder the learning process. For example lecturers could be careful to introduce students to a variety of examples of metric spaces and to point out the differences between them and the more familiar Euclidean space. From our analysis of the course materials, it seems that the lecturer worked hard on this by using examples from a wide class of metric spaces, but we see that the effects of previous experience still persist.

Wawro et al. (2011) reported on the ways in which students' definition of subspaces in a Linear Algebra course were integrated into their concept image of the concept. They found that students often had both geometric and algebraic aspects in their concept image and they saw that encouraging students to work with the definition was successful in overcoming potential cognitive conflicts or inconsistencies. One possible way forward is to employ the Defining as a Mathematical Activity framework of Zandieh and Rasmussen (2010) which aims to provide a means of creating rich links between concept images and concept definitions. An approach like

this may be fruitful in helping students explore the meaning of, for example, a boundary in metric spaces.

This study has given us some information about the definitions that the students in this study use and the components of their concept images in metric space topology. We believe that this information would be useful to lecturers when planning and delivering courses in this area. We make no claim that our results are generalizable to all topology students; however in the spirit of Fischbein (1989) we hope that the findings could be at least used to alert learners to tacit models or aspects of their concept images that may be limiting their understanding and reasoning. The study presented here was relatively small in scale and it would be interesting if it could be extended to students in other universities to see if additional conceptions of openness appear.

We have further data about students' concept definitions and concept images of the notion of distance in a metric space and we hope to report on this soon.

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Comparaison entre l'évolution historique ayant mené aux développements limités et leur pratique d'enseignement au début de l'université : Entre syntaxe et sémantique

Rahim Kouki¹, Fatma Belhaj Amor² et Yassine Hachaichi³

Université de Tunis El Manar, Institut Préparatoire aux Etudes d'Ingénieurs El Manar, rahim.kouki@gmail.com

Ce texte vise à présenter les résultats d'une étude épistémologique préalable sur l'enseignement et l'apprentissage des développements limités au début de l'université. Deux analyses didactiques y sont menées : l'une, de nature historico-épistémologique porte sur l'évolution mathématique des dimensions sémantique, syntaxique et sémiotique ; et l'autre, institutionnelle, est consacrée à l'exploration des programmes, des manuels et des photocopiés de cours pour en décrypter les visées et les caractéristiques didactiques.

Les principaux résultats dégagés invitent à privilégier didactiquement les approches de modélisation ainsi que la construction via la dialectique outil/objet du concept de développement limité.

Keywords: développements limités, sémantique, syntaxe, sémiotique.

OBJET ET CADRE DE LA RECHERCHE

L'une des principales raisons qui nous a amenés à conduire cette étude réside dans le fait que la plupart des recherches didactiques conduites au niveau de la transition lycée/université ont rarement traité d'une façon explicite l'enseignement et l'apprentissage du concept de développement limité même si elles ont mentionné la pertinence de cet objet comme étant un outil très puissant dans plusieurs domaines d'application comme le calcul de limite et l'étude locale d'une fonction, la modélisation mathématique, l'étude de phénomènes physiques etc. (cf. Ghedamsi, (2008) & Praslou, (2000)).

Notre étude n'est pas totalement indépendante des ces recherches qui ont été généralement centrées sur les difficultés du passage de l'analyse algébriste à l'analyse formelle Artigue (1998), Bloch (2012) et Ghedamsi (2008).

En effet, Bloch (2012) a montré l'existence des ruptures dues à ce passage qu'elle a interprété comme un 'saut conceptuel' et qui est interprété par Haddad (2012) comme un manque de collaboration entre les niveaux d'enseignement secondaire versus supérieur ce qui a mis en évidence une complexité dans l'élaboration des nouveaux concepts au début de l'université.

Les outils d'analyse didactique adoptés tout au long de nos investigations épistémologiques et didactiques font référence d'une part, à la sémantique logique et en particulier aux dimensions sémantique / syntaxique développées dans les travaux

de Durand-Guerrier (1996), Bloch & Gibel (2011) et Kouki & Ghedamsi (2012), aussi bien à la fin du secondaire qu'au début de l'université, et qui ont montré la pertinence de la prise en compte de ces dimensions dans les analyses didactiques.

D'autre part, notre étude s'est appuyée sur les résultats de plusieurs travaux didactiques dans le domaine de l'enseignement de l'analyse réelle comme ceux de Hitt (2004) et Haddad (2012) qui ont aussi montré l'apport que présente l'étude en termes de registres de représentations sémiotiques d'un objet mathématique au sens de Duval (1993).

Nous avons choisi de faire appel au jeu de cadres et en particulier à la dialectique outil / objet développée par Douady (1984, 1986), puisque nous faisons l'hypothèse que la dimension objet du développement limité mérite d'être mieux explicitée au niveau du processus de sa transposition didactique.

Dans un premier temps, nous délimitons le contour de l'objet développement limité à travers une analyse historico-épistémologique par une étude des différents types de techniques qui ont contribué à l'élaboration et par la suite à la genèse de cet objet de savoir.

Dans un deuxième temps, nous présentons les principaux résultats d'une étude didactique du programme officiel, de manuels et de photocopiés de cours en vue de confronter les différentes dimensions de l'objet développement limité comme savoir avec celles qui sont employées lors de son enseignement.

GENESE DES DEVELOPPEMENTS LIMITES

Nous rejoignons le point de vue de Sierpinska (1989) qui pense qu'il faut s'intéresser à l'histoire et l'épistémologie d'un concept mathématique qui nous permet de connaître convenablement son intérêt comme un savoir à enseigner. En effet, elle explique que:

« L'analyse épistémologique sert avant tout à comprendre (nous le soulignons) les concepts mathématiques dont l'enseignement nous intéresse. [...] Comprendre un concept, c'est aussi savoir pourquoi et quand il est devenu important ou fondamental en mathématiques. » (Sierpinska, 1989)

Nous focaliserons notre étude sur les moments importants de l'évolution du concept de développement limité par la présentation, ainsi que par l'interprétation, des différentes méthodes élaborées et développées par les mathématiciens des différentes civilisations du début du XVII^e siècle jusqu'à la fin du XIX^e siècle, qui ont marqué l'évolution des concepts qui sont en étroite liaison avec notre objet d'étude et ce, par la prise en compte des différentes dimensions d'analyse adoptées ci-dessus.

Au XVII^e siècle, les vitesses, les quadratures, les tangentes, maxima et minima sont les aspects des problèmes de différentiations dont l'approche géométrique est

l'approche la plus dominante chez les mathématiciens et les physiciens. En effet, Bourbaki (1984) écrit que:

« Ces quadratures font l'objet de nombreux travaux, de Grégoire de Saint-Vincent, Huygens, Wallis, Gregory; le premier croit effectuer la quadrature du cercle, le dernier croit démontrer la transcendance de e; chez les uns et les autres se développent des procédés d'approximation indéfinie des fonctions circulaires et logarithmiques, les uns de tendance théorique, d'autres orientés vers le calcul numérique, qui vont aboutir bientôt, avec Newton, Mercator, (...), J.Gregory, puis Leibniz, à des méthodes générales de développement en série. » (Bourbaki, 1984, p.226)

Tout au long de cette période l'approche géométrique est l'approche la plus dominante. En effet, Fermat, Descartes, Wallis, Newton et Leibniz se sont appuyés essentiellement sur l'approximation, au cours de leurs études du «problème des tangentes». Certains d'entre eux ont élaboré les deux premiers termes du développement de Taylor. Ils se sont intéressés aussi à la méthode cinématique.

Taton (2004) explique cette méthode est

«...équivalente à notre méthode élémentaire de détermination de la tangente à une courbe définie paramétriquement.» (Taton, 2004)

Dahan et Peiffer (1986) ont détaillé «le problème des tangentes» par la mise en valeur des travaux de Torricelli, Roberval, Descartes, Fermat et Barrow. Ce dernier

«...est le premier à reconnaître clairement que le problème des tangentes est l'inverse du problème des quadratures, et vice versa. Aucun de ces auteurs n'a reconnu la généralité et l'importance du lien qui fait aujourd'hui l'objet du théorème fondamental du calcul différentiel et intégral.» (Dahan et Peiffer, 1986, p.188)

Le statut de l'objet mathématique «tangente» prend son étendue la première fois dans les travaux de Torricelli et Roberval où

«... les problèmes de quadrature ont une origine très ancienne, les tangentes ne seront étudiées qu'au milieu du XVIIe siècle(...) La méthode élaborée par Archimède pour construire la tangente à la spirale se nourrit de considérations cinématiques. Elle a été étendue, au XVIIe siècle, dans les travaux de Torricelli et dans ceux de Roberval sur le mouvement des trajectoires.» (Ibid., p.185)

Au début du XVIIe siècle, l'objet développement limité est connu par la notion de développement en séries infinies dans le calcul infinitésimal. Sa genèse est due à certains problèmes physico- mathématiques.

Dans un premier temps, la résolution du «problème des tangentes» au début du XVIIe siècle, ramène à la genèse du statut objet de «la tangente» par la détermination de son équation par une méthode dynamique développée par Torricelli et Roberval à

partir du concept de vitesse et des mouvements d'une part, et d'une méthode géométrique (ou la détermination des premiers termes de développement de Taylor) par Fermat, Descartes et Barrow, d'autre part.

« *Toricelli et Roberval considèrent les courbes engendrées par la composition de deux mouvements, dont on connaît les vitesses. La vitesse résultante sera la diagonale du parallélogramme des vitesses des deux mouvements qui engendrent la courbe. La droite ayant la direction de la diagonale sera la tangente à la courbe au point P. Cette méthode dynamique permet de déterminer les tangentes à beaucoup de courbes, mais la définition de la tangente qui y opère repose sur des concepts physiques et n'est pas applicable à toutes les courbes.* » (Taton, 2004, p.185-186)

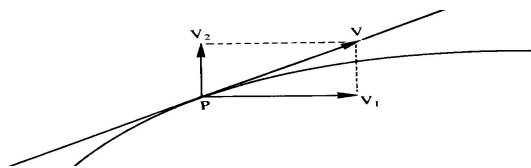


Schéma 1

Ceci nous permet de dire que la démarche développée dans la recherche de l'équation d'une tangente est du type graphique et algébrique articulant les dimensions sémantique et syntaxique dans le cadre de la géométrie.

Le deuxième temps de la genèse de notre objet d'étude s'est réalisé par la détermination de développements en séries infinies (des cas particuliers de sinus, cosinus, tangente et logarithme ($\ln(1+x)$) qui s'est effectuée par des techniques géométriques développées par Mercator et Leibniz en articulant les dimensions sémantique et syntaxique relativement aux registres algébrique, géométrique et graphique.

« *Nicolas Mercator (...) Dans sa Lograthmotechnia, publié à Londres en 1668, il trouve l'aire de l'hyperbole en réduisant d'abord en série géométrique, puis en intégrant terme à terme suivant la méthode de Wallis. Ce dernier trouvait d'ailleurs la même année des résultats analogues qu'il publiait en 1670. La méthode a un succès foudroyant et, en quelques années, James Gregory, Newton, Leibniz s'y distinguent.* » (Taton, 1961, p.239)

Il parvient ainsi à la formule connue par son nom qui est :

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \text{ et } \int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

De son côté, Newton a trouvé les mêmes développements en séries infinies, indépendamment de Leibniz, à partir de l'utilisation de sa méthode cinématique «les vitesses des mouvements». Ainsi, la figure prend une place importante pour la genèse des développements en séries infinies qui sont connus de nos jours par les développements limités usuels avec une absence du reste.

Dans le traitement des problèmes de la mécanique céleste, Newton a utilisé comme technique la méthode de Wallis pour découvrir son développement du binôme qui est connue, de nos jours, par la formule du binôme de Newton.

Il a utilisé son développement du binôme comme une technique préférée pour le calcul de certains cas particuliers des développements en séries en posant :

$$P = c^2 \text{ et } Q = \frac{x^2}{c^2} \text{ où } m = 1 \text{ et } n = 2$$

$$(c^2 + x^2)^{\frac{1}{2}} = c + \frac{1}{2}c \cdot \frac{x^2}{c^2} + \frac{1 \cdot (-1)}{2 \cdot 4} c \left(\frac{x^2}{c^2}\right)^2 + \frac{1 \cdot (-1) \cdot (-3)}{2 \cdot 4 \cdot 6} c \left(\frac{x^2}{c^2}\right)^3 + \dots = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{10c^5} - \dots$$

puis il obtient $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{x^8}{128} + \dots$ pour $c = 1$.

Le troisième moment se réalise par la détermination de développements en séries infinies d'une fonction quelconque, même avant l'apparition de cette dernière notion, qui est connue la première fois sous la formule connue de Gregory-Newton. Tandis que, la formule de Taylor est due à la méthode d'intégration par parties de John Bernoulli puis par la méthode «des différences finies» de Taylor qui nécessite une mobilisation du registre de l'analyse. Dans cette époque, la formule de Taylor devient un outil puissant pour résoudre certains problèmes physico-mathématiques et surtout pour les calculs des approximations. Comme Taylor n'a pas étudié rigoureusement le reste de sa formule, Maclaurin, Euler et Lagrange se sont intéressés à refaire cette formule par leurs méthodes analytique au XVIIIe siècle. Lagrange renforce la confiance en cette formule par son approche numérique en utilisant «la théorie des séries» pour déterminer des valeurs approchées d'un transcendant. Ce qui montre bien la pertinence de l'articulation des différents registres numérique, analytique et algébrique dans le cadre de l'analyse, et confirme la nécessité de la prise en compte des dimensions sémantique et syntaxique dans le traitement des objets mathématiques.

La rigueur en mathématiques s'organise par la genèse du concept de «limite» et c'est d'Alembert qui a donné un nouvel aspect à l'analyse. De ce fait, Cauchy, le père de la rigueur, refait les démonstrations des différentes écritures de la formule de Taylor par une étude de sa convergence. De même, Abel a étudié rigoureusement la formule du binôme de Newton.

Dans une dernière étape de sa formulation historique, le développement limité d'une fonction au voisinage d'un réel issu de la formule de Taylor est devenue un cas particulier du développement asymptotique développé par Poincaré en 1886.

Après ce bref récit des principales étapes historiques, nous résumons ci-dessous les différentes phases avec les caractéristiques des différentes techniques :

Méthode	Période	Type de technique
La détermination de la tangente et les deux premiers termes du développement limité	Début du XVIIe	Technique dynamique (Torricelli et Roberval). Technique analytique et géométrique des sous tangentes de (Fermat, 1637) Technique des extrema des deux premiers termes du développement de Taylor actuel (Fermat, 1637-1638) Technique d'approximation géométrique des courbes algébriques (Descartes, 1638) Technique géométrico-algébrique (Isaac Barrow)
Les cas particuliers de développements en séries infinies	A partir de 1665	Technique géométrique Mercator (1668) Techniques de Newton Techniques de Leibniz
Le développement en séries infinies d'une fonction quelconque et la formule de Taylor	Fin du XVIIe	Technique algébrique de Newton Techniques de détermination du développement de Taylor
L'étude rigoureuse de la formule de Taylor et la formule du binôme de Newton	1823 1826	Techniques analytiques des différentes écritures de la formule de Taylor (Cauchy, 1823) Technique algébrique (Abel, 1826)
Développement asymptotique de Poincaré (1886)	1886	Techniques analytiques de la détermination des développements limités et asymptotiques

Table 1: Les principales phases de la genèse des développements limités

Au cours de la formulation du concept de développement limité d'une fonction d'une variable réelle, les mathématiciens ont exploité différentes techniques dans différents cadres et registres algébrique, analytique, graphique, numérique et géométrique en articulant les dimensions sémantiques et syntaxiques qui joignaient les courbes aux tangentes aux équations en passant du numérique au graphique à l'algébrique et à l'analytique etc.

L'approche géométrique a pris une place importante dans l'étude des premiers termes de la formule de Taylor et dans la détermination des développements en séries infinies (développements limités usuels). Nous pouvons ainsi dire que dans cette période, ces mathématiciens articulaient les deux dimensions sémantique dans l'interprétation graphique et syntaxique dans la manipulation des différentes techniques du calcul formel et algébrique (équations, expressions analytiques).

On ne peut pas parler du concept de développement limité sans faire retour sur les relations de comparaison de fonctions via la notion d'équivalence et de négligeabilité qui est en étroite liaison avec le théorème fondamentale de l'analyse qui suppose que si $f = o(g)$ au voisinage d'un point a on a la même relation entre leurs primitives nulles en a . En effet, Bourbaki confirme que sa genèse est due à

«...P. du Bois-Reymond [94 a et b] qui, le premier, aborda systématiquement les problèmes de comparaison des fonctions au voisinage d'un point, et, dans des travaux très originaux, reconnut le caractère «non archimédien » des échelles de comparaison, en même temps qu'il étudiait de façon générale l'intégration et la dérivation des relations de comparaison, et en tirait une foule de conséquences intéressantes [94 b]. Ses démonstrations manquent toutefois de clarté et de rigueur, et c'est à G. H. Hardy [147] que revient la présentation correcte des résultats de du Bois-Reymond: sa contribution principale a consisté à reconnaître et démontrer l'existence d'un ensemble de « fonctions élémentaires », les fonctions (H), où les opérations usuelles de l'Analyse (notamment la dérivation) sont applicables aux relations de comparaison.» (Bourbaki, 1984, p.254)

L'étude historique nous a permis de repérer les différentes écritures symboliques du type syntaxiques, utilisées dès la genèse du concept de développement limité, dans les divers problèmes développés traitant des situations du domaine de la physique et celui des mathématiques.

Nous allons conduire une analyse des programmes, des manuels, des photocopiés de cours et des résultats d'une enquête auprès des enseignants qui pourrait nous permettre de saisir les conditions de diffusion de ce domaine de savoir de l'analyse réelle à l'échelle de l'université.

ETUDE DES PROGRAMMES, DES MANUELS ET DES POLYCOPIES DE COURS

Nous avons choisi d'étudier la transposition des développements limités dans les programmes et les manuels de la première année de l'enseignement universitaire et plus précisément au niveau des classes préparatoires section mathématiques et physiques. Nous les supposons un lieu idéal de l'enseignement des mathématiques en général et des développements limités en particulier puisqu'elles sont supposées enseigner des mathématiques fondamentales et appliquées afin de donner une

formation théorique et pratique la plus adéquate aux futurs ingénieurs qui auront à modéliser dans des domaines articulant les mathématiques et la réalité.

L'étude des recommandations générales du programme tunisien de l'analyse réelle montre qu'elles articulaient les dimensions sémantique et syntaxique dans différents registres analytique, algébrique, graphique, géométrique et numérique. En revanche dans le contexte lié au concept de développement limité, il y a une absence de l'exploitation de la dimension sémantique au niveau du statut objet des développements limités d'une part, et de la mobilisation du calcul numérique et du graphique, d'autre part.

<p>DL à l'ordre n d'une fonction au voisinage d'un point.</p> <p>Opérations algébriques sur les DL : Somme, produit; développement limité de $u \rightarrow \frac{1}{1-u}$, application au quotient.</p> <p>Existence d'un DL à l'ordre k pour une application de classe C^k: formule de Taylor-Young.</p>	<p>Les étudiants doivent savoir déterminer sur des exemples simples le DL d'une fonction composée. Aucun résultat général sur ce point n'est exigible des étudiants.</p> <p>Les développements asymptotiques sont à étudier sur quelques exemples simples. Toute étude systématique est exclue; en particulier la notion générale d'échelle de comparaison est hors programme.</p> <p>Il convient de donner un exemple où f admet un DL à l'ordre 2 en un point sans être deux fois dérivable en ce point.</p>
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Table 2 : Contenu du programme relatif à l'enseignement des développements limités

En ce qui concerne les quatre manuels [1] et les trois polycopiés de cours [2] analysés, nous remarquons qu'ils suivent une même stratégie «de type magistral» dans l'enseignement des développements limités, où ils commencent par une présentation théorique de cet «objet» purement syntaxique à partir de définitions, propriétés et théorèmes dans un registre analytique. Puis ils passent directement au statut «outil» purement analytique en l'utilisant dans le calcul de limite, l'étude locale d'une fonction (que ce soit au voisinage d'un point ou au voisinage de l'infini), utilisation d'outil dans lequel ils articulent les deux dimensions sémantiques et syntaxiques par la mobilisation des registres analytique, algébrique et géométrique.

En revanche, la place de la figure géométrique pour justifier l'intérêt des développements limités comme modèle d'outil (de technique) d'approximation locale graphique et numérique d'une fonction méritent d'être mieux explicités : elles sont presque absentes dans les différents supports de cours analysés.

Ces résultats d'analyse mettent en question la divergence entre l'évolution historique qui a mis en lumière l'objet développement limité avec les différentes techniques décrites, et celle des techniques recommandées et enseignées actuellement.

En effet, les statuts «outil» et «objet» du développement limité s'articulaient tout au long de sa genèse dans différents registres (analytique, algébrique, numérique, géométrique et graphique). En revanche, l'analyse du processus de transposition de ce savoir montre l'absence d'un travail au niveau des registres graphique et numérique.

CONCLUSION

Les analyses didactiques ont montré que la genèse historique développements limités est étroitement liée aux différents cadres et registres de représentations sémiotiques et qu'il y avait une dialectique dynamique entre les statuts outil et objet du concept de développement limités et des différents concepts qui sont en étroite liaison avec ce concept mathématique. Ceci a permis aux différents mathématiciens d'avancer sur leur recherche et leur mise d'un nouvel objet qui par la suite est devenu un outil fondamental dans le calcul d'approximation locale aussi bien en mathématiques que dans les domaines de la physique, de la mécanique etc. En revanche, du côté institutionnel, la pertinence de la prise en compte de l'articulation entre les points de vue sémantique et syntaxique qui n'est pas prise en compte aussi bien par les programmes que par les manuels de cours. Le travail numérique et le travail graphique, permettant de montrer la pertinence de ce concept surtout pour la formation des futurs ingénieurs méritent d'être mieux explicités.

Une étude articulant les éléments de notre réflexion en didactique, et principalement en matière de raisonnement mathématique, avec les pratiques enseignantes pourrait nous renseigner davantage sur les priorités des apprentissages, les manques ainsi que les opportunités qui pourraient s'offrir dans le cadre d'une réflexion sur l'apprentissage des approximations locales et développements limités à l'université.

NOTES

1. Deux manuels français et deux manuels Tunisiens.
2. Des photocopiés de cours de trois instituts préparatoires aux études d'ingénieurs en Tunisie.

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Revenir au sens de la notion de limite par certaines de ses raisons d'être : un chantier pour le début de l'analyse à l'université

Marc Rogalski

Laboratoire de Didactique André Revuz, Université Paris-Diderot ; Laboratoire Paul Painlevé, Université des Sciences et Technologies de Lille et CNRS ; Institut Mathématique de Jussieu-Paris Rive Gauche, Université Pierre et Marie Curie et CNRS, marc.rogalski@imj-prg.fr

Résumé : Nous proposons une réflexion épistémologique sur la notion de limite articulant trois dimensions : l'approximation d'un nombre, la formalisation, des aspects « méta » du début de l'analyse. A partir de ces choix nous proposons, selon diverses possibilités, de coordonner un ensemble de situations didactiques permettant d'aborder les notions de limite avec les étudiants.

Abstract: We propose an epistemological thought about the concept of limit along three points of view: approximation of numbers, formalisation, « meta » viewpoints at the beginning of analysis. Then we present various possibilities for coordinating didactical situations intended to provoke students' work on the notion of limit.

Keywords: limits, approximations, formalisation, meta, university level.

INTRODUCTION

Les problèmes didactiques de l'analyse enseignée à l'université ont donné lieu à plusieurs travaux, depuis les années 1980. Outre les analyses figurant dans (Cornu 1983) et (Sierpiska 1985) concernant les « obstacles épistémologiques » propres au concept de limite, le travail fondamental de (Robert 1982) dégageait les diverses conceptions de la notion de limite qu'on trouve chez les étudiants, et leurs efficacités différentielles pour la résolution de problèmes dans le domaine.

Depuis, de nombreuses publications se sont centrées sur les difficultés d'accès des étudiants à la logique mobilisée pour formaliser l'idée de limite, voir par exemple (Grenier-Boley et al. 2015, Litim, Zaki et Benbachir 2015, Mamona-Downs 2001). Certaines de ces approches insistent sur la nécessité de cette formalisation pour *prouver* des assertions ressenties par les étudiants comme « évidentes » mais résistantes. On peut citer à ce propos les scénarios proposés dans (Robert 1983), (Robinet 1983), (Lecorre 2015), (Rogalski 2015). Pour les deux premiers, on peut voir dans (Grenier-Bolley et al. 2015) des reprises récentes, ainsi que la présentation dans (Bridoux 2015).

D'autres présentations récentes comme celles de (Bloch 2000) ou (Rogalski 2015) insistent plus sur l'idée d'approcher la limite à un ordre donné. Dans cette optique on peut aussi s'inspirer de travaux de l'approche (Hauchart et Schneider 1996). Notre objectif est de se situer dans une perspective plus vaste s'appuyant sur un retour aux « *raisons d'être* » de la notion de limite et non plus sur un enseignement fondé sur la définition formalisée. A partir de réflexions épistémologiques, nous

essaierons de voir comment il pourrait être possible d'articuler plusieurs situations qui dégageraient une notion de limite dont les composantes logiques et heuristiques deviendraient *nécessaires* et *naturelles* pour les étudiants eux-mêmes. Nous visons donc une reformulation de l'organisation mathématique du champ conceptuel de la notion de limite.

I. UNE RÉFLEXION ÉPISTÉMOLOGIQUE ARTICULÉE SELON TROIS DIMENSIONS

Nous proposons de dégager trois dimensions qui peuvent nourrir chez les étudiants la conceptualisation de la notion de limite : l'approximation, la formalisation, le rôle des « approches méta » autour de la convergence et des débuts de l'analyse.

I.1. L'approximation des nombres définis par des limites

Plusieurs aspects apparaissent ici comme complémentaires.

- La mesure des grandeurs et la construction de *nombres réels par la mesure*.
- La construction de *nombres réels par la résolution d'équations $f(x) = 0$* .
- *L'approximation de nombres λ (a priori non rationnels, tels $\sqrt{2}$, π , $e...$) par des suites de rationnels u_n ; cette approximation appelle tout de suite deux idées fondamentales :*

(a) l'objectif de *se fixer a priori un ordre d'approximation* de λ aussi bon qu'on le veut, ou qu'on en ait besoin (trouver 6 décimales de $\sqrt{2}$, par exemple...) ;

(b) *le choix du terme de la suite (son numéro n) pour que l'approximation à l'ordre souhaité soit réalisée, lorsqu'on a une suite « naturelle » (trouver des entiers n permettant d'approcher $\sqrt{2}$ à 10^{-6} près par le terme u_n de la suite de Héron) .*

Ces trois approches amènent naturellement à poser la question de la construction, de l'existence, de l'unicité et des propriétés de l'ensemble des nombres réels.

I.2. La formalisation du concept de limite

Nous pensons qu'il faut dégager à ce propos deux *raisons d'être* d'une telle formalisation, pour leur nécessité et leur efficacité.

- La réponse aux deux questions concernant l'approximation.

(a) D'une part *on se fixe* un ordre d'approximation arbitraire, noté ε , du nombre λ avec donc comme *but* de réaliser l'inégalité $|u_n - \lambda| < \varepsilon$ (et on peut le faire choisir par les étudiants sur des exemples du I.1.).

(b) De l'autre *le choix de n* : les étudiants peuvent, d'abord sur des exemples, voir que si l'on cherche à approcher un nombre *irrationnel* λ par une suite de *rationnels*, alors si l'on cherche des ordres d'approximation de plus en plus précis, ce ne pourra être que *pour des n de plus en plus grands*. De plus certains exemples à travailler par les étudiants montrent qu'il faut distinguer entre « des n de plus en

plus grands » (mais difficiles sinon impossibles à déterminer : cas des valeurs d'adhérences) et « tous les n à partir d'un certain rang N » (cas de la limite, où on peut souvent dans la pratique déterminer un tel N), avec de plus la difficulté qu'on pourrait dans le premier cas approcher plusieurs nombres différents à la fois.

- L'efficacité de la formalisation classique, issue *nécessairement* du premier point, se voit dès que l'on veut avec cette notion de limite entrer dans des problèmes d'analyse et faire des preuves, à commencer celle de l'unicité de la limite (impossible dans la formulation « des n de plus en plus grands »), ou des premiers résultats de l'analyse.

I.3. Le rôle des aspects « méta » mobilisés dans les débuts de l'analyse

Il s'agit là de voir comment l'usage du concept de limite dans les débuts de l'analyse met en œuvre des démarches nouvelles. Pour ce faire, nous proposons d'avoir en vue (en perspective plus ou moins proche) *un certain degré d'expertise* des étudiants, car l'opérationnalité des savoirs sur la limite est nécessaire à leur compréhension. Cela demande certaines démarches « méta ». Nous entendons par ce terme l'organisation, dans l'enseignement, à la fois d'un discours des enseignants et l'utilisation de situations, aptes à faire comprendre aux étudiants l'origine et les buts de l'introduction de certains concepts, et les modalités de leur fonctionnement, voir (Robert et Robinet 1996). En voici des exemples pour l'usage de la convergence en analyse (voir aussi II.2.).

(a) Un prototype d'approche « statique » d'une recherche de limite : « soit $\varepsilon > 0$ (l'ordre d'approximation choisi), je veux réaliser l'inégalité $|u_n - \lambda| < \varepsilon$, pour cela je cherche une condition suffisante sur $n \dots$ ».

(b) *Le raisonnement à ε près*, fondé sur un critère d'égalité entre nombres par la petitesse arbitraire de leur écart, et avec l'analogue pour les inégalités. La première occurrence en est l'unicité de la limite, mais certains énoncés de base de l'analyse se démontrent ainsi (voir II.3.A). La nécessité de ce fonctionnement provient de ce que les limites sont rarement explicites.

(c) La méthode de *découpage de ε* et le sens du quantificateur \forall dans la limite.

(d) Le caractère opératoire des aspects « méta » de la notion de limite, et l'intérêt de souvent revenir au sens du ε - N sans se limiter à des énoncés de limite automatique.

II. DES PROPOSITIONS DE MISE EN ŒUVRE ARTICULANT DES SITUATIONS POSSIBLES

Nous proposons des exemples de situations pour les trois dimensions que nous avons repérées, en essayant de les coordonner.

II.1. Quelques situations d'approximations de nombres définis comme mesures de grandeurs ou solutions d'équations

Nous reprenons ici quelques exemples classiques. Le but est de faire une sorte de raccourci, au niveau universitaire, de la démarche présentée dans (Hauchart et Schneider 1996).

II.1.A. Approximation du périmètre du cercle unité

On a noté 2π le périmètre (dont l'existence provient d'une intuition géométrique) du cercle unité. Notant a_n la longueur du côté du polygone régulier P_n à 2^n côtés, inscrit dans le cercle unité, avec $a_2 = \sqrt{2}$, on fait calculer aux étudiants l'expression de a_{n+1} en fonction de a_n . Ils en déduisent une relation de récurrence entre les périmètres de P_n et P_{n+1} , qu'on fait exploiter numériquement pour montrer des approximations de 2π . On peut alors énoncer l'irrationalité de π , en renvoyant à plus tard une preuve classique. On peut aussi dans cette partie utiliser les polygones circonscrits, et aussi étudier l'aire du disque unité, voir (Rogalski 2001, ch. 3).

II.1.B. Retour et prolongements sur $\sqrt{2}$

C'est encore l'occasion de jouer sur le changement de cadres, entre géométrie : diagonale du carré, et algèbre/analyse : résolution de l'équation $x^2 = 2$. Cette dernière peut donner plusieurs preuves classiques de l'irrationalité de $\sqrt{2}$ (son existence allant de soi au début), mais la preuve géométrique est aussi instructive. On peut mettre en œuvre un algorithme de dichotomie, ou mieux de « décatomie », pour faire apparaître un grand nombre de décimales successives de $\sqrt{2}$.

Le changement de cadre amène les étudiants à transformer peu à peu un rectangle de base 1 et hauteur 2 en des rectangles successifs de même aire 2 mais se rapprochant de plus en plus d'un carré, de côté nécessairement $\sqrt{2}$. L'idée est de transporter la moitié de l'aire en excédent par rapport au carré de côté 1, de manière à construire un rectangle d'aire 2, de hauteur plus petite que 2 et de base plus grande que 1, donc plus proche d'un carré : un rectangle de hauteur $3/2$ et de base $4/3$. On peut alors dévoluer aux étudiants la tâche de recommencer la même opération sur ce nouveau rectangle, etc, et construire ainsi une suite de rectangles de plus en plus proches d'un carré, et ayant toujours une aire égale à 2. On renvoie au cadre numérique l'approximation obtenue dans le cadre géométrique, sous la forme d'une suite de nombres rationnels : $u_0 = 2$, $u_{n+1} = (1/2)(u_n + 2/u_n)$. C'est la méthode de Héron pour approcher $\sqrt{2}$. On peut alors faire évaluer aux étudiants une « vitesse de convergence quadratique » de la suite de Héron, pour qu'ils voient que « le nombre de décimales exactes double à chaque itération ».

On peut aussi faire montrer par les étudiants que pour tout rationnel positif p/q (approchant $\sqrt{2}$) on a $|p/q - \sqrt{2}| \geq 1/(6q^2)$: une mesure de la difficulté à approcher $\sqrt{2}$ par des rationnels.

Enfin, pour une autre suite intéressante convergeant vers $\sqrt{2}$, voir (Ghedamsi 2008).

II.1.C. *L'aire de la spirale d'Archimède par encadrement*

On se propose de calculer l'aire S intérieure à la courbe d'équation en coordonnées polaires $\rho = c\theta$, où c est une longueur et où $0 \leq \theta \leq 2\pi$. Comme le montre un dessin, on peut découper l'intervalle de variation de l'angle $[0, 2\pi]$ en n parties égales par les points $\theta_k = 2k\pi/n$, $0 \leq k \leq n$. On encadre alors l'aire du morceau de spirale entre θ_k et θ_{k+1} par les aires des deux secteurs de cercles de rayons $c2k\pi/n$ et $c2(k+1)\pi/n$ et de même angle $2\pi/n$. On somme, et on demande aux étudiants de majorer en valeur absolue l'écart entre S et $4c^2\pi^3/3$: ils obtiennent comme majorant $4c^2\pi^3/n$. La discussion sur la valeur possible de S peut faire apparaître l'idée que $4c^2\pi^3/n$ peut être rendu aussi petit qu'on veut par n assez grand, et qu'ainsi on a l'égalité $S = 4c^2\pi^3/3$. On a là un premier raisonnement à ε près : pour tout $\varepsilon > 0$ on a l'inégalité $|S - 4c^2\pi^3/3| < \varepsilon$, car $4c^2\pi^3/n < \varepsilon$ si n est assez grand. Pour d'autres exemples classiques de mesure de grandeurs, voir (Hauchart et Schneider 1996, Rogalski et al. 2001, ch. 7).

II.1.D. *Quelques propriétés d'approximations du nombre e*

On peut partir de la suite $u_n = 1 + 1/1! + 1/2! + 1/3! + \dots + 1/n!$, faire montrer qu'elle est majorée et que la suite $v_n = u_n + 1/(n.n!)$ est décroissante, et les étudiants étant persuadés à ce point de l'existence du nombre e situé entre les deux suites et approché par elles, pointer la non évidence de ce résultat admis, en renvoyant au II.1.E. On peut alors faire montrer que e est irrationnel, ce qui renforce la nécessité d'une preuve d'existence (en fait d'une construction puis d'une preuve).

II.1.E. *Construire les nombres réels*

A ce point, il faut choisir de faire apparaître chez les étudiants *une demande forte de clarification de ce que sont les nombres réels, et de leurs propriétés*. Cela peut se faire de deux façons. D'une part, par un ensemble d'activités de discussion avec les étudiants sur des exercices mettant en évidence pour eux à quel point ils ont des conceptions floues et souvent contradictoires sur la question ; on peut pour cela se reporter aux activités développées avec les étudiants à l'université de Lille dans les années 1984-1996, et qu'on trouvera dans (CI2U 1990, p. 163-170). D'autre part, des activités montrant la nécessité de l'existence de la borne supérieure ou du théorème des intervalles emboîtés pour prouver des résultats (souvent géométriques) intuitifs. On peut proposer au moins trois exemples : le théorème des valeurs intermédiaires par dichotomie (avec l'idée à ce moment « naïve » de continuité) ; l'activité présentée dans (Rogalski et al. 2001, ch.3) pour déterminer la limite de $(\sin x)/x$ quand $x \rightarrow 0$; l'approche du nombre e évoquée ci-dessus.

Nous proposons alors de répondre au besoin de clarification sur les nombres et à la nécessité de leur "complétude" par une ébauche de construction, par les développements décimaux illimités, comme dans (Rogalski et al. 2001, Annexe 3), dans l'esprit de (CI2U 1990, p. 163-170).

II.2. Un exemple emblématique pour la formalisation de la convergence

Nous proposons l'étude de la suite $u_n = 2\cos n$, essentiellement par voie numérique (une étude théorique prouvant le résultat conjecturé par informatique peut être renvoyée à plus tard). Il s'agit de se convaincre numériquement de la propriété fondamentale de cette suite :

Propriété A. *Soit λ un nombre quelconque de $[-2, +2]$. Etant donné un nombre $\varepsilon > 0$ donné, arbitraire (l'ordre d'approximation choisi pour λ), on peut trouver des entiers n aussi grands qu'on veut tels qu'on ait $|u_n - \lambda| < \varepsilon$.*

Les tests numériques¹ montrent deux choses : d'une part, on peut ainsi approcher $\lambda = \sqrt{2}$, mais aussi $\lambda = \sqrt{3}$, par exemple (et bien d'autres !) ; d'autre part, on est incapable de prévoir pour quels entiers n on pourrait ainsi approcher $\sqrt{2}$, et si on augmente n , c'est de $\sqrt{3}$ (ou d'un autre nombre de $[-2, +2]$) qu'on pourrait s'approcher ! La suite $u_n = 2\cos n$, quoique permettant d'approcher $\sqrt{2}$, est ainsi très malcommode pour cet objectif . On veut donc lui préférer une suite telle celle de Héron définie (en 11.1.B .) par $v_0 = 2$, $v_{n+1} = (1/2)(v_n + 2/v_n)$; comme nous l'avons vu, elle vérifie la propriété suivante :

Propriété B. *Etant donné un nombre $\varepsilon > 0$ donné, arbitraire (l'ordre d'approximation choisi de $\sqrt{2}$), on peut trouver N tel que pour tout entier $n \geq N$ on ait bien $|v_n - \sqrt{2}| < \varepsilon$. De plus, seul le nombre $\sqrt{2}$ peut être approché de cette façon par cette suite.*

De plus, les inégalités que nous avons vues sur la suite v_n nous permettent effectivement de trouver un nombre N (dépendant de ε) permettant cette approximation de $\sqrt{2}$ à un ordre fixé $\varepsilon > 0$. L'unicité du nombre que peut ainsi approcher cette suite à des ordres arbitrairement petits est conséquence d'un raisonnement à ε près : si α et β sont deux tels nombres, et $\varepsilon > 0$, on peut approcher le premier à ε près par tous les entiers supérieurs à N_1 , et le deuxième par tous les entiers supérieurs à N_2 , donc si on a $n \geq \max(N_1, N_2)$, on a $|v_n - \alpha| < \varepsilon$ et $|v_n - \beta| < \varepsilon$, donc $|\alpha - \beta| < 2\varepsilon$. Comme l'ordre de proximité de α et β est ainsi arbitrairement petit, ceux-ci sont égaux. Les étudiants peuvent voir que c'est le fait que les inégalités sont vraies pour tous les entiers n assez grands qui assure ainsi l'égalité $\alpha = \beta$, ce qui ne serait plus vrai avec la propriété A.

Les étudiants sont ainsi amenés à avoir de bonnes raisons pour choisir ce que va être la définition formelle finalement retenue pour le concept de limite d'une suite : **choix**

¹ Un programme *maple* très simple permet, pour $x \in [-2, +2]$, $\varepsilon > 0$, N et p dans \mathbb{N} , de déterminer les n dans $[N, N+p]$ tels que $2\cos n$ approche x à ε près. Les étudiants peuvent alors voir, par exemple, que dans la plupart des tranches de 100.000 termes consécutifs on trouve environ 4 termes de la suite qui approchent $\sqrt{2}$ à 10^{-4} près...

arbitraire d'un ordre d'approximation ε , possibilité d'approcher à cet ordre pour tous les entiers n suffisamment grands ($n \geq N$).

Reste alors à faire travailler les étudiants sur plusieurs aspects logiques de la définition retenue, de préférence à l'occasion de plusieurs exercices nécessitant l'un ou l'autre de ces aspects : le rôle des *quantificateurs* ($\varepsilon/2$, notation N_ε , rôle de $\max(N_1, N_2)$) ; le voisinage de l'infini ; le cadre graphique avec *la bande* $\mathbb{N} \times]\lambda - \varepsilon, \lambda + \varepsilon[$; l'*implication* et sa négation (écrire « v_n n'a pas γ pour limite ») ; des inégalités *suffisantes*, en général loin d'être nécessaires ; le fait de parfois exiger d'avoir déjà choisi ε assez petit, ou un premier entier N_1 assez grand ; le rapport avec la formulation : « pour *un nombre fini seulement* de valeurs de n , u_n n'appartient pas à l'intervalle $] \lambda - \varepsilon, \lambda + \varepsilon[$ » ...

A ce point, il est essentiel de relier la propriété de la borne supérieure dans \mathbb{R} et le théorème des intervalles emboîtés à des théorèmes de convergence de suites (suites monotones bornées, suites adjacentes, avec retour sur quelques exemples traités),

II.3. Des situations pour l'usage du « méta » dans les débuts de l'analyse

L'objectif est de développer chez les étudiants un certain nombre de principes d'action leur permettant de résoudre des problèmes non triviaux typiques de ce qu'on fait en analyse, et dont certains serviront de pierre de touche pour illustrer comment divers aspects de la notion de limite sont utilisés.

II.3.A. Limite de fonction, cas de la dérivée, dichotomie

D'abord, il semble utile d'étendre rapidement la notion de limite aux fonctions (et plutôt la limite pointée, à savoir $\lim_{x \rightarrow a, x \neq a} f(x)$, la seule qui corresponde aux problématiques naturelles en analyse, par exemple l'introduction de la dérivée), même si on peut en reporter l'étude détaillée à plus tard. La reprise de l'exemple de la limite de $(\sin x)/x$ en 0 ($x \neq 0$) paraît indiquée à ce point, pour montrer le rôle de la borne supérieure dans la définition nécessaire de la longueur de l'arc de cercle, comme dans (Rogalski et al. 2001, ch. 3).

La notion de dérivée étant rappelée, on peut alors montrer directement *le théorème de base sur la monotonie* : si $f'(x) > 0$ sur un intervalle, alors f y est strictement croissante (on raisonne par contraposée et dichotomie, avec une suite d'intervalles emboîtés $[a_n, b_n]$ convenables se réduisant à un point c réel² où la dérivée serait négative ou nulle). L'idée du raisonnement par l'absurde et de la dichotomie étant proposée par l'enseignant, les étudiants devraient se voir chargés de conclure... Le théorème sur la croissance au sens large est un corollaire facile par *un raisonnement à ε près* : si $f'(x) \geq 0$ sur $]a, b[$, la fonction $g(x) = f(x) + \varepsilon x$ vérifie $g'(x) > 0$...

II.3.B. Des activités pour renforcer la formulation en ε - N et le raisonnement statique

Voici quelques exercices pour mettre en valeur le « méta » du point de vue « statique » sur la convergence des suites.

(a) Le *précepte d'action* : on dit « soit $\varepsilon > 0$, on veut réaliser $|u_n - \lambda| < \varepsilon$, cherchons des inégalités suffisantes $n \geq N \dots$ ». Applications immédiates avec les théorèmes de convergence automatique (somme, produit, etc).

(b) Un travail approfondi sur *le théorème de Césaro* : la méthode du découpage d'une somme de n termes en deux, en choisissant déjà $n > N_1$, de sorte que la somme des N_1 premiers termes soit majorée par $\varepsilon/2$, puis prendre n assez grand pour que le reste de la somme soit majoré à son tour par $\varepsilon/2$.

(c) Utiliser la méthode précédente avec la suite $\sum_{0 \leq k \leq n} 1/k! - (1+1/n)^n$.

(d) *Le théorème d'encadrement à ε près* : « si $v_n \leq u_n \leq w_n$, et si $v_n \rightarrow v$ et $w_n \rightarrow w$, alors pour $\varepsilon > 0$ donné on a $v - \varepsilon < u_n < w + \varepsilon$ pour n assez grand ». C'est une version « désalgorithmisée » du théorème des gendarmes. On peut l'utiliser pour étudier $u_1 = 1$, $u_{n+1} = \sqrt{[n/(n+1)+u_n]}$ (pseudo-récurrance) : si $n \geq N$, on a $1 - \varepsilon \leq n/(n+1) < 1$; on introduit les suites récurrentes encadrantes, définies pour $n \geq N$, par $v_N = u_N = w_N$, $v_{n+1} = \sqrt{[1-\varepsilon+v_n]}$ et $w_{n+1} = \sqrt{[1+w_n]}$; leurs limites sont évidentes, et on applique le théorème d'encadrement à ε près, en jouant sur le quantificateur universel.

II.3.C. Vers l'expertise : enseigner une méthode d'étude de suites numériques

Nous préconisons alors d'augmenter le degré d'expertise des étudiants en enseignant explicitement une méthode d'étude de la convergence éventuelle de suites numériques. Une telle méthode a pour but de rendre effectivement opérationnels les savoirs sur la convergence, en permettant ainsi d'étudier des problèmes plus difficiles, et donc d'améliorer, pour les étudiants, le sens de ces savoirs. Nous renvoyons à (Rogalski et Rogalski 2015) et à (Rogalski 1990).

III. QUELQUES QUESTIONS DIDACTIQUES

Nous avons présenté ci-dessus un schéma d'*organisation mathématique* apte, pensons-nous, à faire comprendre par des étudiants de première année d'université des *raisons d'être* de la notion de convergence, en particulier celles de la formalisation de cette notion et de son sens logique. Nous pensons ainsi que l'aspect formalisateur, unificateur et généralisateur (FUG) entraîné par l'enseignement traditionnel, partant directement de la version formelle peu motivée de la notion de

² L'exemple de la fonction définie sur \mathbb{Q} par $f(x) = x1_{1,\infty} + \sqrt{2}1_{\sqrt{2},\infty}(x) + (x-1)1_{1,\sqrt{2},+\infty}(x)$ montre qu'une fonction sur \mathbb{Q} peut être dérivable de dérivée 1 sans être croissante, car \mathbb{Q} ne possède pas le théorème des valeurs intermédiaires.

limite, pourrait être dans une large mesure évité. Il nous semble que les ingénieries présentées dans (Robert 1983), (Lecorre 2015), (Grenier-Boley et al. 2015), (Rogalski 2015) qui ont pour but de faire « demander par les étudiants » une formalisation de la notion de convergence, et peuvent donc faciliter sa présentation, sont cependant loin d'en épuiser les raisons d'être.

Reste maintenant à présenter des scénarios didactiques précis pour les différentes étapes de l'organisation proposée. Il s'agit là d'un travail futur d'envergure. Néanmoins, certaines de ces étapes ont déjà été travaillées dans des situations d'enseignement. L'un des points principaux que l'on peut dégager de ces expériences (parfois difficiles à évaluer car insérées dans des projets globaux longs d'améliorer une année entière d'enseignement, avec la présence de multiples paramètres dont il est souvent délicat de différencier les influences) est le rôle important à faire jouer aux « ateliers » ou « travail en petits groupes » tel qu'il est décrit dans (Robert et Tenaud 1989) et (CI2U 1990 p. 49-55), ou à des activités de « débat scientifique » tel que proposé dans (Legrand 1993). Pour ce qui est de l'enseignement d'une méthode d'étude de suites numériques, des détails sont présentés dans (Rogalski et Rogalski 2015). Sur les questions de mesure de grandeurs classiques, on peut trouver des indications didactiques dans (Hauchart et Schneider 1996).

Ces scénarios et leurs articulations mériteraient d'être testés sous la forme d'une ingénierie longue couvrant une grande partie du programme d'analyse d'une première année d'université (niveau L1).

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Densité de \mathbf{D} , Complétude de \mathbf{R} et analyse réelle

Première approche

Viviane Durand-Guerrier¹ et Laurent Vivier²

¹Université de Montpellier, Institut Montpelliérain Alexander Grothendieck, France

viviane.durand-guerrier@umontpellier.fr

²Université Paris Diderot, Laboratoire de Didactique André Revuz, France,

laurent.vivier@univ-paris-diderot.fr

Nous présentons ici les premiers résultats d'un travail de recherche en cours visant à étudier la question des interrelations entre la conceptualisation des nombres réels et l'appropriation des principaux concepts de l'analyse en début d'université. Les données analysées proviennent d'un questionnaire préliminaire visant à identifier des connaissances clés relatives à la topologie de l'ordre de \mathbf{R} d'une part, aux limites de la visualisation en analyse d'autre part. Nous présentons brièvement les analyses a priori et a posteriori des quatre questions relatives à ces deux thèmes.

Mots clés : complétude de \mathbf{R} , analyse réelle, topologie de l'ordre, densité versus continuité, conceptualisation.

INTRODUCTION

Dans cette proposition de communication, nous présentons la première étape d'une recherche en cours dont le principal objectif est d'étudier finement les relations entre la conceptualisation du continu (c'est-à-dire l'ensemble \mathbf{R} des nombres réels comme complété de l'ensemble \mathbf{D} des nombres décimaux ou de l'ensemble \mathbf{Q} des nombres rationnels) et l'appropriation par les étudiants des principaux concepts d'analyse réelle enseignés dans les programmes de licence. Nous mettons au cœur de nos analyses les aspects mathématiques, épistémologiques, cognitifs et didactiques et leurs interrelations.

Nous avons conduit une étude épistémologique circonscrite de l'élaboration du concept de continu (au sens de la complétude de l'ensemble des nombres réels) à partir en particulier des travaux de Bolzano, Dedekind, Cantor et Cauchy qui nous amène à faire l'hypothèse déjà émise par Bloch et al. (2007, 117) qu'une appropriation adéquate des principaux concepts de l'analyse est liée de manière intrinsèque à une conceptualisation adéquate du concept de nombre réel, en particulier en ce qui concerne la complétude de l'ensemble des nombres réels (Durand-Guerrier, 2012). Or, l'enseignement secondaire français, depuis de nombreuses années, ne propose plus de travail substantiel sur la construction des nombres, ceci s'étant encore accentué depuis 2009 avec la disparition dans les programmes de toute référence explicite aux ensembles de nombres. De ce fait, les connaissances sur les nombres des étudiants arrivant dans l'enseignement supérieur sont extrêmement fragiles (Vivier, 2015, Vergnac et Durand-Guerrier, 2014). Les

études empiriques montrent en particulier une identification de la nature des nombres à leur écriture, une méconnaissance des critères permettant de catégoriser les nombres à partir de leurs écritures décimales et une confusion entre le concept de densité et le concept de continuité pour les ensembles de nombres (Vergnac & Durand-Guerrier, 2014). Bien que de nombreuses recherches aient été conduites sur les difficultés rencontrées par les étudiants dans l'apprentissage des nombres réels d'une part (par exemple : Bronner 1997, Bergé 2010), dans l'apprentissage des premiers concepts d'analyse (par exemple : Cornu 1991, Ghedamsi 2008, Gonzales Martin et al, 2014, Vandebrouck, 2011) ou de topologie (Bridoux, 2011), le rôle spécifique joué par les connaissances sur les nombres dans l'apprentissage des principaux concepts et théorèmes de l'analyse¹ a été peu étudié à notre connaissance.

La première étape de notre projet vise à aller au delà de ce constat par l'identification de connaissances dont nous faisons l'hypothèse qu'elles jouent un rôle clé en analyse réelle. Pour répondre à cette question de recherche, nous avons proposé à des étudiants de début d'université un questionnaire dont nous présentons ici les questions relatives à la topologie de l'ordre, à la densité de \mathbf{D} et à la complétude de \mathbf{R} d'une part, aux limites de la visualisation en Analyse d'autre part. Dans cette première étude, nous avons mis l'accent sur la propriété suivante : « l'ensemble \mathbf{D} des nombres décimaux est un ensemble dense dans lui même (entre deux décimaux différents, il y a toujours un décimal différent des deux précédents) qui n'est pas complet vis à vis de l'ordre naturel sur \mathbf{D} (ordre lexicographique). Ce choix est motivé par l'hypothèse selon laquelle la distinction entre la propriété de densité et de continuité est peu visible pour les élèves et les étudiants arrivant à l'université, d'une part parce que dans le curriculum français secondaire, elle ne fait pas l'objet d'un enseignement explicite ; d'autre part parce que la visualisation de cette propriété est problématique, les représentations graphiques non discrètes renvoyant directement à notre intuition du continu porté par le tracé d'une ligne sur une feuille sans lever le crayon (Longo, 2002). Pontille et al (1996) rend compte de travaux d'élèves résolvant un problème de point fixe dans l'ensemble \mathbf{D} des nombres décimaux illustrant cette difficulté : ceux-ci sont en effet confrontés à la question de la représentation graphique de la courbe représentative d'une fonction définie sur l'intervalle $[0 ; 1] \cap \mathbf{D}$. En ce qui concerne le point de vue ensembliste sur les nombres décimaux, ceci se traduit par le saut conceptuel du passage des ensembles discrets de nombres décimaux $D_n = \{N/10^n ; N \in \mathbf{N}\}$, dont le nombre de chiffres non nuls après la virgule est borné par un entier donné n , à l'ensemble \mathbf{D} des nombres décimaux, qui n'est ni discret (i.e. pas de successeur), ni continu et qui met en jeu le passage du fini à l'infini potentiel. Plusieurs des questions que nous présentons font

¹ Citons par exemple les concepts de limite, essentiel en analyse, de continuité (lié notamment au théorème des valeurs intermédiaires, aux théorèmes de point fixe,...), de variation et de dérivation (les problèmes d'optimisation), l'accumulation et l'intégration (notamment pour les primitives et le calcul de grandeurs).

$\{2,5\}$ ou $\{d\}$ ou encore $[2,2]$ et $[2,5 ; 2,5]$. Mais il est possible qu'un singleton ne soit pas perçu comme un intervalle d'où une réponse négative à c , ce qui montre une possible compréhension correcte de la densité de **D**. Néanmoins, cette nécessité d'avoir deux extrémités différentes peut entraîner des réponses erronées telles que $]0,1 ; 0,3[$ (ou du même type de réponses dans D_n) ou encore $[0,999... ; 1]$ ou $[0,4999... ; 0,5]$. Cette dernière réponse est correcte puisqu'il s'agit d'un singleton, mais elle peut aussi être proposée en pensant qu'un nombre ayant une infinité de chiffres n'est pas un décimal (une variante possible avec un intervalle semi-ouvert).

Au d , la réponse correcte est « non » ou « impossible » avec un argument de densité : entre deux décimaux il y a toujours un décimal (plusieurs, une infinité) ; cet argument peut être plus ou moins explicite. Mais des réponses peuvent apparaître en cohérence avec la question c : $[0,2 ; 0,3]$, $[0,999... ; 1]$ ou $[0,4999... ; 0,5]$ (il apparaît ici deux représentations d'un même décimal ce qui devrait permettre de lever l'ambiguïté du c). Un dessin peut être fait pour illustrer ou justifier.

La densité dans le registre des représentations graphiques

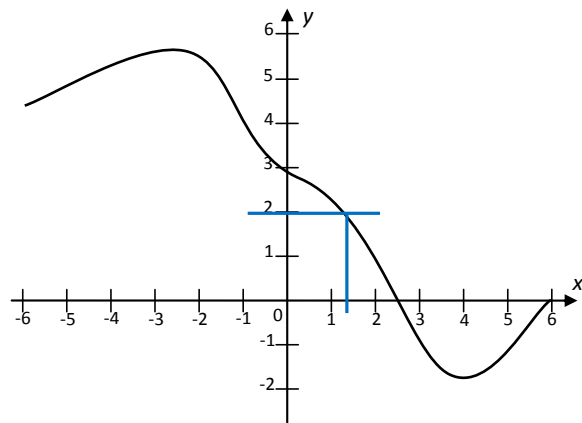
Q5 – On donne ci-contre la représentation graphique d'une fonction f définie et continue sur $[-6 ; 6]$.

L'équation $f(x)=2$ admet une solution :

- a. dans **N** b. dans **D**
 c. dans **Q** d. dans **R**

Pour chacune des questions il fallait répondre en cochant une case :

- Vrai Faux On Ne Peut Pas Savoir



Dans cette question, nous cherchons à tester la compréhension de la densité en acte dans le registre des représentations graphiques qui, comme nous l'avons mentionné plus haut, ne permet pas de distinguer visuellement une courbe définie sur un ensemble continu (comme **R**) d'une courbe définie sur un ensemble dense.

Graphiquement, l'équation $f(x)=2$ admet une unique solution, dont on peut déterminer une valeur approchée. La tâche est classique, travaillée depuis la classe de troisième (grade 9) en France ; nous faisons l'hypothèse qu'elle ne pose pas de problème et donnera lieu à des traces graphiques. La question qui nous intéresse ici concerne l'existence de solutions selon l'ensemble dans lequel on résout l'équation.

a. Deux réponses correctes possibles : Faux, car on trouve environ 1,2 sur le graphique (distance à **N** marquée) et ONPPS car le graphique n'est pas assez précis (il ne démontre rien). La réponse Vrai n'est pas attendue

b. La réponse correcte ONPPS peut être justifiée par la valeur approchée donnée par le graphique ou de manière moins précise comme « on ne connaît pas le nombre » ou « il faudrait connaître $f(x)$ ». La réponse Vrai est possible par une confusion nombre réel/nombre décimal ou par confusion entre un nombre et ses valeurs approchées (« la solution est 1,2 »). A l'opposé, la réponse Faux peut être proposée avec comme justification « la solution est un nombre réel ».

c. On s'attend aux mêmes types de réponses malgré des éventuelles différences de traitement entre décimal (écriture décimale) et fraction (écriture fractionnaire).

d. Seule la réponse Vrai est attendue ; on relèvera si la justification de la réponse fait appel ou non au Théorème des valeurs intermédiaires (TVI).

Réponses des étudiants

Pour les items *a* et *b* de Q1, il n'y a, effectivement, aucun problème (sauf pour un étudiant). On relève, surtout pour *b* et *d*, des problèmes de compréhension ou d'écriture pour les intervalles comme prévu. Pour la réussite, on compte 20 étudiants pour l'item *c* et 16 pour l'item *d* (en comptant large), soit moins de la moitié des étudiants, alors que, respectivement, 5 et 7 étudiants ne donnent pas de réponse à ces items. Il est à noter que 10 étudiants avancent des arguments liés à la densité comme « infinité de chiffres/décimales » ou « \mathbf{D} est dense dans \mathbf{R} » ou « \mathbf{D} est dense ».

Pour les réponses incorrectes liées à la densité, ce sont 5 étudiants en *c* et 9 en *d*, soit un quart des étudiants ce qui en fait la cause première d'erreur. On relève 3 étudiants qui répondent dans \mathbf{D}_0 ou dans \mathbf{D}_1 à *c* et *d*, de manière cohérente. On trouve aussi des réponses avec des fractions (1 en *c* et 4 en *d*) ainsi que des réponses avec des écritures décimales illimitées (1 en *c* et 2 en *d*).

Pour Q5, la réussite est, comme prévu, plus importante : 26 étudiants répondent correctement, soit plus des deux tiers, et tous les étudiants donnent une réponse. L'argument du TVI est avancé par 11 étudiants (avec, toutefois, souvent des oublis dans les hypothèses et sans mentionner explicitement le théorème) dont 2 étudiants avancent également l'imprécision du graphique (les graduations).

Mais il y a également 6 étudiants qui répondent que la solution est dans \mathbf{D} ainsi que 2 qui affirment qu'elle est dans \mathbf{D}_1 (la solution est égale à 1,2 ou 1,3). Il est à noter que sur ces 8 étudiants (presque un quart), il y en a 5 qui ont donné des réponses erronées à Q1 en lien avec la densité.

Finalement, ce sont un tiers des étudiants qui présentent des difficultés à appréhender la densité de \mathbf{D} dans au moins une des questions posées, que ce soit pour les intervalles – essentiels en analyse – ou pour la résolution graphique d'une équation.

DÉPASSER LA VISUALISATION PREMIÈRE EN ANALYSE

Une des spécificités de la visualisation en analyse est que ce processus doit être contrôlé par des connaissances théoriques. Trois questions du questionnaire étaient

spécialement destinées à tester ce point. Nous en présentons deux ici qui s'appuient sur les signes donnés par un instrument : les questions demandent de contrôler la visualisation première, notamment sur l'égalité de deux nombres, les instruments ne donnant en général qu'une valeur approchée, laissant la valeur exacte à l'interprétation du sujet. Les principales connaissances en jeu sont les approximations décimales d'un nombre réel et les conditions d'application du TVI selon les propriétés de la fonction.

Analyse a priori rapide des questions

Approximation décimale versus valeur exacte d'un nombre réel

Q2 – On a calculé sur un tableur deux nombres réels a et b . Le tableur a une précision de 12 décimales et il donne le même résultat pour a et pour b , à savoir : 2,718281828459.

Que peut-on en déduire sur les nombres a et b ? Entourez la ou les bonnes réponses (les sept réponses proposées étaient données à la suite).

Pour chaque question, nous indiquons les justifications envisagées a priori.

$a=b$: deux nombres égaux jusque la douzième décimale sont égaux ; s'ils avaient été différents, le tableur aurait donné des résultats différents ; égaux car *presque égaux* (on néglige les autres décimales) ; pas d'autres décimales que ce que donne la machine ?

$a \neq b$: deux lettres différentes, donc ils sont différents malgré les 12 premières décimales (cf. deux points A et B en géométrie qui, souvent, suppose implicitement que A et B sont distincts, par contrat implicite) – peu attendu ; il peut aussi y avoir des arguments avec des décimales.

$a > b$ et $a < b$: non attendus car privilégier l'un des nombres ne semble pas naturel.

$|a-b| < 10^{-12}$: réponse correcte, interprétation des décimales en fonction de leurs valeurs et estimation/majoration du reste des décimales (inconnues)

$|a-b| = 10^{-12}$: les deux nombres diffèrent exactement de la valeur de la douzième décimale, pas d'estimation, d'encadrement ou de majoration du reste

Aucune réponse ne convient : pour que les questions soient complètes, que l'on puisse toujours répondre – éventuellement un étudiant qui pense à l'association de deux réponses, notamment pour $|a-b| \leq 10^{-12}$.

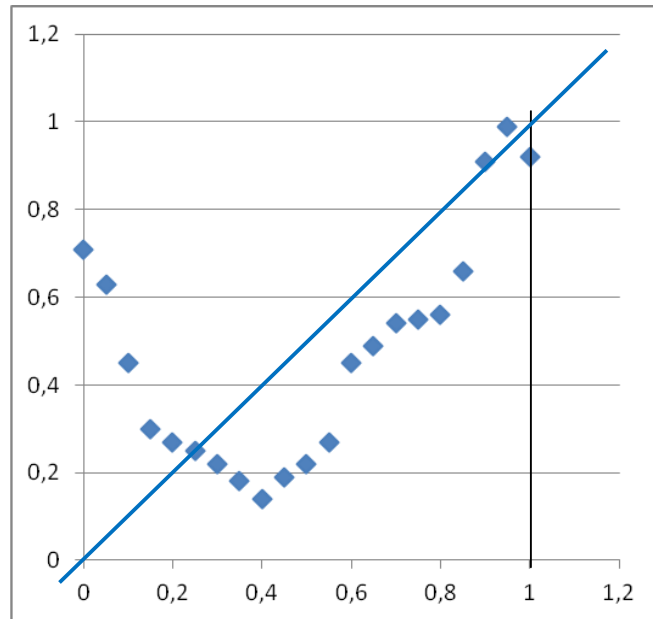
Un dessin possible pour la réponse correcte (aussi pour $|a-b| = 10^{-12}$?)

Utilisation d'une approximation d'une fonction pour étudier ses propriétés

Q4 – On a entré dans un tableur une fonction f qui est définie sur $[0;1]$ à valeurs dans $[0;1]$. On obtient la table de valeurs ci-dessous avec un pas de 0,05. Les données ont été placées automatiquement par le tableur dans le graphique ci-dessous.

On rappelle qu'un point fixe d'une fonction f est une solution de l'équation $f(x)=x$.

x	$f(x)$
0,00	0,71
0,05	0,63
0,10	0,45
0,15	0,30
0,20	0,27
0,25	0,25
0,30	0,22
0,35	0,18
0,40	0,14
0,45	0,19
0,50	0,22
0,55	0,27
0,60	0,45
0,65	0,49
0,70	0,54
0,75	0,55
0,80	0,56
0,85	0,66
0,90	0,91
0,95	0,99
1,00	0,92



a- On suppose que f est strictement décroissante sur $[0 ; \sqrt{3}/4]$, strictement croissante sur $[\sqrt{3}/4 ; 19/20]$ et strictement décroissante sur $[19/20 ; 1]$.

b- On suppose que f est continue sur $[0 ; 1]$.

En a et b, il était demandé de cocher une case pour ces trois affirmations :

- (1) f n'admet aucun point fixe : Vrai Faux On Ne Peut Pas Savoir
- (2) f admet exactement un point fixe : Vrai Faux On Ne Peut Pas Savoir
- (3) f admet plusieurs points fixes : Vrai Faux On Ne Peut Pas Savoir

Le graphique est donné (sans les droites $y=x$ ni $x=1$), pour faire gagner du temps à ceux qui voudraient le faire (bien sûr, cela anticipe l'idée de faire un graphique) et éviter les erreurs. D'autre part, un tableur donne facilement un tel graphique.

Les nombres sont donnés avec un format de deux chiffres après la virgule pour faire penser aux approximations faites par le tableur qui ne donne que deux décimales (et avoir un format de nombre différent des autres questions). Au cas a, la valeur $\sqrt{3}/4=0,433\dots$ est choisie pour être vraisemblable et *sérieuse* (avec la racine carrée), tout comme pour $19/20=0,95$, mais le fait de mettre une fraction renvoie moins facilement à une valeur approchée (ce qui veut être testé dans le tableau).

On peut penser que c'est le graphique qui servira d'appui pour les réponses, plus que la table de valeurs, sauf peut-être pour le point fixe approché $(0,25 ; 0,25)$. Néanmoins, la tâche est moins classique que celle de la question 5 et les traces

graphiques sont moins attendues : les points sont-ils reliés ? si oui par une courbe continue ou non ? la droite $y=x$ est-elle tracée ? le carré $[0 ; 1]^2$ est-il délimité ?

Pour le cas a , la réponse correcte est « On Ne Peut Pas Savoir » pour chacune des trois affirmations. Pour le cas b , les réponses correctes sont Faux – Faux – Vrai.

Nous faisons l'hypothèse que la réponse « Faux » à l'affirmation (1), pour les deux cas a et b , sera majoritaire et ceci pour deux raisons : d'une part pour des raisons de cohérence (avec une réponse Vrai en 2) ou 3)), d'autre part parce que la valeur 0,25, du tableau de valeurs, peut sembler correspondre à un point fixe.

En ce qui concerne l'affirmation (3), nous faisons l'hypothèse que certains étudiants utiliseront le TVI pour conclure y compris dans le cas a) ce qui conduit à la réponse Vrai qui n'est pas correcte dans ce cas ; l'utilisation du TVI conduit à la réponse correcte en b). Pour cela, les étudiants peuvent soit s'appuyer sur le graphique, soit localiser trois points fixes avec la table de valeurs ($[0,2 ; 0,3]$; $[0,85 ; 0,9]$; $[0,95 ; 1]$). Nous regarderons si le théorème est mentionné ou bien s'il utilisé *en acte*. Nous regarderons également si la précision « exactement 3 points fixes » apparaît. Nous interprétons la réponse similaire Vrai pour les cas a et b comme une *continuité naturelle*, ce qui ne serait pas surprenant vu que la grande majorité des fonctions rencontrées par les étudiants sont continues.

Réponses des étudiants

Tous les étudiants répondent à Q2, mais seulement 22 répondent correctement ce qui montre une difficulté à percevoir la différence entre valeurs exactes, valeurs approchées et degré d'approximation. Il ne semble pas y avoir de lien avec Q1 qui est centrée également sur les nombres : 11 étudiants répondent correctement à Q1 et à Q2, ce qui répartit les étudiants en 3 sous-groupes équivalents. L'égalité est donnée, sans autre réponse, par 4 étudiants qui présentent une réponse issue d'une visualisation directe. Mais on relève également 3 étudiants qui donnent des réponses contradictoires avec plusieurs cases cochées et surtout 8 étudiants qui affirment qu'« Aucune réponse convient », peut-être parce qu'ils n'arrivent pas à identifier une autre réponse – dont un étudiant qui affirme que la 13^{ème} décimale peut être différente, compté dans les réponses correctes.

Pour la question 4, seuls 5 étudiants répondent correctement aux deux items, ce qui apparaît extrêmement faible (8 répondent correctement en a et 15 en b). La variété des réponses est grande et il est difficile de toutes les indiquer, nous ne donnons que les erreurs les plus courantes. La continuité apparaît comme naturelle pour 6 étudiants, sans prise en compte des hypothèses ; nous faisons l'hypothèse que ceci s'appuie sur la visualisation d'une courbe *nécessairement* continue passant par les points du graphique, en lien avec des pratiques scolaires. La table numérique de valeurs, est une source de réponse, par une visualisation directe, pour beaucoup d'étudiants malgré le fait qu'ils s'agissent de valeurs approchées : à l'affirmation 2, un seul point fixe, 12 étudiants répondent Vrai en a et 6 répondent Vrai en b , dont 4

répondent Vrai au deux items. On peut penser que pour ces étudiants, la réponse s'appuie sur l'identification du *point fixe* (0,25 ; 0,25) dans la table de valeurs.

CONCLUSION

Dans cette communication, nous nous proposons de rendre compte des tous premiers éléments d'un projet de recherche visant à étudier les relations entre conceptualisation des nombres réels et apprentissage de principaux concepts de l'analyse en début d'université. Nous avons centré nos premières analyses sur la prise en compte par les étudiants de la densité de \mathbf{D} dans les réponses à quatre questions soumis à 35 étudiants de première année d'université scientifique. Les premiers résultats montrent que pour près d'un tiers des étudiants, cette propriété de \mathbf{D} n'est pas perçue, soit que les étudiants identifient l'ensemble \mathbf{D} à l'un de ses sous-ensembles D_n , soit qu'ils ne considèrent pas de propriété intermédiaire entre le discret et le continu. Ceci est problématique pour les activités ordinaires en analyse en début d'université. En outre, nous faisons l'hypothèse que ceci est un obstacle à une conceptualisation adéquate des principaux concepts de l'analyse. Cette hypothèse s'appuie sur les résultats de l'analyse épistémologique montrant que l'élaboration de l'ensemble des nombres réels comme complété pour la topologie de l'ordre de \mathbf{Q} ou de \mathbf{D} (selon la construction théorique choisi) est précisément liée au fait que la densité d'un ensemble dans lui-même ne garantit pas la complétude qui est nécessaire pour pouvoir démontrer le théorème des valeurs intermédiaires. Nous nous proposons d'approfondir cette question en complétant les questionnaires par des entretiens et des observations in situ de situations posant explicitement cette question comme la situation du *point fixe* de Pontille et al. (1996). Au cœur de ce problème se trouve la question de l'existence ou non d'un point fixe pour une fonction croissante d'un ensemble ordonné borné dans lui-même. Cette existence est garantie (Théorème de Katsner-Tarski) dès lors que l'ensemble est un treillis complet, ce qui est le cas pour une partie finie de l'ensemble \mathbf{N} des entiers naturels et pour un intervalle fermé de l'ensemble \mathbf{R} des nombres réels. Dans l'ensemble \mathbf{D} des nombres décimaux et pour l'ensemble \mathbf{Q} des nombres rationnels, le théorème n'est pas vérifié : on peut trouver des contre-exemples qui mettent en évidence le fait qu'un ensemble dense en-lui même n'est pas nécessairement continu. Nous faisons l'hypothèse que ce problème est un bon candidat pour construire une situation didactique. Ceci est argumenté dans un texte soumis pour un numéro spécial de la revue IJRUME² et la prochaine étape de notre projet. La question de la visualisation en analyse sera également approfondie notamment avec le modèle des Espaces de Travail Mathématique de Kuzniak.

² Durand-Guerrier V. Conceptualization of the continuum: an educational challenge for undergraduate students, soumis pour le numéro spécial de la revue IJRUME faisant suite au séminaire de Oberwolfach (décembre 2014).

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Students' visualization of functions from secondary to tertiary level

Fabrice Vandebrouck, Séverine Leidwanger

Université Paris-Diderot, Laboratoire de Didactique André Revuz

IMJ-PRG, Université Paris Diderot, Université Sorbonne Paris Cité

In this paper, we report a test which was proposed to students entering University (more than 500 students). The test was built to help teachers identify students' strength and weakness in some important mathematics topics, especially limits of functions. The test's analysis shows some specific abilities of students which surely can be used to introduce new knowledge involving the local perspective and formalism at the beginning of the university.

Key words: mathematics, functions, university students, activity, visualization, local perspective

In this article we want to investigate one problem that arises in the transition between secondary school and university concerning the concept of functions. We make an attempt to introduce specific students' activities with functions (called DWP), similar to those introduced by Duval (1999) about figure in the geometrical frame. The results may suggest that university teachers can built on these specific activities, to introduce some better students' understanding, involving the local perspective on functions, with its formalism and its relations to the other ones (global, point-wise).

NON-ICONIC VISUALIZATION AND DECONSTRUCTIONS WITH PERSPECTIVES OF FORMULAS

Rogalski (2008) and Vandebrouck (2011) have considered the notion of perspectives. In fact, different perspectives can be adopted concerning functions: a point-wise perspective – associated to function values in some particular points - a global perspective – ability to appreciate some global properties of functions such as variations, parity... - and a local perspective – ability to appreciate some local properties such as behavior near a point or near infinity.

The present paper aims to understand how students deal with perspectives on functions which are only given with their algebraic formulas. We examine the way perspectives can be useful when students have to compute some limits of functions given by their algebraic representations (formulas). The current practices of teaching in secondary schools in France don't give a qualitative vision of functions and reinforce tasks belonging to the algebraic frame (computations of limits with algebraic rules which are more or less demonstrated, of derivative...). These practices seem to erase the perspectives which can be adopted on these objects.

For our focus, we introduce the notion of *deconstruction with perspective* (DWP) of a formula in a similar way Duval (1999) has introduced the dimensional deconstruction of a figure in the geometrical setting. The dimensional deconstruction is a specific activity with geometrical figure linked to the ability to identify objects of dimensions 0, 1 or 2 in a whole complex figure (in 2 or 3 dimensions). In a similar way, the DWP is an activity which is specific of the analysis setting as we will explain below. This notion has been already introduced in Kuzniak and al (2015). It can also be applied for graphs of functions as Vivier does about tangents of curves, however without using this new terminology of deconstruction (Montoya Delgadillo & Vivier, 2015).

As the dimensional deconstruction does in the geometrical frame, the DWP supposes first of all a non-iconic visualization of the formula. We use the concept of visualization also introduced by Duval (1999) in the geometrical setting, but as Duval says, visualization can be produced in any register of representation. Duval distinguishes two type of visualizations: the iconic and the non-iconic. The latter involves some highlights, a global apprehension of the representation, may be a kind of classification, and some embarked properties.

We shall now give two examples of DWP, one about global DWP of algebraic formula and one about local DWP.

As it was focused in Vandebrouck's previous papers (Vandebrouck, 2011), only for experts formula can represent a function from a global perspective. For instance, the formula $x^2 + \sqrt{x} + \exp(x)$ represents a growing function on \mathbb{R}^+ . The non-iconic visualization of the formula by an expert allows him to identify three terms x^2 , \sqrt{x} and $\exp(x)$, each term representing a growing function on \mathbb{R}^+ . This decomposition of the formula $x^2 + \sqrt{x} + \exp(x)$ into three growing functions can be named a *decomposition with global perspective*. For students, interpreting an algebraic formula as a function from a global perspective seems only natural for elementary functions \exp , \ln , x^2 , \sqrt{x} , whose global properties – variations for instance - are well known. For more complex algebraic formulas, the most natural perspective is the point-wise one: non experts are only able to have an iconic visualization, using the formula as a dark box, associating $f(x)$ to x .

We notice that the *decomposition with global perspective* of a formula is more complex than identifying sums, products, quotients, several factors and so on, which is only a usual algebraic decomposition. For instance, the algebraic deconstruction of the formula $x^2 + \sqrt{x} + \exp(x)$ is the *sum* $\{x^2 / \sqrt{x} / \exp(x)\}$. It is well done when we want to derivate or integrate the formula. We postulate that the algebraic decomposition doesn't suppose a non-iconic visualization, that is to say the non-iconic visualization is more complex. Many students are not able to visualize the function in a non-iconic way. To show this function is growing, they only identify the three algebraic terms of

the sum – algebraic decomposition, iconic - and then compute the derivative as a sum, which is positive on \mathbb{R}^+ .

The second example (about local DWP) is about computing a limit of a function. For instance, the formula $(x^2+3x+1)/\ln(x)$ represents a function on \mathbb{R}^+ . Let's compute the limit of the function at $+\infty$. As experts, we adopt a non-iconic visualization of the formula and we are able to operate a *decomposition with local perspective*. Near $+\infty$, the function is equivalent to $x^2/\ln(x)$ – we must forget some negligible terms, a difficult activity. Moreover $\ln(x)$ represents a negligible function compared with x^2 . So the limit of the function represented by $(x^2+3x+1)/\ln(x)$ is $+\infty$. Near 0^+ , we do the same kind of local DWP. Of course, such decompositions have their limitations for students; as experts, we have some expert knowledge about sum, product and quotient of equivalent functions.

The issue in this paper concerns the students' ability to enter in such DWP after their algebraic practices at secondary school. That is to say we wonder in which way secondary teaching still allow students to develop such reasoning with DWP. If students are able to operate *decomposition with global and local perspective*, we suppose that they are more fluent with function in their formalism (local) at the beginning of the university: $f \sim g, f = o(g)$ and so on.

METHODOLOGY

In order to answer to this question, we had the opportunity to analyze answers of a diagnostic within the EVALAC¹ project at University Paris Diderot. All students entering university in scientific teaching were asked to answer an online questionnaire including 5 limits randomly selected among the 21 limits given in annex 1. 513 students answered the questionnaire, coming directly from secondary school. The limits were chosen among an IREM group by teachers from secondary schools and universities. Several issues about limit of functions were chosen such as algebraic classical rules were no sufficient to answer them.

Moreover, 6 students were interviewed while answering the questionnaire. All of them were students from Terminale S class in Lycees (grade 12, last course of the secondary school before a scientific baccalaureate). The focus of the questionnaire dealt with the cognitive way students answered questions about the limits. The question they had and the answers they gave are given in annex 2.

Statistics are given in annex 1. Not all students were from scientific baccalaureate classes so we can only take into account the highest or weakest percentages.

In order to analyze a priori the limits of functions, we draw on the task-analyzing tools (grills of complexity) proposed by Robert (1998): Do the limit calculations call

¹ <http://www.ldar.univ-paris-diderot.fr/EVALAC>

only for immediate applications of algebraic rules (direct substitution and algebra of limits), or, on the contrary, do they call for adaptations (especially for indeterminate form), sub-tasks (apply an algebraic rule to clear the indeterminate form for instance) and/or necessity for students to recognize other knowledge to be used (using DWP for instance)?

RESULTS AND EXAMPLES OF STUDENT RESPONSES

The first general observation is the rather poor rate of correct answers. This can be partly explained by the fact that not all students passed a science baccalaureate, even though they were highly predominant. This observation restricts the interpretations that can be made. Indeed some students who passed the test were not skilled enough with the theoretical knowledge about limits - such as the definition of a limit or the algebra' rules about limits.

The second general observation is the fact that the task-analyzing tool can't explain a lot of the results. Indeed, most of them are not significantly better when tasks are easier according to our a priori analysis. This observation is reinforced when we have identified that some algebraic rules which could be applied directly. The best success rates are on $(x^2-1)/(x+3)$ at $+\infty$ (indeterminate form, 81%) and on $1/(x+1)$ at $+\infty$ (that is not an indeterminate form, 85%). The limit of $(x-1)/(x+1)$ at $+\infty$ (indeterminate form) collects only 52% of correct answers even though students can directly apply a rule.

A last remark is the well-known difficulty linked to the idea that x is always positive. For instance, 22% of students think that the limit of $\exp(-x)$ at $-\infty$ is 0.

Student's algebraic difficulties

The first six questions were about functions $\exp(-x)$, $\ln(1+x)$ et $\ln(1/x)$. There basically are compositions of limits or substitutions. There is no decomposition to operate. The iconic visualizations can be sufficient enough because students can substitute and compose limits very algebraically. However, the results (between 50 and 70%) are not significantly higher than average.

The four next limits are about the functions $\exp(x)-x$, $\exp(-x)\ln(x)$ and $\exp(x)(1-\sqrt{x})$. Students can also manipulate algebra rules about limits. Here, we observe that the existence of an algebraic indeterminate form ($\infty-\infty$, $0 \times \infty \dots$) is not a criteria of difficulty. For instance, the limit of $\exp(x)-x$ is better found at $+\infty$ (78%) than at $-\infty$ (55%) whereas only the first one is an indeterminate form $\infty-\infty$ which has to be cleared. In the same way, the limit of $\exp(-x)\ln(x)$ at $+\infty$ is better found (70%) than the limit of $\exp(x)(1-\sqrt{x})$ at $+\infty$ (56%), whereas only the first one is an indeterminate form $0 \times \infty$.

The computation of the limit of $(\exp(-x)-1)/(\exp(-x)+1)$ at $-\infty$ is also interesting. Students have to identify a quotient. It is an algebraic deconstruction, traditional in

algebraic activities, as we already said (students used to apply such deconstructions, especially for derivative computation). Then they have to identify that it is an indeterminate algebraic form belonging to the category ∞/∞ . Students should then divide the numerator and the denominator by $\exp(-x)$, the dominating term. This factorization requires algebraic and functional knowledge. For instance $1/\exp(-x)=\exp(x)$. We observe that the percentages of answers are very scattered, with only 30% of good answers.

Concerning limits of rational functions at $+\infty$, students know the algebraic rule of factorization which are traditional. However, as in the previous example, these kind of factorization are not so easy for them. The limit of $(x-1)/(x+1)$ at $+\infty$ is only succeeded with 52% of good answers. One can identify some effects of wrong rules application, for instance $\infty / \infty = 0$ (it surely justifies 20% of them answer 0) or $\infty / \infty = \infty$ (19% of them answer ∞). For instance, student B says for the function $(2x-2)/(x+1)$ at $+\infty$ « *at the top, + infinite, at the bottom, + infinite, so + infinite, we have it in the array of indeterminate forms* ». So the algebra of limits seems not to be very well known. It can explain some big mistakes.

Students' ability to adopt some DWP – first kind of evidences

It seems that for expressions with the exponential function, such as $\exp(x)-x$ and $\exp(-x)\ln(x)$ near $+\infty$, students are able to identify that the exponential function dominates. For instance, student B says « *exp is very powerful, very fast, if I replace x by a great number...* » and student F, who must have answered the limit of these two functions says about $\exp(x)(1-\sqrt{x})$: « *the square root of x goes to + infinite, so minus square root of x goes to - infinite, +1, it's again - infinite, the exponential grows faster than square of x so it wins, so + infinite* ». And about $\exp(x)-x$ the same student says: « *+ infinite, it's the same answer, exp(x) goes to + infinite, -x goes to - infinite but the exponential goes faster so + infinite* ». In fact, this student failed in the algebraic deconstruction: he didn't see that it was first a sum and then a product. But his main argument is based on the exponential domination.

For the limit of $\exp(x)(1-\sqrt{x})$ at $+\infty$, student F doesn't take into account if the algebraic form is indeterminate or not. He doesn't see that it is not an indeterminate form and his knowledge about the exponential domination leads him to a mistake. In the general results, we really observe a great failure with this calculation of limit: 44% of the students answer $+\infty$ as the limit of $\exp(x)(1-\sqrt{x})$ at $+\infty$. It seems that in all computation of a limit, the exponential function dominates. Moreover, as we mentioned earlier, students have difficulties with the signs of the quantities.

For the limit of $(\exp(-x)-1)/(\exp(-x)+1)$ at $-\infty$, we can assume that many students do not have enough knowledge to operate the awaited algebraic manipulations. Some of them seem to operate a deconstruction with local perspective to visualize – in a non-iconic way - that the function behaves like 1 near $+\infty$. For instance, student C says « *exp(-x)-1 goes to + infinite, exp(-x)+1 goes to + infinite, so it goes to 1, -1 et +1 we*

can neglect them...». Moreover, 26% of students answer -1, which can be explained by the traditional mistake of sign (x is always positive) and $\exp(-\infty)=0$. 26% of students again answer 0, may be here again because they visualize that the exponential function dominates, even if it is a quotient, but it is a major hypothesis.

Student's amalgam between point wise and local perspective

Let's come back to the six first questions. Students may have an amount of skills that work both with substitution method (for continuous functions) and with real limits in a generalized algebra. It seems that functions are always continuous over $[-\infty; +\infty]$ with extended values such as $\exp(-\infty)=0$; $\exp(+\infty)=+\infty$; $\exp(0)=1$; $\ln(+\infty)=+\infty$; $\ln(0)=-\infty$; $\ln(1)=0$. It is an implicit extension of \mathbb{R} to the extended real number line \mathbb{R} barre. The interviewed students do not seem to distinguish between « it is » and « it tends » which are used indifferently; as student A says about $\ln(1+x)$ « it is a composed form $\ln(u(x))$, $1/x$ goes to 0^+ , it's a formula from the course, we put $X=1/x$, $\ln(X)$ it gives minus infinite when X tends to 0^+ » and after « it's the same, it's a composed form, $X=-x$, x goes to 0 so X goes to 0 and $\exp(0)=1$ ». For this student, there is no distinction between a substitution or a numerical composition in a continuous function (point-wise perspective) and a real limit (local perspective).

This phenomena also appear when students seem to operate some DWP: for instance, student C says for the limit of $\exp(-x)\ln(x)$ at $+\infty$ « exponential at - infinite, it's 0, $\ln x$ goes to + infini, and 0 times infinite it's 0 ». We can call this phenomena double DWP – point wise and local. As another example, computing the limit of $\sin(x)/x$ at 0, student C says « $\sin(0)$ it's 0, and x it only goes to 0, so the limit is 0 ». He operates a point-wise decomposition of the numerator (as if x equal 0) and a local decomposition of the numerator (x goes to 0). Such reasoning can explain some qualitative wrong rules teachers of the IREM group have confirm, for instance « 0 over something equals 0 ». These wrong rules can justify that 35% of students answer 0 for the limit of $(x^2-1)/(x-1)$ at 1 and 30% answer 0 for the limit of $(2x-2)/(x+1)$ at 1.

In the same way, qualitative rules such as « something over 0 gives infinite » and « something over infinite gives 0 » are also associated with juxtaposed point-wise and local decomposition. We can correlate these rules with strong rates of success when they are right and strong rates of failure when they are wrong. For instance, if we avoid students' traditional problems with signs, we can say that 82% of students (53%+29%) answer correctly an infinite ($-\infty$ or $+\infty$) for the limit of $(x^2-1)/(x+3)$ at -3 and 60% of them (37% + 23%) answer correctly an infinite for the limit of $(2x-2)/(x+1)$ at -1^+ . Student C says for $(2x-2)/(x+1)$ at -1^+ « the more we divide by one 0, the greater it is. As it is negative (the numerator) it is minus infinite ». The application of such rules can also explain that 31% of students answer $+\infty$ for the limit of $(x^2-1)/(x-1)$ at 1^+ , which is a wrong answer.

Students' ability to adopt some DWP – second kind of evidences

As we have already said, it seems that many students are unable to deal with the application of algebraic rules to clear indeterminate forms (for instance factorization by the dominant terms in a quotient or identification of a basic algebraic identity – remarkable - for the case $(x^2-1)/(x-1)$ at 1^+). However, some of these students seem more comfortable with DWP: for instance students are less successful in the calculation of the limit of $(x-1)/(x+1)$ at $+\infty$ (52% of good answers) than in the one of the limit of $(x^2-1)/(x+3)$ at $+\infty$ (81%). In the same way, student F says « x^2 goes to infinite faster than x , so the limit is $+\infty$ ». Furthermore, 37% of the students answer $+\infty$ for the limit of $(2x-2)/(x+1)$ at $+\infty$, which is wrong. The origin of the mistake can be found by students' focusing on the qualitative argument $2x$ grow faster to $+\infty$ than x . Concerning student D, he doesn't succeed in applying an algebraic rule in a right way. However he still finds the right answer, stating « *it is twice more above than below* ». We can clearly say that he has operated a DWP instead of calculating with difficult algebraic rules.

SYNTHESE

The confrontation to the results of a real test over students is not so easy, especially when the framework of the test is not stable. For instance not all students have the same knowledge about limit. The conclusions of this paper have to be confirmed and refined. However these conclusions seem original. The notions of non-iconic visualization and decompositions with point-wise, global or local perspectives seem enough robust to explain and characterize specific students activities in the analysis setting.

Our paper suggests that students have algebraic abilities which are weak in order to compute limits: they have difficulties to identify the kind of indeterminate forms ($\infty-\infty$, $0 \times \infty$...). Indeed, there is no significant difference of results whether the algebraic form is determinate or not. Students also find difficulties with algebraic rules and algebraic manipulations to clear indeterminate forms.

Moreover, it seems that they amalgam point-wise and local perspectives, embedded in algebraic procedures. They have developed a specific knowledge about a generalized algebra ($\exp(-\infty)=0$; $\ln(0)=-\infty$...). In this specific mathematical area, local limit calculations and point-wise substitutions are mixed. In consequence, students are able to amalgam point-wise substitutions and decompositions with local perspective on the same formula.

However, a vicious circle may become a virtuous circle. Without a sufficient work involving the perspectives on functions – mostly by algebraic calculations, a lack of graphical tasks and coordination of the two registers, the internalization of few elementary functions... - students do not understand properly the technical rules they are asked to remember and apply. In particular, they are not able to identify which

forms are indeterminate or not. Moreover, they have difficulties in algebraic calculus (for instance isolate commons factors in complex expressions). Consequently, it seems they may have developed qualitative knowledge which appear near to the DWP we have introduced above.

This conclusion helps to explain why the limit of $(x^2-1)/(x+3)$ at $+\infty$ is better succeeded than the one of $(x-1)/(x+1)$. It helps to explain why many students find that the limit of $(2x-2)/(x+1)$ at $+\infty$ is $+\infty$, which is wrong. It helps again to explain that near half of them answer $+\infty$ for the limit of $exp(x)(1-\sqrt{x})$ at $+\infty$, considering surely that the exponential function dominates and not operating the algebraic rule. This ability to operate such (sometime partial) global and local decompositions instead of algebraic operations, substitutes perhaps knowledge about the algebra of limits. This algebraic knowledge appears very technical for students, and it doesn't have any meaning for them – surely because it is not relied to perspectives.

Of course, this new kind of knowledge about DWP is not totally operational. There never is institutionalization about it during classroom. Students use these decompositions but with mistakes, without any mastery. This leads them to good answers as well as big mistakes. We also observe some decompositions with different perspectives on the same formula, with the automatic rules we have listed at the end of the previous paragraph. May be the teaching in secondary school could built on these new kind of knowledge instead of developing algebraic skills with less and less meaning for students?

ANNEX 1

$exp(-x)$	$+\infty$	0	70%	CA	$(exp(-x)-1)/(exp(-x)+1)$	$-\infty$	1	30%	CA	
		$+\infty$	11%				$+\infty$	16%		
		$-\infty$	11%				0	26%		
		1	10%				-1	26%		
	$-\infty$	0	22%			$1/(x+1)$	$+\infty$	0	85%	CA
		$+\infty$	65%	CA				$+\infty$	8%	
		$-\infty$	7%					1	7%	
		1	6%				$+\infty$	-1	8%	
								1	52%	CA
	0	0	16%			$(x-1)/(x+1)$	$+\infty$	0	20%	
		$+\infty$	10%					$+\infty$	19%	
		$-\infty$	9%					0	-1	73%
		1	65%	CA			1		9%	
$ln(1+x)$	$+\infty$	0	19%			0	14%			

		$+\infty$	76%	CA			$+\infty$	3%	
		$-\infty$	5%				2	10%	CA
	0^+	0	68%	CA	$(x^2-1)/(x-1)$	1^+	0	35%	
		$+\infty$	23%				1	23%	
		$-\infty$	9%				$+\infty$	31%	
$\ln(1/x)$	$+\infty$	0	23%				$+\infty$	81%	CA
		$+\infty$	12%				0	14%	
		$-\infty$	50%	CA	$(x^2-1)/(x+3)$	$+\infty$	-1/3	5%	
		PDL	14%				$+\infty$	29%	
	$-\infty$	0	18%				$-\infty$	53%	CA
		$+\infty$	55%	CA			8	16%	
		$-\infty$	27%						
$\exp(x)-x$	$+\infty$	0	11%				$+\infty$	2	56%
		$+\infty$	78%	CA			$+\infty$	37%	
		$-\infty$	11%		$(2x-2)/(x+1)$	$+\infty$	-2	7%	
	$+\infty$	0	70%	CA			$-\infty$	37%	CA
		$+\infty$	16%				$+\infty$	23%	
		$-\infty$	14%				0	30%	
$\exp(-x)\ln(x)$	$+\infty$	$-\infty$	56%	CA			-4	9%	
		$+\infty$	44%						
	$+\infty$	-1	60%	CA					
		$+\infty$	8%		$\sin(x)/x$	0^+	0	20%	
		0	18%				1	20%	CA
		1	13%				NL	32%	
$(\exp(-x)-1)/(\exp(-x)+1)$	$+\infty$	$+\infty$					$+\infty$	26%	
		$-\infty$							
		1			$\sin(x)$	$+\infty$	NL	51%	CA
							$+\infty$	12%	
							1	37%	

NL means No limit - CA means Correct Answer
 Second column is the point where the limit is asked, third column is the four propositions.

ANNEX 2

Student A

$\ln(1/x)$	$+\infty$	$-\infty$	CA
$\exp(-x)$	0	1	CA
$\sin(x)$	$+\infty$	NA	
$(\exp(-x)-1)/(\exp(-x)+1)$	$-\infty$	1	CA

Student D

$(x^2-1)/(x-1)$	1^+	2	CA
$(x^2-1)/(x+3)$	-3^-	$-\infty$	CA
$(2x-2)/(x+1)$	$+\infty$	2	CA
$\exp(x)-x$	$-\infty$	$+\infty$	CA

$(\exp(-x)+1)$			
No data			

Student B

$\exp(-x)$	$-\infty$	$+\infty$	CA
$(2x-2)/(x+1)$	-1^+	-4	
$(2x-2)/(x+1)$	$+\infty$	$+\infty$	
$\exp(-x)$	0	1	CA
No data			

Student C

$\exp(-x)$	$-\infty$	$+\infty$	CA
$(2x-2)/(x+1)$	-1^+	$-\infty$	CA
$\sin(x)/x$	0^+	0	
$\exp(-x)\ln(x)$	$+\infty$	0	CA
$(\exp(-x)-1)/(\exp(-x)+1)$	$+\infty$	1	CA

$(x-1)/(x+1)$	$+\infty$	1	CA
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Student E

$\ln(1/x)$	$+\infty$	$-\infty$	CA
$\sin(x)$	$+\infty$	NL	CA
$(x^2-1)/(x+3)$	$+\infty$	$+\infty$	CA
$\exp(x)-x$	$-\infty$	$+\infty$	CA
$1/(x+1)$	$+\infty$	0	CA

Student F

$\sin(x)/x$	0^+	NL	
$(\exp(-x)-1)/(\exp(-x)+1)$	$-\infty$	0	
$\exp(x)(1-\sqrt{x})$	$+\infty$	$+\infty$	
$\exp(x)-x$	$+\infty$	$+\infty$	CA
$(x^2-1)/(x+3)$	-3^-	$-\infty$	CA

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Angles, trigonometric functions, and university level Analysis

Carl Winsløw¹,

¹University of Copenhagen, Faculty of Science, Denmark, winslow@ind.ku.dk

The transition from calculus to analysis presents well known challenges to many students. In line with Chevallard's "paradigm of questioning the world", we consider that part of the difficulty is a failure to question (explore) what the formal theory of analysis means for "familiar objects" from calculus. As a significant case, we consider the familiar notions of "angle", cosine and sine. We first examine some "advanced" viewpoints on these objects coming from analysis, then the viewpoints an advanced student could produce on these notions, based on knowledge from an analysis course in which resources for a rigorous account were indeed available.

Keywords: calculus, analysis, angles, sine, cosine.

INTRODUCTION: QUESTIONING IN UNDERGRADUATE ANALYSIS

This paper is motivated by observations and reflections on the calculus-analysis transition which we have studied for several years (Gyöngyösi, Solovej, & Winsløw, 2011; Winsløw, 2008). In particular, we have observed that students do not easily relate the contents of analysis courses (formal theory of continuity, integration, function spaces etc.) to *calculus* as encountered in high school and the first few courses at university. Even evident relations, such as the use of concrete cases to question general assertions on continuity or convergence of various types, are far from easy to establish; for many students, the more abstract theory seems to constitute a universe in itself. Our original motivation for this problem was the hypothesis that supporting students' work with such "concrete-abstract" relations could be used to prevent some of the widespread failure which students experience in the more abstract courses. In (Winsløw & Grønbæk, 2014) we investigated the potentials and obstacles of the opposite transition, as pursued in so-called *capstone* courses for future teachers. Finally, in (Winsløw, 2016) we investigated the potential of exercises explicitly *questioning* key definitions in the setting of abstract courses. The didactical significance of "questioning" was further developed, based on (Chevallard, 2012); the idea is for students to meet and develop some of the more fundamental questions which gave rise to an otherwise unmotivated abstract definition, such as the definition of uniform continuity of functions; such a questioning often leads to recover relations between the abstract theory with classical and concrete problems, and in particular, to consider the role of definitions within a theory - indeed, that they are not mere arbitrary conventions, but in fact often essential elements of crucial mathematical advances.

In the present paper, we investigate a concrete case which touches upon all three aspects: the relation R between *the problem of giving a rigorous definition of sine*, and *the material covered in a first rigorous course on analysis* (including, as a

minimum, a theory of integration of functions on an interval and along a rectifiable curve in the plane). We begin with a historical background section, referring essentially to Klein's ideas on R and similar relations (Klein, 1908/1932). We proceed to a study of how this problem is treated in two textbooks for analysis courses of the type mentioned above, the aim being to identify the relations which such courses could aim to develop. Finally, we report on an interview with an advanced and successful student of mathematics who, in particular, served as a teaching assistant on a course of the type mentioned, in view of identifying an "upper bound" of the relations of type R which typical students in that course could develop.

BACKGROUND: PLAN A AND PLAN B

Classical analysis can be described as the study of *functions*, with an emphasis on so-called elementary functions, of which the most important are polynomials, power functions $x \mapsto x^a$, exponential functions $x \mapsto a^x$, the logarithmic functions \log_a , and trigonometric functions (chiefly sin and cosines). Felix Klein (1908/1932, pp. 77-85) considered that in the history of mathematics, as well as the school discipline, we may identify two possible "Plans" (one might also say, visions or strategies) for developing a subject; and he used the construction and study of elementary functions as a main case to illustrate these plans.

What Klein calls *Plan A* is a *compartmentalized approach* to mathematics, which emphasizes precise and purified work within certain small "areas" of mathematics, which are hardly related among each other:

Plan A is based upon a more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part for itself, with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields (ibid., p. 78).

For the case of the elementary functions, this corresponds to constructing and studying different classes of functions separately. For instance, the general exponential function $x \mapsto a^x$ is constructed based on progressive extension in terms of x (from integers to real exponents), taking care of explaining all details in the process. In another compartment, namely analytic geometry, one studies trigonometric functions in relation to triangles and circles. Naturally, in school versions of this plan, the level of detail and precision may vary. In any case, the two functions classes arise from quite distant universes. It is only students who continue to study mathematics at university who will learn of an almost "magical" relation among them, when the complex versions of these functions appear (and with them, Euler's formula $e^{ix} = \cos x + i \sin x$). The most famous examples of "Plan A" in the history of mathematics itself include, of course, Euclid's elements, and more generally the axiomatic method with its massive influence in all of modern mathematics.

Plan B, by contrast, involves a more holistic approach which emphasises and exploits connections between different sectors, sometimes at the expense of strict rigor:

... the supporter of Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon another. He prefers, therefore, the methods which open for him an understanding of several fields under a uniform point of view. His ideal is the comprehension of the sum total of mathematical science as a great connected whole (ibid., p. 78).

Klein observes that in the history of mathematics, “Plan A” and “Plan B” both appear during fruitful periods of research, in analysis as well as in other areas. For instance, the initial developments of the calculus took place much according to Plan B, led by Leibniz and Newton; later, a progressive move towards Plan A occurred, as Cauchy and others gave classical analysis the solid foundations we know today.

Klein strongly recommends including “Plan B” as a strategy for presenting mathematics to students, and laments the exclusive use of Plan A at school (in his days). We note, as an aside which we will not further treat in this paper, that Klein also briefly mentions a “Plan C”: essentially, “pursuit of algorithms”. It could be considered as vastly more important today as in the days of Klein, which had no computers to fuel it; but we leave Plan C here, as it appears less relevant to the teaching analysis in a university program on pure mathematics. In such programs in general, Plan A prevails to the extent they consist of specialised modules on a few specific sectors, with little explicit links among them or to mathematics at large.

PLAN B IN CALCULUS

An introduction of exponential and trigonometric functions following Plan B could, according to Klein (ibid., pp. 155-169), be based on what he calls ‘quadratures of simple curves’ - in modern terms, functions given by definite integrals, and their inverses. It thus requires a solid calculus background. In outline, his proposal is:

$$\ln(x) = \int_1^x \frac{dt}{t}; \exp(x) = \ln^{-1}(x); a^x = \exp(x \ln(a))$$

$$\arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}; \sin(x) = \arcsin^{-1}(x); \cos(x) = \sin(x + \arcsin(1)).$$

Klein points out many analogies and relations between the two cases. Note, however, that one specific difficulty remains in the second case: to prove that the integral converges at ± 1 . It can be solved using a limit argument on the identity (ibid., p. 168)

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = 2 \int_0^x \sqrt{1-t^2} dt - x\sqrt{1-x^2} \quad (-1 < x < 1)$$

as the right hand side is clearly meaningful when $x = \pm 1$.

The definition of arcsin could also be related to a formula encountered (and perhaps to some extent explained) in calculus classes, concerning the length of the graph of a function between two points on the graph. Using it on $f(y) = \sqrt{1-y^2}$ for $y \geq 0$, one finds that the length of the graph from $(1,0)$ to $(f(y), y)$ is exactly $\arcsin(y)$ as defined above. We can therefore interpret $\arcsin(y)$ for $y \geq 0$ as the length of a unit circle arc from $(1,0)$ to the unique point (x, y) on the unit circle with $x \geq 0$. Besides

explaining the strange name “arc sine”, it also shows that this function gives a kind of “angle” of circle points (in the first quadrant directly, then by extension). Moreover, sine takes the “angle” ($\arcsin(y)$) to the second coordinate (y) of the point on the circle that is situated at that angle, as explained in a common school approach.

PLAN A IN SECONDARY SCHOOL

Klein’s approach is certainly not common in secondary school. Instead, various Plans A appear for the case of introducing cosine and sine, with three distinct contexts (Demir & Heck, 2013, p. 120):

- First, trigonometry appears as a “toolbox” for solving triangle problems in plane geometry, where cosine and sine are defined as certain ratios of sides in a right triangle, accompanied with something like Figure 1;
- Then, in the setting of analytic geometry, cosine and sine are defined as coordinates of the intersection of a ray through the unit circle, accompanied with something like Figure 2;
- Finally, cosine and sine emerge as functions through tables and graphs like Figure 3, with a discussion of function properties such as domain, range, zeros, period etc.

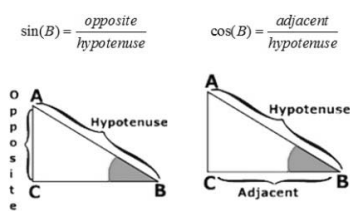


Figure 1: Triangle context.

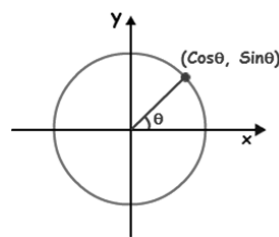


Figure 2: Cartesian context.

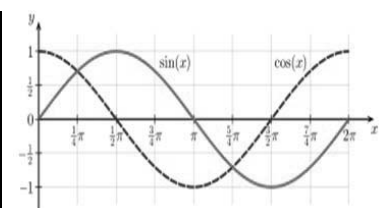


Figure 3: Function context.

Naturally, some efforts are being made to relate the three contexts, which usually appear in the order of the figures; elements of plan B can be seen in such efforts. We should note here that while Figure 1 and 2 are really used as central ostensives for different definitions (which are consistent, as is proved by appealing to similarity), Figure 3 is not presented as a definition, but more as an illustration of the definition shown in Figure 2; the postulate character of the graph is the main obstacle addressed by Demir and Heck (2013). No doubt, the graphs are instrumental - along with the symbolic ostensives, tables, and the use of usual terminology related to functions - to institutionalize the “function status” of cosine and sine. But some mystery remains:

The sine and cosine functions may have been defined, but the graphs of these real functions remain mysterious or merely diagrams produced by a graphing calculator or mathematics software. The complex nature of trigonometry makes it challenging for students to understand the topic deeply and conceptually. (Demir & Heck, 2013, p. 119)

While we do agree with the authors that difficulties remain, a “deep understanding” may not result from simply multiplying the ostensives produced with dynamic software, since none of the “definitions” given are really basing the functions on firm mathematical ground. In fact they ultimately relate on the meaning of *angle*, naturalised since primary school as a “measure” of the space between two crossing line segments (for instance, sides in a triangle). Among the mysterious operations which usually accompany the passage from the triangle context to the Cartesian context is the unmotivated change of “unit” for this “measure”, as “degrees” are replaced by “radians”. The historic reasons for the appearance and rejection of the bizarre convention to associate, for instance, the magnitude 90^0 with a right angle, are certainly interesting, and there is no shortage of other units and notations which are or have been commonly used for “angles”. But whether or not these variations (or the sudden passage to radians) are questioned, they only distract from the heart of the matter: the meaning(fullness) of “angle measure”. It remains a mere postulate, with no mathematical grounds, that a real number may be associated to *any* pair of crossing lines, as somehow a “measure” of the “width” of the “space” between them. Of course, turning to Klein’s proposal (for secondary school) may indeed put trigonometric functions on a firmer basis, assuming (as he did) that integrals are given a more than superficial treatment at that level. However, this may at best rescue the functions as such, not the geometric interpretation shown in Figure 1 and 2, and in particular the more fundamental notion of angle measure. In fact, angle measure is nothing else than arc length in the special case of a unit circle, and just as the notion of area, it remains materially or sensually based in school for the simple reason that any mathematical definition depends on the topology of the real numbers.

GERMS OF PLAN B IN UNIVERSITY ANALYSIS

Standard calculus courses maintain, and do not question, the material approach to angles and trigonometric functions. Based on two different but typical texts from typical first courses on real analysis, we shall now see that in such courses, a deeper questioning is at least possible. This means that we may seek new meanings and explanations of two claims which are both plausible and familiar to the students:

- (I) every point on the unit circle S^1 corresponds uniquely to a real number (“angle”) and these numbers give rise to a natural “distance” on S^1 (and hence to a measure of the “width opened between rays”);
- (II) the map from angles to coordinates of the corresponding point in S^1 gives rise to two well-defined real functions (“cosine”, “sine”) with the usual properties.

Of course, a variety of formal and informal approaches to these questions exist, especially if we ask *only* for definitions of sine or *only* for an explanation of what angles mean; a quick web search will convince the reader that both themes are of great public interest. What we focus on here are coherent contributions to (I) and (II) which could be developed in undergraduate analysis courses.

Complex analysis approach

Rudin (1986, p. 1) begins his seminal book “Real and Complex Analysis” as follows:

This is the most important function in mathematics. It is defined, for every complex number, by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \quad (1)$$

With the notations $e = \exp(1)$ and $e^z = \exp(z)$, he *defines* $\cos(t) = \operatorname{Re}(e^{it})$ and $\sin(t) = \operatorname{Im}(e^{it})$ for $t \in \mathbb{R}$, and proves the usual functional properties based on (1). The geometric interpretation of these functions, shown in Figure 2, follows from the Cartesian representation of complex numbers. However, the meaning of “angles” is not clarified. Some ideas can be extracted from statements proved after the above definition, including the fact (ibid., p. 2) that “The mapping $t \mapsto e^{it}$ maps the real axis onto the unit circle”. Combined with other elements of the exposition, including a definition of π , it is in then easy (but not done) to prove a converse: for every point z on the unit circle, there exists a unique number $a \in [0, 2\pi[$ such that $z = e^{ia}$. This enables a possible definition of the angle $A(z)$ corresponding to the point z (in fact, the angle between the ray from 0 to z , and the ray from 0 to 1). We note that in complex analysis, $A(z)$ is usually called an argument of z . However, this does not really answer question (I) above: why do arguments provide a kind of *distance* on the circle? To do so, we would need to relate the above definitions to a rigorous theory of curve length in the complex plane. In particular, a theorem (not a definition) of curve length would then give us that when $0 \leq s < t < 2\pi$, the length of the arc from e^{is} to e^{it} is $\int_s^t |ie^{ix}| dx = t - s = A(e^{it}) - A(e^{is})$. And then, finally, we also have an answer to (II), as $\cos(a) = \operatorname{Re}(e^{ia})$ and $\sin(a) = \operatorname{Im}(e^{ia})$ for $a \in [0, 2\pi[$.

The topological notion of curve length in the complex plane is essentially the same as that of curve length in \mathbb{R}^2 (as covered in vector analysis, cf. the next subsection). The above approach requires, *in addition*, a solid background on series and their convergence properties. It has the merit of relating exponential and trigonometric functions from the outset, including the characteristic differential identities. So, assuming that all prerequisites are in place, it certainly holds potential for a “Plan B” approach to (I) and (II).

Real analysis approach

A more minimal approach to question (I) and (II) can be found in the textbook (Eilers, Hansen, & Madsen, 2015), written for a second semester course on real and vector analysis at the University of Copenhagen. The textbook includes an appendix, including one section on “Trigonometric functions”, which builds on chapter 7 of the book. The appendix refers to that chapter as basis for stating that there is an “angle mapping” $\gamma: \mathbb{R} \rightarrow S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ which is *natural* in a sense explained in the next paragraph; this map is then used to define cosine and sine (as the first and second coordinate function of γ), and to prove the most important

identities and inequalities involving those functions, including inequalities which can be used to find the derivatives (this, however, is not even mentioned as an exercise).

The meaning of *natural* only appears from a sequence of definitions and theorems in the main text (ibid., pp. 223-227, all quotes translated from Danish by the author). First some notation is introduced: for a continuous curve $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^m$ and a partition D of $[a, b]$ consisting of points $a = t_0 < t_1 < \dots < t_k = b$, we put

$$\ell(D) = \sum_{j=1}^k \|\mathbf{r}(t_j) - \mathbf{r}(t_{j-1})\|.$$

A figure illustrates how this measures the total length of line segments between points on the curve corresponding to the partition, and how this can be interpreted as a lower bound of what is intuitively the “length” of the curve. Then follows the precise definition of curve length, ultimately in terms of the usual distance in \mathbb{R}^m :

Definition 7.14. For a continuous curve $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^m$ with bounded, closed parameter interval, the curve (or arc) length is given as

$$\ell = \sup\{\ell(D) \mid D \text{ is a finite partition of } [a, b]\}$$

If $\ell < \infty$, the curve is said to be *rectifiable* (ibid., p. 223).

It is then proved that if \mathbf{r} is piecewise C^1 , then it is rectifiable, with $\ell = \int_a^b \|\mathbf{r}'(t)\| dt$. Under the further assumption that \mathbf{r} is smooth (i.e. $\mathbf{r}'(t) \neq \mathbf{0}$ for all t), it is proved that there is an interval $[c, d]$ and a strictly increasing C^1 -function $\varphi : [c, d] \rightarrow [a, b]$ such that $\tilde{\mathbf{r}} = \mathbf{r} \circ \varphi$ is a *natural* parametrization defined on $[c, d]$, meaning that $\int_u^v \|\tilde{\mathbf{r}}'(t)\| dt = v - u$ whenever $c \leq u \leq v \leq d$. In particular, $d - c = \ell$ where ℓ is the curve length of \mathbf{r} . In words: there is a reparametrization of \mathbf{r} , such that the length of any curve segment is the distance between the corresponding parameter values.

The text gives three examples, including that after piecing together some smooth C^1 -parametrization \mathbf{r} of S^1 , traversing S^1 once from $(1,0)$ to $(1,0)$ in the usual direction, we may use the above to construct a *natural* reparametrization γ of \mathbf{r} . It is not made explicit that \mathbf{r} has finite length, and one can then *define* π as half of that length.

The appendix does not explain why γ is called an *angle map*, for instance, how it may be used to define the angle between two points on S^1 . The example in the main text is likely to strike the students as trivial, even if the above “example” concludes by postulating that \mathbf{r} can be used to define cosine and sine, referring to the appendix.

We observed (and co-developed) a course based on this textbook in the spring of 2015. Indeed, the overall emphasis of the course followed a Plan A, laying out theoretical foundations for real and vector analysis, with a strong emphasis definitions, theorems and proofs. The appendix was not covered by the lectures or in the exam requirements, and was only briefly used for an exercise in the first week. The above material from Chapter 7 (and more) was covered in one lecture during the sixth week. No exercises set for students addressed the problem of defining angles; the pieces are not brought together. So, even if the book and its material holds

potential for a complementary Plan B related to (I) and (II), it thus seems likely that students will not realize that the theorems on curve length can be used to solve two related mysteries of their past teaching. We now turn to investigate this further.

A STUDENT'S DEVELOPMENT OF PLAN B, BASED ON PLAN A

To gauge whether the second approach could after all be noticed or at least excavated for some students having taken the course considered above, I interviewed a master level student who had served as a teaching assistant in this course twice, and is within the top 10% tier in the mathematics programme at Copenhagen. This student's knowledge would likely be an upper bound of what younger students would retain or be able to reconstruct from the course, as regards the questions of angles and trigonometry. The interview was semi-structured, based the following questions:

- What is your favourite definition of the function sine?
 - o Follow-up questions according to the definition chosen, leading to:
- What is your favourite definition of angles? How do they relate to sine?
 - o Follow-up questions for instance on arc length, if referring to circle arc
- What mathematical resources does the course (described above) provide to elucidate the previous questions? (textbook at hand to look up points)

The advanced student (AS) begins answering the first question by tracing the graph of sine (Fig. 4), then enumerates “a lot of properties” of the function (Fig. 5). AS realises that it is “certainly not a definition”. I insist on AS giving one. AS then gives the Cartesian description (Fig. 6). After a slight confusion, AS identifies x (in Fig. 6) as the appropriate angle in the triangle.



Figure 4: A graph.

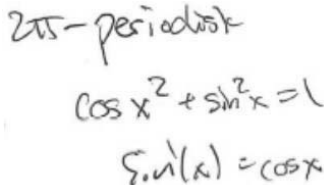


Figure 5: Properties.

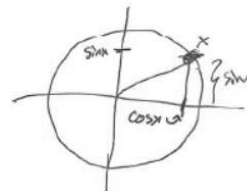


Figure 6: Diagram.

The following dialogue follows (some redundancies are left out, marked by //):

CW: Then, the question arises, the angle, what mathematical object is that? // Can you give like a mathematical definition of that?

AS: It determines like how far two lines are from each other. // [AS draws two crossing lines and says there are four angles, two different, and I repeat the previous question]

AS: In the old days, you used your compass and your protractor, and then later when you have to compute them, there were, you had these smart things where you get hold of cosine and sine to compute them, the angles...

CW: Yes, but you have just used angles to define sine and cosine...

AS: Oh yes, precisely, then it comes backwards again, so that it not so good...

CW: So, my thousand dollar question, could An0 [*the course*] help us with that?

AS soon recalls this “fantastic exercise” where you had two functions, and it turns out they are cosine and sine. After a few minutes, AS finds it in the section on curve integrals; the exercise begins with “a natural parametrization” $\mathbf{r}(t) = (C(t), S(t))$ of the unit circle, and asks the students to prove that $(C'(t), S'(t)) = \pm(S(t), -C(t))$. However, AS does not notice, at first, the assumption of \mathbf{r} being *natural*, and when I point it out and ask what it means, AS looks it up in the index of the book. This leads to the definition mentioned above. AS reads the definition for a while, then says:

AS: the parameter values, they should, if we subtract them from each other // the curve length should be like one minus the other // the curve cannot make like strange crossings, that must create a mess, I think...

CW: // you get such a length preserving map from an interval onto the curve. How could that help us with the question about sine, cosine and angles?

AS: That’s a good question.

I point to the appendix on trigonometric functions. AS recalls that they did do an exercise from there but “otherwise we did not look at it”. AS does not recall the part on the “angle map” γ and has no idea where γ comes from. I point out the reference to the main text and the explicit mention that γ is a natural parametrization. AS returns to the main text and looks at Definition 7.14. Paging a bit, AS finds the theorem on the existence of natural parametrizations of smooth C^1 -curves. I ask if AS could verify the conditions for the circle. Supported by a hint (“could you parametrize the circle without using sine and cosine?”), AS comes up with the parametrization $(\sqrt{1-t^2}, t)$, for the circle in the first quadrant. AS says this can be differentiated many times so it is C^1 . AS does not recall the definition of “smooth”, but quickly looks it up, and then verifies it for the parametrization above. Going back to the theorem, AS concludes that then we can construct the angle map.

CW: Using that, can you give a precise definition of what an angle is?

AS: Not immediately...

CW: What should it be, if you look at the unit circle? //

AS: It has something to do with the arc length //

AS tries to find out what the arc length is for γ and writes down the formula $\int_0^x \|\gamma'(t)\| dt$. As seen above, the meaning of “natural curve” is not familiar to AS. It also seems to confuse AS that the values of γ are clearly not angles (but points), in spite of the name “angle map”. After some neutral circling around these matters, I ask AS if we could make “a function from points on the circle to arc length”. AS suggests that $\gamma(t)$ should be mapped to t in some sense. As this is very close to a satisfactory answer and as the agreed time is almost up, I briefly show how to formalize this last point, and wrap up the conversation.

The 45 minute conversation outlined above suggests that the angle question would be challenging for average students and require more support than the course offers.

CONCLUSION

The formal definition (7.14 quoted above) of curve length in real analysis is a rigorous answer to questioning the intuitive idea that “nice, bounded curves have length”, relating that idea to the only case in which a proper, non-analytic definition can be made, namely for the length of line segments. When introduced to “radians”, high school students do encounter a link between curve (circle) length and angles; but the meaning of curve length is never questioned (in the sense of Chevallard, 2012) and the link to angles may soon be forgotten. The organization of the textbook (Eilers et al., 2015) and the corresponding course does not really highlight this link, and the experiment with AS suggests that the approach outlined in this paper is too ambitious for the course. Identifying viable “local Plan B’s” could contribute to diminish the “gap” which students otherwise experience between rigorous analysis and their previous knowledge (from calculus and other areas) - not to speak of their future practice, for instance if they become high school teachers.

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Comparaison de schémas de genèses didactiques de définitions, le cas de la limite d'une suite.

Renaud Chorlay¹ (*LDAR Paris-Diderot*)

Cécile Ouvrier-Bufferet² (*CEREP, URCA*)

Nous proposons un poster dans le cadre du colloque INDRUM comme moyen et occasion pour lancer un projet de collaboration scientifique sur le thème des genèses didactiques de définitions en analyse, en particulier de celle de limite de suite.

Le projet s'appuie sur une double expertise. D'une part, Cécile Ouvrier-Bufferet est didacticienne, ses travaux portent principalement sur l'épistémologie et la didactique de l'activité de définition ; les terrains empiriques jusqu'ici explorés dans le cadre de cette problématique se situaient à la transition secondaire-supérieure, dans des domaines relevant principalement des mathématiques discrètes, et donc peu dépendant du reste du cursus mathématique « classique » des étudiants à l'époque (Ouvrier-Bufferet, 2006 & 2011). D'autre part, Renaud Chorlay s'appuie sur l'histoire des mathématiques pour nourrir une réflexion épistémologique (Chorlay, 2011 & forthcoming) et didactique sur l'enseignement de l'analyse, tout en travaillant à l'explicitation de cadres robustes de collaboration entre deux champs disciplinaires distincts (à savoir : didactique et histoire des maths (Chorlay & Hosson, forthcoming)). Sur ces bases, des éléments de convergence permettent de dessiner les contours d'un projet collaboratif. Parmi ces éléments : la centration sur les définitions non pas comme éléments statiques d'ensembles textuels déductifs, mais sur l'activité de production de textes définitoires et sur l'activité même de définition ; l'usage d'allers-retours entre les pratiques savantes et les pratiques d'enseignements ; le souci d'articuler explicitement des composantes historiques, épistémologiques et didactiques.

Le cas d'une définition formelle de la limite (finie ou infinie) d'une suite nous semble permettre un tel travail. Il est à peine nécessaire de citer deux raisons fondamentales : premièrement, l'importance de cette notion dans l'analyse du supérieur (par opposition au *calculus*) ; deuxièmement, l'existence d'une abondante littérature didactique sur la question, dont nous regardons certains éléments comme des acquis. Les conclusions des travaux de Ouvrier-Bufferet (2013) soulignent l'intérêt que représente le champ de l'analyse pour étudier les processus définissants. Au sein de l'analyse, le cas des limites de suite nous permet de comparer deux schémas de genèses didactiques de définitions, dont nous ne pouvons indiquer ici que quelques grandes lignes.

Le premier schéma de genèse de définition consiste en un appui sur une familiarité avec la notion de convergence de limite pour produire des prévisions

¹ ESPE de l'Académie de Paris, renaud.chorlay@espe-paris.fr

² Université de Reims Champagne Ardenne et ESPE de Reims, cecile.ouvrier-bufferet@univ-reims.fr

relatives aux suites ayant une limite et des évaluations d'affirmations les concernant. Ces types de tâches conduisent à des moments de différenciation conceptuelle ; à l'identification de propriétés des suites ayant une limite ; puis à l'identification de propriétés caractéristiques (définitions possibles).

Le second schéma de genèse est celui affiné dans les travaux de Ouvrier-Bufferet (2013) : il s'appuie sur des conceptions emblématiques de l'activité de définition à un niveau épistémologique, et sur une modélisation de cette activité en accord avec le travail de mathématiciens contemporains (épistémologie contemporaine). Ce schéma est en particulier construit autour de moments de travail sur la définition, mais aussi avec des types d'énoncés définissants spécifiques (définitions-en-acte, zéro-définitions notamment) qui permettent de baliser la genèse d'un concept (*a priori*, et pour l'analyse de procédures).

Le poster présentera, outre ces éléments structurant du projet d'ensemble, les premiers éléments d'une ingénierie didactique construite selon le premier schéma de genèse. Seront présentés : une liste d'assertions à évaluer ; des éléments d'analyse *a priori* (pour un niveau Terminale – 1^{ère} année d'université) ; une comparaison avec le travail de Lecorre (2014) ; une analyse via des outils pour l'étude d'une activité de définition (Ouvrier-Bufferet, 2013).

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Didactical implications of using various methods to evaluate $\zeta(2)$

Margo Kondratieva

Memorial University, mkondra@mun.ca

Keywords: multiple proofs, interconnecting problems, Basel problem.

Mathematics instruction may benefit from using *interconnecting problems*, defined as problems that: allow various solutions at both (relatively) elementary and more advanced levels; can be solved by various mathematical tools from different mathematical branches; and, can be used in different courses (Kondratieva, 2011).

This research project uses the following interconnecting proof-problem: Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Five solutions are considered, via (1) Euler's representation for $\frac{\sin x}{x}$; (2) integral calculus and trigonometry; (3) reduction to a double integral; (4) Fourier series; and, (5) Cauchy's Residue Theorem. Solution (1) has a historical value, (2) and (3) are suitable for calculus courses, while solutions (4) and (5) require more advanced techniques from analysis. Known as the Basel problem, or evaluation of $\zeta(2)$, this problem has played a stimulating role in mathematical research from the 17th century to this day. It might play a similar role in the teaching of mathematics.

Multiplicity of approaches is important in mathematics research, and therefore, in training future researchers. According to Sir Michael Atiyah, "any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions - they are not just repetitions of each other." (Interview in EMS Newsletter, Sept. 2004, p. 24; <http://www.ems-ph.org/journals/newsletter/pdf/2004-09-53.pdf>) From the educational perspective, studying multiple solutions contributes to a learner's cognitive development through formation of related crystalline concepts (Tall, Yevdokimov, Koichu, Whiteley, Kondratieva, Cheng, 2012, p. 20). In particular, within the "Structure of Observed Learning Outcomes" (SOLO) model proposed by Biggs & Collis (1982), the fundamental UMR-cycle of concepts' construction includes three levels: (U) uni-structural, (M) multi-structural and (R) relational. The SOLO taxonomy focuses attention upon structure of learners' responses. At the U-level, the learner shows familiarity with only one solution. At the M-level, several approaches are used by the learner, without any relation perceived between them. At the R-level, the learner is able to compare, relate and integrate different approaches and ideas. Thus, the use of interconnecting problems may bring the learner to the M-level and help to create the environment in which learners can develop connections while moving towards the R-level. Local UMR-cycles occur in all SOLO-modes of operations, ranging from sensori-motor and iconic to concrete symbolic, formal and post-formal, thus making this framework applicable for analysis at the post-secondary level. A desirable outcome of a UMR-cycle is an extended abstract response, which comprises possible generalizations and extensions.

It emerges from the relational understanding achieved by the learner at the R-level, and often signifies a learner's transition to the higher SOLO-mode of operation. Thus, familiarity with as many as possible approaches, each of which highlights different aspects, properties and contexts where a (interconnecting) problem is considered, may prove vital for learners' cognitive growth.

The interconnecting problem used in this study provides the richness of mathematical context suitable for the emergence of aforementioned UMR-cycle. This preliminary study examines the following questions: (I) To what extent students who have completed a Bachelor degree with a major in mathematics are familiar with each of the five solutions? (II) What are mathematics instructors' views on the possibility to include these solutions into existing university courses? Eight graduate students and 8 instructors of mathematics (all from Memorial University), who have (respectively) completed or taught calculus and analysis courses responded to a survey. The participants were asked to read each of the five solutions and identify whether the solution is (a) familiar (b) accessible (c) connected to other solutions. The participation in the project was voluntary and anonymous. All students showed either U-level or M-level response with at most 3 different solutions recognized as being familiar. While all five solutions were found accessible by all students, advanced solutions (4) and (5) were more popular than solutions (1) - (3). This situation is in agreement with instructors' responses. They favoured solutions (4) and (5) for inclusion in corresponding analysis courses as opposed to presence of solutions (1) - (3) in any courses. The instructors felt that (4) or (5) is a more natural fit, while the other proofs would require extra time for explaining technical details in a course with an already busy curriculum. This challenge outweighed the advantage of giving a broader and more complete picture based on historical material and illustration of alternative techniques, let alone making connections between various methods.

The goal of my paper is to attract readers' attention to this rather not uncommon situation within undergraduate mathematics teaching and hopefully initiate some shift towards a more balanced approach based on inclusion of interconnecting problems in many levels of the mathematics curriculum and linkage of their various solutions.

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Introduire les réels à la transition lycée-université: Questions épistémologiques et horizon didactique

Viviane Durand-Guerrier, *Laboratoire IMAG*, vdurand@math.univ-montp2.fr

Denis Tanguay, *Université du Québec à Montréal*, tanguay.denis@uqam.ca

Mots clés : nombres réels ; densité ; continuité ; complétude

PROBLÉMATIQUE

Le présent projet de recherche est né du constat que les élèves-étudiants de fin de lycée et de début d'université n'ont des connaissances que lacunaires sur les nombres réels. L'étude de Vergnac et Durand-Guerrier (2014) montre que la majorité des élèves de seconde sondés a une conception du nombre qui l'assimile à son « écriture » (cf. aussi González-Martín et al., 2014), l'écriture décimale conférant seule un véritable statut de nombre dans leur entendement. Très peu de ces élèves savent « comment définir un nombre réel », aucun ne propose une représentation à partir de la droite, les réponses les plus fréquentes étant celles qui invoquent l'intervalle $]-\infty, +\infty [$ ou des esquives plus vagues encore, telles « l'ensemble de tous les nombres ». Ces réponses à la même question sont encore majoritaires en Terminale et en licence, et la référence aux points d'une droite ne s'y manifeste que dans une proportion de 7%.

Cette étude confirme par ailleurs les constats que font les enseignants, à l'effet que la plupart des étudiants ne font pas la différence entre nombres décimal, rationnel et irrationnel, avec des réponses qui vont jusqu'à l'incohérence quand des développements décimaux infinis sont en cause (cf. aussi Bronner, 1997). Tout porte à croire que les représentations ensemblistes ou même géométriques — à partir de la droite repérée — ne suffisent pas à problématiser la différence entre le « dense » (les décimaux et les rationnels sont denses par rapport à l'ordre \leq) et le « continu » : la connaissance, même étayée d'une preuve, qu'un nombre comme $\sqrt{2}$ est *irrationnel* n'est ni liée à la notion d'incommensurabilité, ni intégrée à la conceptualisation du continu, et est en fait « vécue » dans un monde mathématique à part. Bergé (2010) a pour sa part montré que les étudiants universitaires qu'elle a sondés ont de la complétude des réels une connaissance non opérationnelle, c'est-à-dire qui ne s'appuie pas sur un ensemble de situations/problèmes que la complétude résoudrait.

Hypothèses et questions de recherche

Nous faisons l'hypothèse qu'une conceptualisation adéquate des réels est un préalable nécessaire à un abord réussi des cours d'analyse à l'université. L'idée que les ensembles denses \mathbb{D} et \mathbb{Q} puissent laisser des « interstices » sur la droite réelle est contre-intuitive, et nous faisons l'hypothèse que la continuité suggérée par le tracé de la droite cristallise l'idée d'une dichotomie entre le *discret* et le *continu* qui donne

toute la place à la *densité*, alors confondue avec la continuité. Nous nous intéressons donc à la genèse conceptuelle des réels, telle qu'elle pourrait être soutenue dans l'enseignement : doit-on envisager une forme ou une autre de reconstruction des réels en classe ? Si oui, comment selon les niveaux scolaires considérés ? À quels concepts, objets et preuves donne-t-on accès selon les approches adoptées ?

ENCADEMENT THÉORIQUE ET MÉTHODOLOGIQUE

Dans la présente affiche, nous exposerons les analyses épistémologiques et mathématiques préalables à l'élaboration des dispositifs de collecte de données, ces dispositifs étant prévus pour constituer, dans un premier temps un questionnaire destiné à des étudiants d'université, dans un second temps des situations et séquences d'enseignement à expérimenter en classe. Ces considérations mathématico-épistémologiques préalables s'inscrivent donc dans la phase d'*analyse a priori* (Artigue, 1988) d'une élaboration méthodologique encore en cours.

Le corpus principal comme contenu central de l'affiche

Nous avons ainsi repéré et examiné quatre démarches typiques, qui seront synthétisées et organisées en réseau conceptuel dans le cœur de l'affiche, dont des éléments de la problématique occuperont le côté gauche.

1. Une construction de \mathbb{R} via les *coupures*, issue du texte original de Richard Dedekind (1872), dans la version traduite et commentée par Sinaceur (2008).
2. Une construction comme ensemble des suites de Cauchy de rationnels, quotienté par la relation : $\{a_n\} \sim \{b_n\} \Leftrightarrow \{a_n - b_n\} \xrightarrow{n \rightarrow +\infty} 0$. Initiée par Cantor, nous l'avons jugée insuffisamment achevée dans la source originale et nous reprenons plutôt les versions qu'en donnent Burril (1967) et Lelong-Ferrand & Arnaudière (1977). À partir de constructions analogues, ces deux ouvrages concluent sur la complétude de façons très distinctes. La comparaison des deux bifurcations est en soi intéressante et riche de significations, et fera l'objet d'un encart dans l'affiche.
3. Une construction qui associe les réels aux développements décimaux illimités propres (i. e. sans queue infinie de 9). Elle est examinée dans les notes d'un cours rédigées par M. Herzlich (2013), lui-même inspiré d'un manuel de D. Perrin.
4. Chez Choquet (2002) et chez Dieudonné (1963), des constructions qui s'appuient sur des théories algébriques relativement sophistiquées et tenues pour acquises, et où au moins une propriété centrale de la continuité est donnée comme axiome.

Conclusions

Il s'agit de repérer, à travers ces différentes approches, ce qu'elles privilégient comme objets, représentations et avenues de conceptualisation, comment elles articulent les éléments centraux que sont l'ordre, les développements décimaux infinis, la densité, et les 3 ou 4 théorèmes classiques équivalents où se déclinent la complétude et la continuité. L'horizon des questions que posent les transpositions (didactiques) de ces démarches en classe complètera l'affiche (en bas et à droite).

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TWG2: Modelling and mathematics in other disciplines

Relevant knowledge concerning the derivative concept for students of economics - A normative point of view and students' perspectives

Frank Feudel

University of Paderborn, Germany, feudel@math.uni-paderborn.de

The concept of derivative plays a major role in economics. A proper understanding of the concept and its application in economics is therefore important for students of economics. In this paper two perspectives on the relevant knowledge concerning the derivative for students of economics are presented: a normative point of view based on literature and the students' perspectives identified in an empirical study. It can be seen that the students' perspectives differed from the normative point of view. An interesting result is that, although emphasized in the course, the students of economics seemed to consider the economic interpretation of the derivative and the corresponding mathematical background knowledge to be much less important than pure mathematical procedures like differentiation rules.

Keywords: derivative, economics, concept image, economic interpretation.

INTRODUCTION AND BACKGROUND

The concept of derivative is one of the very important mathematical concepts used in economics. Therefore, students of economics should have an adequate concept image in the sense of Tall and Vinner (1981) of the derivative to be able to deal with the concept in economics in a reflective manner. The study presented in this paper is part of a larger research project (my PhD-Thesis, supervisor: Prof. Dr. Rolf Biehler) at the Centre for Higher Mathematics Education in Germany (khdm, www.khdm.de) about the understanding of the derivative by students of economics. This research project has the following three research questions:

1. Which understanding of the concept of derivative do students of economics need to have?
2. Which understanding of the concept of derivative do students of economics have before attending any mathematical course at university?
3. Which understanding of the derivative do students of economics have after the math course, especially concerning the use of the derivative in economics in the example of marginal cost?

In this paper the focus mainly lies on the question what knowledge concerning the derivative concept is relevant for students of economics (question 1). Besides a normative point of view based on literature (which also serves as theoretical framework of the study presented in this paper) the students' of economics perspectives are the main issue in this paper. If knowledge is not considered to be relevant by them, they probably will not have that knowledge later on. So the results are also relevant for question 3.

LITERATURE REVIEW ON RELEVANT KNOWLEDGE FOR STUDENTS OF ECONOMICS CONCERNING THE CONCEPT OF DERIVATIVE

The mathematical concept of derivative

According to Zandieh (2000) the concept of derivative is connected with three other mathematical concepts that she calls layers of the derivative:

1. The concept of ratio/rate (relevant for understanding the difference quotient as the first step for getting from a function f to its derivative function f')
2. The concept of limit (relevant when taking the limiting process of difference quotients)
3. The concept of function (relevant for the transition from the single value of the derivative $f'(x_0)$ to the derivative function f')

Each of the concepts can be seen as a process-object pair. For the layer of limit for example, the process is the limiting process, and the object is the value of the limit. Furthermore, Zandieh (2000) mentions multiple representations for the derivative that students ought to know: a) graphically as the slope of the tangent line at a point, b) verbally as the instantaneous rate of change, c) physically as speed or velocity, or d) symbolically as the limit of the difference quotient. These representations should be part of an adequate concept image of the derivative after the math course.

Furthermore, the students are supposed to know the connections between the derivative and other mathematical concepts like monotonicity or convexity to be able to use the concept for finding maximal or minimal values of economic functions.

The economic interpretation of the derivative

Students of economics also need to give an interpretation of the derivative in economic contexts (mostly with discrete units in the independent variable). In economics, the derivative is often interpreted as the absolute change of the values of the function if the independent variable of the function increases or decreases by one unit. In case of a cost function C , for example, the derivative $C'(x)$ is often interpreted as the additional cost while increasing the production from x to $x+1$ units (Schierenbeck, 2003). However, that additional cost for the next unit (exactly calculated by $C(x+1)-C(x)$) actually represents a different mathematical object. The derivative $C'(x)$ as a mathematical object represents the rate of change of the cost function C at the point x while $C(x+1)-C(x)$ is the absolute change of the cost while increasing the output x by one unit. Both objects differ in its numerical value and in the corresponding unit (if the output is measured in units per quantity and the cost $C(x)$ is measured in Euro, the unit of $C'(x)$ would be Euro per unit of quantity).

Although $C'(x)$ and the additional cost are different mathematical objects, they are connected via the approximation formula $C(x+h)-C(x) \approx C'(x) \cdot h$ for h close to 0 . This formula can either be derived from the symbolic representation of the derivative as

limit of the difference quotient by using the approximation aspect of the limit (Çetin, 2009) or from the property of the derivative being the slope of the tangent line that is the best approximating linear function of C near the point x (Danckwerts & Vogel, 2006). Because $h=1$ can be considered as small in economics the numerical values of $C(x+1)-C(x)$ and $C'(x)$ are often close for cost functions, which justifies the interpretation of $C'(x)$ as additional cost while increasing the output x by one unit.

The knowledge concerning the economic interpretation of the derivative is not included in the framework of Zandieh (2000) directly. One could extend the framework with an extra column “economics” like it was done by Roorda, Vos, and M. (2007). But the interpretation of $C'(x)$ as the additional cost $C(x+1)-C(x)$ still does not match one of the resulting layers (average cost $\Delta C/\Delta x$, derivative $C'(x)$, derivative function C') directly. For getting from the derivative $C'(x)$ to the additional cost $C(x+1)-C(x)$ one would have to go backwards from $C'(x)$ to the average cost per unit again and then to the additional cost $C(x+1)-C(x)$ by specifying the interval as just one unit. However, the differences in the units between $\Delta C/\Delta x$ and $C(x+1)-C(x)$ would still remain (the first term is a rate, the second is not).

The usual approach to the economic interpretation of the derivative in math courses for students of economics” that is found in many math books for students of economics, e.g. Sydsæter and Hammond (2009) or Tietze (2010), and that is also the approach in the course in which the study presented in this paper takes place, is different and avoids the layer of the average cost. Instead of starting with average cost, this approach starts with the derivative as a pure mathematical concept (with all the representations mentioned by Zandieh directly). Afterwards, the approximation formula $f(x+h)-f(x)\approx f'(x)\cdot h$ for h close to 0 is derived by the above mentioned approximation arguments, and then the argument, that $h=1$ is small in economics comes into account that finally justifying the identification of $C'(x)$ and $C(x+1)-C(x)$. For that approach Zandieh’s framework should rather be extended by approximation aspects of the derivative (like for example presented in Serhan (2009)) than with another representation “economics” containing the classic layers.

After the math course, the students of economics should know the above mentioned differences between the derivative as a mathematical concept and its economic interpretation as additional cost (differences in the numerical values and the unit), but also know the connection between $C'(x)$ and $C(x+1)-C(x)$ to justify the economic interpretation of the derivative, which is normally not done in books of economics (e.g. (Wöhe & Döring, 2010)) and should therefore be aim of the math course.

A STUDY ABOUT THE STUDENTS’ OF ECONOMICS PERSPECTIVES ON RELEVANT KNOWLEDGE CONCERNING THE DERIVATIVE

Aim of the study

The aim of the study was to find out which of the aspects of the expected knowledge concerning the derivative, that were mentioned above, are considered to be important

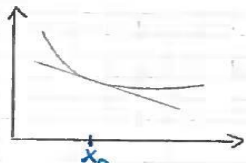
by students of economics. If students do not consider knowledge as relevant, they will probably soon forget it shortly after the exam (or even never absorb it).

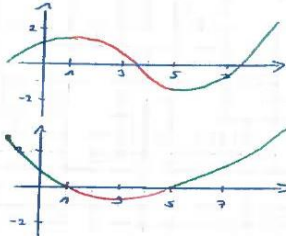
Data Collection

In January 2015, three weeks before the final exam, the students of economics at the University of Paderborn were given the homework to write a “concept summary” about the relevant knowledge concerning the derivative (which they would use when preparing for the exam) in the math course. Since the task was given as homework, the use of books and the lecture notes was allowed. The students had practiced writing such concept summaries, called concept bases in the course (Dietz, 2015), for the concept “relation” in the tutorials two weeks before and were therefore familiar with the given task. They should know that concept summaries should contain the definition of the concept, examples and counterexamples, visualizations, important statements involving the concept, and applications. So this summary should contain the definition of the derivative, the aspects of the concept image, and the economic interpretation. The task was given to them just after having dealt with the derivative in the math course. So all the relevant knowledge mentioned above was covered in the course. Since the task was voluntary, only 146 students handed a solution in (in a course with over 700 students).

Data Analysis

The summaries were analyzed with the help of quantitative content analysis. Different parts of the summaries were assigned to different categories, which were mainly deduced from the intended knowledge mentioned above: representations of the derivative (Zandieh, 2000), connections to other concepts like monotonicity, the economic interpretation of the derivative (Tietze, 2010) and relevant mathematical background knowledge especially the approximation aspect of the derivative (Çetin, 2009). During the coding process, the layers in Zandieh’s framework were first coded separately, but later aggregated for this paper due to limited space. Some categories like differentiation rules, often found in the analysis, were then also included into the scheme. In the end, this led to a system of 12 categories (table 1).

Category	Description	Prototypical example or examples
Definition	The formal definition of the derivative is mentioned.	$f'(x) = \lim_{h \rightarrow 0} \frac{f(x) - f(x_0)}{x - x_0}$
Slope of tangent line	The geometric interpretation as slope of the tangent line (or shortly as slope of the function at one point) is mentioned – either verbalized or illustrated with the help of a visualization.	<ol style="list-style-type: none"> $f'(x_0)$ is the slope of the tangent line t at the graph of f at the point $(x_0, f(x_0))$. 

Rate of Change	The interpretation of the function as (local) rate of change of a function is mentioned (verbalized as rate of growth by the lecturer). A part of the summary is already sorted into that category if the interpretation of the difference quotient as average rate of change is mentioned.	<p>1. The derivative is the limit of growth rates and represents therefore the local rate of growth of a function.</p> <p>2. $\frac{\Delta f}{\Delta x} = \frac{\text{(absolute) growth of the values of the function}}{\text{(absolute) growth in the independent variable}}$</p>
Derivatives of elementary functions	Elementary functions are written down together with their derivative.	$f(x) = ax + b \quad f'(x) = a$ $f(x) = x^p \quad f'(x) = px^{p-1}$ $f(x) = e^x \quad f'(x) = e^x$ $f(x) = \ln(x) \quad f'(x) = \frac{1}{x}$ $f(x) = \sin(x) \quad f'(x) = \cos(x)$ $f(x) = \cos(x) \quad f'(x) = -\sin(x)$
Differentiation rules	The rules of differentiation like the product rule, the quotient rule, or the chain rule are mentioned.	$(f + g)' = f' + g'$ (Sum-Rule) $(\lambda f)' = \lambda f'$ (Factor-Rule) $(fg)' = f'g + g'f$ (Product rule) $(f(g(x)))' = f'(g(x))g'(x)$ (Chain rule)
Algebraic Example	A concrete sample function (differing from elementary functions) is written down with its derivative.	$f(x) = 4x^3 + 3x^2 + 26$ $f'(x) = 12x^2 + 6x$
Derivative and monotonicity	The connection between the derivative and monotonicity for differentiable functions is mentioned or is clearly visualized.	<p>1. f is increasing $\Leftrightarrow f' \geq 0$ f is strictly increasing $\Leftrightarrow f' > 0$</p> <p>2. </p>
Derivative and convexity	The connection between the derivative and convexity for functions being two times differentiable is mentioned.	$f'' \geq 0 \Leftrightarrow f$ is convex $f'' > 0 \Rightarrow f$ is strictly convex

Derivative and extreme values	It is mentioned that the derivative is used for finding extreme values of a function.	Use: determining the maximum and the minimum value of a function
Approximation aspect	It is mentioned that the derivative can be used for approximation – either verbalized or with an approximation formula.	1. $\Delta f \approx f'(x_0)\Delta x$ 2. Approximate determination of the values of a given function
Term “marginal” means derivative	It is mentioned (either with an example of an economic function or in general) that the term “marginal” means to take its derivative.	1. Let K be a cost function. The derivative K' is called marginal cost. 2. Marginal means derivative
Economic interpretation	An economic interpretation of the derivative in an economic context is mentioned.	$K'(21) = \frac{1}{3}$ means: If one increases the output from 21 units of quantity by one unit, the cost increase round about $\frac{1}{3}$ units of money.

Table 1: Categories of the students’ summaries concerning relevant knowledge of the derivative (that has been addressed in the course “Mathematics for students of economics” at the University of Paderborn)

For each concept summary, a category occurring in the summary was coded with “1”, a missing category was coded with “0”. It is important to mention in addition, that it did not matter for the coding process if there were any mistakes in the summary. For example, a wrong connection between monotonicity and the derivative like “ f is strictly increasing $\Leftrightarrow f'(x) > 0$ ” was nevertheless coded with “1” in the category as “Derivative and monotonicity” because the connection between the derivative and monotonicity is still considered as important even if it is not known correctly.

The concept summaries were later coded again by a student, who successfully completed the course one and a half year ago, to check reliability. All categories have been proven to be reliable ($\kappa > 0.8$ for all categories) except for the category “approximation aspect”, where the student first wrongly sorted the notation $f'(x) = \frac{df}{dx}$ in that category. After recoding, this category was also reliable with $\kappa = .97$.

Results of the study

Some aspects of the derivative seem to be more important than other aspects to the students of economics. Concerning the category-scheme related to the aspects of the derivative covered in the course (table 1), the percentages of students whose summary contained a certain aspect of the derivative can be found in figure 1.

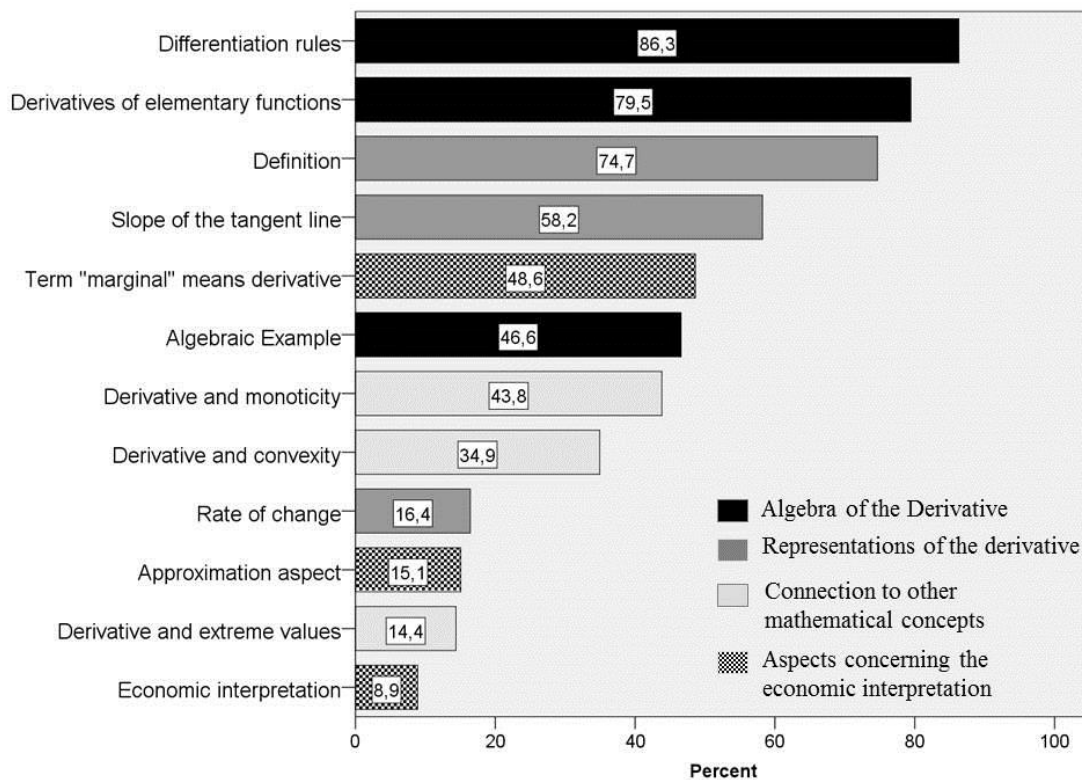


Figure 1: Percentage of students, whose summary contained the different aspects of the derivative addressed in the course “Mathematics for students of economics” (N=146)

These percentages yield several interesting direct findings:

- i. The calculation of derivatives by using algebraic rules is considered as being most important.
- ii. The geometric representation as slope of the tangent line is clearly preferred in comparison to the representation as rate of change
- iii. Only about 75% of the students included the definition in their summary after the course “Mathematics for students of economics”.
- iv. The economic interpretation of the derivative and the corresponding mathematical background knowledge concerning the approximation aspect of the derivative seemed to be least important, although these aspects are particularly relevant for students of economics

Ad i: This is not a surprising result and coincides with the often mentioned result that students are able to differentiate but often do not understand the concept of the derivative (see for example Orton (1983)). A possible explanation for our students could be that the students probably often experienced at school that calculus mainly consists of calculating derivatives with the help of the differentiation rules and using those calculations to find extreme values and turning points of functions.

Ad ii: The students of economics clearly preferred the geometric interpretation as slope of the tangent line in comparison to rate of change. This result shows that

although also studying mathematics for application like engineering students, students of economics do not seem to appreciate the representation as rate of change unlike engineering students (Maull & Berry, 2000). A reason might be that using graphical arguments, when dealing with functions, is very common in books of economics (Wöhe & Döring, 2010). This suggests that justifying the economic interpretation by using the “best approximation property of the tangent line” could reach more students than using rate of change arguments or symbolic arguments.

Ad iii: Interesting about that finding is that although the students were explicitly told that they ought to know the definition in the exam about 25% did not seem to consider the definition as important. A possible reason might be that definitions of mathematical concepts had rarely been part in exam tasks at High School. The students considering the definition to be unimportant will probably not be able to solve any task involving the definition of the derivative. They will especially not be able to justify the economic interpretation of the derivative by de-encapsulating the limiting process behind the derivative and using the resulting approximation formula $C(x+h)-C(x)\approx C'(x)\cdot h$ for h close to 0, as intended in the course.

Ad iv: This is the most interesting result. The only aspect concerning the use of the derivative in economics that many students considered to be important seems to be that the term “marginal” in economics means to take the derivative of an economic function (e.g. of a cost function). The detailed interpretation of the derivative and the relevant background knowledge for understanding it (approximation aspect of the derivative) were seen as very unimportant. This was surprising since these aspects were emphasized in the lecture and the tutorials very much. The students even had to work on problems involving these aspects by themselves. A possible reason could be that the economic interpretation seemed trivial to them and therefore did not have to be explicitly learned. Another reason could be that they cared more about procedures and vocabulary needed when dealing with economic functions in economic contexts rather than about understanding the mathematical background. Especially a justification of the identification of the derivative $f'(x)$ with the value $f(x+1)-f(x)$ with approximation arguments, used when interpreting the derivative in economic contexts, might not have been seen as necessary to them for using that interpretation (although it definitely is from a mathematical point of view).

Result iv also gives a possible explanation why students often perform poorly on tasks involving the use of the derivative for approximation, even at the end of a course (Bingolbali & Monaghan, 2008). Maybe they did not recognize that aspect to be relevant knowledge concerning the derivative and never even tried to learn it.

Limitations of the study

The students were asked to write a concept summary they could use when preparing for the exam. The task was formulated that way to motivate them really writing it and handing it in because that task could not be made obligatory for all students

because there was not enough staff in the course to correct the summaries for all of the students. Therefore, expectation concerning exam tasks could have influenced the results. Maybe some aspects would have less and others more often occurred if the students were asked to write a summary about the derivative that they could use as a reference in their later courses of economics.

DISCUSSION AND CONCLUSION FOR FUTURE RESEARCH

From the results of the study it can be clearly concluded that the students' of economics perspectives on the relevant knowledge concerning the derivative differed from the normative point of view (based on literature). In detail two results can be seen from this study:

1. The students of economics considered procedural knowledge concerning the derivative to be more important than conceptual knowledge even although they were told that the conceptual knowledge (the definition, examples, visualizations, connections to other concepts, or applications) is required for the exam.
2. The students of economics did not consider the mathematical background concerning the use of the derivative in economics as important learning material even although that background was emphasized in the course.

From the first result the following conclusion can be drawn: If conceptual knowledge is an important goal in a math course, the students must experience the importance of the conceptual knowledge by themselves already during the semester, e.g. through an adequate proportion between tasks involving conceptual knowledge and tasks involving procedural knowledge in the exercises. This is even more important for math service courses, in which many students just plan to pass the exam with 50% of the points (at least half of the tasks should then involve conceptual knowledge).

The second result concerning the fact that the economic interpretation of the derivative (and the relevant mathematical knowledge to understand that interpretation properly) was not considered to be important by many students of economics, even although it was emphasized in the course very much, was surprising. That result is a problem because students not having that knowledge will not be able to work with the derivative in economics in a reflective manner. Several reasons for the felt unimportance are possible, e.g. (felt) triviality of the economic interpretation, expectation that the economic interpretation would not occur in the exam, interest only in the procedures and vocabulary when dealing with economic functions in economic contexts and not in the mathematical background knowledge, or even no interest in economics itself because the study subject "economics" was mainly chosen because of an expected high salary.

The reasons for the felt unimportance of mathematical background knowledge directly related to the own study subject in comparison to pure mathematical procedures could be a starting point for future research. Similar phenomena might

occur in case of other mathematical concepts or procedures used in economics (e.g. elasticity, differentials, or Lagrange's method), but can also occur in other math service courses. Only if the reasons for a felt unimportance of the mathematical background knowledge concerning the use of mathematical concepts in other sciences are discovered, adequate conclusions can be drawn so that students might feel a need to understand the mathematical background properly that enables them to use the concepts in a reflective manner.

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SRP design in an Elasticity course: the role of mathematic modelling

Ignasi Florensa¹, Marianna Bosch², Josep Gascón³, Marta Mata¹

¹Escola Salesiana Universitaria de Sarrià in Barcelonan, Spain, iflorensa@euss.es;

²IQS School of Management, Univ. Ramon Llull, Barcelona, Spain;

³Dep. Matemàtiques, Univ. Autònoma de Barcelona, Spain

We present the design process of a Study and Research Path (SRP) in a course of General Elasticity, which is part of a degree in Mechanical Engineering. General Elasticity is a field in which mathematical modelling is very much present. In fact, it cannot be conceived without mathematics. We take the observation of several didactic facts associated to the actual pedagogical and didactical organization of the course as starting point. The SRP design tries to overcome these didactical problems by proposing a possible new rationale for teaching General Elasticity.

We have carried out an a priori analysis of the SRP in order to evaluate to what extent the generating question is substantial enough to act as the main motivation of the study community. A systematic plan to collect data during experimentation is also presented.

Keywords: Mathematical modelling, Anthropological Theory of the Didactic, Study and Research Path, Engineering.

MATHEMATICS AS A SERVICE SUBJECT: MODELLING

The central point of our research is the role played by mathematics in engineering courses, more particularly those with a high load of mathematics. The paper presents the design of a Study and Research Path (SRP) in a third-year General Elasticity course of a Mechanical Engineering Degree.

In this context, the third ICMI study (Howson et al 1988) presents different reflections on mathematics as a service subject. One of the central ideas of the different papers of the study is that teaching mathematics to non-mathematicians (as a service subject) should highlight the capacity of mathematics to solve the practical problems of the domain. This “capacity to solve the problems of the domain” is closely related to the ability of mathematics to model systems of the domain and, consequently, to solve the problems associated (Romo, 2014).

The integration of mathematical modelling into current educational systems has been tackled by numerous investigations but still remains a major challenge. Many examples of mathematical modelling in various domains of engineering education exist: modelling acoustic properties of materials (Hernández, Couso, & Pintó, 2014) or the works of engineering teaching in US high schools (English & Mousoulides, 2011). Numerous theoretical approaches agree on the need to incorporate mathematical modelling in mathematics and engineering teaching in consequence. As a result, some new curricular approaches try to introduce mathematical modelling in certain some university degrees (Gould, Murray, & Sanfratello, 2012) (Dangelmayr & Kirby, 2003). Some studies consider that mathematics in engineering play such an important role that

engineering could not exist without them. Because of this strong interdependence between mathematics and engineering the classical modelling cycle approach cannot be applied in this case (Bihler, Kortemeyer, & Schaper, 2015).

However, many institutional constraints and limitations appear when designing and implementing modelling devices in university teaching institutions (Barquero, Bosch, & Gascón, 2010). The institutional ecology plays a crucial role in the study of these conditions and constraints. The Anthropological Theory of the Didactic (ATD) framework enables us to describe these conditions and constraints affecting the implementation of mathematical modelling in scholar institutions, especially at university level. The necessary conditions for mathematical modelling at the undergraduate level have been studied in the case of first-year students of a business administration degree (Barquero, Serrano, & Serrano, 2013).

Mathematical modelling appears to be central in the ATD framework: the ATD postulates that “most of the mathematical activity can be identified to some extent [...] with a mathematical modelling activity” (García, Gascón, Ruiz, & Bosch, 2006). This means that mathematical activity can only be understood as a collective modelling activity. The modelling activity in the ATD framework is not limited to non-mathematical systems but includes intra-mathematical modelling as a key notion. Algebraic modelling of geometry can be seen as an example of this intra-mathematical modelling.

Garcia et al. (2006) state that the modelling process emerges from an initial generating question: from this starting point the collective work will generate a collective answer to the question. This answer can be seen as a sequence of interconnected praxeologies. Intra- or extra- mathematical models play a central role: models are used to obtain results and to understand the modelled phenomena.

WHY A STUDY AND RESEARCH PATH IN AN ELASTICITY COURSE?

Beyond the mathematical role played by mathematics as a service subject and the importance of mathematical modelling, a second motivation justifies the adoption of a SRP for the General Elasticity course. Until the last academic year this course was structured in mixed theory and problem sessions, and practical sessions. The latter included six 2-hour sessions on the following topics:

- Tensile test in three different metals (AISI 304 Stainless Steel, SR 275 Structural Steel and T6061 Aluminium).
- Charpy test in three different metals (AISI 304 Stainless Steel, SR 275 Structural Steel and T6061 Aluminium).
- Finite Element Method (FEM) simulation of a tensile test (using SolidWorks™ simulation as software).
- Oral presentation about failure criteria in different family materials.

During the practical sessions in the past two academic years three didactic facts were observed. First, a *thematic autism* in the sense of Barbé et al (2005) explicitly appeared.

This means that all four activities were ‘lived’ as independent by the students even if the activities were closely connected. For example: FEM simulation (3rd session) simulated the real test carried out in the 1st session. The second didactic fact is related to the role played by the computer during the FEM simulation. Students introduced geometrical data, loads and meshing conditions to obtain the required results. Important difficulties appeared when they tried to understand “how the computer solved the problem” and “validating the results obtained”. The students tended to validate all the results without any validating process. Both factors can be understood as a “black box” phenomenon: computer simulation is not understood by students and thus hinders them when judging the adequacy of the results obtained. And thirdly, we detected a clear absence of rationale in the four practical sessions. Both for students and for lecturers the presence of these sessions was more due to its “classical” character in elasticity than to a well-founded and justified didactic choice.

It seems that the adoption of a SRP based on a substantial enough generating question may partially overcome these limitations. The choice of the generating question emerges from the question “Why is General Elasticity taught in engineering?” which necessarily leads to the missing rationale. Once this question is posed, it is clear that the main reason to teach the subject is to provide engineers with tools enabling them to design any part of a machine working under an elastic regime. The connection between themes comes up immediately. To begin with a specific issue, the two lecturers teaching the subject agreed to start the SRP with the generating question: “How to choose one material (with unknown mechanical properties) from a set of three and design a part for a bike given in advance (brake lever, crank, gear, and bike lock key)?”

PROFESSIONAL PRACTICE FOR ENGINEERS

Some approaches, state that mathematical education as a service subject in the engineering context should also take into account the professional practice of the collective addressed. Two studies in particular highlight the role played by mathematical modelling in the professional practice of engineers (Kent and Noss, 2003) (Gainsburg, 2006). Both studies consider mathematical modelling as a paramount in the professional practice of engineers. However Gainsburg (2006) claims some crucial aspects of mathematical modelling needed to be taken into account. Firstly, she states that the “centrality” of mathematical modelling in professional practices is not identified in all research works. Gainsburg states that this phenomenon may be caused because many studies:

...focus on solution-generating activity (or even, ironically, on the comparison with school-type math) has prevented these researchers from detecting and reporting other activities that might count as modelling such as describing, interpreting and explaining quantitative relationships and patterns, making predictions or developing reusable solution methods. (Gainsburg, 2006, p 6).

The second aspect highlighted by Gainsburg and Kent & Noss is that the development of mathematical models is usually carried out by mathematics specialists and that the use of these models rarely calls for advanced calculation techniques. For example, civil engineers use basic mathematics 95% of the time: multiplication, division and understanding of statistics (Kent & Noss, 2003).

A third aspect to be taken into account is the important variety of models mobilised by engineers depending on the degree of abstraction. Engineers are usually able to work with received models but in some specific cases (because of their complexity or uniqueness) the model has to be adapted to the local reality: an adaptation process is left to the engineer. In this case, two particular challenges emerge: on the one hand practitioners have to *understand the phenomenon to be modelled* (which usually remains inaccessible to the engineers) and on the other hand the *model has to be kept in track* (practitioners have to be explicitly aware of the assumptions and hypotheses of the model used).

The last aspect addressed by both studies is the preponderant role computers play in the professional practice of engineers. Both studies agree that computers have caused profound changes in the professional practice of engineers. These computer-based technologies seem to have reduced routine calculations but increased the need to solve more complicated, non-routine problems. In addition, a new phenomenon might appear: the “black box” effect we mentioned earlier. That is why only the input to the computer and the given output are explicit and the calculations done by the computer remain implicit. This implicitness makes it difficult for the users of computer simulation to question the results obtained.

GENERAL ELASTICITY COURSE

Taking into account the different aspects presented in the previous sections a SRP on general elasticity has been designed and it will be experimented in two big groups (between 20 and 25 students) in September 2015 and January 2016. One of the groups will be taught by one researcher in didactics and the other one by a teacher with no didactic training. The students will work in the mechanical laboratory during eight 2-hour sessions. The laboratory is equipped with a universal tensile test machine, a Charpy test machine, computers with simulation software and two 3D-printers. Each large group of students will be divided in groups of 4 or 3 students: each small group will have one specific part to be designed.

Each group is asked to design a specific part of a bike. At the end of the eight sessions they will be asked to write a final report addressed to a fictional “bike design company”. The report must include:

- Specific dimensions of the part including its dimensional plans,
- Estimated loads
- Justification of the choice of the material
- Estimated strains that it will suffer while being used
- The adopted safety factor for stresses and strains

- Justification of the results regarding the computer simulation and the mathematical model used
- Final cost of the whole design process calculated by using the prices in table 1. If the students decide to carry out another test that is not available in the laboratory (and in the price list) a price will be decided by the teacher as long as its adequacy is justified by the group.

The requirement of explicitness of these aspects are expected to partially “enlighten” the existing “black boxes” such as computer simulation and mathematical models.

During the first session each small group of students receives three samples of different metallic materials, whose mechanical properties are totally unknown to the students. Then students are asked to write a first partial report that will be delivered after the first week. It shall include:

- Time planning for the whole design phase
- Initial budget
- First questions that have emerged and that are planned to be solved during the following week.

After this first report, a weekly report will be generated by the students. The content of the weekly reports is intended to collect data from the dynamics of the activity. In order to collect this kind of data the proposed content was:

- An updated time planning
- The questions that the team planned to ask during the week
- A description of the tasks carried out even if obtaining wrong results
- The obtained and validated answers that they have obtained (and how) from the questions of the week and derived questions.
- New questions for the next

Test / Material	Price
Tensile test	75 €/specimen
Hardness test	170 €/specimen
3d printer	0,25 €/printed cm ³
Engineer	50 €/h
Computer Amortization (including software licences)	1,25 € / h
Charpy test	85 €/specimen
Specimen	5 €

Table 1: Price table for different operations

EXPECTED MOBILIZED KNOWLEDGE

As an *a priori* analysis of the SRP, we have studied what kind of knowledge is expected to emerge when the students work on the design process. As a partial representation of this mobilised knowledge a question-answer map has been used (Figure 1). This tool was already used when modelling knowledge geneses from a generating question (Winsløw, Matheron, & Mercier, 2013) (Jessen, 2104).

From a mathematical point of view, the model used (generalized Hooke's Law) in general elasticity concern the use of tensors as well as their diagonalization and the eigenvectors and eigenvalues associated. In fact, the stress level of a point of a solid is characterized by a symmetrical *stress tensor* including three normal stresses and three shear stresses. The strains in a point are also characterized by a symmetrical *strain tensor* (formed by 3 normal strains and 3 shear strains). Both tensors are related (under an elastic regime) by the generalized Hooke's Law. Diagonalization of both tensors is a crucial point for two reasons. First it is a matter of economy: the stress level of a point, when diagonalized, is described by only 3 scalars (instead of 6). Secondly, the principal stresses and strains delimit all the possible tensional states and provide the conditions for failure.

Apart from mathematical knowledge many others aspects are expected to emerge. One aspect expected to appear is related to the need to estimate loads in order to feed the computer simulation as well as to establish the safety factor for the stresses and strains obtained. A second aspect is related to the use and limitations of the computer because the students have not done any previous course on FEM simulation. A FEM course is available for students as a four year subject.

Time planning and budgeting are aspects that usually remain outside the scholar knowledge related to General Elasticity but they are very much present at the professional practice of engineers. This is the main reason why students are asked to create and update on a weekly basis and to manage a limited budget defined in the first session.

Another aspect that has been considered in this *a priori* analysis is the mesogenetic level, specifically the media – milieu dialectics. The SRP has been designed in order to enrich the sources of information and the validation devices used by the students. The design of a bike part will ask students to validate its shape design, the choice of the material as well as the level of acceptable strains. The devices that students will choose validate these decisions are expected to enrich the milieu of the students comparing it with the previous sessions where any validation was used further than teacher's correction.

DATA COLLECTING PLAN

In order to collect all relevant data systematically a data collecting plan has been designed. First of all, the weekly reports of each group appear as a key document to be analysed. We expect these reports to include a wide range of information from both a

pedagogical and didactic level including practical aspects (such as organizational issues) to content-related aspects (such as the chronogenesis: question – answer dynamics).

As part of the plan a set of interviews will be included in the collected data. Three actors of the SRP will be interviewed: the non-researcher teacher, new students, and students retaking the subject. In fact, the opinion of the non-researcher teacher and of the students retaking the subject are particularly significant because they ‘lived’ the practical sessions when the didactical facts presented in the second section were observed.

Finally the students will fill out a survey evaluating the most difficult and easy aspects faced during the project, to what extent the project helped them integrate the different parts of the subject and which strong points and weak points they identified during the course.

EXPECTED RESULTS

The first aspect to be considered is how the expected articulation of different fields of knowledge has been reached. The main tool to evaluate this articulation will be to analyse the weekly reports and the questions and answers that will appear explicitly stated. The degree of transversality of the questions and their degree of interdependence can be used as factors to be taken into account.

A second aspect concerning to which extent the “black boxes” associated to computer simulation have been enlightened has to be measured. The empirical data that will enable the measurement of this aspect is the final report. The students are required to justify not only the computer simulation options but also the mathematical model underlying the model.

Another crucial point to be evaluated is the viability of the SRP. In this case the institutional conditions and constraints hindering (or facilitating) the development of the SRP experimented have to be studied. The nature of these conditions and constraints can be diverse: from practical aspects such as difficulties in the use of the laboratory equipment by the students to the rigidity of the time structure of the sessions (8 two-hour sessions during 4 weeks).

The expected results consist in measuring to what extent the observed problematic didactic facts will be partially overcome.

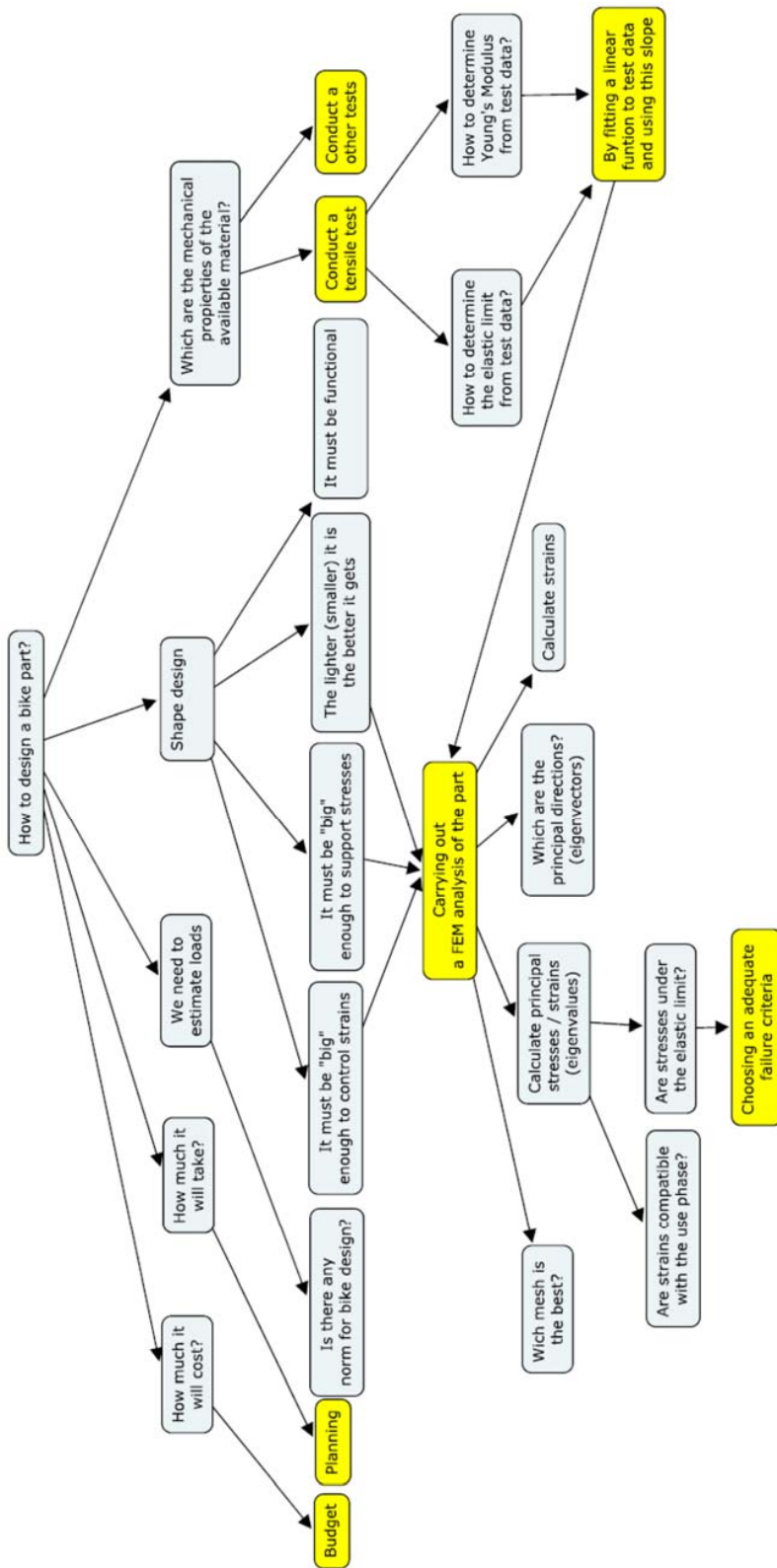


Fig 1: A priori question-answer map

AKNOWLEDGEMENTS

Founded by the Spanish Government EDU2012-39312-C03-01 and Obra social La Caixa - Universitat Ramon Llull

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Teaching Calculus in engineering courses. Different backgrounds, different *personal relationships*?

Gisela Hernandez-Gomes and Alejandro S. González-Martín

Université de Montréal, Département de Didactique, Canada,
gisela.hernandes@umontreal.ca; a.gonzalez-martin@umontreal.ca

In this paper we seek to further investigate whether having different backgrounds influences Engineering teachers' views of Calculus and shapes their opinion of how the subject should be taught, and whether these views affect their actual teaching practices. Our research is based on an institutional perspective and employs Chevallard's Anthropological Theory of the Didactic (ATD), in particular the notion of personal relationship, and we analyse the possible impacts of institutional choices on an individual's practices. Our data seem to indicate that even when they occupy the same position in the same institution, teachers with different academic backgrounds hold quite different personal relationships with the contents of their Calculus course, and that this has a significant impact on their practices.

Keywords: calculus, engineering, university teacher education, personal relationship, Anthropological Theory of the Didactic (ATD).

INTRODUCTION

In many science and technology programs, Calculus is among the first courses taught. It is considered one of the most important early courses in engineering, allowing students to subsequently study and model real problems in ways that can be applied to their professional lives. Despite this, Calculus instructors often emphasise the application of techniques, the memorisation of definitions and the manipulation of formulae, rather than the acquisition of notions that are directly relevant to the practice of engineering. This can result in students failing Calculus and abandoning their professional ambitions (Ellis, Kelton, & Rasmussen, 2014). Regarding this issue, Christensen (2008, p.131) has pointed out that “it can be quite difficult to connect the abstract formalism of mathematics with the necessary applicable skills in a given profession”, and that this could create a “gap in the students' ability to use mathematics in their engineering practices”.

In general, Engineering courses are organised into two main groups: general science courses such as mathematics, chemistry and physics; and technical courses, which are specific to each branch of Engineering. Under this system, students in their first years of study may be unable to see where and when they will practically apply the mathematics and physics they are learning. In addition, they may find it challenging to recognise and apply this knowledge in later courses. As Harris, Black, Hernandez-Martinez, Pepin, Williams, & TransMaths (2014, p.334) conclude, “mathematics should be embedded with the engineering principles being taught. There [is] a danger that when mathematics becomes isolated from its use in engineering, the opportunity to foster a perception of its use-value in the wider sense [is] lost.” Research on the

teaching and learning of Calculus and analyses of students' difficulties have spurred growing interest in teachers' practices (Rasmussen, Marrongelle & Borba, 2014), which opens a new avenue of research in postsecondary mathematics education.

As part of this trend, Pinto (2013) recently analysed two lessons on infinitesimals given by two different teaching assistants – each with a different level of experience – using the same lesson plan. The analyses show that their different beliefs, objectives and levels of confidence in various resources resulted in two substantially different lessons. For the author, “a more specific contribution of this study refers to the ways in which teachers assistants' pedagogical content knowledge, or lack of it, affected the lessons” (p. 2424). This suggests that the teaching practices of university instructors are highly influenced by their own experience. Regarding this issue, in Hernandez Gomes & González-Martín (2015a), we studied the vision of mathematics held by engineering mathematics teachers with different academic backgrounds to understand how these backgrounds shaped their teaching practices. Our data revealed differences in the way these teachers approach topics such as mathematical rigor and approximation. Using tools drawn from the Anthropological Theory of the Didactic (ATD), and specifically the notion of *personal relationship*, we analysed data from interviews with engineering students studying under those teachers (Hernandez-Gomes & González-Martín, 2015b). Our results seem to indicate that elements of the teachers' *personal relationship* with mathematics emerge in the students' interviews, in particular those elements pertaining to rigor and estimations. These preliminary results motivate our current research agenda. We seek to further investigate whether having different backgrounds influences Engineering teachers' views of Calculus and shapes their opinion of how the subject should be taught, and whether these views affect their actual teaching practices.

THEORETICAL FRAMEWORK

Some teachers receive their initial training in one faculty but eventually teach in another, while others bring their professional experience into the classroom. We wish to determine whether these different backgrounds influence teaching practices and the way instructors prepare courses, and believe an institutional approach is appropriate for this purpose. We therefore applied tools from Chevallard's (1999) ATD. According to ATD, an *institution I* (in a broad sense) is a social organisation which allows, and also imposes on its *subjects*, ways of doing and thinking proper to *I* (Chevallard, 2003, p.82). Human activity can be modelled in terms of *praxeologies*, which are defined by the types of tasks carried out, the techniques that allow tasks to be completed, a discourse to justify the techniques used, and a theory that explains and justifies the discourse. The type of tasks and techniques allowed or promoted by an *institution* – together with the discourses that justify these techniques – have an impact on the individuals who belong to the institution.

A *subject* is defined as every person x who occupies any of the possible positions p offered by *I*. For our purposes, we may use the example of a faculty of engineering (I_1) which offers several positions, including teacher (in various departments) and

student. The types of tasks, as well as the techniques and discourses available, are different for these two positions. It is possible for the same individual to have occupied the position of student and teacher in the same faculty and to have worked as an engineer at a firm (I_2); in this way they will have been exposed to new *praxeologies*, that is, types of tasks, techniques, and discourses. It is also possible for an individual to have been educated (position of student) in a faculty of Mathematics (I_3) and subsequently work as a teacher (a different position) in a faculty of Engineering (I_1). These situations, among others, lead to the idea of *personal relationship*. If we define an *object* as any entity, material or immaterial, that exists for at least one individual, then every subject x has a *personal relationship* with an object o . This *personal relationship* develops as a result of the interactions that x has with o in different institutions I , where x occupies a given position p , solving tasks where o is put into play or developing discourses where o plays an important role. The *personal relationship* includes elements such as ‘knowledge’, ‘know-how’, ‘conceptions’, ‘competencies’, ‘mastery’, and ‘mental images’ (Chevallard, 1989, p.227). All subjects in a position p within I are influenced by the *institutional relationship* with o ($R_I(p, o)$). This *institutional relationship*—which is defined as the relationship with o which should ideally be that of the subjects in position p within I —remodels subjects’ *personal relationship* with o . However, this may result in conflicts: a subject could have a *personal relationship* with an object that is at odds with the *institutional relationship* with that object. For instance, students entering university often have a *personal relationship* with functions, mostly crafted through their experiences in school and everyday life, which is not always compatible with the formal vision of functions they encounter in rigorous mathematics courses.

These tools allow us to model situations such as the ones that are the focus of our research. For instance, an individual who studies limits in a faculty of mathematics will develop her or his *personal relationship* with limits under the restrictions of $R_M(s, l)$. This *personal relationship* may be different than that of an individual who studies limits in a faculty of engineering and is subjected to $R_E(s, \lambda)$ (of course, it is also arguable that the position of each student, s , is different in each faculty). If these two individuals go on to teach limits in a faculty of engineering, they will be subjected to the *institutional relationship* $R_E(t, \lambda)$. This will further influence their *personal relationship*, which we conjecture has already been shaped by their different learning experiences. This situation can be more complex if the individuals work/have worked as engineers in addition to teaching in a faculty of Engineering, or if they teach/have taught in other faculties as well. We believe that ATD can offer an interesting lens through which to observe and analyse these phenomena and identify differences between teachers’ *personal relationships*, which might explain their divergent practices and the various choices they make in preparing courses.

METHODOLOGY

We interviewed six university teachers with different academic backgrounds, who teach Calculus in engineering programs at two different private universities in Brazil

(A and B). All six had been teaching Calculus in Engineering for at least fifteen years. Prior to the interviews, we sent them a questionnaire to collect information on their academic and professional background, which allowed us to classify their profiles (Figure 1):

Figure 1: Profile of the six teachers

T1	T2	T3	T4	T5	T6
<ul style="list-style-type: none"> •Female •Bachelor of Mathematics •Master of Mathematics Education •Doctorate of Mathematics Education 	<ul style="list-style-type: none"> •Female •Bachelor of Mathematics •Master of Mathematics •Doctorate of Mathematics 	<ul style="list-style-type: none"> •Female •Bachelor of Mathematics •Master of Space Eng. and Technol. •Doctorate of Mechanical Engineering 	<ul style="list-style-type: none"> •Male •Bachelor of Mathematics •Master of Theology •None 	<ul style="list-style-type: none"> •Female •Bachelor of Electrical Engineering •Master of Electrical Engineering •Doctorate of Electrical Engineering 	<ul style="list-style-type: none"> •Male •Bachelor of Electrical Engineering •Master of Mathematics Education •Doctorate of Mathematics Education

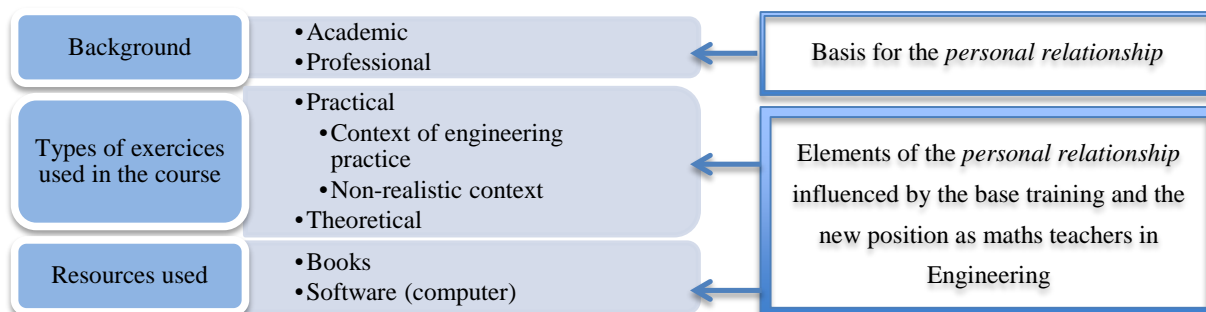
We are currently analysing data from our sample, the results of which will form the basis for future publications. For this paper, we have chosen to focus on the interviews with teachers T3 and T6. This is because T3’s profile could be considered typical for teachers in engineering faculties, and because T6’s background, while also typical, is augmented by postgraduate studies in postsecondary mathematics education—on topics introduced in his Calculus courses—which could explain important differences in his *personal relationship* with Calculus and its teaching.

Both instructors teach Calculus in first-year engineering courses; T3 teaches at universities A (15 years) and B (8 years), and T6 teaches only at university B (27 years). T3’s entire professional career has been as a university instructor. T6, in addition to teaching, worked as an electrical engineer early in his career, spending two years as both an engineer and a university instructor before focusing on teaching exclusively. At both universities, they teach a sixth-month course entitled Calculus I. The course covers functions, limits and derivatives, and ends with rate of change and optimisation problems.

All the interviews were conducted at the teachers’ workplace in a room with only the interviewer (first author of this paper) and interviewee present, on a day chosen by the interviewee. The interviews covered topics chosen to reveal the teachers’ vision of Calculus notions, how they use these notions, their background, their course and exercise preparation methods, their teaching practice, and whether the professional aspirations of their students play a factor in their approach to teaching. We sought to identify aspects of the teachers’ *personal relationship* with Calculus and its teaching, and pinpoint the origin of elements that influence this *personal relationship*. The interviews took place in September 2015, and were audio recorded and transcribed. After completing the transcription, we assigned codes to the answers and explanations of the teachers, allowing us to classify the data and facilitate our research. In particular, elements that could be linked to some type of ‘knowledge’,

‘know-how’, ‘conceptions’ and ‘mastery’, and which can be related to specific *praxeologies*, were used as indicators to guide our analyses. Figure 2 lists the elements we discuss in this paper.

Figure 2: Final categories and subcategories



DATA ANALYSIS

We asked both teachers how their academic background influences their course preparation, their choice of student exercises, and the types of books and resources they select for their course. Their first responses were transcribed as follows:

T3: In fact, I think what helps a lot is this attitude [that engineers have] toward applications. A student in the Engineering courses, he doesn't want a lot of theory, he wants to know how he will use these concepts in practice in his life. Obviously, [he also] needs to know about the concept, where it came from [...] to build his understanding, build all that... for instance, for modelling. [...] Now, this heterogeneous training is not just about thinking, solving, proving, it also is effective in engineering courses, I have no doubt. [...] I have this bias of an engineer, not being an engineer [because of her background in mathematics] [...] I strongly believe it influenced my training. I believe that a mathematician, a pure mathematician, has a different view of mathematics, of Differential and Integral Calculus. [...] Does a mathematics course have to be the same in Engineering [as it is for mathematicians]? I don't believe it has to be as rigorous [...] A mathematician who teaches Calculus, he doesn't think about the applications. [...] He's not thinking about temperature going up or down, or about controlling an air conditioner. An engineer, he's much more preoccupied with this.

T6: It has an influence, yes. For instance, when you're facing a problem you have to solve. What do I always say to my students? You're going to be engineers. What does an engineer do? He solves problems. [W]hat is a problem? Then, I make a drawing [...] and I say: "Here you have a right triangle with sides measuring 3, 4, and 5, [...] what is the area of this triangle?" Everybody [will know the answer, but if] you have a triangle, this cathetus measures 3, this hypotenuse measures 5, and I'm not giving you the measurement of the other cathetus, and I ask you the same question. [...] Then, you're going to first calculate this cathetus to get the answer. Now, how do you calculate this? Then, here you have your problem, you need to stop and think. [...] And it's the same in any other situation. You have a

problem to solve. What will you do? You'll link your islands of knowledge, you'll try to articulate, try to make links among them with what you already know [and] what you don't know. [...] Here, my training as an engineer carries a lot of weight at this point. Basically at this point.

There are two common elements in their responses. First, they both believe their training has influenced their vision and practices. Second, they both seek to foster critical thinking and instil in their students the 'skills an engineer needs', which may be a by-product of their Engineering training. However, T3's response more clearly reveals elements that seem to derive from her various educational experiences: her mathematics training could have shaped her view that students must 'know about the concept' – typical in mathematical *praxeologies* – and her engineering training could explain her belief in the need for real applications (she also provides specific examples, which we interpret as indicators that she has participated in *praxeologies* involving them). Paradoxically, T6 seems to favour solving problems using mathematics in a way that diverges from an engineer's daily practice; it appears his knowledge and know-how do not come from actual engineering *praxeologies*.

Differences are also apparent in the teachers' choice of resources for their course. T3 supplements the Calculus course book adopted by the department with other books. She also employs Winplot software to help students visualise notions, and acknowledges that students today have access to the Internet and its resources at home. However, T6's attitude is quite different:

T6: Then, it's like this, this is basically a course where all teachers of Calculus use this book. We *follow* the book. The idea is to follow the book. The student missed a lesson... [...] he goes here in the book [...] and he'll see the lesson we gave. How do I prepare my lesson? The way we follow this book [...] I, specifically, use exactly the same definition that appears in the book, I write it on the blackboard, I discuss that definition with the students.

T6 added that from time to time he uses a data projector to show students graphs or approximations, but just "to make things [...] more impactful visually. To make it cooler". If we consider T6's choice of resources as indicators of his conceptions and mastery, his *personal relationship* with Calculus seems to be closer to that of a student, which could explain his almost exclusive use of a single textbook. Although he is an engineer himself, the fact he worked in the field for just two years leads us to conjecture that his *personal relationship* with Calculus is derived mainly from his experience as a student, solving most tasks while relying heavily on a textbook.

When questioned about the types of exercises (practical, theoretical, problem solving) they use in their course, the teachers again revealed some interesting differences:

T3: They are more practical. [...] Some problems, and when I have application problems in engineering, I think this type of exercise is quite interesting, and can illustrate how to apply that concept in an application, in [engineering].

T6: When you get to the part about [...] functions and limits [it is] basically theoretical, so it goes like: Calculate the limit; find the inverse function; [...] Sketch the graph [...]. It's later, in derivatives [where] we can proceed to determine maxima and minima, and rate of change problems. [...] They are more practical problems, with practical application. But before that, they are quite conceptual: Do this, do that. But there, from rate of change on, we have... there's an inversed cone [...] being filled at a rate of some cubic meters per minute, what is the rate of change of the height in relation to time, if the height measures x meters?

Once again, we see that T6's *personal relationship* with Calculus seems reduced to what is presented in his textbook and evokes, as with the examples of the triangle, tasks that seem more related to a mathematician's *praxeologies* than an engineer's. He does not question engineers' need to master the basic theoretical tools regarding limits, and he does not seem to have knowledge or mastery of application problems where functions could be applied—for instance, modelling problems—which could be useful for future engineers.

When asked about specific examples of exercises that apply to engineering practices, the teachers' responses were:

T3: Let's think about the lesson on maxima and minima of a two-variable function. You can calculate the [...] tangent plane to a given point, and you can exemplify this with a spherical surface, calculating the shortest distance. You can give an example of a satellite in orbit, and then you calculate the shortest distance [...] from the position of the antennas. [...] Then, you get to connect the theory and obviously some applications. Obviously you make some approximations [...] because [...] you will not consider [...] all those principles that you should obviously consider in a real job or a simulation, but you use a practical example to illustrate this concept.

T6: As we are in [the first years of Engineering], you have applications [of a different type]. [...] [The exercises] are generic. [...] I won't... give specific applications [...] [Because it's the first semester, they are [students] who aren't yet at the professional level. So, [the exercises] are more generic, everyday situations, that anyone, from any field, would be able to work on that situation or problem.

Again, we see clear differences in their *personal relationships* which seem related to different *praxeologies*. T3 showed evidence of knowledge and know-how relating mathematical content to an application in engineering. In another point in the interview, she added that when a student asks her how a given notion will be used in practice, "I may not have thought beforehand of a direct application, or maybe there's no practical example in the book I use. But when a student asks that question, I take five seconds to think and tell him: 'Look, in this situation you're going to use this. You will use it in this application'". T3 also reveals an awareness of how the notions she teaches in her Calculus course will be applied in the more advanced courses of

her university's engineering program. We believe that T3's postgraduate engineering studies introduced her to specific *praxeologies* that enriched her *personal relationship* with Calculus, and that this has had an impact on her practices. Conversely, T6 states it is not possible to give examples of concrete applications in his first-year course, showing a lack of knowledge and a (likely) limited repertoire of applications for the content he teaches; this is probably due to his not having participated in specific *praxeologies* that put these notions into practice. It seems that his repertoire of applications comes solely from the textbook he uses, and that these applications are mostly mathematical and disconnected from the field of engineering. In his case, it is not possible to draw a direct line between his postgraduate training in mathematics education and his teaching practices, at least with regard to the practical needs of engineering students.

In addition to the practical application of Calculus, the interviews explored the importance that the teachers assign to theorems and demonstrations. Whereas T3 acknowledged that they help students understand a particular notion, T6 answered:

T6: Proof? Prove the theorem? No! [...] I don't do any. I don't. There are some things I usually show them, for example, the limit when x tends to zero of $\sin(x)/x$ is one. Why is this limit 1? [...] Is there an analytical proof for this? There is. Will I do it for you? I won't. Why? Because there is no interest. [...] There are situations like that here... easy to convince, you get a little table, with some values close to zero, you calculate the sine, you divide one by the other, and you see it gets close to one. But there are other situations where it's not as easy to convince [students]. And then, you say: "Guys, let's not worry about this, let's move on."

Once more, we see a significant difference between T6's and T3's *personal relationship* with notions of Calculus. Their positions regarding rigor are also quite different: it seems that T3's position is influenced by her background in mathematics and *praxeologies* that demand demonstrations, whereas T6's position seems to stem from his experience as an Engineering student.

Finally, we asked the teachers about their opinion and use of technology in their courses, in particular the use of computers in their Calculus courses and in the professional practice of engineers. In general, T3 seems to see computers as powerful tools when properly used, whereas T6 seems to think that computers do not provide students with meaningful benefits.

T3: We have very powerful computational tools. But the computer doesn't do anything on its own. Who programs it? Then, you do the programming, and you have to interpret the result. Because if you do not know the concept, [imagine] you get the result with a negative volume. And you go to your boss. A negative volume? But who created the program? It was an engineer, and he didn't realise the volume cannot be negative? He only used integrals, more integrals, he used the mathematical tool, used the computer and ...? And what do you get?

T6: Before going to the computer, I'd try to develop something [...] build some "gadget". [...] I think it gets more attention from engineering students than the computer. Computers today are just appliances. And it's fake. [...] For as much as it simulates, it's simulating, it's not reality. And I think reality is more... concrete than virtual reality. [...] I, as an engineer, I have a lot of this stuff. To convince me of something, that what the computer says is real... [...] I think engineers are more convinced of things this way. Not with the computer. I guess.

T3 demonstrates her belief in the need to properly apply mathematical notions and results. She stresses the importance of being able to interpret results, and evokes some know-how about programming and practical engineering cases. This might come from her postgraduate training, where she engaged in *praxeologies* using computers to build neural networks and for engineering purposes. On the other hand, T6 clearly shows his scepticism towards computers, which might be due to the fact he did not use them in his professional career, or because they were not a part of his own undergraduate engineering education.

FINAL REMARKS

Our data indicates that T3 and T6, although they occupy the same position in the same institution, hold quite different *personal relationships* with the content of their Calculus courses – due to their participation in different *praxeologies* throughout their academic and professional paths – and that this has a major impact on their practices. We have more data on these two teachers in the other categories we used to construct the interviews, which could help us better understand their vision of Calculus (and their teaching methods) and pinpoint possible origins of this vision. We expect that this data, together with the data from the other four teachers, will shed light on the various phenomena that influence university teachers' visions and practices. We are also aware of the limits of our research, and the fact that teachers' *personal relationships* may be influenced by factors outside their academic and professional experience. However, the model provided by the notion of *personal relationship* could be used to analyse influences originating outside academic and professional institutions and further illuminate university teachers' practices. At this point we do not intend to account for these elements.

Our work contributes to recent research on university teachers' practices and education (Rasmussen *et al.*, 2014). We are developing tools to further our study of phenomena already identified in Hernandes Gomes & González-Martín (2015a), which seem to have an effect on students' learning (Hernandes Gomes & González-Martín, 2015b). In the context of engineering courses, these tools can be used to examine the various possible profiles of Calculus teachers, and contribute to the debate on the type of mathematics most useful for engineering students. In line with Pinto's results (2013), our data also indicate that teachers with different training and experience may teach the 'same' content in different ways; the lack of teacher training may contribute to this variety of visions and explain why university teachers seem to craft their pedagogical *praxeologies* based on knowledge, conceptions, and

mental images imported from *praxeologies* present in their academic and professional experience. By analysing the complete data from interviews with all six teachers, we expect to pinpoint influential elements that can be traced to teachers' academic and professional backgrounds. This will be the focus of further research.

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Engineering students' use of visualizations to communicate about representations and applications in a technological environment

Ninni Marie Hogstad, Ghislain Maurice Norbert Isabwe, and Pauline Vos

University of Agder, Norway

Research about learning styles show that many engineering students are visually inclined. Therefore, we study engineering students' use of visualizations to communicate while solving mathematical problems. Based on a framework for mathematical representations, visualizations and mediation, we set up an explorative study with the visualization tool Sim2Bil, which combines a simulation of two cars, velocity graphs, and an input for velocity functions. We asked three engineering students to create a simulation for the cars with certain conditions and studied their visualizations related to verbal, graphical and symbolical representations. The visualizations were Sim2Bil-based, paper-based and gestures and these were all used in different stages of the problem solving.

Keywords: digital visualization tool, engineering education, kinematic simulation, representation, visualization.

INTRODUCTION

Our research is situated within mathematics education in engineering studies. Research has shown that many engineering students have a tendency to perceive and process information visually, for example in forms of pictures, diagrams and flow charts (Felder & Brent, 2005; Hames & Baker, 2015). Therefore, it is important to study aspects of visualizations in mathematics education for engineers.

Technology has offered new ways for visualizations in mathematics education by, for example, allowing users to construct and drag geometrical shapes and graphs of functions. As such, technology changes mathematical practices (Artigue, 2002; Hegedus & Moreno-Armella, 2014). Also, future developments in education may lay in expanding technology-supported collaborative work between students (Lowyck, 2014). In this sense it is important to study visualizations in how engineering students communicate with each other during problem solving.

The focus of the study is on how engineering students use visualizations in their mathematical communication. We carried out an explorative study on students working on a mathematical task within a kinematical context and with the digital visualization tool *Sim2Bil* (described in 'Methods'). Especially, we focus on how they socially create and communicate meaningful visualizations.

Within engineering education an important mathematical topic is calculus, in which students learn about concepts such as derivatives and integrals. These concepts are useful for modelling certain phenomena in an engineer's practice. In our study we focus on *integrals*. Previous studies conclude that technology can support students in

their graphical approach to integrals (Berry & Nyman, 2003; Swidan & Yerushalmy, 2014). In these studies, the tasks they used included graphical representations and the integrals were not connected to contexts. To supplement these studies, we opted for using a kinematical context for integrals, because kinematical concepts such as *velocity* and *acceleration* are basic phenomena in engineering (and in physics). Also, digital technology enables objects to move on screen. Therefore, we wanted students to use mathematical concepts for speeding up or slowing down objects. So, students were to use integrals for the simulation of movement within a technological environment.

The aim of our study is to investigate how engineering students use visualizations in their communication about representations and applications within a kinematical simulation context.

THEORETICAL FRAME

In our study students' interactions with mathematics, Sim2Bil and each other are analyzed through a socio-cultural perspective. We will explain key terms within our study: visualizations, representations and mediation.

Visualization has been used differently in the literature. Presmeg (2006) describes visualization “to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics” (p. 206-207). According to her, when someone constructs a visual arrangement (e.g. draws a graph of a function), there is a visual image in the mind guiding this creation (p. 206). The socio-cultural approach taken in this paper offers a different perspective. Visualizations organize and structure information in a spatial way, and thereby offer a means of communication. Visualizations may be mental, but they also are a social, complex system for communication.

The term *representation* is often defined by using the word represent in itself. Janvier (1987) introduced four representations for functions: 1. situations, verbal description, 2. tables, 3. graphs, and 4. formulae. Zandieh (1997) adapted the naming of representations to the derivative: 1. graphical (slope), 2. verbal (rate of change), 3. physical (velocity, acceleration) and 4. symbolical (difference quotient). Using and adapting these terms to the concept of integral, we can say that an integral has different representations. First, an integral can be represented verbally, for example, by stating “an integral can be calculated by an anti-derivative” or “an integral is a limit of a summation”. Second, an integral can be represented graphically as area under a graph. Third, an integral can be represented symbolically: $F(x) = \int_0^x f(t) dt$.

These mathematical representations (verbal, graphical, and symbolical) have been developed by mankind through a long cultural history. They are cultural artefacts.

When set within a kinematical context, an integral can have special interpretations. To capture the kinematical context, Zandieh (1997) introduced the term ‘physical

representation'. We will adapt this by speaking of kinematical applications. Kinematics is a sub-discipline of physics, in which motion is studied through concepts, such as *velocity* and *acceleration*. Within a kinematical context, if one has a function $v(t)$ for the velocity of an object, then an integral models the distance travelled $s(t)$. This kinematical application of an integral is not mutually exclusive to the three mathematical representations. Instead, stating that "the integral of the velocity yields the distance travelled" is a verbal representation, which could also be expressed graphically or symbolically. Engineering students will need to learn about such representations and applications, and about which one is needed when. Hong and Thomas (2015) point out that students need to develop considerable flexibility in this, in order to do well in their studies.

Finally, we want to explain the term *mediation*. In our interpretation of the social world around us, cultural tools play a role in mediation. Tools are important resources and function as media, through which we communicate knowledge in the social context (Vygotsky, 1978). Cultural tools within this study are, for example, Sim2Bil, gestures, language and mathematical representations. Tools have a dual nature: material and ideal. Tools are material in that they are embodied in artifacts. Language is material in the configuration of writing, sound waves or as neuronal activity (Cole, 1993). Tools are ideal in that they contain "in coded form the interactions of which they were previously a part and which they mediate in the present (e.g. the structure of a pencil carries within it the history of certain forms of writing)" (op.cit., p. 249).

In the technological environment we created, the students can use mediating tools in different ways for their communication, and we will focus on visualizations. Our research question is: *how do engineering students use visualizations to communicate about representations and applications in a technological environment?*

METHODS

The methodological approach seeks to document students' communication using Sim2Bil. The object of study is the interaction between students. We set up a small-scale controlled environment, which was organized outside the normal lectures. The students volunteered to take part in our experiment.

Participants

The participants were three engineering students from our university and indicated by pseudonyms: Sam, Erik and Tom, 20-25 year old. The students had almost finished their first year, which included courses in calculus, linear algebra and physics (including kinematics). Sim2Bil was not familiar to them.

Sim2Bil

Figure 1 shows the digital visualization tool Sim2Bil. There are four windows. In the top left is a window where two cars can drive from a starting line to a finish line.

One car is green, one car is red. This is the *simulation window*. In the bottom left is a window with two graphs. One graph is for the velocity-time function of the green car and the other graph is for the red car. This is the *graph window*. In the bottom right are the velocity functions for each of the cars. Here, parameters for a 3rd degree polynomial can be inserted to create a velocity function. This is called the *formula window*. The fourth window includes a menu, which is unused in the study.

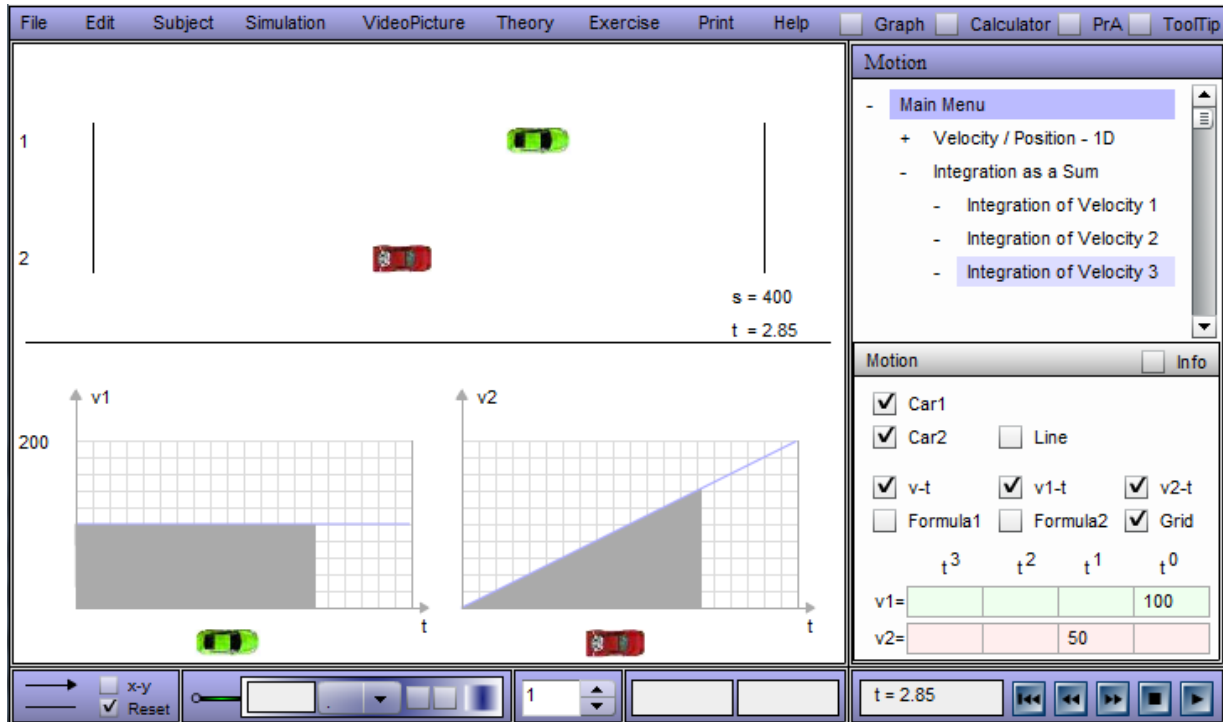


Figure 1: The interface of Sim2Bil for a kinematic simulation

A user of the tool can fill parameters for the velocity functions for each of the cars. By pressing the *Start*-button (bottom right corner), the cars will start running from the starting line to the finish line. The distance between start and finish is set to be 400 meters. The areas under the graphs will be shaded grey in an animation, whereby the shaded areas increase while the time runs (it is the same scale for both graphs). These areas represent the distance travelled, but this is not communicated explicitly.

By default, there were already two functions given in the formula window:

$$v1=100 \quad (\text{for the green car})$$

$$v2=50t \quad (\text{for the red car})$$

With these given functions a user can start the simulation by pressing the *start*-button. A user then sees how the cars run differently in the simulation window and at the same time sees two different graphs in the graph window. This is an easy introduction to the functionalities of the tool. To summarize: in the tool the symbolical representation generates simultaneously the graphical representation and the kinematical simulation.

The task

The students were asked to press the *Start*-button to see the cars run. The default setting made the cars run with different speeds, but they finished at the same time. This ‘finishing together’ constitutes a certain condition for the cars’ trajectories. The students were asked to fulfil a similar condition in following task:

Find the velocities of the green car and red car (v_1 and v_2 respectively) so that v_2 is half of v_1 when they reach the finish line simultaneously at 4 sec. Can you prove that your answer is correct?

The task was offered on paper in a verbal representation. Symbolically, the conditions for the simulation of the cars may have been written as:

At t_{finish} you should have $s_1=s_2=400$ and $v_1=2*v_2$.

The task is open to different approaches, and therefore we expected it to generate collaboration between students. A trivial approach is to reverse the velocity functions of the cars. If the students suggested this, they would be asked to find other ways. It was anticipated that students would see relations between the distance travelled by the cars (in the simulation window), the shaded areas under the graphs (in the graph window), and the integral of the velocity functions (in the formula window). The representations can have different roles. The *simulation window* shows animated distance and velocity, and offers validation whether both cars get to the finish line at the same time. The *graph window* can, amongst others, offer information on the velocity at the finish line. The *formula window* asks for performing symbolic manipulations up to 3rd degree functions, which constrains possible solutions.

The tool was set up on a laptop in front of the students. Additionally, students had availability over paper, a pen and their own handheld calculator.

Data collection techniques

To capture the students’ interaction with Sim2Bil and how they communicated with each other using verbal expressions, visualizations and gestures, the students’ work was video recorded with two cameras. One camera was directed at the students and their writings. The second camera was directed at the computer screen to capture mouse movements and students’ input within the interface. The first author of this article was present, introduced the interface and task to the students. Students’ notes were collected at the end. The students were not given a specific time for their work and it turned out that they spent 26 minutes on this task.

Data analysis

For the data analysis multimodal transcriptions (including written and spoken words, other writings, and gestures) were applied to the video recordings. The transcripts were coded and put into categories. We studied the instances, in which the students

used visualizations in their work. The visualizations will be related to the different representations they used.

FINDINGS

The three students Sam, Erik and Tom were sitting from left to right. Erik pressed the *Start*-button in Sim2Bil and the three students watched the two cars starting together, running at different velocities, but finishing together again.

The students started to discuss the task. To meet the conditions in the task, Erik started explaining the cars' paths by making a gesture of how the cars would run. He used one hand for each of the cars, as if he was replaying the simulation window. Figure 2 shows this gesturing.

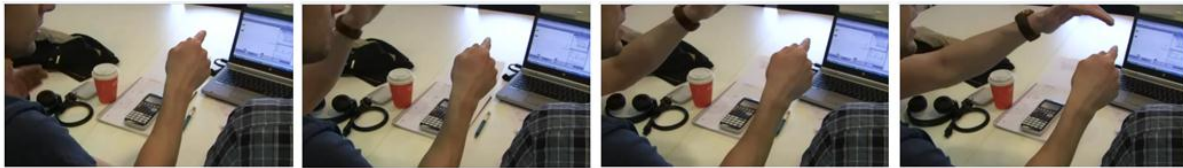


Figure 2: Excerpt of Erik's sequence of gestures of the cars' movement

He explained verbally the cars' paths:

- 5 Erik: The red car should, the red car should speed up and then it should [his hands simulate the cars] slow down and then should this, the green car catch up. And then when they reach the finish line it should be half of the speed as the green car is when they drive into the finish line.

So, at the start Sim2Bil mediated a kinematic situation by showing the cars running in the simulation window. This kinematic situation was mimicked by Erik. Gestures and a verbal description of the kinematics were used by him and mediated to his peers how the cars' would need to run.

While Erik and Tom started a discussion on the conditions, Sam started to calculate on his own. After some minutes Sam wanted to verify his calculations. He asked Erik to insert parameters into the formula window in Sim2Bil:

- 58 Sam: Write one half
[Erik writes in a wrong number, and Sam corrects him]
- 67 Erik: One half. There I got one half
- 68 Sam: No, then it's wrong [looking at the graphs at the screen]
- 69 Erik: I tried to take minus fifty there [pointing at the table on screen] and then it will go like that. So, they will reach at the same time, but it will stop there. It shall be half the speed when it stops there. So, it must have a [making a gesture of a parabola, see Figure 3]

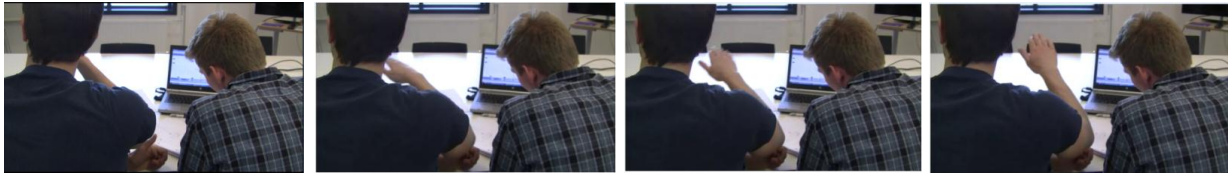


Figure 3: Excerpt of Erik's sequence of a gesture of the parabola

In this episode, Sam came up with velocity functions (which were incorrect) and he suggested using Sim2Bil for verifying. They observed the graphs of his functions within the graph window, and this alone served as verification. They didn't need to see the cars running in the simulation window, but merely a visualization of the functions through its graphical representation. Thus, the graphical representation in Sim2Bil mediated whether Sam's functions met the conditions.

In 69 Erik gestures a parabola, which is another visualization of a graphical representation. So, Erik's parabolic gesture mediated a new idea.

After discussing more, Erik suggested to simplify the problem by letting one of the cars run with constant speed. They let the green car have the same velocity as it had at the beginning of the work ($v_1=100$). Tom drew a sketch of two graphs (one for each of the cars), for which the enclosed areas needed to be equal to fulfil the conditions in the task. Figure 4 shows Tom's sketch, which is a visualization that mediates his thoughts to his peers. Here is what they said and drew:

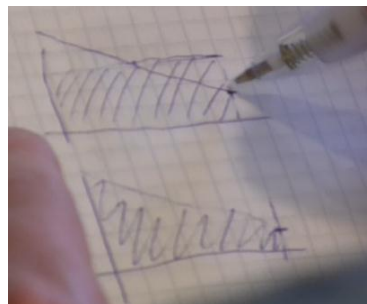


Figure 4: Tom's visualization of graphs

- 148 Tom: And then you have the other one [draws the bottom graph in Figure 4]. That should just, that should end on half of here right [draws a point to the right of each graphs]? So, it should hit here. And then it should begin up here [draws an oblique line], and it shall have the same area [shades area under bottom graph]
- 149 Sam: Like the other one
- 150 Tom: Like the other one. Ehm, so then I mean that you can just draw a line from the middle there then. Up like this I was about to say. So the area here, so you can find out how high up it must start. And the slope on this one here, because it should, because... yes. You only need the same area on this one if you think trigonometrically then
- 156 Sam: But what it is about is to find one...the area and a function for the y [the oblique line]

Figure 4 and 5 show the geometrical approach. Erik started from a constant velocity for one car, and took the distance driven as area under the graph (a graphical representation of integrals). This was a rectangle. Thereafter, they cut a triangle off to make a trapezium (a rectangle + a triangle) for the second car. So, Tom's first draft helped to develop and communicate the approach of using area calculation for rectangles and a triangle (they did not explicitly say that they wanted to avoid anti-derivatives). Then Sam filled in details to make a detailed graphical representation, see Figure 5. From there they created a velocity function for the second car.

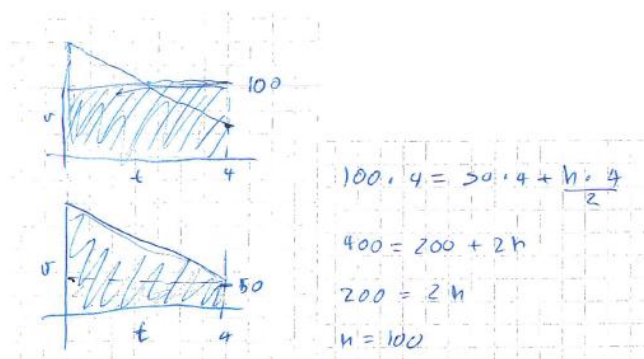


Figure 5: Calculating areas

The students ended with a graphical representation of two functions and the areas under their graphs. With the vertical v -axis, horizontal t -axis and proportionally measured values, this has the characteristics of a culturally mediating artefact. This picture could have been mediated by Sim2Bil, because the students could have copied the graph from the tool. Also, it could have been mediated in the courses that they had followed. To reach this stage, they first needed a visualization in the shape of a sketch, then a detailed drawing before they could reach a symbolical representation. Thereafter, they could fill in the parameters into the formula window of Sim2Bil to verify their symbolical representations. They were cheering for the cars when these finished exactly together at the finish line.

CONCLUSION AND DISCUSSION

We have studied engineering students' use of visualizations to communicate about representations and applications while using Sim2Bil. We observed three different media for visualizations:

- Sim2Bil-based visualizations: the cars running in the simulation window and the graphs of the functions in the graph window
- paper-based visualizations: sketches and detailed drawings
- gestures: mimicking the kinematic application (showing how the cars would run), and indicating a parabolic graph.

Although the task could have existed without Sim2Bil (by symbolical calculations), the tool was central in the task by offering visualizations. At the start, it served to introduce the students to the task, showing two cars run different trajectories, and finishing at the same time. Midway and at the end, the students used the tool for discussing and verifying, whereby both the graph window and the simulation window were used. So at the start, the tool mediated the mechanism for the kinematic simulation, while later the tool mediated the verification by giving feedback on the inserted parameters. Also, the mediating tool could be observed to be material and ideal. For example, the *Start*-button is a material area on which to click (and no real button) and it is ideal as a familiar sign developed through a cultural history.

The paper-based visualizations were used by the engineering students with the aim to produce symbolical representations for the formula window. For this, they needed to make sketches for organizing, discussing, and elaborating their ideas. So, the paper-based visualizations were tools in the mediation to find symbolical representations.

Finally, the gestures were mainly used to mediate ideas at the beginning of the session, when the students were exploring and explaining to each other the conditions of the task. Table 1 summarizes how the students' used visualizations in their communication about representations and applications of the integral. One cell remains empty: the students did not use gestures to produce symbolical representations.

		Mathematical representations			Kinematical applications
		Verbal	Graphical	Symbolical	
Visualizations	Gestures	explore explain	explain		explore explain
	Paper-based	discuss organize	elaborate	produce	discuss, organize elaborate
	Sim2Bil-based	discuss	verify	enter	introduce discuss, verify

Table 1: Use of visualizations in communicating about representations and applications

Of course, the results in this study are affected by the functionalities of Sim2Bil. If another digital tool had been offered, and a different task had been asked, the visualizations used by the students would have been different. Nevertheless, we see a variety in visualizations used by the students. This tells us that an environment such as Sim2Bil offers rich opportunities to visualize mathematics (e.g. the cars moving), and to connect different mathematical representations and applications. This sheds light on how we could enrich technology-based problem solving with visual aspects, not only for engineering students, but also for other students in higher education.

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Une approche fréquentiste des probabilités et statistiques en première année d'Université au Vietnam dans un cursus non mathématique

Jean-baptiste Lagrange¹ and Bui Ahn Kiet²

¹LDAR, Université Paris-Diderot, France, jb.lagrange@casyopee.eu;

²Cantho University, VietNam

Au Vietnam, l'enseignement des probabilités donne une place importante à une approche « classique » basée sur la loi de Laplace, et les statistiques inférentielles sont vues comme une application. De plus, l'enseignement ne tient pas compte de l'utilisation des logiciels, génériques ou consacrés aux statistiques. Ceci entraîne des difficultés ayant une signification particulière pour les étudiants d'autres disciplines que les mathématiques, qui doivent être préparés à la compréhension de phénomènes aléatoires et à une approche statistique dans la vie réelle. Cet article vise à évaluer la possibilité d'introduire des innovations « viables » dans l'enseignement des probabilités et statistiques au Vietnam au niveau universitaire, ainsi que les améliorations que ces innovations apportent à la compréhension des étudiants.

Keywords: approche fréquentiste, probabilités et statistiques, R, tableur, modèles.

INTRODUCTION

Au Vietnam comme dans d'autres pays, les manuels d'enseignement et le programme en probabilités et statistiques à l'université sont élaborés selon cet ordre: l'analyse combinatoire, la formule de Laplace, les fréquences relatives dans des essais répétés, le calcul sur des événements, les variables aléatoires et puis les statistiques descriptives et inférentielles. La formule de Laplace s'applique à une loi de probabilité uniforme sur un univers fini. Elle permet de calculer la probabilité d'un événement comme le quotient du nombre d'éventualités dans cet événement par le nombre total d'éventualités dans l'univers. Dans le cas où la distribution modélise un phénomène aléatoire (par exemple un lancer de dé) la loi de probabilité est présumée à partir du principe de raison suffisante¹. L'étude des fréquences relatives dans des essais répétés débouche sur la loi des grands nombres selon laquelle les caractéristiques statistiques d'un échantillon aléatoire se rapprochent d'autant plus des caractéristiques de la population dont l'échantillon est issu, que la taille de l'échantillon augmente. Formalisée en théorème, la loi forte des grands nombres est le fondement théorique des statistiques inférentielles.

L'approche choisie par les manuels et les programmes, privilégie l'analyse combinatoire comme fondement du calcul de probabilité, donnant peu de place à l'aléatoire. Dans le chapitre sur les fréquences relatives, un nombre limité d'exemples d'expériences répétées est donné, mais il n'y a pas de tâches qui confronteraient les étudiants à l'aléatoire, telles que l'observation des fluctuations de façon à estimer la

taille d'échantillon nécessaire pour l'estimation d'un caractère avec une incertitude donnée. Ceci entraîne des difficultés :

- Même pour des tâches «classiques», de nombreux étudiants ont des difficultés avec l'analyse combinatoire et ne peuvent donc calculer avec succès les probabilités théoriques. Ils n'ont aucun moyen pour confronter leur raisonnement et leurs calculs à la réalité, parce qu'il n'y a pas de lien réalisé avec des fréquences empiriques. La relation entre les fréquences empiriques et la probabilité théorique d'un événement reste superficielle.
- Le lien entre probabilités et statistiques inférentielles est fait sur un plan théorique très difficile pour ces étudiants, et dans la plupart des cas ils peuvent seulement appliquer des formules sans comprendre les théories sous-jacentes.
- Les étudiants ne sont pas vraiment confrontés à des phénomènes aléatoires, et manquent ainsi des occasions pour une compréhension profonde de la théorie des probabilités et de ses liens avec les questions statistiques de tous les jours, comme par exemple la taille de la population à retenir pour un sondage.
- Les manuels et les programmes ne tiennent pas compte de l'utilisation croissante des logiciels, génériques ou consacrés aux statistiques, les étudiants s'éloignent donc de la réalité pratique dans leur monde professionnel.

Ces difficultés ont une signification particulière pour les étudiants d'autres disciplines que les mathématiques, qui doivent être bien préparés pour la compréhension de phénomènes aléatoires et une approche statistique dans la vie réelle et le monde professionnel, plutôt que d'une approche purement mathématique. L'objectif général de l'étude est de remédier à ces difficultés et, plus précisément, d'évaluer la possibilité de la mise en œuvre de deux innovations, l'une liée à une "approche fréquentiste", et l'autre à la simulation, à l'intérieur d'un cours existant destiné aux étudiants d'autres disciplines que les mathématiques.

REVUE DE LITTERATURE ET BUTS DE L'ETUDE

Il existe des preuves de recherche solide que les conceptions erronées au sujet des probabilités ne disparaissent pas à la suite de l'enseignement traditionnel centré sur des définitions formelles, des règles et des procédures. Bien que les étudiants puissent apprendre des règles et procédures en probabilité et même dans le cas où ils obtiennent des réponses exactes aux tests mathématiques, ces mêmes étudiants se méprennent souvent à propos des idées et des concepts de base et ignorent souvent les règles lors de leur propre jugement sur les événements incertains. Surtout pour les étudiants universitaires, les conceptions erronées peuvent aussi apparaître comme le résultat de l'enseignement reçu sous la forme de théories explicites utilisées en dehors de leur champ d'application. Une intervention pédagogique conçue spécifiquement pour éliminer les conceptions erronées des étudiants sur les probabilités est nécessaire pour que des améliorations tangibles et stables dans les concepts des étudiants puissent être obtenues. La participation active des étudiants dans la construction des connaissances,

la confrontation avec de grands échantillons, l'utilisation de la simulation informatisée sont des outils pour cet objectif².

Nous retenons aussi que l'idée de fréquence relative d'un événement et les modèles de comportement à long terme jouent un rôle très important. Nous souhaitons introduire une « approche fréquentiste » de la probabilité d'un événement basée sur l'observation de la convergence des fréquences relatives pour cet événement dans les essais aléatoires répétées de façon à sensibiliser les étudiants à la fois sur le caractère imprévisible à court terme des phénomènes aléatoires et la régularité à long terme que décrit la probabilité. De plus, cette approche fréquentiste sera utile en fournissant une approximation d'une probabilité réelle, grâce à un échantillon suffisamment grand : les étudiants seront en mesure de confronter des résultats empiriques obtenus par l'observation des fréquences, et les résultats théoriques obtenus par l'approche classique. L'observation d'une divergence pourra les aider à prendre conscience d'une conception erronée.

L'utilisation de la simulation est cohérente avec une approche fréquentiste, parce que l'observation de la convergence des fréquences relatives n'est généralement pas possible sur les données réelles. En outre, la simulation sert à construire un modèle d'un phénomène aléatoire et donc une profonde compréhension de la situation aléatoire sous-jacente, et prépare ainsi une « approche classique ». Selon la littérature, par exemple Garfield, Chance & Snell (2000), un tableur comme Excel, et un langage de programmation comme R offrent des possibilités pour cela. Avec un tableur les étudiants peuvent construire simplement une simulation et en visualiser directement les résultats. Ils peuvent facilement produire et utiliser de nombreux échantillons. R est un logiciel professionnel et donc son usage aide les étudiants à se préparer pour de nouvelles utilisations dans leur vie professionnelle. Par rapport à un tableur, la simulation dans R peut être effectuée sur de très grands échantillons. Les questions de recherche découlent de ces considérations :

- Quelles sont ces tâches et les techniques liées à une "approche fréquentiste», et comment améliorent-elles l'enseignement/apprentissage des probabilités, en particulier en ce qui concerne les conceptions erronées et les modèles inadéquats des situations aléatoires ?
- Comment connecter cette approche fréquentiste et l'approche classique? Surtout, comment construire un milieu (Brousseau 1997) adéquat et mettre en œuvre des contrats didactiques appropriés à cette connexion?

MÉTHODOLOGIE

Notre étude est exploratoire, en ce sens que nous avons mis en place des séances expérimentales, conçues et menées par le second auteur de cet article dans le cadre de sa thèse, et que ces séances servent aussi bien à mettre en évidence des apports des innovations mises en place, qu'à repérer des occasions manquées qu'il s'agit d'analyser comme des points d'attention pour de futures implémentations.

Les séances expérimentales

Les séances ont été menées à l'Université de Can Tho dans une classe de 30 étudiants de différents cursus : économie, ingénierie, informatique, agriculture... Ces étudiants suivent le cours probabilités et statistiques de première année organisé sur une série de 45 leçons de 50 minutes sur 15 semaines. Trois leçons ont été regroupées pendant quatre semaines pour organiser quatre séances expérimentales de deux heures et demie. Les quatre séances expérimentales correspondent à quatre problèmes aléatoires. Dans chaque problème, nous considérons un ou plusieurs événements, ou une variable aléatoire et posons une question liée à la probabilité de ces événements ou à l'espérance de la variable. Nous choisissons ces problèmes parce que dans chacun d'entre eux, les données en jeu ne sont pas évidentes pour les étudiants et peuvent être un sujet de débat. Pour chaque problème, la construction d'une simulation doit apporter deux contributions différentes: d'abord pour obtenir des fréquences relatives ou des moyennes d'un événement pour un nombre donné d'essais, et ensuite pour développer un modèle qui sera aussi utile pour un calcul théorique de la probabilité. La mise en œuvre commune des quatre séances suit cette analyse : les étudiants font d'abord quelques expériences pratiques pour se familiariser avec le problème puis l'enseignant lance une discussion sur la question en jeu. Ensuite, les étudiants construisent une simulation et observent les fréquences ou les moyennes et leurs fluctuations en vue d'étudier la question. La dernière étape est le calcul mathématique classique pour confirmer la réponse.

Les problèmes

1. Le problème des deux dés : Soit X la somme des nombres apparus après le lancement de deux dés équilibrés. La question est de comparer $P(X = 7)$ et $P(X = 8)$. Ces chiffres ont été choisis parce un malentendu commun est que ces probabilités sont égales, la paire (4; 4) étant comptée deux fois dans la numérotation des événements élémentaires.
2. Le problème du lapin et de la tortue : Lancer un dé juste, si le nombre 6 apparaît alors c'est la victoire de lapin; si le numéro 6 n'apparaît pas alors la tortue fait un pas. Continuer à lancer le dé jusqu'à la victoire du lapin ou jusqu'à ce que la tortue fasse 6 pas et gagne. La question est de décider qui a plus de chance de gagner. La probabilité de gain de la tortue est proche de $1/3$, mais ne peut être estimée sans un calcul ou une simulation. Le nombre de pas impose une simulation plus élaborée que dans le cas précédent.
3. Le problème des canards (Engel, 1990) : Il y a cinq chasseurs et cinq canards. Chaque tir d'un chasseur vise un des cinq canards au hasard et ne le rate jamais. La question est la moyenne du nombre de canards survivants quand les cinq chasseurs tirent simultanément. Le nombre 5 a été choisi afin que la taille de l'univers soit suffisamment grande pour que le simple comptage des éventualités soit irréaliste.

4. Le problème de Monty Hall : Il y a trois portes, derrière l'une d'elles se trouve une voiture et derrière les deux autres se trouve une chèvre. Le joueur est autorisé à choisir l'une des trois portes et celle-ci n'est pas ouverte. Puis l'animateur ouvre l'une des deux autres portes. Dans le cas où la voiture est derrière une des deux portes non choisies, l'animateur ouvre toujours la porte sans la voiture. Ensuite, le joueur peut rester sur son premier choix ou changer pour sélectionner l'autre porte qui n'est pas ouverte. La question est de décider si, pour gagner la voiture, le joueur a avantage à garder son choix original ou à passer à l'autre porte. Il s'agit d'une situation complexe mettant en jeu un raisonnement sous hypothèse. Les personnes non averties ont tendance à penser que changer de porte ne donne pas un avantage.

Logiciels et simulation

Pour présenter l'usage des logiciels dans les séances expérimentales, nous utilisons la notion de « fonctionnalités didactiques d'un outil numérique » (Cerulli et al., 2006) définie par trois éléments clés :

1) Un ensemble de fonctions de l'outil. Comme tous les tableurs, Excel est un logiciel basé sur des formules. Nous pouvons voir l'évaluation des formules dynamiquement mises à jour sur l'écran de l'ordinateur. La répétition se fait par la copie des formules dynamiques selon les lignes ou colonnes. Il n'y a pas de façon commode d'exprimer une condition d'arrêt et le nombre de répétitions est limité dans la pratique. Excel possède également des fonctions standard pour les statistiques descriptives et un générateur pseudo aléatoire. L'affichage graphique est utile pour la présentation des données recueillies à partir de simulations. Excel a également la touche F9 qui est utilisée pour recalculer toutes les formules, en donnant de nouvelles valeurs pour les données pseudo aléatoires. R est un langage de programmation fonctionnel. R peut être utilisé par des lignes de commande ou par programmation de fonctions structurées. Dans R la structure principale de la répétition est la boucle for. R dispose de fonctions pour les vecteurs (moyenne, unique) utiles pour les traitements statistiques. Le générateur pseudo aléatoire est appelé par l'intermédiaire de la fonction `sample(a: b, n, repl)`. Le paramètre `repl` contrôle le «remplacement». Le rééchantillonnage peut être effectué en réexécutant.

(2) un objectif éducatif. Conformément à la loi des grands nombres, Excel et R peuvent être utilisés pour simuler une situation aléatoire et recueillir des données afin d'approcher la probabilité théorique d'un événement ou la moyenne théorique d'une variable aléatoire en augmentant la taille de l'échantillon. Les résultats de la simulation dans Excel ou R seront utilisés pour confirmer ou infirmer le calcul théorique de probabilité et aider les étudiants à mieux contrôler ces calculs. C'est la dimension pragmatique. La simulation dans le contexte de l'enseignement peut avoir aussi une dimension conceptuelle. Premièrement, les étudiants peuvent prendre conscience de la diminution de la fluctuation des fréquences relatives ou des moyennes statistiques lorsque l'échantillon augmente de taille et comprendre les probabilités ou les espérances théoriques comme des limites. Deuxièmement, pour

effectuer une simulation, les étudiants devront construire un modèle leur donnant un meilleur contrôle dans le calcul des probabilités théoriques.

(3) Les modalités d'utilisation de l'outil. Dans chaque séance expérimentale, les étudiants feront d'abord une simulation avec Excel, puis une simulation avec R, avant de faire un calcul théorique. Ils vont ainsi construire un premier modèle sur tableur et obtenir des résultats sur des échantillons limités, puis adapter le modèle ou en créer un nouveau avec R et obtenir des résultats sur des échantillons plus grands. Les modèles seront utiles pour le calcul théorique, et les résultats des simulations seront confrontés aux valeurs obtenues par calcul théorique.

EXPERIMENTATION

Pour chaque séance, nous donnons quelques indications sur le déroulement, puis une analyse résumée des observations réalisées par Bui (2015).

Séance expérimentale 1 : La somme des deux dés

L'enseignant amène les étudiants à se familiariser avec la réalisation d'une expérimentation concrète avec deux dés et d'une collecte des données. Il exploite les données recueillies afin que les étudiants voient la nécessité d'une plus grande taille de l'échantillon afin de pouvoir comparer les probabilités d'obtenir respectivement 7 et 8. Puis les étudiants effectuent des simulations en utilisant le tableur Excel sous la direction de l'enseignant. La fluctuation des fréquences conduit les étudiants à voir la nécessité d'une plus grande taille de l'échantillon. Puis, l'enseignant présente le logiciel R pour la simulation. A la fin de la séance, il fait discuter les étudiants sur l'intérêt de la simulation.

Les étudiants sont sensibles aux fluctuations et font le lien avec la taille de l'échantillon sans cependant les exploiter directement pour comparer les deux probabilités en jeu. Concernant la taille de l'échantillon lors des simulations, l'enseignant et les étudiants se limitent à deux tailles : 1000 (trop petite) et 100.000 (assez grande); l'occasion a été manquée de questionner la taille de l'échantillon, en vue de déterminer une taille optimale permettant de discriminer les deux probabilités en jeu ($P(S = 7) > P(S = 8)$ dans 95% des cas). Dans le tableur, les étudiants ont des difficultés avec la fonction *ALEA* qui appelle le générateur de nombre aléatoires : ils ne comprennent que chaque appel déclenche un nouvel appel et non une référence à la valeur antérieure. La fonction *sample* de R donnant directement un échantillon, ils ne rencontrent pas cette difficulté dans la seconde simulation.

Cette séance expérimentale témoigne des conceptions erronées des étudiants dans les différentes phases. Ces conceptions erronées sont déstabilisées par la confrontation aux fréquences obtenues par simulation sur un large échantillon, plutôt que par une réflexion sur le modèle développé pour cette simulation. En particulier, dans la dernière phase, lors du calcul de probabilités théoriques, les fréquences obtenues par la simulation sont utilisées pour invalider la conception erronée d'un espace

constitué de paires (non ordonnées). Cependant, le modèle développé pour la simulation n'est pas exploité, alors qu'en considérant par exemple les données produites par le tableur, les étudiants pourraient voir que les sommes 4+3, 3+4 et 4+4 apparaissent avec la même fréquence. L'enseignant institutionnalise ce rôle pragmatique de la simulation, plutôt qu'une réflexion sur le modèle construit dans les phases de simulation.

Séance expérimentale 2 : Le lapin et la tortue

Cette séance marque une progression dans la complexité et n'est pas facile pour de nombreux étudiants. Après la première phase d'expérimentation avec un dé, l'enseignant et les étudiants ne reprennent plus la question initiale (le gagnant le plus probable) qui est ainsi oubliée au profit d'une centration sur la probabilité de chaque événement (le lapin gagne, la tortue gagne), d'abord approchée de façon fréquentiste, puis calculée de façon théorique. Ceci résulte d'un contrat didactique orienté vers la recherche de valeurs de probabilités et minimisant les questions statistiques.

Comme dans la séance précédente, les étudiants rencontrent des difficultés avec le tableur. Le modèle implémenté avec le tableur, est conforme à la situation dans le sens où le processus est arrêté après un gagnant, ce qui ne se réalise pas facilement dans l'organisation de la feuille de calcul. Cette condition d'arrêt n'est pas non plus commode à implémenter dans R, et un autre modèle est proposé qui revient à lancer le dé 6 fois, et à conclure que le lapin gagne s'il existe un 6 parmi les nombres obtenus. Ce modèle est à nouveau proposé par l'enseignant pour le calcul théorique de façon à guider les étudiants dans l'utilisation de la formule de Laplace, mais il est rapidement abandonné après qu'un étudiant propose d'utiliser la formule de multiplication des probabilités. Ainsi des modèles différents sont utilisés dans les différentes phases. Ce changement de modèle simplifie la programmation dans R et pourrait simplifier le calcul théorique, mais il ne fait pas l'objet d'une discussion en classe : le nouveau modèle n'est pas comparé au modèle précédent et l'équivalence des modèles n'est pas discutée³. C'est encore une occasion manquée, et comme dans la séance expérimentale précédente, le rôle pragmatique de la simulation est privilégié par l'enseignant aux dépens d'une réflexion sur les modèles utilisés dans la phase de simulation.

Séance expérimentale 3 : la chasse au canard.

Les étudiants ne rencontrent pas beaucoup de difficultés dans l'utilisation des fonctions Excel et R pour simuler. Comme dans les séances précédentes, l'enseignant souligne la nécessité d'augmenter la taille de l'échantillon appropriée à la simulation pour le passage à l'utilisation de R. De cette façon, les étudiants proposent une très grande taille d'échantillon sans qu'il y ait une discussion sur une taille appropriée. Cette analyse confirme que la simulation peut fournir un environnement riche pour la réflexion des étudiants. Cependant, il semble que l'enseignant puisse faire mieux en posant la question de la taille minimale de l'échantillon pour une précision donnée.

Les stratégies de calcul de la moyenne diffèrent dans les deux simulations : avec le tableur, la moyenne est calculée directement en globalisant les tirages (nombre total de canards tués divisé par le nombre de tirs successifs). Avec R, les étudiants calculent d'abord les fréquences empiriques de chacune des éventualités. Puis, après la présentation de la fonction moyenne par l'enseignant, les étudiants abandonnent les fréquences et calculent directement la moyenne comme avec le tableur. Pour le calcul théorique, l'enseignant conduit les étudiants à calculer la distribution de probabilité de la variable nombre de canards tués en confrontant aux fréquences empiriques obtenues avec R comme préalable au calcul de l'espérance. Le fait de ne pas discuter sur la possibilité de calcul direct de l'espérance sans passer par la distribution de probabilité est une autre occasion manquée : en effet, la discussion sur ce sujet pourrait porter l'attention des étudiants sur l'espérance mathématique dans des tirages répétés (somme des espérances dans chaque tirage).

Séance expérimentale 4 : Le problème de Monty Hall

A partir des expérimentations sans logiciels, les étudiants comprennent la situation et reconnaissent que le choix de changer de porte va apporter plus de chance de gagner. Les simulations qui suivent avec des tailles d'échantillon plus grandes aident les étudiants à éliminer les intuitions erronées et à quantifier plus précisément les probabilités de gain pour chaque choix (garder la même porte ou changer). Dans chaque cas, les simulations sont faites pour chacun des deux choix. Implicitement, le choix de la porte ouverte par l'animateur dans le cas où le joueur a choisi la porte gagnante, est équiprobable.

La première simulation avec Excel conduit à des formules assez complexes, mais les étudiants réussissent à présenter de nombreuses solutions différentes. Un étudiant remarque que dans le cas où le joueur garde la même porte, il gagne si et seulement s'il a choisi la porte gagnante au premier choix. Ceci simplifie notablement la simulation et est utilisé par la plupart des étudiants dans la simulation avec R. En revanche, pour le calcul théorique, l'enseignant engage les étudiants vers un calcul utilisant des probabilités conditionnelles conditionnées par le choix de l'animateur. Ces probabilités conditionnelles sont calculées en appliquant les connaissances académiques déjà enseignées dans le cours mais ne répondent pas en fait à la même question que celle posée et conduisent à la même valeur seulement parce que l'équiprobabilité est supposée pour le choix de l'animateur (Grinstead and Snell 2005, p.139). Une occasion de discuter de modèles, reliant la simulation et les probabilités théoriques a été manquée également dans cette situation.

DISCUSSION ET PERSPECTIVES

Tâches et techniques

La simulation apporte de nouvelles tâches associées à des questions statistiques. Le choix a été fait dans les séances expérimentales de proposer ces tâches avant les tâches classiques. Nous avons observé que les étudiants vérifient systématiquement

leurs résultats par rapport aux résultats empiriques, en particulier dans la troisième séance où cinq probabilités théoriques sont calculées, et aussi pour déstabiliser les idées erronées ou les faux modèles. Cette vérification apparaît comme une technique intégrée dans des programmes d'action. Cela signifie que les étudiants articulent la simulation avec des techniques classiques déjà enseignées, afin d'obtenir un meilleur contrôle de ces techniques. La pratique de cette technique «mixte» semble avoir un effet positif relativement à des conceptions erronées : l'observation montre que les étudiants questionnent leurs modèles à partir de données de simulations. Elle a aussi des limites : un étudiant peut corriger un modèle faux afin d'obtenir une valeur théorique cohérente avec les données obtenues par la simulation sans que le modèle ainsi produit soit juste.

Milieu et contrats didactiques

Dans les séances expérimentales, la simulation peut être considérée comme un milieu à deux niveaux :

1. Un milieu d'action. Les fréquences et leurs fluctuations peuvent être considérées comme des rétroactions que les étudiants obtiennent lors de la construction et de l'exécution des simulations sur l'ordinateur, qui renforcent leur compréhension des relations entre fréquences et probabilités. Implicitement, la simulation permet aux étudiants de construire des modèles «en action» qui déstabilise les conceptions erronées. La simulation joue également un rôle en tant que milieu de l'action dans le calcul des probabilités théoriques lorsque les étudiants vérifient systématiquement leurs résultats par rapport aux fréquences empiriques.

2. Un milieu de réflexion. Nous avons vu qu'au cours de chaque séance, plusieurs modèles de la situation aléatoire ont été considérés, mais qu'ils ne sont pas discutés. Par ailleurs, nous avons vu que la fluctuation n'est pas étudiée de façon précise en regard des questions posées initialement : l'enseignant favorise en réalité une dimension pratique de la simulation, en insistant sur un échantillon de taille appropriée afin de motiver l'utilisation de R, plutôt que de discuter sur une taille pertinente à l'égard de la question statistique. Ceci montre que potentiellement, la simulation contribue à créer un milieu de réflexion, qui n'est pas exploité : l'existence de plusieurs modèles devrait être l'occasion de discuter leur équivalence ; discuter la taille de l'échantillon, en relation avec la question posée devrait être une préparation à la statistique inférentielle. Nous avons ainsi repéré plusieurs « occasions manquées » qui peuvent être interprétées comme une sous-estimation de cette dimension « réflexion » du milieu.

Les questions posés dans les 4 séances expérimentales sont problématiques pour les étudiants et motivent à construire des simulations. Cependant nous remarquons que dans ces séances, dès qu'ils passent à la simulation et aux calculs, les étudiants et l'enseignant ne se réfèrent plus à la question initiale. Ainsi, ces questions initiales

gènèrent une motivation afin que les étudiants cherchent des valeurs probabilistes, plutôt qu'une véritable enquête statistique. Nous interprétons cela comme la manifestation d'un contrat didactique orienté vers une utilisation pratique de la simulation pour approcher les valeurs probabilistes plutôt que pour enquêter sur la question en jeu, sous l'influence d'un contrat didactique « traditionnel » favorisant le calcul des probabilités théoriques. Cette analyse critique permet de proposer une reprise de chacune des 4 séances expérimentales de façon à mieux exploiter leur potentiel en mettant davantage l'accent sur les questions probabilistes et en prenant en compte davantage la simulation comme un milieu de réflexion sur les modèles de situations probabilistes.

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¹ « ...puisque à cause de la similitude des faces et du poids absolument semblable du dé, il n'y a aucune raison (nulla sit ratio) qu'une des faces soit plus encline à tomber qu'une autre.. » Jacques Bernoulli, *Ars conjectandi*.

² Pour une synthèse sur les conceptions erronées, voir Batanero & Sanchez (2005).

³ Pour une discussion sur l'équivalence de modèles dans la simulation d'une expérience aléatoire répétée avec loi d'arrêt, voir Parzysz (2009).

Quelle synergie entre mathématiques et physique au sein de l'enseignement universitaire ?

Michel ROLAND

Université Catholique de Louvain-la-Neuve (UCL), Institut de Recherche en Mathématiques et en Physique (IRMP), roland.debled@skynet.be, michel.roland@student.uclouvain.be, Belgique.

Pour qu'une interdisciplinarité physico-mathématique trouve sa place au lycée, il est souhaitable que son utilité soit démontrée et exploitée lors du parcours du futur enseignant. Or nous constatons, à partir d'exemples et d'enquêtes, qu'un fossé apparaît entre le cursus des mathématiciens et des physiciens. C'est pourquoi, dans cet article, sont développées des approches comparées de concepts démontrant leur complémentarité. L'exemple de la différentielle, décrit comme un obstacle à la mathématisation de la physique¹, est exploité afin de le transformer en un outil de conceptualisation offrant une approche duale des notions et de leurs applications pour une meilleure compréhension des élèves en fonction de leurs parcours.

Mots Clefs: vitesse, dérivée, différentielle, cadre, inférence.

INTRODUCTION

Historiquement, la relation entre les mathématiques et la physique fut bénéfique pour leur développement et permit l'éclosion de théories. Actuellement, la spécialisation engendre une approche différente de concepts communs par les mathématiciens, les physiciens et les ingénieurs. Cette différence se marque dans l'enseignement des mathématiques et de la physique. Au sein des universités francophones de Belgique, nous avons relevé des difficultés de communication entre professeurs. Ces difficultés engendrent la création de cours séparés de mathématique ou de physique en première année, avec un cours de mathématique assuré par un physicien et plus un mathématicien. Un cours de physique en ingénieur débute par une présentation des mathématiques utiles pour ce cours en ne renvoyant plus à celui de mathématiques. Cette différence de langages et les relations unissant ces deux disciplines sont synthétisées dans Karam et al, 2015, en voici un extrait :

Since their beginnings in the ancient world, physics (natural philosophy) and mathematics have been deeply interrelated, and this mutual influence has played an essential role in both their developments as illustrated in the quotations above. However, the image typically found in educational contexts is often quite different. In physics education, it is usual to find mathematics being seen as a mere tool to describe and calculate, whereas in mathematics education, physics is commonly viewed as a possible context for the application of mathematical concepts that were previously defined abstractly. This dichotomy creates significant learning problems for the students... This problem demands

¹ Karam and others (2015)

a systematic research effort from experts in different fields, especially the ones who aim at informing educational practices by reflecting on historical, philosophical and sociological aspects of scientific knowledge.

En outre, fin des années 90, des études se sont penchées sur la désaffection des étudiants pour les études scientifiques dont les mathématiques et la physique. Elles préconisent une modification dans l'enseignement des sciences dont un décloisonnement des disciplines, souhaité en mai 2006 dans un rapport appelé rapport Rolland. Ne serait-il pas temps de tenter un rapprochement entre les deux disciplines et également d'unir leurs atouts respectifs?

Cet article se situe dans la droite ligne de notre recherche sur l'interdisciplinarité physico-mathématique et les obstacles à sa mise en œuvre au Lycée. Comment promouvoir une interdisciplinarité si les enseignants censés l'appliquer n'y ont pas été sensibilisés lors de leur formation universitaire ?

Étudions la mise en place de pistes associant pour chaque obstacle un objectif de dépassement montrant la plus value du recours à l'interdisciplinarité, un objectif-obstacle (Martinand, 1986). Nous avons recours à des dualités possibles entre approche de physiciens ou de mathématiciens, pour la mécanique et l'analyse, fournissant deux chemins pour introduire des concepts ou résoudre des problèmes, évitant en cas d'incompréhension par des élèves, la simple répétition d'une explication à partir d'un changement de cadre (Douady, 1992).

Commençons par une dualité historique autour des concepts de vitesse instantanée et de dérivée mettant en évidence un cadre algébrique et analytique. Le débat sur la notation de dérivée, lagrangienne ou leibnizienne, obstacle conceptuel ou corporatiste, est transformé en un outil pédagogique fournissant une double approche des problèmes, une dualité appliquée.

Une dualité théorique donne un éclairage sur l'inférence déductive et inductive (Barth, 1987). L'une utilisée par des mathématiciens est une réflexion partant du général ou du théorique vers le particulier ou vers une nouvelle vérité théorique; l'autre l'apanage de physiciens part d'une observation particulière pour obtenir une généralité conjecturale.

Terminons par une dualité modélisatrice. L'étude du pendule simple illustre la notion d'obstacle par l'évitement du recours aux équations différentielles ordinaires (EDO) alors même que la loi de la mécanique de Newton renvoie à un tel recours.

Finalement, les concepts clefs sont définis à la fin de cet article dans un glossaire.

DUALITÉ HISTORIQUE ET APPLIQUÉE

Dès la naissance du calcul différentiel, deux visions du concept de dérivée se développent, l'une centrée sur les rapports, inspirée de Leibniz, des Bernoulli, de Varignon,...; l'autre sur les fonctions, inspirée de Leibniz, de Lagrange, de Cauchy,...

Cette évolution donne naissance à deux courants qu'il faut éviter de mélanger sous peine d'engendrer chez les élèves et les étudiants des incompréhensions comme l'indiquait déjà Kac et Randolph (1942) :

Granted, then, that we are not going to dispense with differentials, can we not do something to help students understand what they are? The unsophisticated undergraduate who tries to believe everything he reads and his instructor tells him is hopelessly confused. One day he tries to believe (but does not succeed) that dy/dx is not dy divided by dx . The next day he may have momentary comfort when he learns that is true after all that dy/dx is dy divided by dx and that $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Analysons la recherche des équations du MRUA (mouvement rectiligne uniformément accéléré) par Benson (1999) utilisé dans différents cours de physique en première année d'université en Belgique francophone.

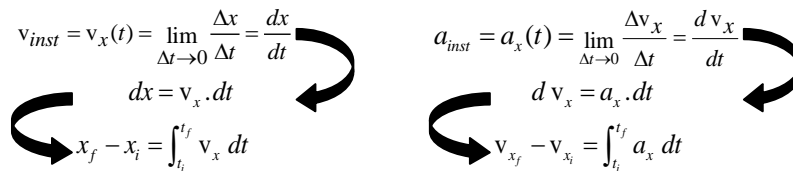


Figure 1 – Recherche des équations du MRUA, Benson

Nous sommes face à l'obstacle épistémologique décrit par Kac et Randolph. L'emploi de la limite ne permet pas de passer de la première à la deuxième ligne. La dérivée n'est pas un quotient de différentielles. Pourtant, des physiciens l'interprètent comme tel, le justifiant pour des raisons empiristes. Dans son cours, Richard Feynman (2014), prix Nobel de physique en 1965, renvoie à l'écriture observée en deuxième ligne :

Les physiciens aiment l'écriture $ds = v dt$, car par dt ils veulent dire Δt dans des circonstances dans lesquelles il est très petit ; sachant ceci, l'expression est valable à une bonne approximation ... La quantité ds/dt que nous venons de trouver est appelée « la dérivée de s par rapport à t (ce langage aide à garder trace de ce qui fut changé), et le processus compliqué pour la trouver est appelé prendre la dérivée ou dérivation. Les ds et dt qui apparaissent séparément sont appelés des *différentielles*.

Il n'est pas anodin de constater que Feynman utilise le terme quantité. En effet, historiquement, il se place dans le cadre défini notamment par Varignon (1698) que nous appelons **algébrique**:

Règle générale. Des vitesses, Des temps, Des Espaces. $y = \frac{dx}{dz}$, ou $dz = \frac{dx}{y}$, ou $dx = y dz$.

Il s'agit d'un rapport voulu adimensionnel, afin de s'abstraire du sens physique des grandeurs, conformément à la théorie d'Euclide (Varignon, 1707) :

Il est ici à remarquer que l'espace et le temps étant des grandeurs hétérogènes, ce n'est point proprement elles qu'on compare ensemble dans le rapport qu'on appelle vitesse, mais seulement les grandeurs homogènes qui les expriment, lesquelles sont ici, et seront

toujours dans la suite ou deux lignes, ou deux nombres, ou deux telles autres grandeurs homogènes qu'on voudra.

Cependant, le recours à la limite renvoie au cadre défini par Cauchy (1823) que nous appelons **analytique**:

Il en sera de même en général; seulement, la forme de la fonction nouvelle qui servira de limite au rapport $\frac{f(x+i) - f(x)}{i}$ dépendra de la forme de la fonction proposée $y = f(x)$.

Pour indiquer cette dépendance, on donne à la nouvelle fonction le nom de *fonction dérivée*, et on la désigne, à l'aide d'un accent, par la notation y' ou $f'(x)$.

Nous constatons ainsi un mélange des deux cadres par Benson. L'emploi du rapport est-il imposé pour la recherche des équations du MRUA ? A-t-on vraiment besoin de la dérivée pour cette recherche, la règle de Merton n'est-elle pas suffisante ?



Figure 2 – Règle de Merton, formalisme actuel : $v_m = \frac{v_f + v_i}{2}$

Pour garder un rapport de grandeurs, il faut définir la notion de différentielle. A quel moment l'introduire ? Comment assurer une réciprocité des objets de savoir entre le monde mathématique et le monde physique ? Qui prendra en charge l'introduction du concept permettant à l'autre d'utiliser l'outil-objet au sein de sa propre discipline ? Nous entrons ici dans la théorie de l'action conjointe de Gérard Sensevy (2011).

De même, comment justifier le passage de la ligne 2 à la ligne 3 ? L'écriture $x'(t) = v(t)$ permet une intégration des deux membres de l'égalité par rapport à t mais dans le cas $dx = v dt$ que fait-on ? S'agit-il d'une intégration à partir de bornes différentes (en x et en t) ? Avec quel découpage ? Quelle définition de l'intégrale ?

Retrouvons-nous cette distinction d'approches dans la résolution de problèmes ?

La cinématique se prête au changement de cadre et ses fenêtres conceptuelles (Douady, 1992). Pour de plus amples explications sur le sujet, nous invitons le lecteur à prendre connaissance de notre article dans *Épistémologie et didactique*, à paraître.

Développons brièvement un des exemples. Un problème a été soumis à 96 étudiants de première année universitaire (mathématique, physique, chimie, ingénieur civil et de gestion) (réussite 8,4%) et ensuite aux 511 élèves des olympiades de physique (réussite 16,6%). Voici le problème posé :

Le conducteur d'un train roulant à 100 km/h aperçoit, à 85 m, sur la même voie, le fourgon d'un train roulant dans le même sens que lui, à 28 km/h. Il bloque aussitôt les freins, ce qui entraîne une décélération de 2 m/s^2 . Y aura-t-il collision ? A quelle distance minimale doit être aperçu le fourgon afin d'éviter la collision ?

Une analyse des erreurs commises par les étudiants et par les 62 finalistes (réussite 33,9%) des olympiades sur un problème semblable figurera dans notre thèse.

Voici le tableau reprenant une synthèse des méthodes utilisées par les étudiants.

CADRE ALGÈBRIQUE	CADRE ANALYTIQUE
FC : relation algébrique	Fenêtre Conceptuelle (FC) : relation numérique
Aucune méthode de ce type pour ce problème, mais bien dans les problèmes de rencontre ou de poursuite pour le MRU, voir analyse proposée dans le cadre de notre recherche doctorale.	
FC : vitesse relative	FC: fonction numérique
Vitesse relative : $\Delta v = v_B - v_A$ Temps (vitesse relative nulle) : $t = \frac{\Delta v}{a}$ rapport algébrique Règle de Merton : $v_m = \frac{v_f + v_i}{2}$ Distance minimale : $x_0 = t \cdot v_m$	$\left. \begin{aligned} x_A &= v_A \cdot t - \frac{at^2}{2} \\ x_B &= x_0 + v_B \cdot t \end{aligned} \right\} x_A(t) = x_B(t)$ $v_A \cdot t - \frac{at^2}{2} = x_0 + v_B \cdot t$ $\Leftrightarrow \frac{at^2}{2} - (v_A - v_B) \cdot t + x_0 = 0$ $\Delta = (v_A - v_B)^2 - 2 \cdot a \cdot x_0 \Leftrightarrow t = \frac{(v_A - v_B) \pm \sqrt{\Delta}}{a} \Leftrightarrow x(t) = \dots$ $\Delta = 0 \Leftrightarrow x_0 = \frac{(v_A - v_B)^2}{2a}$

L'existence d'une solution graphique est prouvée par le théorème des accroissements finis de Lagrange. Ce théorème indique également que rechercher la distance revient à rechercher le moment où les deux trains ont la même vitesse, un changement de cadre.

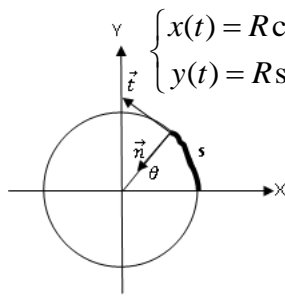
Revenons aux infiniment petits, notés dx , souvent utilisés en physique par des ingénieurs et des physiciens. N'est-il pas problématique de travailler sur ces derniers ?

Référons-nous à l'étude réalisée dans les années 90 à l'Université Paris-Diderot, *Procédures différentielles dans les enseignements de mathématiques et de physique au niveau du premier cycle universitaire*. Nous y trouvons un exemple d'erreur d'analogie entre la recherche du volume d'une sphère et de sa surface par un découpage en tranches infinitésimales. Il y a un problème d'approximation lors du passage à la limite dû au paradoxe de Schwarz. Il est important de vérifier la validation de l'approximation choisie sous peine de commettre une erreur lors du passage à la limite. La surface des cylindres infinitésimaux ne tend pas vers la surface de la sphère.

DUALITÉ THÉORIQUE

Analysons un autre exemple, la recherche de la formule de l'accélération centripète.

Soit une particule tournant à une vitesse quelconque autour d'un point à une distance R . Recherchons la position à chaque instant du point sur le cercle, ensuite sa vitesse par la dérivée et finalement son accélération par la dérivée seconde.



$$\begin{cases} x(t) = R \cos \theta(t) \\ y(t) = R \sin \theta(t) \end{cases} \quad \begin{cases} \dot{x} = -R \sin \theta \dot{\theta} \\ \dot{y} = R \cos \theta \dot{\theta} \end{cases} \quad \begin{cases} \ddot{x} = -R \cos \theta \ddot{\theta} - R \sin \theta \dot{\theta}^2 \\ \ddot{y} = -R \sin \theta \ddot{\theta} + R \cos \theta \dot{\theta}^2 \end{cases} \quad \vec{a} = R\dot{\theta}^2 \vec{n} + R\ddot{\theta} \vec{t}$$

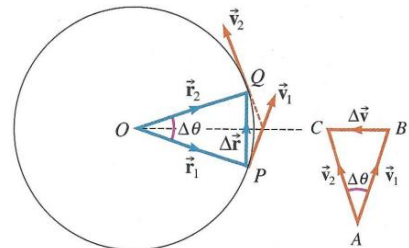
Par définition du radian: $s = R\theta$ $\vec{a} = \frac{\dot{s}^2}{R} \vec{n} + \ddot{s} \vec{t}$

Nous déduisons de la formule générale, le cas particulier du mouvement circulaire uniforme: $a = \frac{v^2}{R}$

Il s'agit donc d'une *inférence déductive* (Barth, 1987).

Dans l'extrait de Benson repris ci-dessous, l'auteur part du cas particulier du mouvement circulaire uniforme, par exemple, un objet tournant sur une platine, une observation :

La figure représente une particule se déplaçant à une vitesse constante v sur un cercle de rayon r . Il s'agit d'un **mouvement circulaire uniforme**. Supposons que, durant un court intervalle de temps Δt son vecteur position tourne d'un angle $\Delta\theta$, et que le déplacement de la particule $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, soit vertical. Comme \vec{v} est toujours



perpendiculaire à \vec{r} , les directions de ces deux vecteurs varient selon le même angle durant un intervalle de temps quelconque. Sur le diagramme vectoriel de l'équation $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, nous notons que $v_2 = v_1 = v$. La direction $\Delta\vec{v}$ est horizontale et radiale vers l'intérieur, et confondue avec la bissectrice de l'angle $\Delta\theta$ à l'intérieur du cercle. Les triangles OPQ et ABC sont deux triangles isocèles ayant les mêmes angles. (Pourquoi ?)

Donc, $\frac{|\Delta\vec{r}|}{r} = \frac{|\Delta\vec{v}|}{v}$ et nous tirons que $|\Delta\vec{v}| = \left(\frac{v}{r}\right)|\Delta\vec{r}|$. Puisque $|\Delta\vec{r}| \approx v\Delta t$, nous voyons que

$\frac{|\Delta\vec{v}|}{\Delta t} \approx \frac{v^2}{r}$. D'après la définition $\vec{a} = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta\vec{v}}{\Delta t}\right)$, nous trouvons que le module de

l'accélération centripète est $a_r = \frac{v^2}{r}$.

Benson a recours au rapport de similitude entre deux triangles semblables. Il s'agit d'un rapport adimensionnel. Il généralise ensuite en sommant les accélérations centripète et tangentielle ($a_t = \frac{dv}{dt}$), obtenant ainsi l'accélération pour un mouvement circulaire non uniforme. Il aurait dû valider cette conjecture. Il s'agit donc d'une *inférence inductive* (Barth, 1987).

En comparant les deux approches, nous constatons que l'inférence inductive utilise une démarche arbitraire de la pensée imposant une validation à posteriori.

De même pour l'inférence déductive, l'utilisation des dérivées, un passage à la limite évitant la manipulation d'« infiniment petits », renvoie au **cadre analytique**.

Pour l'inférence inductive, l'utilisation du rapport de similitude et d'approximations (« court intervalle de temps », un infiniment petit) avant le passage à la limite renvoie

au **cadre algébrique** du rapport de grandeurs, les Δ selon Feynman (2014). Cette méthode est semblable à la justification erronée du passage entre la ligne 1 et la ligne 2 de la figure 1 utilisant les Δ plutôt que les d .

DUALITÉ MODÉLISATRICE

La recherche des équations du MRUA renvoie à des équations différentielles du premier ordre avec intégration. Le danger est la mise en place d'une méthode de « séparation des variables » avec une dissimulation de la composée ou du changement de variables comme l'indiquait Poincaré (1904) et Hadamard (1923).

En 1996, un groupe de travail, composé de trois professeurs et de trois étudiants, s'est penché sur des difficultés rencontrées par les étudiants de la faculté des sciences appliquées de l'UCL à la lecture de photocopies. Le travail a été repris sous le nom de *Dialogues concernant deux sciences*. En voici des extraits :

Nathanaël – Pour résoudre l'équation différentielle $y'' = \frac{a^2 - y^2}{y^2}$ (3.1) certains procèdent comme suit. Ils remarquent qu'elle est bien équivalente à $\frac{dy}{dx} = \pm \frac{\sqrt{a^2 - y^2}}{y}$ (3.2)... Les variables se séparent, disent-ils, et l'on a $\frac{y dy}{\sqrt{a^2 - y^2}} = dx$. Donc $x = \int y(a^2 - y^2)^{-\frac{1}{2}} dy$

Nathanaël – Quel sens faut-il donner à l'écriture $\frac{y dy}{\sqrt{a^2 - y^2}} = dx$? On semble multiplier par un dx qui pour moi est mystérieux.

Ce type de séparation n'est pas possible dans le cadre des EDO du second ordre. Il est en effet impossible de multiplier directement par $(dx)^2$ à cause du d^2y .

Lors d'interviews, nous avons constaté au sein de l'université de Mons une difficulté de communication entre mathématiciens et physiciens. Le mathématicien ne comprenait pas comment on ne parlait pas d'EDO lors des cours de dynamique.

Ainsi, au cours d'un exposé *La transposition didactique et son « triangle » : le pendule simple comme exemple* (séminaire Fondements et notions fondamentales, 2015), nous avons présenté une analyse d'approches différentes du pendule simple.

En dernière année du lycée est étudié le mouvement harmonique. La plupart des livres scolaires s'appuient sur l'idée d'établir une concordance entre trois types de mouvement (le mouvement circulaire uniforme (MCU), le mouvement rectiligne sinusoïdale (MRS), le mouvement harmonique) telle que présentée notamment dans le livre écrit par Hecht (1999). En s'appuyant sur les dérivées des fonctions trigonométriques, il est possible de caractériser le type de force engendrant un mouvement harmonique. Le lien entre le pendule simple (MRS) et le mouvement harmonique n'est pas prouvé de manière rigoureuse. Ils s'appuient sur une inférence inductive, par exemple, en observant la courbe décrite par une bouteille de sable percée en mouvement pendulaire sur une feuille défilant en MRU.

Pourquoi alors que les élèves connaissent la décomposition des forces et la loi de la mécanique de Newton n'y a-t-il aucun recours à l'EDO associée ?

Pour Benson, la situation est encore plus complexe. L'auteur tourne en rond. Il part du mouvement harmonique générale $y(t) = A \sin(\omega t + \phi)$, recherche la dérivée première et seconde pour comparer la position et l'accélération, obtenant ainsi l'EDO. Il affirme alors que l'élongation en est **une** solution. Sans prouver l'unicité, il part de l'observation du mouvement bloc-ressort affirmant que son élongation est sinusoïdale.

Il suffirait pourtant d'écrire la loi de Newton, d'utiliser la linéarisation et de rechercher une fonction dont la dérivée seconde est opposée à elle-même. L'unicité serait prouvée sur base de théorèmes de mathématiques rencontrés en fin de lycée. Il est dommage de ne pas renvoyer aux EDO par la loi de la mécanique au sein de certaines universités.

Notons également, qu'à l'UCL, les ingénieurs et les physiciens ne devront plus obligatoirement suivre un cours d'EDO. Pourtant de nombreuses modélisations, permettant des avancées significatives en physique et en compréhension de phénomènes physique, s'appuient sur les EDO et sur les EDP.

Au niveau universitaire, il serait également intéressant de s'interroger avec les étudiants sur l'erreur commise par la linéarisation du pendule simple. Dans de nombreux ouvrages, on retrouve un renvoi à une marge d'erreur sans la moindre explication. S'agit-il de l'erreur commise en comparant le $\sin(x)$ et x ou celle commise en comparant la solution de l'équation linéaire à celle non-linéaire?

CONCLUSION

Une réflexion sur l'utilisation de dx/dt comme un rapport et l'utilité d'un tel recours erroné n'est pas assez pris en compte dans l'enseignement ainsi que la problématique du dx , un infiniment petit. Il s'agit pourtant d'un obstacle épistémologique comme le décrivent Kac et Randolph (1942). En effet, les notations $\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = x'$ et $dx = x' dt$ ne renvoient pas au même cadre théorique, la dérivée ou la différentielle.

Par contre, le recours au rapport, dans un cadre algébrique défini, permet la résolution de problèmes en utilisant le concept de vitesse relative. Tandis que le recours à la fonction, dans un cadre analytique, permet la résolution de ceux-ci en utilisant le concept de vitesse instantanée. Nous obtenons une approche double des problèmes permettant à chaque élève de s'approprier une résolution adaptée à son esprit.

De même, cette double approche permet de s'adapter au parcours de l'élève en tenant compte de ses connaissances. Au lycée, il n'est pas possible d'avoir recours pour la recherche de la formule de l'accélération centripète à l'inférence déductive mais bien à l'inférence inductive s'appuyant sur des observations de phénomènes particuliers.

C'est pourquoi, nous nous sommes intéressés, comme le montre cet article, à l'étude comparée des approches de concepts, de problèmes et d'applications par des

mathématiciens et des physiciens. Il est important que lors de leur formation, les futurs enseignants prennent connaissance d'approches différentes afin de les exploiter dans leur enseignement. A l'université Paris-Diderot, un dialogue entre différents professeurs des deux disciplines a été mis en place afin d'éviter des ambiguïtés lors des cours qui n'occasionnent que des difficultés supplémentaires pour les étudiants. Des séances communes ont été réalisées montrant la complémentarité des approches sur divers thèmes pour les étudiants. Malheureusement, cette tentative s'est avérée infructueuse concernant la séance sur la différentielle. En conclusion, nous sommes encore très loin d'être parvenu à retrouver une synergie didactiquement profitable.

GLOSSAIRE

Un **cadre** est constitué des objets d'une branche des mathématiques, des relations entre les objets, de leurs formulations éventuellement diverses et des images mentales associées à ces objets et ces relations. (Douady, 1992)

Le **cadre algébrique** renvoie aux objets suivants : les rapports dimensionnels ou adimensionnels de nombres représentés par des lettres (Δx , dx , a , v , ...) et les opérations algébriques associées.

Le **cadre analytique** renvoie aux objets suivants : les fonctions numériques et les opérations sur ces dernières.

Le **changement de cadres** est un moyen d'obtenir des formulations différentes d'un problème qui sans être tout à fait équivalentes, permettent un nouvel accès aux difficultés rencontrées et la mise en œuvre d'outils et de techniques qui ne s'imposaient pas dans la première formulation.

Une **fenêtre conceptuelle** est l'ensemble des objets, des outils et des relations mobilisés pour analyser l'énoncé du problème ou de la situation, ou pour développer une stratégie de résolution, quels que soient les cadres dont ils relèvent. (Douady, 1992)

Une **inférence** est une opération mentale qui consiste à sélectionner des composantes d'une entité complexe, à en retenir quelques-unes et à en négliger d'autres. Elle s'applique à la fois à un raisonnement inductif et à un raisonnement déductif. (Barth, 1987)

L'**inférence inductive** infère une règle à partir d'une information limitée, à partir de l'observation des faits particuliers, des exemples. On infère du général (théorique) à partir du particulier. Il s'agit d'hypothèses ou de conjectures à valider par la suite.

L'**inférence déductive** est la conclusion exacte à partir d'une vérité donnée. On infère du particulier à partir du général ou du théorique. L'inférence est nécessairement vraie.

L'**interdisciplinarité physico-mathématique** est un instrument curriculaire, pédagogique et didactique visant, au sein d'une des deux disciplines, mathématique ou physique, la construction et l'intégration de savoirs, l'appropriation de concepts et une

modélisation du réel à partir de connaissances disciplinaires provenant de l'autre discipline. Elle implique une collaboration entre mathématiciens et physiciens.

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A model from signal analysis to design linear algebra activities

Rita Vázquez¹, Maria Trigueros² and Avenilde Romo-Vázquez³

¹Universidad Autónoma de la Ciudad de México, ²Instituto Tecnológico Autónomo de México, ³Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada, México.

In this paper we present a didactical activity based on modelling an engineering problem known as Blind Source Separation (BSS) and the results of its implementation in a linear algebra course in a Mexican university. The problem had previously been analysed from an institutional point of view, carrying out the notion of the matrix map $T(x)=Ax$. In the frame of APOS theory we propose a genetic decomposition for this concept, in order to analyze students' constructions related to it, and at the same time, using the BSS context as a reference to connect mathematical constructions with a real life situation.

Keywords: modelling, engineering, signals, BSS, linear algebra.

INTRODUCTION

In this paper we present a didactical activity based on a real engineering problem known as *Blind Source Separation*, and the results of its implementation in two groups in a Mexican university. In previous research The Anthropological Theory of didactics (ATD) (Chevallard, 1999) was used to analyze the mathematical concepts embedded in this engineering context from an institutional point of view. As a result, we identified that the notion of linear transformation and its matrix formulation, play an important role in modelling this problem. We also reported that some other notions such as that of signal and sampling are important concepts in the training of engineering students and that they can be introduced in the teaching of mathematics for engineering students.

We consider that if these notions are introduced early in the mathematics curriculum for engineering students, they can become a tool for reducing the gap between what students learn in mathematics courses and the way they should apply this knowledge to solve real problems in the context of their profession; an issue that has been widely studied (Kent y Noss, 2002).

In order to achieve this goal, we selected the notion of a matrix linear mapping and its inverse in the context of signal separation as the starting point to design a didactic activity for an introductory linear algebra course for engineering students.

The analysis presented in (Vázquez, Romo-Vázquez & Trigueros, 2015) was intended to study the context known as Blind Source Separation (BSS), a problem from Signal Analysis that was first established in order to study motion decoding in vertebrates (Comon & Jutten, 2010).

In that study, using the notions of praxeology and institution from ATD we distinguished, in first place, the mathematical tasks and techniques involved in formulating and solving BSS when the institution of reference is Signal Analysis. Then we identified mixed praxeologies considering both mathematical knowledge and professional practice that could be didactically transposed to an introductory Linear Algebra course.

In general terms, BSS is about separate information that is measured in form of signals. The problem consist in retrieving n source signals $\mathbf{s}=(s_1,s_2,\dots,s_n)$ that are mixed under a linear model $\mathbf{A}\mathbf{s}=\mathbf{x}$, when only the observed signals \mathbf{x} are known. The research on separation methods has generated a whole research area known as Independent Component Analysis (Comon & Jutten, 2010). There is a rich variety of applications on BSS: studying the brain information obtained with electroencephalograms and other biomedical signals, processing satellite images, as well as radioastronomy, sound, GPS or interfered signals, are a few examples.

As a result of the analysis we found that the notion of signal (defined in engineering courses as a function) can elicit students' reflection on the relationship between the type of functions (and their graphs) commonly studied in a Calculus course and those needed in signal analysis, and differences in their graphical representation. In particular, the sampling of a signal relates the concept of *function* with that of *vector*. The mathematical model for BSS, which works as an input-output system in the form of a linear system of equations, makes clear, from the start, the need to relate concepts such as linear system, linear transformation and the map $\mathbf{x}\rightarrow\mathbf{A}\mathbf{x}$.

A rich body of literature exists about students' difficulties when learning Linear Algebra concepts and also about the use of modelling in the learning of concepts in this discipline, particularly using APOS theory (Trigueros, Oktaç & Manzanero, 2007), (Possani, Trigueros, Preciado & Lozano, 2010). Also, the map $T:\mathbb{R}^m\rightarrow\mathbb{R}^n$, defined by $T(\mathbf{x})=\mathbf{A}\mathbf{x}$, can be regarded as an extension of the notion of a function of several variables which has been studied by Martinez-Planell & Trigueros, 2010) using the same theoretical framework. However, there are important differences between these two types of functions as $\mathbf{A}\mathbf{x}$ also entails the matrix-vector product. Taking results of previous studies into account and considering that the use of a non-mathematical problem can help students to make sense of these new functions and to abstract the main mathematical ideas involved in their construction, we designed a modelling activity based on BSS context. The goal of this part of our research project was to add an analysis of the constructions needed in the learning of the concepts of linear transformation and its inverse using APOS Theory to results identified in the praxeological analysis of the context to design a modelling activity, and a set of activities to introduce students to both, the ideas related to BSS as an engineering problem and matrix transformations.

Our research questions were: What are the constructions involved in relating transformations with matrices? What are the constructions needed in the learning of

inverse transformation and inverse matrix? Does the use of an extra-mathematical situation play a role in favouring those constructions?

THEORETICAL FRAMEWORK

APOS Theory was used as a framework to study students' constructions (Arnon, et al. 2014). It intends both to model the way students learn advanced mathematical topics in order to design teaching sequences that can be proved to be effective in terms of students' learning, and to analyze the knowledge that students display when solving a specific activity at a particular moment in time. When using APOS theory researchers take into consideration students' previous knowledge. The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition (GD) – a description of specific mental constructions a person may make in the process of understanding mathematical concepts and their relations. Students work collaboratively in groups discussing and responding to specific tasks contained in the pre-designed activities. Different kinds of activities, which have particular aims, are carefully developed based on the GD. In some activities students need to perform actions on objects and reflect on them. Other tasks have as a goal to interiorize those actions into processes. Reflection on how and why they work helps students abstract their main characteristics, take control over them and flexibly use them. There are also tasks, which intend to make students reflect on the process and be aware of it as a totality so that they can apply new actions to it. When this happens the process is encapsulated into an object. Tasks are also designed to help students be aware of the relations among actions processes and objects and also on the relations to other concepts. The theory refers to these collections as schemas. Schemas evolve as new relations between new and previous actions, processes, objects and other schemata are constructed and reconstructed.

METHODOLOGY

For the design we proposed a GD for the matrix map $T(\mathbf{x})=A\mathbf{x}$ and its inverse: The construction of the map T involves the interiorization of actions of evaluation of a map on different points in a vector space. This process can be encapsulated into an object when properties of the map are studied. A schema for transformations as functions involves the construction of the domain and range sets as objects and the construction of a relation where the process of real valued function is coordinated with the transformation process to consider both as functions that differ in their domain and range. The transformation process is coordinated to the product of matrix and vector process into a new process where specific transformations can be described in terms of this product. Actions on matrices and vectors and on interchanging the order of factors are involved in the construction of a process construction of $A\mathbf{x}$. The coordination of the function schema, euclidian space R^n , and the process for the product $A\mathbf{x}$, results in the coordination of the vector resulting from the product $A\mathbf{x}$ as a vector image of T . The construction of solution set of a linear system of equations as an object is necessary to pose and solve questions about the possibility to find the preimage of a vector in the range of T , and to find an inverse of

A and the inverse of a transformation as objects. Actions on the map T and its inverse to validate the rule $TT^{-1}=I$ and determine its unicity on the domain of T help in encapsulation of these transformations as objects. The didactical activity we propose in this paper intends to foster these constructions, using BSS as a modeling frame of reference.

A first design of the modeling of the source separation problem in an audio context was previously tested with a pilot group. The designed activity was probed with 24 students in a linear algebra group. The analysis of the results of this experience, and in particular, the modeling process constitutes the focus of this paper. Five sessions of two hours each were used in this experience. This group had previously worked with activities based on APOS theory to construct a linear system of equations schema. The teacher had also introduced Gauss-Jordan elimination method, and the product of a matrix and a vector, by means of dot product. At the time we tested the activity, matrix product, inverse matrix and linear transformation had not been introduced. Students worked in teams of six, first discussing within their team and later exposing their conclusions to the whole group. In the first two sessions they worked in the construction of a mathematical model for BSS and linear maps $x \rightarrow Ax$. Next the experience focused on the inverse of the map. The last two sessions intended to introduce the inverse of a transformation and of a matrix through activities based on the GD. All the sessions were observed by the researchers and video-recorded. In the next section we present some representative activities from the design used in the classroom, and results from the analysis of the obtained data.

DISCUSSION AND RESULTS

First part: the modelling activity

As an initial problematic situation, we posed the problem: *Let's suppose we are spying an important meeting. We have put some recorders in some places of the office where the meeting is hold. We also have a map of the seats of the important people we want to spy, but we cannot see them. After the meeting, the only thing we have are the recordings of the conversation -most of the time there are more than two people talking at the same time- and a map of the location of the recorders. (We in fact reproduced the sound of several mixed voices in several recordings, placed in different locations of the classroom). How can we indentify, with only this information who is talking and where is she/he seated?*

The purpose of this problem situation was to introduce a context where a mathematical model could be developed to find a response to the problem's question. As it has been recommended in previous modelling based research, one of the first activities when introducing a modelling task is the recognition of relevant variables in the situation. We then asked the group (working on teams of six students) for the types of variables they could identify. The constructions students need to make to propose variables are related to mathematical and non mathematical schemas, such as physics of sound, space, the mechanism of recording, among others. The distance

between sources and recorders, as well as the tone of voice of the speakers were variables that students considered as crucial. Other variables proposed were the noise, the echo at the office, or voice fluctuations. The ability to define the main variables in an open modelling situation is mentioned in (Hamilton, Lesh & Lester, 2008). The next dialogue shows how decisions when considering variables, which is an important competence in engineering education, were taken by students in a team:

Student 1: We think that the shape of the office is important because sound bounces and makes it more difficult to distinguish the speakers. And, it is possible that, when sound wave reaches the wall sounds can cancel each other out.

Student 2: I disagree. If you only have the recording and a 2D map, you cannot take into account the shape of the room. I know it is important but you have to drop it out.

Tutor: So we drop out that variable from our list?

Students 1 and 2: Yes.

In order to establish the instantaneous linear mixture model for BSS the teacher and the researchers proposed a tool named “configuration”. A configuration is a two-dimensional representation of the location of voices (called sources) and recorders (observations). We presented four different configurations (varying the number of sources, observations and distances between them) and asked the question: *in which of these configurations would it be easier for the spy to solve the problem?* Students considered that if number of sources exceeds by far the number of observations, the sound would be “too mixed” in each recorder. They also discussed that, on the other hand, many observations and few sources entails “too much” information and makes the task of separating it difficult. Most of them decided that the best configuration to be able to separate sounds was the one shown in diagram C. Later on, when the matrix form of the model appeared, students’ were able to relate these conclusions to the size of a matrix representing the configuration and the possibility of finding its inverse. During students’ discussion the issue of the speaker’s voice tone also appeared. Students were not sure how to deal with it, so the teacher helped them to consider that the speaker’s voice at an instant could be simplified as a pure tone $y = \sin(2\pi wt)$. She suggested students’ to use an online tool in order to produce pure tones with different frequencies, w . They performed actions of changing frequency and simulated different distances between sound (source) and receptor (ear). From this exploration students were able to construct a mathematical model of the sound received at an instant by a recorder. This model related sound amplitude to distance from the source as an inverse proportionality. Finally, students were able to develop a linear mathematical model for each observation as a linear combination of pure tones. As a closing discussion, together with students, the teacher proposed a common notation for a set of instantaneous sources as a vector $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{x} = (x_1, \dots, x_m)$ for observations, where each s_i corresponds to a pure tone. She also named \mathbf{s} the input and \mathbf{x} the output of the mixing system.

Part two: the construction of matrix mapping $T(\mathbf{x})=A\mathbf{x}$

The goal of the second part of the experience was to investigate the constructions needed to relate the notion of the particular map $T(\mathbf{x})$ to an input-output system, its matrix form, and the system of linear equations associated for each $\mathbf{b}=T(\mathbf{x})$.

The signal configuration tool was used to probe students' constructions on domain and range of a map. They were given the diagram below and asked:

- Consider the mixing transformation that maps sources to observations in configurations A, B and C. What is the domain and range of each of them?
- Suppose that sources in diagram C correspond to three pure tones with frequencies 440Hz, 660Hz and 880Hz, respectively. What is the image under the mixing map of the sources at instant $t=2$ seconds?
- Write the mixing matrix for each configuration.

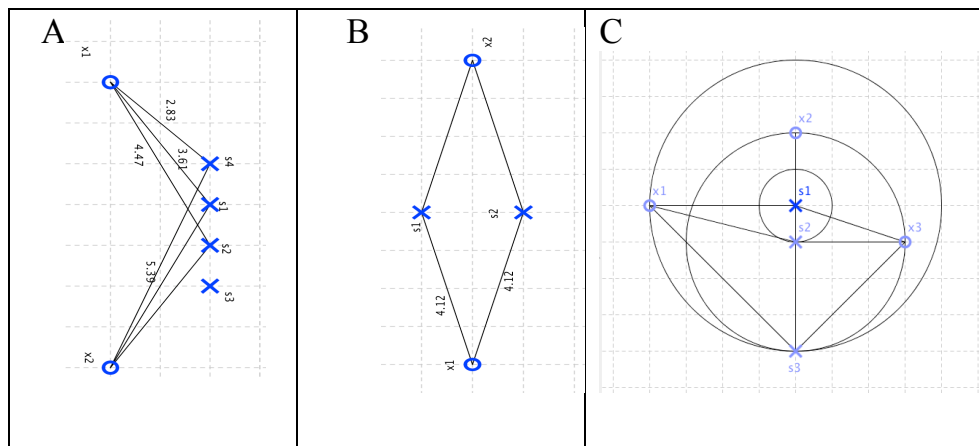


Figure 1: Three signal configurations to explore the mixing map $T(\mathbf{x})=A\mathbf{x}$

The first question probes students' constructions related to domain and range of a map. The construction of a consistent schema of these concepts requires a previously constructed schema of function (Martínez-Planell & Trigueros, 2010). As the mixing map relates vectors in \mathbb{R}^n (where n is the number of sources) to vectors in \mathbb{R}^m (m the number of observations) there is not a geometric representation of domain or range for $n > 3$ (see diagram A). Students were asked to assign a vector \mathbf{s} to a vector with the purpose of helping them reflect on their actions and interiorize them into a vector function process. Some students struggled to associate domain and range with Euclidean vector spaces; it was necessary to recall the representation of signals as their value at some instant, for them to recognize each of the sources as a real number, and therefore, an element in the domain as a vector formed by n values of those sources. Once this was done, students easily identified \mathbb{R}^4 as the domain and \mathbb{R}^2 as the range of the mixing map in figure A; they were able to find on their own the domain and range for transformations depicted in figures B and C. They were able, as well, to relate to the previous work with the model and to conclude that separation was easier when the dimension of both domain and range are the same. Furthermore,

students coordinated this process to a previously constructed process for domain of single-valued functions into a domain process as the set where both types of functions are defined; necessary to answer question a). The dialogue of the teacher with Student 3 shows the difficulties found by students who have not constructed function as a process:

Student 3: I know that domain is where variables are ok but here... I can't see here if there are problems with the sources.

Tutor: What do you mean when you say that they are ok?

Student 3: Yes, for example: to obtain the domain of a function I solve the inequality or look at points where the denominator equals zero. But I can't see that in this example, because its $\sin(x)$.

Student 3 showed an action conception of a domain. She had previously worked with real functions of a single variable, and their analytic expressions, solving inequalities or indeterminacies. Regarding the definition of domain as the set where the function is defined, other students included words from the context of the problem:

Student 4: The domain depends on the voices of speakers.

Tutor: How can that be?

Student 4: Umh... I am not sure, but I think that only the voices of speakers are in the domain, the domain cannot include all possible tones.

Tutor: Does that mean that the variables for the mixing map are the pure tones?

Student 4: Yes, I think so.

Student 5: No. I don't think so. The domain is \mathbb{R} ...each source is a function whose domain is \mathbb{R} because sinus function is continuous on all \mathbb{R} .

Student 5's answer shows he has not interiorized yet the notion of variable in the domain beyond the context of single-valued real functions; more actions on different types of functions need to be performed so that students can interiorize them into a process construction of domain. The explanation given by Student 4 shows an interaction between her schema for the external world related to the problem situation and her domain schema. Her schema for domain, however, only contains processes related to elements where it is "useful" to evaluate the function. Some of these obstacles were also found in the case of the range of the function. Once domain and range of the mixing map were institutionalized in a whole group discussion, students went on to solve item b) which they were able to answer pretty fast. We observed that they progressed better when they had a specific value to obtain source vector \mathbf{s} as an arrange of real numbers, which evidences that most of them needed to do more work in order to interiorize domain and range as processes. Most of them reflected a coherent construction of the image under a function of an element in the domain, as they did actions of calculating each value $s_i(2)$, they did also the actions of combining them linearly and grouping them to form the image vector $\mathbf{x}=\mathbf{T}(\mathbf{s}(2))$.

Finally, item c) intended to probe if students' previously constructed structures about matrices and vectors enabled them to recognize the product of a matrix A and a vector \mathbf{s} in the model for the transformation of sources and observations they were developing. We intended to observe if they were able to relate these constructions to BSS contextual elements in order to explain, beyond mathematics, the need for A to have n columns if vector \mathbf{s} is in \mathbb{R}^n . Results obtained showed that effectively, most students had interiorized the matrix form of a system of equations into a process and could coordinate it with a process of coefficient matrix once the $n \times m$ linear system was identified in item b). Students related the size of the matrix A to the BSS context by observing that the number of columns of A must equal the number of sources, and the product results in m observations, so they concluded A has to have m rows by making reference to configurations in each case.

An interesting result emerged when students worked with a symmetrical configuration of 2 sources and 3 observations, where it was asked if –due to symmetry- the information received by x_1 and x_2 was the same. Some of the students answered this question starting by writing the system, then its coefficient matrix and gave a clear explanation in terms of linear dependence of the rows of A , showing they were able to use linear dependence as an object and relate it to the problem.

Part three: A genetic decomposition for A^{-1}

The next step in working with this model was to recognize the inverse matrix as a tool to solve the separating sources problem. It is worth to mention that the design simplifies the conditions of the BSS problem by posing a non-blind problem, where the matrix, if not given, can be deduced.

Students recognized during work with the problem the usefulness of having an inverse map for $T(\mathbf{x})=A\mathbf{x}$ in order to separate sources. A discussion on how to find the inverse map was opened. As the mixture map is represented by a matrix, students assumed the fact that the inverse would also have a same sized matrix representation. This fact was not considered initially in the genetic decomposition and should be part of a refinement.

We need to recall now that matrix product was not yet constructed by students. This was a decision of the researchers because product matrix, seen from a linear transformation perspective corresponds to the composition of maps. Yet, in BSS a composition of mixtures makes little or no sense. A 2×2 matrix representing a mixture transformation was presented to students and the corresponding mixtures were shown in Geogebra (where the sources were hidden); specific values of the mixtures were thus available, and the problem of determining the corresponding input vector, given a specific “output” vector \mathbf{x} chosen by students was posed. Students who showed a process construction of linear system of equations, posed the corresponding system and solved it for two variables. They asked for “more information” and used the mixtures to obtain it. After some work, they arrived to the inverse matrix of A . Other students stated that it was impossible to obtain the input,

or that information given was not enough. These responses evidenced they had constructed action conceptions of linear system and its solution set, as they only referred to the first data given and were able to solve only that specific system. Later, students were asked to perform the same actions on different matrices and different output vectors, selected by researchers in order to explore the conditions for the existence of an inverse transformation or the inverse of matrix A . Some students who demonstrated a process or an object construction of linear system were able to relate the existence of A^{-1} with the linear independence of the rows of A and a suitable number of linearly independent output vectors. Students who showed an action conception were not able to find these general conditions but were able to solve exercises by doing actions and obtaining a conclusion for each given matrix. The following part of the activity was designed to help students construct the property that if the output vectors given were those of the standard basis for \mathbb{R}^n , then the input vectors obtained were the columns of A^{-1} . Students did actions related to this and after reflecting and interiorizing them into a process they were able to construct by themselves the Gauss-Jordan algorithm to obtain the inverse of a matrix. In every case, the reflection on the validity of the argument arose, but, in general, students showed understanding of what they were doing. A small group struggled with the fact that solving the system $Ax=e_1$ then $Ax=e_2$, etc. separately was equivalent to reduce by Gaussian elimination A and I simultaneously, that is the augmented matrix $(A|I)$. These students showed an action construction of the Gauss-Jordan elimination algorithm, in the sense that they can't recognize that variable x is just a label and doesn't modify the solutions of the system. We consider that this result can contribute to refine a genetic decomposition for the schema of linear system equations. The use of the visual interface (Geogebra) allowed the possibility to ask what would happen if, for a specific inverse matrix, calculated from output vectors, output data is changed. Students who had not an object construction of linear system doubted in deciding if the inverse changed. Through a whole group discussion the unicity of the inverse transformation was related to the BSS context, as the coefficients of the matrix, defined by the reciprocal of the distances between sources and observations are invariant. Further insight on this issue will be obtained from questions asked in a mid-term exam, and from semi-structured interviews with students to be conducted.

Finally, a brief presentation on the importance of the BSS problem was shown to students, together with some of its applications. Students proposed different situations where they thought separation of signals could be useful.

CONCLUSIONS

The modelling part of the design is suitable to trigger students' interest; in particular the work with pure tones elicits the use of different registers and makes the notion of vector a useful tool to represent a sampled signal. The BSS context broadens the possibilities to construct coherent schemas of domain, range and function. We propose a genetic decomposition for the matrix map and its inverse. Student's work on its construction showed that an object conception of a system of linear equations is

necessary in order to relate the map with the product of a matrix and a vector, and to see it as an input-output system.

The design allowed students to construct an algorithm for finding the inverse of a matrix without using the matrix product and to explore conditions of existence of the inverse matrix related to linear independence of the rows of A . Research will continue by exploring constructions related to linearity. The adapted model of BSS used for the didactical design presented here, equivalent to $\mathbf{s}=\mathbf{A}^{-1}\mathbf{x}$ seems to be a powerful tool to solve inverse problems in contexts beyond audio signals. Future work on this project will focus on the analysis of a final questionnaire answered by all the students in the group and of interviews conducted with selected students.

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Using the flipped classroom model of instruction to explore teaching and learning activities in mathematical education for engineers: An activity theory perspective

Helge Fredriksen and Said Hadjerrouit
University of Agder, Norway

THE FLIPPED CLASSROOM MODEL OF INSTRUCTION

The main concept of Flipped Classroom (FC) is to invert, or flip, the content of teaching in terms of what is done during the homework phase, compared to the activity in the class with the teacher. Traditionally, the students at higher education attend a lecture, accompanied with tasks to solve at home or in a colloquium arranged by the University. The FC model will alter this completely, giving the students the opportunity to watch the lectures at home using pre-recorded videos. When attending class, instead of listening to a lecture, the students spend the time in a more dialogue-based problem-solving activity. To capture this complexity, the poster proposes a conceptualization of FC within the Activity Theory Framework.

THEORETICAL FRAMEWORK

A variety of theories has been used to conceptualize FC, such as constructivism, ZPD, Bloom's taxonomy, or cognitive load theory. In addition, the research literature uses mostly quantitative methods to analyze the effectiveness of FC in comparison to traditional classroom, or to investigate participants' perceptions of FC. As a result, most studies fail to capture the complexity of the FC model of instruction. The complexity of FC resides in introducing a new instructional culture into classroom, using new mediating artifacts, new rules and new division of labor within the community of teachers and students. Furthermore, in contrast to traditional classroom, FC puts more emphasis on homework, which is better incorporated into classroom activities by means of group work or collaborative activities (Bergmann, 2012). Clearly, to understand the complexity of FC, there is a need for a theory that covers all components of FC. Activity Theory (AT) provides such an overall framework for analyzing human activity through a socio-cultural lens (Engeström, 2010). Due to its adaptability, the theory can be used to conceptualize FC and mathematical activities as a product of social interactions and socio-historical needs (Roth and Radford, 2011). AT is a potentially powerful tool to conceptualize FC as an activity system, and to investigate how engineering students engage in group work activities, and how they use videos and quizzes to shape the mathematical discourse. Furthermore, AT is well suited to analyze the contradictions that arise within the elements of FC. As a result, we use AT as an overreaching theoretical framework to analyze the relationships between the elements of FC (subjects, objects, mediating artifacts, community, division of labor, and rules). To capture the

specificities of FC, we need middle-range theories to analyze the mathematical discourse that evolves in- and out-of-class activities, the types of interactions that occur in classroom, and the way the students use videos and quizzes.

RESEARCH QUESTIONS

Our research questions address both in- and out-of-class activities of FC:

1. What characterizes the students' use of videos and quizzes?
2. To what extent is the knowledge gained from the videos and quizzes integrated into classroom activities?
3. What types of interactions occur in classroom sessions?
4. What types of mathematical discourse emerge from students activities and how does it evolve and change over time?

We are also interested in the general question on how the students and teachers apprehend to the new set of rules that govern FC, that is, the new way of studying using videos and discussion in groups.

METHODOLOGY

We suggest performing an initial pilot study on the cohort following the study year of 2016/2017. Gathering experience from this initial study, while also being engaged in creating the study material (videos, quizzes and in-class activities), we would be ready for the main case study in the study year of 2017/2018. Because we consider this research work through the lens of Activity Theory, the study will focus on qualitative research methods. We will use an ethnographical approach to data gathering by means of video-filming, tape-recording, and interviews, following the same group of students over a whole year (Bryman, 2012). To analyze the data collected, we perform transcriptions to extract the mathematical discourse, explore the interactions in classroom, and other methods to make sense of video use.

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The development of the mathematical discourse of biology students working with mathematical modelling

Olov Viirman

University of Agder, Norway, olov.viirman@uia.no

Mathematical modelling, biology education, discourse

INTRODUCTION

This paper builds on the ongoing pilot phase of a developmental research project (Goodchild, Fuglestad & Jaworski 2013) aimed at increasing biology students' motivation for, interest in, and perceived relevance of studying mathematics through the use of mathematical modelling. The project is a collaboration between two of the four Norwegian centres of excellence in higher education – the Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC) and the Centre for Excellence in Biology Education (bioCEED) – and is motivated by changing demands in undergraduate biology education. The increased relevance of mathematics in biology (e.g. Cohen 2004) has created a need for developing the education of future biologists through a greater integration of mathematics and biology in the curriculum through, for instance, the use of mathematical modelling (Brewer & Smith 2011; Steen 2005). The focus of this paper, however, is on how the mathematical discourse of the students develops as they participate in the project.

MATHEMATICS AS DISCOURSE

In the commognitive framework of Sfard (2008), mathematics is conceived of as a form of *discourse*, that is, a specific type of communication drawing some individuals together while excluding others (ibid, p. 91). Mathematical discourse is distinguished by four characteristics: *word use*, *visual mediators*, *endorsed narratives*, and *routines* (ibid, p. 133-134). Learning is defined as individualizing discourse, becoming more capable of communicating within the discourse. Learning can take place both on the object-level, expanding the existing discourse, and on the meta-level, involving changes in the meta-rules of the discourse, that is, the rules governing the actions of the discursants. Central to meta-level learning is the notion of *commognitive conflict*, which occurs when different discursants act according to different meta-rules (ibid, p. 256). Such conflict is often a necessary aspect of meta-level learning, and identifying and analysing commognitive conflict is important when trying to understand students' difficulties.

THE TEACHING DEVELOPMENT PROJECT

The pilot phase consists of meetings with 12 first-year biology students from the University of Bergen on four occasions during the autumn of 2015. The meetings take place in parallel with the one compulsory mathematics course included in the

undergraduate biology program. During these three-hour meetings a mathematician skilled in mathematical modelling and with extensive teaching experience works with the students, presenting them with modelling tasks intended to bridge the gap between mathematics and biology, on which the students then work in groups. In the first session the teacher also gave an introduction to the notion of mathematical modelling, presenting the modelling cycle as a way of understanding modelling processes. Tasks given to the students included, for instance, estimating the population density of rabbits in an area based on the number of road-kill rabbits along a stretch of highway; and estimating the size of an extinct species of bird through comparing data on dimensions of fossilized bones with similar data from contemporary species of birds. All sessions are audio- and video-recorded, both the group work and the whole-group activities.

PRELIMINARY RESULTS

At the time of writing only three of the four sessions have been conducted, and none of the recordings have been transcribed. Hence, only very tentative observations can be presented here. One observation made concerns the notion of ‘assumption’. The making of reasonable simplifying assumptions was emphasised by the teacher as central to the modelling process. Apparently, the students took this to heart, and when they presented their work on the rabbit task all groups emphasised the simplifying assumptions they had made. However, two of the three groups had made unfounded and more or less random assumptions on the percentage of rabbits hit by cars. This caused some consternation on the part of the teacher, since from his standpoint it basically amounted to assuming what you want to find out. This can be interpreted as a case of commognitive conflict, where the students’ use the language of mathematical modelling discourse in a way that contrasts with the teacher’s use.

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TWG3: Logic, Numbers and Algebra

From ‘monumentalism’ to ‘questioning the world’: the case of Group Theory

Marianna Bosch¹, Josep Gascón², and Pedro Nicolás³

¹IQS School of Management, Univ. Ramon Llull, Spain, marianna.@iqs.edu;
²Dt. Mathematics, Universitat Autònoma de Barcelona, Spain, gascon@mat.uab.cat;

³Pedro Nicolás, Dt. Didactics of Social Sciences and Mathematics, Universidad de Murcia, Spain, pedronz@um.es

One of the main research gestures within the Anthropological Theory of the Didactic (ATD) consists in questioning the mathematical knowledge that is at the core of teaching and learning processes. In the case of university education, and because the knowledge at stake is closely related to scholar knowledge, this questioning might seem less necessary and the tendency is to focus on the possible different ways of organizing its teaching. Through the example of Group Theory we illustrate the methodology of questioning proposed by ATD in two different moments of the evolution of this framework: a first one centred on the structure and dynamics of the different components of mathematical praxeologies; a second one more focused on the raison d’être and functionality of what is taught and learnt.

Keywords: anthropological theory of the didactic, group theory, praxeologies, study and research paths

INTRODUCTION

Together with school Mathematics at secondary level and teacher education, university education has been one of the focuses of our research team during the past 25 years. The reason is partially circumstantial. As members of Mathematics Departments at the university, we are close to our object of study and are even part of it as teachers, what makes empirical studies easier. In addition, facing university teaching difficulties first-hand obviously contributes to putting them at the centre of our research agenda.

During these years, the approach we use, the Anthropological Theory of the Didactic (ATD), has strongly evolved with the introduction of the notion of praxeology (Chevallard et al 1997, Bosch & Chevallard 1999). Looking back at our first investigations concerning a new teaching device called ‘Workshops of Practice’ (Bosch & Gascón 1993, 1994), we can now better appreciate the evolution of the research methodologies and the changes related to where the attention is placed. We have chosen the case of Group Theory to present this evolution, as a way to illustrate two different ways of questioning the mathematical content to be taught: a first one based leaving the global structuring of the content untouched; and a second one requiring a complete deconstruction and reconstruction of the knowledge to be taught. This might help better understand the interrelations between the didactic and

epistemological assumptions made by teachers and by researchers when designing and experimenting new teaching processes.

FIRST STEP: A WORKSHOP OF PRACTICE ON GROUP THEORY

As said before, our starting point is a research project about the implementation of a new teaching device called Workshops of Practice (WoP) in a Mathematics degree in Barcelona in the 1990s. They were proposed as a way to complement the lectures and problem solving sessions, similar to the ‘lab sessions’ of the other Natural Sciences degrees (Physics, Chemistry, Biology, Geology). In the new pedagogical organisation proposed, the only modification was to add a weekly 3 hours session devoted to ‘practical work’ to each subject taught in the degree, each subject being assigned 3 or 4 sessions. What to do in these sessions? What kind of mathematical work appeared as most suitable?

Our proposal was to use the WoP to overcome the two-fold classical organization of university teaching in ‘lectures’ and ‘problem sessions’, based on an ‘applicationism’ vision of mathematics (Barquero et al 2013): students are first introduced to new concepts and mathematical organisations (lectures) and should afterwards apply them to solve a sample of problems of different types. Using the notion of praxeology (Chevallard 2006), we can consider that, in this traditional didactic organisation of university teaching, the first encounter of students with mathematical praxeologies is made through the main elements of their theoretical block (definitions, properties, assumptions, propositions, theorems and proofs). This is then followed by the exploration of the main types of problems conforming the praxeology at stake, using the techniques derived from the theoretical block previously introduced. The direction is always from the theoretical to the practical block of the praxeologies, from the presentation of new concepts and properties to their use to solve problems.

The work proposed to be done in the WoP was to carry out an in-depth study of a single type of problems (or mathematical phenomenon) taken from the course contents, deep enough to let the development of the techniques used and the raise of new theoretical needs. More concretely, each WoP session asked the students to study a given set of similar cases that could initially be solved with the same technique but also required some more or less important variations depending on the specificity of the case. This kind of work had multiple aims. First, students got a first-hand experience of the emergence of new theoretical needs related to the scope of the techniques used and the limits of the type of problems approached. It showed how the development of the practical block of praxeologies motivates the theoretical block. Secondly, it contributed to giving visibility to the ‘technical work’, which is crucial to mathematics creativity. Finally, it also gave more visibility to the main types of problems that conform the core of the taught subject, thus providing a kind of ‘dual description’ of the subject (in terms of types of problems instead of notions and theorems).

WoPs were experimented during several academic years in almost all the mathematical subjects of a Mathematics degree, from linear algebra to complex analysis. Their

design required a new analysis of the different subjects' contents, usually organised following the logic of the construction of concepts. The example of group theory will help us illustrate it. The subject here is 'Algebra 1' a first year course taught in the second semester, after 'Linear algebra'. Its objective was to introduce the main elements of abstract algebra (groups, rings, fields, morphisms, modular arithmetic, polynomials, etc.) that would be further developed in later courses. An example of a WoP session related to group theory starts with the statement: 'Consider the following groups and establish which ones are isomorphic and which ones are not. When they are not, give reasons. When they are, give a possible isomorphism.' Then a list of 31 groups sorted by their order is given (see figure 1), from order 2 with $(\mathbb{Z}_2,+)$, (\mathbb{Z}_3^*,\cdot) , (\mathbb{Z}_4^*,\cdot) and $(\{-1,1\},\cdot)$, till order 8 with $(\mathbb{Z}_8,+)$, $(\mathbb{Z}_4 \times \mathbb{Z}_2,+)$ $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,+)$ and the dihedral group D4 of the symmetries of a square.

Ordre 2:

- a) $(\mathbb{Z}_2,+)$ b) $(\{1,-1\},\cdot)$ c) (\mathbb{Z}_3^*,\cdot) d) (\mathbb{Z}_4^*,\cdot) e) (\mathbb{Z}_2,\cdot)

Ordre 3:

- a) $(\mathbb{Z}_3,+)$ b) (A_3,\cdot)

Ordre 4:

- a) $(\mathbb{Z}_4,+)$ b) $(\mathbb{Z}_2 \times \mathbb{Z}_2,+)$ c) $(\{1,-1,i,-i\},\cdot)$ d) (\mathbb{Z}_5^*,\cdot) e) (\mathbb{Z}_8^*,\cdot)
 f) $(K4,o)$ amb *K4 el grup 4 de Klein de les simetries d'un rectangle*
 g) $(\{f_1,f_2,f_3,f_4\},o)$ amb $f_1(x)=x, f_2(x)=-x, f_3(x)=1/x$ i $f_4(x)=-1/x$
 h) $(M4,\cdot)$ amb $M4 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \subseteq \mathcal{M}_2(\mathbb{R})$
 i) $(B4,\cdot)$ amb $B4 = \{Id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subseteq \Sigma_4$

Figure 1. Excerpt of the list of finite groups of the workshop of practice

The students had already had some lectures and problem sessions on group theory, with the main definitions, the notion of isomorphism and some examples. The first cases of the workshop thus appeared as non-problematic to them and students could get quickly inside the activity (an unfamiliar situation for most of them). Of course the finding of the isomorphism becomes more difficult as the group order increases. At the same time, having to solve numerous similar cases led many of them to search for properties to skip some steps, for instance by raising the problem of the characterisation of the isomorphism, of the number of groups existing for each order or the properties that are preserved by isomorphism. The very nature of the elements of the set soon seemed to be irrelevant and the strategy of decomposing a group into products of simpler ones sometimes appeared spontaneously.

All these questions, as well as the crucial role played by the group of permutations, were raised during the workshop by the students themselves or by the teacher. Some elements had been introduced in the lectures before the workshop, other were presented in it and treated in detail afterwards. In any case, and even if some of these theoretical elements were frontally introduced by the teacher, it was the technical work done by

the students that appeared as the source of new theoretical needs. At the same time, the WoP contributed to make the general praxeological structure of the taught content explicit to the students: what were the ‘big type of problems’ to approach, what the main techniques to use, what theoretical discourses structured and brought together the whole construction.

THE OFFICIAL RAISON D’ÊTRE OF AN SELF-SUFFICIENT THEORY

From our perspective of researchers in didactics, the aim of the WoPs was to introduce a new device to help students reconstruct the complete local mathematical praxeologies they were supposed to learn. The choice of the lists of problems or cases was based on a previous analysis of the taught subject, its description in terms of amalgams of praxeologies (types of problems, techniques, technological and theoretical discourses) and the search of their *official raison d’être* in the teaching institution, that is types of problems with enough generating power to obtain the main elements of the praxeology.

Given a content such as ‘Group Theory’ (a whole domain or just a theme, depending on the course considered), what does the search of its *official raison d’être* consists of? It requires determining the main types of tasks solved by this theory in the institution under consideration. For example, in the degree of Mathematics at two universities in Barcelona, Group Theory can appear in different subjects (‘Algebra’ or ‘Algebraic structures’, for instance). However, the corresponding content is similar to the one appearing in some standard handbooks:

- Groups, subgroups, homomorphisms.
- Cosets.
- Normality, quotient groups.
- Examples: cyclic, symmetric, alternating, dihedral.
- Isomorphism theorems.
- Group action.
- The Sylow theorems.

After interviewing the teachers in charge of the courses, reading the presentation of contents in the official programmes, consulting the recommended bibliography and the main teaching materials, one notices that the types of tasks concerning Group Theory are *internal* to this theory. The mainly consist in proving that two groups are isomorphic, construction homomorphisms of groups, proving certain properties of given groups or finding examples of groups satisfying certain properties, calculating the index of a subgroup with respect to a given group, proving that a certain subgroup is normal, etc. Therefore, we can claim that, in the institutions under consideration, elementary Group Theory exists for its own sake and is its own *raison d’être*: to study Group Theory is a goal in itself. Of course, we can claim that, in later courses, students will find non-trivial uses of Group Theory, for example in Galois Theory. This is yet another example of the ‘applicationism’ vision of mathematics: elementary knowledge first, its application later (Barquero et al 2013).

There are two main didactic facts associated with this state of things. The first one is that students' risk of missing the answers to important questions concerning Group Theory, to the extent that these answers conform the motivation of the theory and that this motivation is hidden. There are elementary questions (for example, why is the structure of group so interesting?) and more sophisticated ones (for instance, why should one be interested in classifying all groups of a given order up to isomorphisms? or why should one be interested in considering cyclic groups, etc.?). All of them are indeed difficult to answer in a non-trivial way without taking the motivation of Group Theory seriously. The second fact is that this 'Euclidean epistemology', as Lakatos defines it, is usually associated with a 'transmissive' didactic strategy placed in the *paradigm of visiting the works* (Chevallard 2015), where mathematical contents are presented to the students as important works to know (even as monuments to visit), without feeling the exigence of justifying or at least showing their importance.

Whether these two facts (unanswered questions about ultimate reasons and the paradigm of visiting works) are undesirable or not is something beyond the scope of a scientific answer, unless they were proved to cause further established inconvenient facts (for example, as regards the utmost aim of Education). In any case, it is still a challenging didactic problem to understand how and to what extent the study of Group Theory can be differently organised in university institutions. The WoPs tried to modify the traditional Euclidean organisation of contents by 'completing' it with a device aiming at articulating problems and theory by generating needs that could motivate new productions. But they do not question the global organisation of the contents, nor their official (and almost always implicit) *raison d'être*. Furthermore, as in traditional didactic strategies, they keep leaving the responsibility of choosing and posing the problems that are to be studied to the teacher (or to the designer).

Is it possible to go a step further? Can we move from the *paradigm of visiting monuments* to the *paradigm of questioning the world*? What would, in this case, be the transformations needed in the study of Group Theory? Is it possible to find Group Theory as a solution motivated by a certain set of problems? Without prejudging the suitability of any of these two paradigms, the problem seems to be interesting. We will deal with it in the next section.

TOWARDS THE PARADIGM OF QUESTIONING THE WORLD

When moving from *visiting monuments* to *questioning the world*, a new type of didactic analysis is required, focusing not on the *official raison d'être* of Group Theory but on different possible *alternative* ones that could motivate or impel the use of Group Theory as a solution to problematic questions. We are here following a methodology that can seem close to the search for a fundamental situation in terms of the TDS (Brousseau, 1997): What questions can call for the use of the main elements of Group Theory? And what are these elements? What can we do with Group Theory that we cannot do without it? Is there a question that could generate a substantial enough inquiry process so that, at a given moment, Group Theory tools appear to be, if not

necessary, at least highly recommendable? Ideally, the answers to all this kind of problems would give rise to what we call a *reference epistemological model* (Barbé et al 2005; Bosch & Gascón 2006), which would provide an alternative reconstruction of Group Theory to help approach problems related to its teaching and learning.

Before sketching some possible initial steps in the epistemological analysis of Group Theory with the aim to illustrate our methodology, let us mention some connected investigations. Some authors have introduced and investigate the so-called FUGS notions, namely, notions that introduce a new formalism which allows to *unify* and generalise (and, consequently, to *simplify*) several mathematical techniques and previous notions (Robert 1998). An example of such a notion would be that of group. Some of these authors advocate the following two thesis, which are directly related to our work. On the one hand, the FUGS character is more frequently attribute to the Mathematics of post-compulsory education than to the Mathematics of compulsory education (Robert 1998). On the other hand, models in terms of situations in the sense of Brousseau (1997) are considered unable to guarantee the genesis of FUGS notions (Robert et Robinet 1996; Robert 1998, Dorier 1995) because of their missing of a ‘meta’ level.

Concerning the first thesis, we believe that, after specifying the meaning of terms like ‘unifying’ and ‘generalising’, everyone would admit that some local praxeologies appearing in basic Mathematics, like numeral systems and measure of quantities, have a strong FUGS status. We also think that the argument supporting the second thesis should be revised, especially when considering ‘situation’ in the broad sense of alternative epistemological model for the school reconstruction of mathematical pieces of knowledge. Distinctions such as the ‘structuralist praxeologies’ proposed by Hausberger (2013) might seem more productive.

Let us go back to the epistemological analysis mentioned before. It is well-known that the role played by Group Theory in many fields of Science and Art is linked to the notions of *symmetry* and *invariant* (Weyl 1952): the classification of geometries suggested by Félix Klein (1924/2004), the study of solvability by radicals of a polynomial in Galois Theory, the classification of molecular structures, the classification of bidimensional ornaments,... It is hence shocking that the notions of *symmetry* and *invariant* are missing in practice in the university presentation of elementary Group Theory. Now we wonder: how to integrate these notions at the heart of a possible alternative *raison d’être* of elementary Group Theory?

In all the examples mentioned, the groups under consideration are sets S_X of bijective maps from a set X to itself, and the invariants are properties (expressed in terms of the base set plus, perhaps, an additional structure) reflected in a certain subgroup. When one considers S_X together with the composition of maps, the axioms of groups appear naturally. In this way, it becomes apparent that the binary operation in groups is not commutative in general. It also becomes clear that a subgroup of a group is just a subset containing the identity map and is closed under the composition and under taking inverse maps, because this is typically the case when one considers the subset of S_X

formed by those maps preserving some property expressed in terms of X . After this, and according to many authors (Freudenthal 1973; Burn 1996; Larsen 2013) it seems reasonable to postulate that elementary Group Theory might appear as a highly recommendable solution of problems set out in this context. The following is a specific example in this direction.

Consider the square C with vertices $V_1 = (-1,1)$, $V_2 = (1,1)$, $V_3 = (1,-1)$ and $V_4 = (-1,-1)$ in the Euclidean plane. Consider the set $\text{Sym}(C)$ of symmetries of C , that is to say, the set of isometries of the plane leaving invariant the set of points of the square. It is easy to prove that among the elements of $\text{Sym}(C)$ we find the counter-clockwise rotation R with centre $(0,0)$ and right angle, and its powers R^2 , R^3 and R^4 which happens to be the identity map $R^4 = I$. It is also easy to check that the reflection T_x (with axis $y = 0$), the reflection T_y (with axis $x = 0$), the reflection T_{13} (with axis the line passing through the vertices V_1 and V_3) and the reflection T_{24} (with axis the line passing through the vertices V_2 and V_4) are also elements of $\text{Sym}(C)$. The problem that appears is the following: Is the set $D := \{I, R, R^2, R^3, T_x, T_y, T_{13}, T_{24}\}$ the list of all the elements of $\text{Sym}(C)$?

Let us use elementary Group Theory to prove that $D = \text{Sym}(C)$. First, we notice that each element f of $\text{Sym}(C)$ induces an element $s(f)$ of the set S_4 of bijections from $\{1, 2, 3, 4\}$ to itself. Moreover, the map which sends f to $s(f)$ is injective and preserves the composition of maps. For example, we have: $s(I) = I$; $s(R) = (1234)$; $s(R^2) = (13)(24)$; $s(R^3) = (1432)$; $s(T_x) = (14)(23)$; $s(T_y) = (12)(34)$; $s(T_{13}) = (24)$; $s(T_{24}) = (13)$. It is easy to check, from the very definition, that S_4 (in fact S_n for any natural number $n > 1$) is closed under composition, has a neutral element with respect to the composition law, and every element has an inverse with respect to the composition law. Moreover, $\text{Sym}(C)$ can be regarded as a subset of S_4 containing the identity map, closed under composition, and closed under taking the inverse map (even if this last condition is not necessary in the case of finite groups).

After these considerations, if in our list D we had forgotten, for instance, to include element R^3 , we could claim this list to be incomplete. However, D can be regarded as a subset of S_4 containing the identity map, closed under composition, and closed under taking the inverse map. Thus, we still cannot prove that this list is exhaustive since we might still have missed some element of $\text{Sym}(C)$.

The strategy could be a different one. By using elementary Group Theory (more precisely, Lagrange Theorem), we know that the number of elements of $\text{Sym}(C)$ is a divisor of the number of elements of S_4 . Imagine we have proved that S_4 has $4 \cdot 3 \cdot 2$ elements. Since D is contained in $\text{Sym}(C)$ and D has 8 elements, we can say that $\text{Sym}(C)$ has at least 8 elements and at most 24 elements. But now, according to Lagrange Theorem, we know that the order of $\text{Sym}(C)$ divides 24 (since $\text{Sym}(C)$ is isomorphic to a subgroup of S_4) and it is divided by 8 (since D is a subgroup of $\text{Sym}(C)$). Therefore, the only options for the number of elements of $\text{Sym}(C)$ is either 8 or 24. But this last option is impossible, since the permutation (23) cannot be in the image of map s . Indeed, the fact that $(23) = s(f)$ for some f in $\text{Sym}(C)$, is not compatible with the fact that f preserves the Euclidean distance, since the distance between $f(V_1) =$

V_1 and $f(V_2) = V_3$ is the square root of 8, which, in turn, is different from 2, the distance from V_1 to V_2 .

Another problem solved by elementary Group Theory is the classification of the types of symmetries of polygons. Indeed, after the study of the group of symmetries of the regular polygon of n vertices and the corresponding lattice of subgroups, one can classify the types of symmetry of the polygons of n vertices. For instance, the knowledge of the structure of the group $\text{Sym}(C)$ (together with the knowledge of the lattice of subgroups) is useful to answer questions concerning the groups of symmetries of convex quadrilaterals, because every such symmetry is also a symmetry of the square. Moreover, this study gives rise to a classification of the quadrilaterals depending on their type of symmetry. This classification corresponds to the structure of the lattice of subgroups of $\text{Sym}(C)$ and could be more productively exploited in the secondary school Geometry curriculum (Gascón, 2004). Similarly, the study of the group of symmetries of the regular hexagon enables a classification of the possible types of symmetries of the convex hexagons. In general, the group of symmetries of the regular polygon of n vertices, together with its lattice of subgroups, can play the role of a *mathematical model* of the *mathematical system* formed by the different types of symmetries of the convex polygons of n vertices.

We have presented two problems of the same type which can be set out without explicitly mentioning groups but which can easily be solved with elementary Group Theory. This is good, but it is, by no means, guarantee of success in our research of possible *raison d'être* for Group Theory. Others can be found, starting from more extra-mathematical questions, such as the kinship of the indigenous Australian Warlpiri, that also have a structure of a dihedral group (Asher, 2002) or the symmetries of molecules. There are still many open questions left: How promising is this type of problems? For example, is it substantial enough to motivate the study of, for instance, the isomorphism theorems? What about the study of Sylow Theorems? If our motivating type of problems does not give rise to Sylow Theorems, should one consider the possibility of eliminating this from the official program? What if our motivating type of problems requires more knowledge about, for example, geometry, than the one considered in the official program of the degree of Mathematics? This kind of inquiry will soon start questioning the whole domain of Group Theory and, beyond, its relationships with other mathematical domains and the order in which students are supposed to deal with them.

Going further in our inquiry also supposes to consider the important didactics research that has been carried out in the domain (Dubinky et al 1997; Nardi 2000; Lester 2013 among others) and to look at the dimensions of the teaching and learning processes that have been questioned as well as those that have been taken for granted. To what extent, for instance, is the traditional organisation of contents (and the predominance attributed to concepts in detriment of problems) put into question? How are the solid teaching strategies anchored in the pedagogical paradigm of visiting work taken into account by the new inquiry-based teaching proposals?

CONCLUSIONS

We will not pursue the inquiry about possible *raison d'être* of Group Theory any further. Our aim was to illustrate, with this specific case, how the research about the Workshop of Practices carried out within the ATD was only based on a partial questioning of the mathematical content taught at university level and of the didactic devices that supported their teaching. A first questioning of the praxeologies that compose the taught mathematical organisation named 'Group Theory' led to identifying some types of problems the in-depth study of which enable students to feel new needs and raise theoretical and practical questions more or less guided by their teacher. However, the proposal of the WoPs took for granted the *raison d'être* that the teaching institution – here the university – assigned to the content at stake. And, more importantly, it assumed the fact that its *raison d'être* should remain implicit, without even formulating the question of its determination, not to speak about its determination by the own students.... Moving from the 'monumentalistic' approach underlying the WoPs to the paradigm of 'questioning the world' requires locating the *raison d'être* of the contents that are to be taught and learnt, and the reason of their learning, at the core of the study process. We can even say that the search of this *raison d'être* needs to be incorporated in the teaching and learning process itself. It does not appear unreasonable to start an inquiry-based teaching process with the very question: 'What is Group Theory for?'

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Approfondissement du questionnement didactique autour du concept de "borne supérieure"

Faïza Chellougui

Université de Carthage, Faculté des sciences de Bizerte, Tunisie,
chellouguifaiza@yahoo.fr

Ce travail s'inscrit dans le cadre des recherches sur l'imbrication des éléments de logique dans un raisonnement mathématique dans une perspective didactique. Dans la présente communication, je présente une formalisation logique des objets et des structures qui interviennent dans la définition d'un objet sensible enseigné à l'université : la notion de borne supérieure. L'étude didactique s'appuie sur un entretien avec des étudiants autour de la notion de borne supérieure. Cette étude a fait apparaître, d'une part, des phénomènes didactiques liés à l'alternance des deux types des quantificateurs, et d'autre part des difficultés dans la mobilisation de la définition des objets et des structures qui illustrent un problème majeur dans le processus de conceptualisation.

Mots clés: quantificateur universel, quantificateur existentiel, calcul des prédicats, formalisme logique, borne supérieure.

INTRODUCTION

De nombreux travaux de recherche en didactique des mathématiques (Selden & Selden, 1995 ; Bloch, 2000 ; Dubinsky & Yparaki, 2000 ; Durand-Guerrier & Arsac, 2003 ; Chellougui, 2004 ; Durand-Guerrier, 2005) ont montré que la complexité de la structure logique des expressions mathématiques quantifiées engendre souvent des difficultés spécifiques dans la gestion et la manipulation des quantificateurs par les étudiants : difficultés d'interprétation du vocabulaire logico-mathématique ; lacunes d'ordre opératoire ; difficultés dans la manipulation des énoncés complexes à quantifications multiples. Toutefois, en accord avec Duval (1995) et Durand-Guerrier (1996), j'adopte l'hypothèse selon laquelle une majorité d'étudiants scientifiques de première année universitaire maîtrisent la plupart des raisonnements classiques faisant appel seulement aux caractères propositionnels, à savoir le Modus Ponens et le Modus Tollens.

Dans cette communication, je propose une analyse didactique en choisissant un objet enseigné à l'université, il s'agit de la notion de borne supérieure. Cette analyse permet de proposer une formalisation logique des objets et des structures qui interviennent dans la définition de l'objet borne supérieure. Je présente ensuite les résultats obtenus avec des étudiants dans une situation d'entretien.

FORMALISATION DE LA NOTION DE BORNE SUPERIEURE DANS LE CALCUL DES PREDICATS

Dans ce qui suit, je désigne par (E, \leq) un ensemble E muni d'une relation d'ordre total \leq et par A une partie de E . Je propose une formalisation des objets et des structures mathématiques qui entrent en jeu dans la constitution du concept de borne supérieure. Je parle de propriété d'objet lorsqu'elle s'applique à un élément de E , par exemple : *être un majorant d'une partie donnée A de E* , et de propriété de structure lorsqu'elle s'applique à une partie de E , par exemple : *être une partie majorée de E* . Je propose de formaliser selon le calcul des prédicats, d'une part les propriétés des objets : *être un majorant (resp. minorant) de A* ; *être un plus petit élément (resp. plus grand élément) de A* ; *être la borne supérieure d'une partie majorée de E* ; et d'autre part, les propriétés des structures : *admettre un plus grand élément* ; *être une partie majorée de E* . Avant d'aborder cette formalisation, je commence par présenter un cheminement vers une définition de la notion de borne supérieure, en adoptant comme définition de référence celle de Schwartz (1991).

La borne supérieure, un objet mathématique complexe

Schwartz (1991) propose la définition suivante de la structure "*admet une borne supérieure*" donnée entièrement dans un langage naturel, où E désigne un ensemble totalement ordonné :

On dit qu'une partie A de E admet une borne supérieure si l'ensemble de ses majorants admet un minimum, et ce minimum est appelé borne supérieure de la partie considérée. La borne supérieure est donc le plus petit majorant ; tout élément qui majore A majore aussi sa borne supérieure. (Schwartz 1991, p.83)

Dans cette définition interviennent les définitions de la structure "*admet un minimum*" et des deux objets "*majorant*" et "*plus petit élément*", ainsi qu'une nouvelle structure "*ensemble des majorants*". Je fais l'hypothèse que la présence de ces différents éléments est une marque de la complexité de cette notion, qui se traduira en particulier par des difficultés prévisibles de l'interprétation entre objets et structures. Autrement dit, je peux considérer que l'étude de la syntaxe logique de la définition de la notion de borne supérieure est un indicateur de complexité pour la production d'un objet répondant à cette définition.

Je propose de mettre en évidence cette complexité par le schéma suivant :

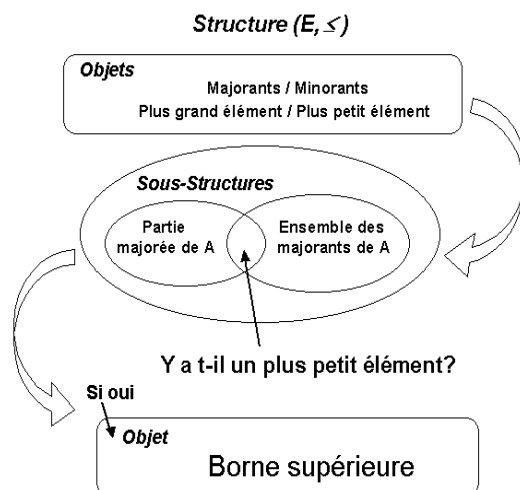


Schéma. – Complexité de la notion de borne supérieure

Ce schéma révèle une complexité de la structuration logique du fait que la définition imbrique des propriétés d'objets et de structures. Cette complexité fait émerger la nécessité d'un formalisme opératoire indiquant clairement à quoi s'appliquent les énoncés et à quel niveau on se situe.

Formalisation des objets et des structures

Objet *majorant / minorant*

Étant donné un élément y de E et une partie A de E , dire que " y est un majorant de A " (resp. " y est un minorant de A ") ou encore " y majore A " (resp. " y mineure A ") revient à définir une relation entre un objet mathématique et une structure. On définit ainsi une famille de propriétés d'objets lorsque A parcourt l'ensemble des parties de E . Je désigne cette relation par $M(y,A)$ (resp. $m(y,A)$) qui s'exprime dans le calcul des prédicats par l'énoncé : " $\forall x \in A \ x \leq y$ " (1) (resp. " $\forall x \in A \ x \geq y$ " (1'))

Il s'agit d'une phrase ouverte en y et close en x . En éliminant la quantification bornée¹ dans (1), on obtient : " $\forall x (x \in A \Rightarrow x \leq y)$ " (2) (resp. " $\forall x (x \in A \Rightarrow x \geq y)$ " (2'))

La négation de l'énoncé (2) est donnée par l'expression suivante :

$$"\exists x (x \in A \wedge y < x)" \text{ (3) ;}$$

ce qui signifie que " y n'est pas un majorant de A ", d'où la formalisation de la propriété *ne pas être un majorant de A* qui correspond donc à $\neg M(y,A)$.

¹ La quantification bornée cache l'implication, elle permet de restreindre le domaine de référence (Durand-Guerrier, 2003 ; Chellougui 2004)

Objet plus grand élément / plus petit élément

La propriété d'objet "y est un plus grand élément pour A" (resp. "y est un plus petit élément pour A") s'exprime dans le calcul des prédicats par l'énoncé ouvert en y, clos en x : $y \in A \wedge M(y,A)$ (4) (resp. $y \in A \wedge m(y,A)$ (4'))

Ou encore par l'énoncé : $y \in A \wedge \forall x(x \in A \Rightarrow x \leq y)$ (5) (resp. $y \in A \wedge \forall x(x \in A \Rightarrow x \geq y)$ (5'))

Dans le vocabulaire courant, cet objet est aussi désigné par l'expression "y est un maximum de A" (resp. "y est un minimum de A")

Structure être une partie majorée

La propriété de l'objet *élément majorant* nous permet de passer à la propriété de la structure *être une partie majorée* par l'affirmation de l'existence d'un majorant, au moins. On désigne par $M(A)$ l'expression "A est majorée" qui s'exprime dans le calcul des prédicats par l'énoncé : " $\exists y \forall x(x \in A \Rightarrow x \leq y)$ " (6)

Il s'agit ici d'une phrase close en y et en x, qui permet de définir une propriété de A au moyen d'un prédicat M sur l'ensemble des parties de E.

Structure admettre un plus grand élément / un plus petit élément

Cette propriété permet de définir, dans le calcul des prédicats, la propriété de structure : A admet un plus grand élément (resp. A admet un plus petit élément) :

" $\exists y [y \in A \wedge \forall x(x \in A \Rightarrow x \leq y)]$ " (7) (resp. " $\exists y [y \in A \wedge \forall x(x \in A \Rightarrow x \geq y)]$ " (7'))

Ce qui signifie qu'on peut trouver un élément y qui remplit les conditions suivantes : y élément de A et y élément majorant de A (resp. élément minorant de A).

Formalisation de l'objet borne supérieure

En me référant à Schwartz (1991), je reformule la définition de borne supérieure de la manière suivante : "Pour tout élément α de E, α est appelé borne supérieure de A, noté $\sup A$, si et seulement si α est un majorant de A et α est le plus petit des majorants" ou encore : "Pour tout élément α de E, α est appelé borne supérieure de A, noté $\sup A$, si et seulement si α est un majorant de A et pour tout élément y de E inférieur à α , y n'est pas un majorant de A"

Dans le calcul des prédicats la relation " $\alpha = \sup A$ " s'exprime par l'énoncé suivant :

" $[\forall x(x \in A \Rightarrow x \leq \alpha)] \wedge \forall y[(y < \alpha \Rightarrow \exists x(x \in A \wedge y < x)]$ " (8)

Selon l'usage en mathématique, on le donnera sous la forme de deux énoncés coordonnés : " 1) $\forall x(x \in A \Rightarrow x \leq \alpha)$ et 2) $\forall y [(y < \alpha \Rightarrow \exists x(x \in A \wedge y < x)]$ " (9)

Je peux noter ici, que le premier énoncé 1) exprime : " α est un majorant de A" et le deuxième énoncé 2) exprime : "Pour tout élément y de E inférieur à α , y n'est pas un majorant de A". Cette reformulation permet de ne pas introduire dans le formalisme

logique l'ensemble des majorants ; on coordonne ainsi deux relations entre un objet et une structure. La caractérisation de la borne supérieure, dans l'ensemble \mathbb{R} des nombres réels, est fréquemment formulée de la manière suivante :

"Pour toute partie A de \mathbb{R} , si A est majorée et si l'ensemble des majorants de A admet un minimum, alors il existe un élément unique α de E vérifiant les deux propriétés suivantes : 1') $\forall x (x \in A \Rightarrow x \leq \alpha)$ et 2') $\forall \varepsilon > 0 \exists x (x \in A \wedge \alpha - \varepsilon < x)$ " (10)

Cette caractérisation est une reformulation de l'expression (9) en utilisant dans l'énoncé 2) l'équivalence " $\alpha > y \Leftrightarrow \alpha - y > 0$ " et en remplaçant " $\alpha - y$ " par " ε ".

RESULTATS DES ENTRETIENS AVEC LES ETUDIANTS

L'analyse logique présentée auparavant va nous servir de référence pour analyser les réponses de six binômes d'étudiants de première année², de la section Mathématiques-Informatique de la faculté des sciences de Bizerte au premier semestre de 2002/2003, dans une situation de résolution d'exercices. La manière dont les étudiants s'approprient la notion de borne supérieure constitue la question centrale de cette expérimentation qui s'est déroulée suivant un questionnaire et un entretien. Ce dernier a fait apparaître plusieurs phénomènes dont nous présentons les plus importants.

« Etrange définition » d'un objet majorant

A la question « quelle est la définition d'un majorant d'une partie A de \mathbb{R} ? », on peut en voir la manifestation dans la réponse formalisée, du type (AE)³, donnée par écrit par l'étudiant J du binôme 1, : « $\forall x \in A \exists M \in \mathbb{R} x \leq M$ » (I)⁴

Il est possible que la réponse de J entremêle les définitions d'élément majorant et de partie majorée. Cette réponse relèverait alors d'une confusion entre propriété d'objet et propriété de structure et confirmerait notre hypothèse précédente. Si c'est le cas, J utilise un énoncé du type (AE) alors que c'est un énoncé de type (EA) qu'il a en tête.

Dans la définition formalisée du majorant M , l'étudiant J utilise un énoncé clos, alors qu'une telle définition nécessite un énoncé ouvert. Pour définir l'élément majorant M , l'étudiant introduit l'existence de M alors que la variable M ne devrait être soumise à aucune quantification.

² Les étudiants arrivant à l'université se trouvent au début de la première année en face d'activités mathématiques qui exigent une écriture quantifiée et un certain usage des éléments de logique. Alors qu'au lycée, les symboles logiques ne font pas un objet d'enseignement dans les programmes des mathématiques. Ceux-ci déconseillent explicitement l'utilisation formelle des quantificateurs (Chellougui, 2000).

³ (AE) désigne les énoncés du type : *Pour tout...il existe...* et (EA) désigne les énoncés du type : *Il existe...pour tout...*

⁴ Les formulations proposées du type (AE) sont désignées par (I) et celles du type (EA) sont désignées par (II).

On observe ainsi, un premier phénomène saillant important présent chez tous les binômes. Le tableau suivant récapitule les réponses des six binômes, formé chacun d'une étudiante et d'un étudiant, pour définir un objet majorant :

Binôme	Etudiante	Etudiant
Binôme1	<p>2.J : Soit M un majorant de E, quel que soit x appartenant à l'ensemble E, il existe M appartenant à IR, tel que x est inférieur ou égal à M.</p> <p>M est un majorant, puisque M appartient à IR, tout élément appartenant à IR et supérieur à M est un majorant. J écrit : « $\forall x \in E, \exists M \in \mathbb{R} / x \leq M$ (I) »</p>	<p>1.T : Un majorant : quel que soit M appartenant à IR, quel que soit x appartenant à E donc M est supérieur ou égal à x, M est un majorant de E.</p> <p>3.T : C'est la définition d'un majorant.</p>
Binôme2	<p>2.C : Quel que soit x appartenant à A, il existe un M appartenant à IR, tel que x est inférieur ou égal à M. C écrit : $\forall x \in E, \exists M \in \mathbb{R} / x \leq M$ (I)</p> <p>4.C : Il existe un M appartenant à IR avec quel que soit x appartenant à A on a x inférieur ou égal à M...C'est la définition d'un majorant. C note : « $\exists M \in \mathbb{R}, \forall x \in E, x \leq M$ (II) »</p>	<p>1.K : C'est le plus grand élément.</p> <p>3.K : M s'appelle un majorant de A</p>
Binôme3	<p>F : Quel que soit x appartenant à A, il existe un réel b tel que x est inférieur ou égal à b. On dit que b est un majorant de A. F écrit : $\forall x \in A, \exists b \in \mathbb{R} ; \text{tel que } x \leq b$ (I)</p>	

Binôme4	<p>N : Quel que soit x appartenant à A, il existe un réel M tel que x est inférieur ou égal à M. On dit que M est un majorant de A. N écrit :</p> <p>« $\forall x \in A, \exists M \in \mathbb{R}; \text{ tel que } x \leq M$ (I) »</p>	
Binôme5	<p>2.M : Quel que soit x appartenant à A, il existe y appartenant à \mathbb{R}., tel que ; x inférieur ou égal à y : y est un majorant de A. M écrit :</p> <p>$\forall x \in A, \exists y \in \mathbb{R}; \text{ tel que } x \leq y$ (I)</p>	<p>1.B : On a A inclus dans \mathbb{R}., quel que soit x appartient à A, on a x inférieur ou égal à y ; y c'est un élément qui appartient à \mathbb{R}.. Donc y majore l'ensemble A.</p> <p>3.B : Oui, c'est ça.</p>
Binôme6	<p>2.S : C'est il existe M un majorant, tel que quel que soit x appartenant à A tel que x est inférieur au majorant M. S note : « $\exists M \in \mathbb{R}, \forall x \in A, x \leq M$ (II) »</p> <p>4.S : C'est la définition d'un majorant.</p>	<p>1.A : Majorant [Long silence]</p> <p>3.A : C'est la définition de borne supérieure ?</p>

Tableau : Tableau récapitulatif des réponses des six binômes

Difficultés dans la mobilisation de la définition

Le deuxième phénomène important concerne l'apparition de difficultés dans la mobilisation de la définition. Après la proposition de l'énoncé (AE) par les étudiants, je suis intervenue en proposant l'énoncé (EA) correspondant :

$$\exists M \in \mathbb{R} \quad \forall x \in A \quad x \leq M \quad (\text{II}) \quad (\text{EA})$$

J'illustre ce phénomène par une séquence de la transcription du binôme choisi :

- 1.J : Dans (I), quel que soit x appartenant à A.
- 2.T : Oui, on peut utiliser (I).
- 3.J : Non, non, quel que soit x appartenant à A, il existe M appartenant à \mathbb{R} .
- 4.T : Oui, les deux sont justes. Il existe M appartenant à \mathbb{R} , quel que soit x appartenant à A, x inférieur à M. L'écriture mathématique est différente.

Il y a une différence d'interprétation des définitions pour caractériser un majorant :

- pour l'étudiant **J**, il y a un rejet de l'énoncé (EA) et un choix de l'énoncé (AE).

- pour l'étudiant **T**, les deux énoncés (AE) et (EA) gardent le même sens bien qu'il y ait une certaine différence dans l'ordre de l'écriture des deux quantificateurs.

A la suite de cette séquence, j'ai proposé de décrire graphiquement la situation sur une droite réelle, et également de déterminer un contre-exemple qui contredit le fait que **M** soit un majorant de **A**. En utilisant cette représentation graphique, j'ai voulu montrer au binôme, d'une part la dépendance entre **M** et **x** dans un tel énoncé (AE), et d'autre part le fait que cet énoncé ne traduit pas que **A** est une partie majorée. Les étudiants ont répondu à cette intervention de la manière suivante :

5.J : Il existe **M**, on a dit il existe..... Il existe ici.

(**J** indique un point extérieur à **A** qui est effectivement un majorant)

6.T : On peut dire que **M** n'appartient pas à **A**.

L'étudiant **J** veut introduire l'objet **M** à l'extérieur de **A** : « ...il existe ici... », puisque **M** existe, donc il est introduit. L'étudiant **T** rejette le schéma, en ajoutant une condition sur l'objet **M** « **M** n'appartient pas à **A** ».

A partir de ces deux interventions, je fais l'hypothèse que les étudiants ne se sont focalisés ni sur l'objet **M** ni sur toute la structure de l'ensemble **A**. Il n'y a eu de réaction ni sur la dépendance entre **x** et **M** ni sur la structure de **A**. Par contre, il y a une réfutation de l'objection introduite qui conduit à modifier la définition. Plus loin, l'étudiant **J** affirme :

7.J : Dans (I) **M** peut être un majorant.

Cette affirmation est correcte ; cependant l'énoncé (I) ne joue évidemment pas son rôle de définition. En effet, la formulation proposée dans l'énoncé (I) permet d'introduire un objet, qui peut être ou non un majorant de la partie considérée.

Il y a deux interprétations possibles : **J** veut dire que ce point **M** placé satisfait (I), ou bien, dans le cas où la partie est majorée, on peut choisir **M** de sorte qu'il vérifie :

$$\forall x \in A \quad x \leq M \quad (\text{II})$$

On voit ici les difficultés à manipuler les définitions et les propriétés : ceci renvoie à savoir ce qu'est une définition.

Principaux résultats

La nature de nos analyses reflète sans doute notre perplexité devant les résultats obtenus. En effet, les réponses que certains étudiants ont pu nous donner dépassaient souvent ce que nous pouvions imaginer. Ces réponses ont montré qu'il y a une imbrication entre le vocabulaire, les notations et les concepts, en particulier les relations entre élément et partie.

L'acceptation, par la quasi totalité des étudiants, d'une « définition » ambiguë d'un objet majorant a fait ressortir des phénomènes didactiques saillants qui sont liés principalement d'une part, à des difficultés dans la manipulation et dans la gestion des quantificateurs universel et existentiel, et d'autre part, à un recours aux représentations graphiques qui cachent souvent la structure complexe de la notion de borne supérieure et le jeu entre objet et structure.

CONCLUSION

Le présent travail a essentiellement porté sur le raisonnement mathématique, la question fondamentale étant le traitement de la logique opératoire dans le formalisme mathématique. L'étude de cette question permet de considérer la synthèse épistémologique comme un appui, d'une part pour analyser l'articulation entre logique et mathématique, et d'autre part pour la mise en place d'une étude empirique.

La formalisation logique complète, dans le calcul des prédicats, de la notion de borne supérieure a mis en évidence la complexité de la structure logique. Cette analyse a montré une imbrication entre les propriétés d'objets et les propriétés de structures que la pratique ordinaire concernant l'usage de la quantification en mathématiques tend à masquer (Chellougui, 2004).

L'étude mise en place, centrée sur la manière dont les étudiants définissent l'objet « être un élément majorant » et la structure « être une partie majorée », consiste à observer six binômes d'étudiants en situation de résolution d'exercices suivie d'un entretien. Les résultats didactiques obtenus confirment que la complexité de la structure logique d'un énoncé mathématique fait apparaître des phénomènes didactiques qui sont liés non seulement à des problèmes dans l'articulation des quantificateurs avec l'argument mathématique, mais aussi à des problèmes langagiers et des difficultés à manipuler les définitions et les propriétés. Ces résultats mettent en évidence des difficultés pour les étudiants à mobiliser de manière rigoureuse les énoncés quantifiés. Il s'agit principalement de la non prise en compte de l'ordre dans l'alternance des quantificateurs, de la tendance à mobiliser prioritairement les énoncés sous la forme « Pour tout... il existe... » plutôt que les énoncés « Il existe... pour tout... » et de la pratique courante selon laquelle l'affirmation de l'existence introduit l'objet.

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Procedural and Conceptual Understanding in Undergraduate Linear Algebra

Ana Donevska-Todorova

Humboldt-Universität zu Berlin, Germany

todorova@math.hu-berlin.de

This paper discusses the learning of concepts in undergraduate Linear algebra by pre-service teachers in mathematics. The focus is set on the bi-linear and multi-linear forms on a real vector space, exemplified by the dot product of vectors and determinants, respectively. Moreover, the paper identifies and describes discrepancies between students' achievements regarding the development of procedural and conceptual understanding. They are investigated through two types of exercises, discussing questions and multiple-solution tasks (MSTs), whose solutions differ under three criteria.

Keywords: linear algebra, pre-service teachers, conceptual understanding, multiple-solution tasks.

INTRODUCTION

This paper elaborates the current state of research about procedural and conceptual understanding. It focuses on a content-specific domain about linearity, bi-linearity and multi-linearity in undergraduate Linear algebra. In particular, I investigate pre-service teachers' understanding of these concepts. Further on, I argue that in order a task to be called a multiple-solution task (MST), a minimum of *three concrete criteria* for the diversity of the solutions must be fulfilled. These criteria may vary in different domains of mathematics. In this paper, I try to specify the definition of MSTs by Levav-Waynberg & Leikin (2009), by giving such criteria in the field of Linear algebra.

THEORETICAL FRAMEWORK

The theoretical framework consists of two parts, one referring to research on procedural and conceptual understanding, and, two, dealing with bi-linear and multi-linear forms in university and high school mathematics.

Procedural and Conceptual Understanding

Procedural knowledge, as defined by Hiebert & Lefevre (1986), is consisted of two parts: one, the symbolic mathematical language, and two, the “rules, algorithms or procedures used to solve mathematical tasks” (Hiebert, 2013, p. 3). The authors describe these procedures as subsequent step-by-step instructions that need to be executed when solving a mathematical task. One kind of such procedure is “a problem-solving strategy or action that operates on concrete objects” (Hiebert, 2013, p. 7). *Conceptual* understanding relates to a web of knowledge and is developed through an establishment of many relations between pieces of information or between existing and new

knowledge. It does not have a linear sequential character. Hiebert & Carpenter (1992) explain conceptual understanding as a structured network of concepts, their representations, and properties [1]. In this paper, I would like to specify these two types of understanding according to the content domain of Linear algebra by giving three examples. Namely, knowing how to carry out the Gaussian algorithm can be seen as a procedural understanding and applying it to solve a system of linear equations or to find an inverse of a matrix, thus linking it to other concepts, can be considered as a conceptual understanding. Likewise, procedural knowledge of the dot product of two vectors is the ability to calculate it according to a formula involving the components of both vectors, while a conceptual understanding is a possibility to connect it with projections of vectors and the trigonometric function cosine, so to interpret the obtained scalar geometrically (Donevska-Todorova, 2015). Similarly, procedural understanding of determinants is knowing how to calculate them by the Laplace (cofactor) expansion, for example, and a conceptual understanding of determinants means knowing how to use them for determining the existence of an inverse of a matrix or to interpret them as oriented volumes (Donevska-Todorova, 2012).

Bi-linear and Multi-linear forms in University and School Mathematics

In this section, I refer to concept definitions of bi-linear and multi-linear forms in pure mathematics and I exemplify them by the concepts of the dot product of vectors and determinants, respectively. Afterwards, I discuss the treatment and the importance of the term *linearity* from a didactics point of view.

Definition 1: A *multi-linear form* on a vector space $V(F)$ over a field F is a function $f: V(F) \times \dots \times V(F) \rightarrow F$ that satisfies the following axioms:

1. $\alpha \cdot f(u_1, \dots, u_i, \dots, u_n) = f(u_1, \dots, \alpha \cdot u_i, \dots, u_n)$
2. $f(u_1, \dots, u_i, \dots, u_n) + f(u_1, \dots, u_i', \dots, u_n) = f(u_1, \dots, u_i + u_i', \dots, u_n)$

for every $\alpha \in F$ and $u_1, \dots, u_n \in V(F)$ and any index i . For $n = 2$, the form is called *bi-linear*.

For example, the function $f((x_1, x_2), (y_1, y_2)) = x_1y_2 + x_2y_1$ is a bi-linear form on \mathbb{R}^2 and the determinant of a square matrix of degree n is a n -linear form of its columns or rows.

Bi-linear and multi-linear forms can take values in any vector space since the axioms make sense as long as vector addition and scalar multiplication are defined. Yet, in bachelor studies for pre-service teachers we usually discuss the bi-linear, i.e. multi-linear form on a real vector space, as is also the case in this study. Bi-linear, i.e. multi-linear forms over other fields, e.g. of complex numbers, are not part of this study.

Theorization of *linearity*, e.g. classification of bi-linear and multi-linear forms contributed to the development of the unifying and general theory of Linear algebra (Dorier, 2000). The term *linearity* is one of the central terms in Linear algebra, for the

reason that, it refers to linear combinations, linear (in)dependencies, linear mappings, bi-linear forms, such as scalar products and multi-linear forms, such as determinants, however, “linearity has not become an organizing idea for the students and this seems also to be true for quite a few teachers” (Tietze, 1994, p. 49). The term *linearity* is also used in high school, e.g. linear functions are studied in lower, and then, differentiation and integration in upper secondary education in relation to topics in Calculus. In high school Linear algebra, we teach linear transformations and treat only the concept of bi-linearity, though not the concept of *multi-linearity*. The term *bi-linearity* itself is never explicitly mentioned, nevertheless implicitly studied through the dot product of vectors. This is relevant for the transition from upper high school to university.

Exemplary research studies with a focus on students’ understanding of linear combinations (Possani, 2013) and linear (in)dependence (Bogomolny, 2007) have used different theoretical frameworks. However, it also seems that there is a lack of studies regarding the teaching and learning of bi-linear and multi-linear forms at any level of education.

DISCUSSING QUESTIONS AND MULTIPLE-SOLUTION TASKS

Before I proceed with elaborating the discussing questions and multiple solution tasks, I show a definition of determinants, which was applied during the observed lecture.

Definition 2. The mapping $\det: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is called a determinant if the following hold:

D1: \det is linear in every row.

D2: $rgA < n, \det(A) = 0$

D3: $\det(E_n) = 1$.

The definition axiom D1 means that both of the axioms 1. and 2. in the Definition 1 hold.

Discussion questions

Discussing questions in mathematics education offer a possibility for the students to talk and oral communication may contribute to the investigations concerning the development of procedural and conceptual understanding. When students articulate their thinking orally and by writing, they recall, reflect and consolidate their knowledge, and adequate understanding of concepts develops.

Discussing questions which were used in this study were the following.

Decide whether the following statements are true or false and provide argumentation to support your answer.

($\forall A, B \in \mathbb{R}^{n \times n}$) True or false:

- a) $AB \neq BA$, but $\det(AB) = \det(BA)$
- b) $\det(A + B) = \det(A) + \det(B)$

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- c) $\det(A) = 0$ then A^{-1} does not exist
- d) For $A \in \mathbb{R}^{2 \times 2}$, $\det(A)$ is the oriented area of the parallelogram spanned by $A\vec{e}_1$ and $A\vec{e}_2$.
- e) $\det(A) = 1$ if and only if $A = E_n$
- f) $\det(A) \neq 0$ if and only if for all $b \in \mathbb{R}^n$, the system of linear equations $A\vec{x} = \vec{b}$ has exactly one solution \vec{x} .
- g) $\det(AB) \neq 0$ if and only if A, B are regular
- h) $\det(A) = 0$ if and only if $\text{Kern}A = \{\vec{0}\}$
- i) For $A \in \mathbb{R}^{3 \times 3}$, $\det(A)$ is the volume of the parallelepiped spanned by $A\vec{e}_1$, $A\vec{e}_2$ and $A\vec{e}_3$.

These discussing questions include many concepts in Linear algebra such as matrix, identity matrix, square matrix, invertible matrix, singularity/ non-singularity of a matrix, linear dependence/ independence of vectors, kernel, systems of linear equations, area and volume. Consequently, this wide net of concepts makes them suitable for explorations of both procedural and conceptual understanding.

Multiple Solution Tasks

The mathematical problems which were implemented in this study can be classified as *Multiple-Solution Tasks* (MSTs) (Leikin, Levav-Waynberg, Gurevich & Mednikov, 2006; Leikin & Levav-Waynberg, 2007) because there exist multiple paths towards their solution. Regarding MSTs in Linear algebra, these solutions may be diverse in the sense of different usage of

- (1) *modes of description and thinking of concepts in Linear algebra* (Hillel, 2000; Sierpinska, 2000),
- (2) *properties of concepts in Linear algebra*, and
- (3) *subject-specific strategies or solving tools in Linear algebra*.

I now explain these *three criteria*. First, Hillel (2000) distinguished between *three modes of description* of concepts in Linear algebra: *geometric*, *algebraic* and *abstract*; and further on, Sierpinska (2000) described *three modes of thought*: *synthetic-geometric*, *arithmetic* and *analytic-structural*. Second, properties of concepts may be used for defining them, which is a usual way at university Linear algebra, or for describing them, which is typical for school mathematics. Third, in a concrete MST, it may happen that each subject-specific strategy for one of its solutions corresponds to exactly one mode of description and thinking. It may also be the case that different solutions require different solving strategies which all correspond to the same mode of description and thinking. Arguing through subject-specific strategies and exchanging geometrical and algebraic ideas and vice versa is a powerful tool for problem solving and obtaining deep understanding (Tietze, 1994). All three criteria are closely connected to both the

procedural and the conceptual understanding. An example of such MST according to the three criteria, which is also a part of the discussion in this paper later, is given in Table 1. Yet, such criteria, which make a particular solution diverse from another, do not explicitly appear in the exemplary MSTs by Levav-Waynberg & Leikin (2009) or by Leikin (2007).

MST1: Find the determinant of the matrix $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Write as many solutions as you can.	(1) Mode of Description/ Thinking	(2) Properties of Concepts	(3) Subject-specific Strategy for Problem Solving by the Use of:
Solution 1: $detM = det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \stackrel{def}{=} 2 \cdot 2 \cdot 2 \cdot det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{def}{=} 8 \cdot 1 = 8.$	Abstract/ Analytic-structural	Multi-linearity (Homogeneity axiom 1 in the Definition 2)	the axioms D1 and D3 in the Definition 3
Solution 2: $detM = det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot 2 \cdot 2 = 8$	Algebraic/ Arithmetic	The diagonal property for a triangular matrix	elementary matrix transformations
Solution 3: $detM = det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot 2 \cdot 2 + 0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 - 0 \cdot 2 \cdot 0 - 2 \cdot 0 \cdot 0 - 0 \cdot 0 \cdot 2 = 8$	Algebraic/ Arithmetic		Sarrus rule
Solution 4: $detM = det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - 0 \cdot det \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} + 0 \cdot det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2 \cdot 4 = 8$	Algebraic/ Arithmetic	A determinant of $n \times n$ matrix is as a sum of determinants of n sub-matrices $(n - 1) \times (n - 1)$	Laplace (cofactor) expansion
Solution 5: The determinant of M is equal to volume of 8 cubic units of a parallelepiped whose sides are obtained when each side of the unit cube is stretched twice.	Geometric/ Synthetic geometric		Geometry and linear transformations

Table 1: Example of a Multiple Solution Task in Linear Algebra

The aim of this study was, however, not to ask the students to explicitly provide more than one solution to all problems, but to ask them to offer one, which according to them, is the most rational in the number of undertaken steps and in the time required. I considered such written solutions, in addition to the oral responses on the discussing questions, as sufficient sources for analyzing students' procedural and conceptual understanding in the frame of this study.

RESEARCH QUESTION AND METHODOLOGY

Research Question

What kind of understanding do pre-service teachers show when they learn determinants during an undergraduate course of Linear algebra and analytic geometry?

The investigations aiming to offer answers to this question relate to procedural or conceptual understanding in the way they were described above.

Research Methodology

The research study took place at the Institute of Mathematics at the Humboldt University in Berlin. In their second semester, pre-service teachers take the course Linear algebra and Analytic geometry II. During this course, they study about the dot product of vectors and determinants, among other concepts. The goal of the undertaken observations was to locate and describe types of understanding which these students develop. The *observational protocol* (Creswell, 2013) includes researcher's notes about all undertaken observations and meetings with the lecturer, three teaching assistants and two tutors, who were responsible for the course. *Researcher's notes* consist of *demographic information* (time, place, date and participants), *descriptive notes* (instruction materials from the lecturer and the teaching assistants, and students' assignments) and *reflective notes* (reconstructions of dialogues, discussions and activities, researcher's personal thoughts, detections, ideas, proposals and impressions). The researcher neither took part in the selection of the exercises nor participated in the discussions during the lectures and the exercises sessions. In this way, researchers' influence on the teaching and learning process was eliminated. All information gathered by the observational protocol represents primary material to be analysed further on.

FINDINGS AND DISCUSSION

There are 120 students taking the course. After every lecture, they participate in course exercises and write home assignments each week. I first shortly discuss students' performance on the discussing questions which were part of the course exercises and then on the MST1 (Table 1) which was given as a homework problem.

During the actual learning process, students were allowed time to think about the true/false questions and discuss in pairs, before they articulate their thinking aloud. At least half of the students have stated their opinion about the validity of each of the

statements (with an exception of one statement). In the space constraints of this article, I comment only two out of all nine discussing questions, which I consider were problematic for the students. Firstly, although the majority of the students claimed falsity of the statement b), none of them could give reasons why the statement is incorrect. Neither could they offer examples to show it. This illustrates students' uncertainty about the distinction between operations with matrices and determinants. In this statement, the multi-linearity property of determinants, so axiom D1 in Definition 2, more precisely axiom 2 in Definition 1 comes into focus and it seems that students were not able to explain it orally. Secondly, on the statement d) only three students gave answers, two of them claiming correctness and one of them falseness of the statement. This question is related to geometric visualization of determinants. What seems to be difficult for the students to comprehend is the establishment of a link between the algebraic symbolism of determinants and their geometric interpretation as oriented area of parallelograms or oriented volume of parallelepipeds (statement i). Moreover, it seems that students confused orientation, i.e. a property of determinants changing their sign when any two rows (or columns) switch their places, with elementary matrix transformations (a conclusion derived from the observational protocol). This confusion is classified as a misconception in Linear algebra by undergraduate students (Aygör & Özdağ, 2012).

Out of the five alternative solutions on the MST1 (Table 1), surprisingly, students used only three (Solution 2, 3 and 4), none of them referring the definition axioms or geometry. This shows that, according to criterion (1), students used only arithmetic-algebraic modes of description in their written assignments, by applying the Saruss rule, the Laplace expansion and transformations of matrices, as subject-specific strategies for solving the problem, which meets criterion (3). According to criterion (2), it seems that students did not use the properties which construct the axiomatic-structural definition, rather others, e.g. determinant of a triangular matrix equals the product of its diagonal entries.

In addition to these findings based on the discussing questions and the MST1, I discuss one more MST [2] which it is the following.

MST2. Find the determinant of the matrix $D = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}$.

There exist several adequate ways of solving the task. I consider the one based on the definition axiom D2 (in the Definition 2) the fastest because it is an argument on which I can immediately derive zero value of the given determinant. Analysing students written performance on this task (which was part of the observational protocol), I found out that only 44% of them used this definition. 28% of them used the Laplace expansion and the

rest of the students used matrix transformations. These data, again according to the same criteria (1) and (3), show that students based their solutions on the algebraic mode of description (Hillel, 2000) the arithmetic mode of thinking (Sierpinska, 2000), but not on the abstract-structural or geometric mode. Similar results were derived by analysing students' written works on other problems [3].

The discrepancy between the utilization of the algebraic mode of description and the arithmetic mode of thinking on the one hand; and the geometric and the abstract-structural modes of description and thought, on the other hand, shows students' predominant possession of procedural versus conceptual understanding. In relation to the research question, it seems that, students think of the Laplace expansion as a secure way towards a correct solution, by carrying out a step-by-step sequence of calculations. In contrast to this, the geometric solution does not require computing skills, but visualizations and interpretations; and the abstract-axiomatic one, necessitates decision making and justification, which seem to be cognitively more difficult processes. This means that the students easily accomplish procedures, but face difficulties in shifting between different modes of descriptions (Donevska-Todorova, 2014), changing strategies and connecting more concepts (Donevska-Todorova, 2012a). Linking procedural and conceptual understanding can be accomplished by developing meaning for symbols and applying procedures to solve problems effectively (Hiebert & Lefevre, 1986).

CONCLUSIONS

The findings of this study show that pre-service teachers taking an undergraduate Linear algebra course have some problems when they learn the concept of *multi-linearity*. It seems that they do not understand what does linearity in a row (column) mean. This conclusion derives upon their insufficient argumentation and exemplification about the additive axiom 2. in Definition 1 discovered through the discussing questions; and absence of usage of the homogeneity axiom 1. in Definition 1 when solving MST1. Homogeneity and additive properties of determinants are often confused with matrix operations when students multiply by a scalar or add all entries of the determinant instead of entries in a single row (column).

The investigations on the performance on the MSTs, according to the three criteria, show that students prefer one mode (the algebraic mode of description and arithmetic mode of thinking), a few concepts' properties (not those which construct the axiomatic concept definition) and a few subject-specific strategies (calculi-based procedures, e.g. cofactor expansion). In connection to the research question, this shows that the pre-service teachers participating in the course have developed mainly procedural understanding while their conceptual understanding remains under construction.

This study may contribute to further research by showing how could the theoretical framework about procedural and conceptual understanding (Hiebert, 2013), in addition to the theory about MST (Leikin & Levav-Waynberg, 2007; Leikin, Levav-Waynberg, Gurevich, & Mednikov, 2006) and the theories about multiple modes of description and thinking (Hillel, 2000; Sierpinska, 2000) be used for analyzing students' achievements in undergraduate Linear algebra. In this article, these different theories are integrated through the three criteria for the solutions of MSTs in Linear algebra.

ACKNOWLEDGMENTS

I thank the Humboldt University of Berlin for the Caroline von Humboldt Grant.

NOTES

1. I come to the point of representations and properties of mathematical concepts in the section Multiple-Solution Tasks.
2. The complete task was: Find the determinants of the matrices:

$$A = \begin{pmatrix} 3 & 4 & 6 \\ 1 & -3 & 1 \\ 9 & 0 & -13 \end{pmatrix}, B = \begin{pmatrix} 2 & 7 & 3 \\ -1 & 2 & 2 \\ 3 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The total number of points was 12 and students' average score was 11.5 points.

3. Similar findings were derived, for example, by finding the determinant of C. There were students who used the Laplace expansion, exclusively for all four matrices A, B, C and D.

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Designation at the core of the dialectic between experimentation and proving: a study in number theory

Marie-Line Gardes¹ and Viviane Durand-Guerrier²

¹ESPE de l'Académie de Lyon, Laboratoire Langage, Cerveau et Cognition, UMR 5304 CNRS-Université Lyon 1, France, marie-line.gardes@univ-lyon1.fr; ²Institut Montpellierain Alexander Grothendieck, UMR 5149 CNRS-Université de Montpellier, France, viviane.durand-guerrier@univ-montp2.fr ;

It is well known in mathematics education that students feel strong difficulties for elaborating by themselves mathematical proofs, even when they are involved in meaningful solving research problem activities. There are research based evidences that even in settings where the milieu for validation seems to be rich enough to support the proving process, some students fail to enter appropriately into it. In this paper, we provide some empirical results supporting the following hypothesis: although the designation of objects plays an important role in the heuristic phases, it might not be sufficient to enrol students in elaborating proof in cases the properties of these objects and their mutual relationships are not made explicit.

Keywords: solving research activities, proof and proving skills, gesture, designating an object

INTRODUCTION

It is well known in mathematics education that students feel strong difficulties for elaborating by themselves mathematical proofs. At first glance, one might think that a relevant way to improve proof and proving skills is to involve students in meaningful solving research problem activities (Durand-Guerrier & al., 2012). However, there are research based evidences that even in settings where the *milieu* for validation (Brousseau, 1997) seems to be rich enough to support the proving process, some students fail to enter appropriately into it (e.g. Tanguay & Grenier, 2010). Relying on an epistemological study, we acknowledge that *designation* plays an important role in the proof and proving process, including the heuristic phases. In this paper, we provide some empirical results supporting the following hypothesis: although the *designation of objects* plays an important role in the heuristic phases, it might not be sufficient to get students enrolled in elaborating proof in instances where the properties of the involved objects and their mutual relationships are not made explicit. In the first section, we present a didactical engineering (González-Martín & al., 2014) in Number Theory from which the empirical data were collected, and aiming at fostering the development of students' skills for solving mathematical research problems. In the second section, we present the concept of *gesture* adapted from Philosophy of Mathematics. Introduced by Cavailles (1981), it has been developed in particular by Châtelet (1993) and more recently by Longo (2005), in order to analyse mathematical activity. We first briefly present seven gestures that appear to be relevant for analysing the research process carried out by researchers and students, and then we develop more

on the *gesture* “designating an object”. In the third section, we analyse some empirical data out of Gardes (2013) in order to support the above mentioned hypothesis.

A DIDACTICAL ENGINEERING IN NUMBER THEORY

The didactical engineering that we present below is part of a research project (Gardes, 2013) whose main goal was to study the conditions and constraints pertaining to the transposition of professional mathematicians’ research activity in the mathematical classroom. The didactical engineering has been elaborated from an unsolved problem in Number Theory, the Erdős-Straus conjecture: “the equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ can be solved in positive integers x, y and z for any integers $n > 1$ ”. The general methodology of the research consists in three interrelated studies in mathematics, epistemology and didactics. The mathematical and epistemological analyses included observation, running over three years, of the work from two mathematicians on this problem. This contemporaneous epistemological study along with a more classical study on the context of discovery in mathematics (Gardes, 2013, pp. 108-149) allow the identification of various strategies for entering into and progressing through the research process: *reducing the problem to prime numbers, transforming the original equation while preserving equivalence, constructing effective decomposition, constructing and implementing algorithms*. Relying on Cavailles (1981) and Châtelet (1993), Gardes (2013) adapted the concept of *gesture* in mathematics activity in order to describe, analyse and contextualize the research processes that appeared in the various situations that have been experimented around the Erdős-Straus conjecture in the frame of the research project. This is developed below in section 2.

Several pre-experimentations at different levels (middle school (grade 7 and 9); high school (grade 12) and university (first and third-year)) have been carried out in order to test the suitability of this situation for secondary and tertiary students, and to determine the main characteristic of a *milieu* (Brousseau, 1997) favouring the involvement of students in a genuine and rich problem solving activity on the Erdős-Straus conjecture (the *devolution*, Brousseau, 1997). We have identified the following core elements: 1. For the students: the availability of mathematical and heuristics knowledge and a frequent practice of mathematical research activities; 2. For the didactical engineering itself: several research sessions with a very precise organisation, a formulation of the conjecture involving verbs of action, and availability of programmable calculators or computers. These elements have been taken into consideration when elaborating the main experimentation, in which we tried to control as precisely as possible the didactical conditions and constraints that we had identified. This experimentation took place in spring 2012 during two months with ten students from grade 12 (17 years old) who followed a two-hour-per-week specific “excellence program”, aiming at preparing them for University, in addition to a two-year optional course providing, among other things, a rich mathematical background in Number Theory: *familiarity with congruence and algorithms; Bezout theorem, Gauss theorem, fundamental theorem of arithmetic*, etc. This experimentation consisted of seven two-

hour sessions: one individual research session; four collective sessions devoted to research in small groups (three groups of three or four students); one debate session and one synthesis session.

During the specific weekly sessions of the “excellence program”, the students had been regularly engaged in research activities aiming at developing heuristics, such as recognising the epistemic status of a conjecture and the role of examples and counterexamples in the development of proof and proving; availability of different types of arguments and modes of reasoning; recognising the importance of taking into consideration intermediate and partial results. The students were able to apprehend different aspects of mathematical research: learning from the unsuccessful phases, identifying the diversity of approaches, being able to use various frameworks and to make links between different fields, etc. We hypothesised that, through these didactic experiences, these students elaborated a rather adequate representation of mathematical activity in general. In addition, the core elements mentioned above have been taken into account in the construction of the didactical engineering, so that the initial *milieu* included all the elements that we identified in order to favour the *devolution* of the problem over a long period of time and to foster significant advances and fruitful research developments.

In the next section, we present the concept of *gesture* that we have developed and we examine in particular the fundamental gesture consisting in “*designating an object*”.

THE CONCEPT OF *GESTURE* IN OUR RESEARCH

The concept of *gesture* comes from the philosophy of mathematics (Cavaillès, 1981, 1994; Châtelet, 1993, Bailly & Longo, 2003, Longo 2005). It allows considering different aspects of the mathematician’s work: active dimension of research, central role of intuition in the creative process and dialectical aspects between acquisition of knowledge and development of heuristics, and of skills in proof and proving. In our didactic perspective, we consider the following definition, adapted from Cavaillès and Châtelet:

A *gesture* is an action connecting mathematical objects and which is carried out with intentionality. It is an operation that is accomplished through a combination of signs with respect to the usage rules of these signs. Because they open on possibilities, *gestures* have the power to enhance mathematical creativity (translated from Gardes, 2013, p. 155).

Relying on our epistemological study (both historical and contemporaneous) and on our pre-experimentations, we have identified seven gestures that appear to be relevant for analysing the research process carried out by researchers and students:

- *Reducing the problem to prime numbers*: this is relevant in the Erdős-Straus problem due to the fact that the property is multiplicative;
- *Designating an object* i.e., representing a mathematical object by means of a natural language expression or a symbol;

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- *Introducing a parameter in a mathematical writing*: this allows making visible some relationships between two or more involved objects, without these objects being explicitly referred to;
 - *Constructing and questioning examples*: to frame a method of constructing examples from manipulation of mathematical objects, and to study these different examples in order to generate information;
 - *Making local controls*: checking the different stages of manipulation and combinations of signs in mathematical writings;
 - *Transforming the original equation while preserving equivalence*;
 - *Implementing an algorithm*: translating a mathematical algorithm in a programming language.

In this paper, we examine specifically the *gesture* “designating an object” by means of a language expression or a symbol, a gesture we consider as crucial for heuristics. In particular, when an object is designated by a symbol, it is possible to perform operations involving this symbol, temporarily leaving aside the reference. In our experiments, in some cases this *gesture* has supported theoretical development. In other cases, an adequate designation of an object allowed a reformulation of the conjecture entailing an enrichment of the initial problem. More generally, we observed in many cases that this *gesture* enabled to keep advantage of the experimental aspects of the problem, fostering the back-and-forth between manipulation of objects and theoretical elaborations, or in other words the dialectics between semantics and syntax (Gardes, 2012).

During the heuristic phases of the students’ work, the *gesture* “designating an object” emerged from the manipulation of the involved objects (fractions, integers) in interrelation with the mathematical knowledge at stake. So, it played an important role in the students’ research at several levels: to introduce some mathematical objects they used as tools to advance in research (prime numbers, congruence); to provide intermediate conjectures and to have the various steps of their method written down and formalized. Nevertheless, our *a posteriori* analyses of the students’ work (in particular from Group 3) show that it might be insufficient for getting enrolled in a proving process, as we will see in section 3. We hypothesise that in order to foster the involvement in proving, the *gesture* “designating an object” should make visible the properties of numbers and the relationship between the numbers involved.

In the next section, we specifically examine the phases of proof construction in the students’ work of two groups (group 1 and group 3) to support this hypothesis.

DESIGNATION IN STUDENTS’ WORK

During the experimental part of their work, the students in group 3 have obtained many results, mainly a method of decomposition of $\frac{4}{n}$ for a given value of n (n a prime number), including six identities and the verification of the conjecture of Erdős-Straus for $0 < n < 300$ (Gardes, 2013, pp. 445-449). However, although the group members explicitly recognize the necessity of a proof, they meet resisting difficulties to engage

themselves into a proving process relying on their experimental results. This is in accordance with results from Tanguay and Grenier (2010) concerning the relationships between activities of definition, of construction and of proving in Geometry. We conjecture that the main obstacles to their involvement in the proving process pertain to the choices the students made in order to designate the mathematical objects, choices that do not enlighten the properties of these objects and their mutual relationships, opposite to what appeared in group 1.

We present below elements of the *a posteriori* analysis (Gardes, 2013, pp. 315-475) that support this claim. We will focus on three phases of the didactical engineering: the session 3 in which students share their results and methods; the session 4 devoted to students work in their own small group and the synthesis of group 1 and 3 work, written at the request of the researcher at the end of session 4.

Session 3 - Students share the first results obtained within the three groups

During the third session, the teacher organised exchanges between the students: in each group, students had prepared a synthesis of their work that was being presented to their classmates. Each presentation was followed by a reaction from students of the other groups. Below are some excerpts of the presentation of group 3, and some students' reaction from groups 1 and 2.

Students in group 3 are presenting a method for providing a decomposition of the fraction $\frac{4}{n}$ as a sum of two fractions: $\frac{4}{n} = \frac{1}{y} + \frac{x}{z}$, where x , y et z are non zero natural numbers. They explain that they are looking for the greatest value of $\frac{1}{y}$ satisfying the property “ $\frac{4}{n} - \frac{1}{y}$ is positive”. This value is found by essay-errors with the calculator i.e. students were trailing different numbers in the calculators. Then they try to decompose the fraction $\frac{x}{z}$ as a sum of two fractions with numerator 1 (Egyptian fractions). The other students ask several questions; they try to understand how to find the value of y .

Student (group 3): we take a prime number by chance, for example $\frac{4}{457}$ and we try to find a fraction such that $\frac{4}{457}$ minus for example $\frac{1}{115}$ is positive; if we take $\frac{1}{114}$ it becomes negative. [...] We know that it is the smallest natural number.

Student (group 2): $\frac{1}{115}$, did you find it with the calculator, making test?

Student (group 3): Feeling blindly, we did not manage to do it with greater numbers.

A student in group 2 proposes to consider the integer part of the number $\frac{n}{4}$ in order to determine the closest Egyptian fraction. A student in group 3 replies:

Student (group 3): it would be more logic, we should make tests.

At this point, it seems to the researcher that the explicit designation of the integer part would allow to put forward the method presented by the students of group 2 and to support an involvement in proof and proving. As will be shown below, this is actually the case only for the students in group 1. If we analyse this with the concept of the

situation *milieu*, we can say that for the students in group 2 and 3, although this came after the discussion, the designation was in the *materiel milieu* of the students but for some of them, it was not in their *heuristic milieu* (González-Martín & al., 2014).

Session 4 - Students came back working in their respective groups

Following the discussion, students in group 3 wrote $\frac{4}{n} = \frac{1}{t} + \frac{j}{k}$ where t is the smallest natural number such that $\frac{4}{n} - \frac{1}{t} > 0$, and j and k are non zero natural numbers. This equation allowed them to find relevant values for t by essays-errors using the calculator, and so to determine several decompositions of $\frac{4}{n}$ for given values of n . This equation with four variables (n , t , j and k) is efficient for providing decomposition; however, it does not show possible relationships between some of these variables, in particular the relationship between n and t that had been explicitly stated during the collective discussion through the denotation of the integer part of the number $\frac{n}{4}$.

Opposite, students in group 1 developed this idea of an explicit relationship between n and t through the use of the integer part, and tried to generalize it. To that purpose, they thought of using the notation $E\left(\frac{n}{4}\right) + 1$. They then wrote $\frac{4}{n} = \frac{1}{E\left(\frac{n}{4}\right)+1} + \frac{x}{n \times \left(E\left(\frac{n}{4}\right)+1\right)}$, with x a non-zero natural number. This allowed them to determine the value of t for given values of n .

Seeing the decompositions provided by both groups 1 and 3 (annex 1, annex 2), one might get the impression that the two equations have the same efficiency. The difference appears when we consider the involvement in proving.

Comparing the syntheses of groups 1 and 3

After having written their equation $\frac{4}{n} = \frac{1}{E\left(\frac{n}{4}\right)+1} + \frac{x}{n \times \left(E\left(\frac{n}{4}\right)+1\right)}$, students in group 1 tried to determine a decomposition of the second fraction $\left(\frac{x}{n \times \left(E\left(\frac{n}{4}\right)+1\right)}\right)$ as the sum of two Egyptian fractions. Relying on examples ($n = 29$, $n = 457$, $n = 461$, $n = 4513$), they conjecture that *if $n \equiv 3[4]$ then $x = 1$ and if $n \equiv 1[4]$ then $x = 3$* . As they had already a decomposition of $\frac{4}{n}$ in sum of three Egyptian fractions when n is even, they recognized that with these results, they were covering all cases pertaining to the determination of y . The designation of y by $E\left(\frac{n}{4}\right) + 1$ allowed these students to getting successfully engaged in the proving process, reaching an important intermediate result. Then, they considered the case where $x = 1$ and managed to provide the following general decomposition for $n \equiv 3[4]$: $\frac{4}{n} = \frac{1}{y} + \frac{1}{2z} + \frac{1}{2z}$. They then considered the case $x = 3$, and made a reasoning by distinction of cases, considering the values of z modulo

6: they established a general decomposition in the cases $z \equiv 0[6]$, $z \equiv 2[6]$ and $z \equiv 3[6]$. They failed to find a general decomposition for the two remaining cases¹.

In group 3, students first wrote their equation: $\frac{4}{n} = \frac{1}{t} + \frac{j}{k}$, where t is the smallest natural number such that $\frac{4}{n} - \frac{1}{t} > 0$, and j and k are non-zero natural numbers. They distinguished cases that they studied successively. The first case they studied is $j = 1$; they got the equation $\frac{4}{n} = \frac{1}{t} + \frac{1}{2k} + \frac{1}{2k}$; then they studied the case $j = 3$ and k even, and got the new equation: $\frac{4}{n} = \frac{1}{t} + \frac{1}{k} + \frac{2}{k}$. The third case they studied is $j = 3$ and k odd, with as a first attempt the case where k is a multiple of 5.

According to us, the comparison of the synthesis of these two groups enlightens the role played by the choice of designation made by students in order to transform their equations. In the first group, the designation of y by the symbolic notation $E\left(\frac{n}{4}\right) + 1$ allowed them to enter into fruitful transformations of their initial equation, opening the possibilities of reasoning by disjunction of cases and so getting stabilized results: identification of classes of natural numbers for which we have a general formula providing a decomposition – identification of the remaining cases that needed to be continued in the research. In the third group, although students showed cleverness in finding operative patterns to perform decompositions, they did not manage to determine the classes of natural numbers for which the answer is positive, and those classes for which the question remains open.

CONCLUSION

The *a posteriori* analyses of the work of these two groups show that for both groups, the designation gesture promotes involvement in the proving process. Nevertheless, the work of the third group showed that for these students, although their gestures of designation of objects are fruitful in the heuristic phases, they met difficulties to enter in general proof. At the opposite, the choices of designation made by the students of the first group, by showing explicitly the relationship between n and y , allowed them to establish general intermediate results and to identify the cases that need further studies. This supports our hypothesis that the *gesture* of “designation of object” might not be sufficient to enter successfully in proof and proving when this designation does not make explicit the properties and the involved relationships. This is a point that we should take into account when elaborating the *milieu* of such didactical engineering, aiming at fostering the development of students’ skills for solving mathematical research problems. The way to do this is an open research question.

¹ Currently, the numbers for which the Erdos-Strauss equation does not hold is a part of one of these classes. For details on a synthesis of mathematical results: Gardes (2013, pp.73-103).

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ANNEXE 1: STUDENTS' WORK (GROUP 1) – SYNTHESIS²

We use the following writing:

$$\frac{4}{n} = \frac{1}{y} + \frac{x}{z}$$

If we put $y = \text{ent}\left(\frac{n}{4}\right) + 1$, we have:

$$\frac{4}{n} = \frac{1}{\text{ent}\left(\frac{n}{4}\right) + 1} + \frac{x}{n \times \left(\text{ent}\left(\frac{n}{4}\right) + 1\right)}$$

So, we try to write $\frac{x}{n \times \left(\text{ent}\left(\frac{n}{4}\right) + 1\right)}$ as a sum of two fractions with numerator 1 ie. with the form $\frac{1}{k} \rightarrow x = 1$ or $x = 3$ (even if n is even and therefore not prime number)

- If $x = 1$ then $\frac{4}{n} = \frac{1}{\text{ent}\left(\frac{n}{4}\right) + 1} + \frac{1}{2n \times \left(\text{ent}\left(\frac{n}{4}\right) + 1\right)} + \frac{1}{2n \times \left(\text{ent}\left(\frac{n}{4}\right) + 1\right)}$
- If $x = 3$, we reason modulo 6.

If $z \equiv 0[6]$ or $z \equiv 3[6]$, $\frac{3}{z} = \frac{3}{3k} = \frac{1}{k} = \frac{1}{2k} + \frac{1}{2k}$.

Then $\frac{4}{n} = \frac{1}{y} + \frac{1}{2k} + \frac{1}{2k}$

If $z \equiv 2[6]$ or $z \equiv 4[6] \rightarrow z$ is divisible by 2, then $z = 2k$.

$$\frac{3}{z} = \frac{1}{z} + \frac{2}{z} = \frac{1}{2k} + \frac{1}{k}$$

Then $\frac{4}{n} = \frac{1}{y} + \frac{1}{2k} + \frac{1}{k}$.

If $z \equiv 1[6]$ or $z \equiv 5[6]$, we have not found any possibility to make a decomposition of $\frac{3}{z}$ as a sum of two fractions with numerator 1 (ie. with the form $\frac{1}{k}$).

² Translated by the authors. The original work is in (Gardes, 2013, Annexes, pp.83-84).

ANNEXE 2: STUDENTS' WORK (GROUP 3) – SYNTHESIS³

- If n is even $\frac{4}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{n}{2}}$ (with n even)
- If n is odd and prime
 $\rightarrow \frac{4}{n} = \frac{1}{t} + \frac{j}{k}$ with t the smallest natural number and j, k non-zero natural numbers.

There are 3 cases.

- Cas 1 :

$j = 1$ with $\frac{j}{k}$ an irreducible fraction.

$$\frac{4}{n} = \frac{1}{t} + \frac{1}{2k} + \frac{1}{2k}$$

- Cas 2 :

$j = 3$ with k even.

$$\frac{4}{n} = \frac{1}{t} + \frac{1}{k} + \frac{2}{k}$$

- Cas 3 :

- $j = 3$ and k is a multiple of 5. Thus $\frac{4}{n} = \frac{1}{t} + \frac{1}{2k} + \frac{5}{2k}$.

- $j = 3$ and k is odd

$\frac{4}{n} = \frac{1}{t} + \frac{3}{k}$. The number k is decomposed into prime factors.

Subcase 1:

There is at least one of divider congruent to 2 modulo 3 such as $k = d \times q$

For this case, we note that $\frac{4}{n} = \frac{1}{t} + \frac{3}{n \times t}$.

Put $\frac{d+1}{3} = e$ with e even.

We multiply $\frac{3}{nt}$ and $\frac{e}{e}$

³ Translated by the authors. The original work is in (Gardes, 2013, Annexes, pp.196-197).

A propos des praxéologies structuralistes en Algèbre Abstraite

Thomas Hausberger

Université de Montpellier, thomas.hausberger@umontpellier.fr

Nous introduisons dans cet article la notion de praxéologie structuraliste en didactique de l'algèbre abstraite, en appui sur une étude épistémologique du structuralisme mathématique. Nous illustrons notre propos en détaillant les praxéologies structuralistes en arithmétique des anneaux abstraits, sur la base d'une étude de manuels, et étudions les praxéologies structuralistes développées sur un forum de mathématiques en situation adidactique. Les enjeux didactiques de la notion de praxéologie structuraliste sont discutés en relation avec les difficultés identifiées dans l'enseignement et l'apprentissage de l'algèbre abstraite, à la transition entre Licence et Master de mathématiques.

Mots clefs : algèbre abstraite, praxéologies, structuralisme mathématique, didactique et épistémologie des mathématiques

INTRODUCTION

Les difficultés que rencontrent l'enseignement et l'apprentissage de l'algèbre abstraite à l'Université (notamment les structures algébriques de groupe, d'anneau et de corps) sont reconnues par de nombreux auteurs (Nardi 2000, Durand-Guerrier et al. 2015, Hausberger 2013) et reflètent un problème de « transition » (Gueudet 2008), qui a lieu cette fois, par comparaison avec la transition lycée-université, à l'intérieur d'une même institution.

De nombreux facteurs explicatifs sont à lier à la nature épistémologique particulière du savoir enseigné (le « challenge de la pensée structuraliste », Hausberger 2012) et ses conséquences didactiques que l'on peut analyser au sein du cadre épistémologique des savoirs FUGS (formalisateur, unificateur, généralisateur, simplificateur ; Robert 1987, voir également Hausberger 2012 p. 430). Un fait nouveau par rapport à l'algèbre linéaire, souligné dans Hausberger 2012, est le suivant : l'unification se situe à plusieurs niveaux

- le niveau 1 : une même théorie s'applique à des objets de nature différente ;
- le niveau 2 : la présentation axiomatique des structures en permet un traitement unifié (on se pose à propos des différentes structures le même type de questions que l'on cherche à résoudre avec le même type d'outils) mettant en avant les ponts entre ces structures ;
- le niveau 3 : ce qui était forme (les structures) devient pleinement objet à un niveau supérieur d'organisation, la théorie des catégories ou autre méta-théorie des structures.

Si le niveau 3 ne peut guère être abordé de façon réaliste avant la seconde année de Master, l'enjeu de la pensée structuraliste se situe au niveau 2, ainsi que le mettent en

exergue les manuels d'algèbre abstraite à la suite de l'ouvrage de van der Waerden (1930). Ces manuels témoignent de l'application d'une méthode : “la méthode structuraliste” (voir ci-dessous).

En Théorie Anthropologique du Didactique (TAD, Chevallard 2002), une méthode est un *ensemble de techniques*. La TAD, en effet, “considère que, *en dernière instance*, toute activité humaine consiste à *accomplir une tâche t d'un certain type T*, au moyen d'une certaine *technique* τ , justifiée par une *technologie* θ qui permet en même temps de la *penser*, voire de la *produire*, et qui à son tour est *justifiable* par une *théorie* Θ . En bref, toute activité humaine met en œuvre une organisation qu'on peut noter $[T/\tau/\theta/\Theta]$ et qu'on nomme *praxéologie*, ou *organisation praxéologique*” (loc. cit.). Alors que l'algèbre abstraite apparaît souvent, du point de vue de l'apprenant, comme un ensemble de questions et de tâches isolées dont on a saisi ou non l'astuce en permettant la résolution, nous faisons l'hypothèse que la mise en évidence des techniques structuralistes permet d'éclairer les pratiques en algèbre abstraite, de faire apparaître leurs raisons d'être et d'en fonder l'unité.

Barbé et al. (2005) montrent comment le modèle épistémologique dominant du Calculus dans l'enseignement secondaire conditionne les organisations mathématiques dédiées à l'étude des limites de fonction. De même, dans le contexte de l'algèbre abstraite, il s'agit d'étudier comment le modèle épistémologique du structuralisme mathématique se traduit en termes d'organisations praxéologiques dans l'enseignement des structures algébriques et la résolution de problèmes en algèbre abstraite.

La constitution d'un modèle praxéologique de référence pour l'algèbre abstraite vise également à éclairer le problème de transition suscité par cet enseignement, dans l'esprit du travail de Winsløw (2006), qui, étudiant les praxéologies en analyse au sein de l'institution Université, constate que la transition du concret à l'abstrait que requiert la transition secondaire-supérieur se traduit, au niveau praxéologique, par le développement de nouvelles praxéologies dont le bloc praxique est construit sur le bloc du logos d'une praxéologie que l'étudiant maîtrise déjà. Si l'algèbre abstraite provient d'une réécriture conceptuelle de l'algèbre classique selon la méthodologie structuraliste, comment cette réécriture se traduit-elle en termes praxéologiques ? Y trouve-t-on des spécificités par rapport aux praxéologies en algèbre linéaire enseignées lors des deux premières années de licence ?

Nous ne répondrons pas à toutes ces questions dans cet article mais nous poserons les fondements de la notion de praxéologie structuraliste que nous introduisons en didactique de l'algèbre abstraite, en appui sur une étude épistémologique du structuralisme mathématique. Nous détaillerons plus particulièrement les praxéologies structuralistes en arithmétique des anneaux abstraits puis nous analyserons les praxéologies développées par un collectif d'apprenants sur un forum de mathématiques à propos d'une question portant sur la structure algébrique des nombres décimaux (voir également Hausberger 2015a et 2015b pour d'autres analyses de ce corpus). Ces données empiriques permettront de donner corps à la

notion de praxéologie structuraliste, fondée épistémologiquement, et de montrer sa pertinence dans l'analyse didactique des praxéologies en algèbre abstraite.

ETUDE ÉPISTÉMOLOGIQUE DU STRUCTURALISME MATHÉMATIQUE

La notion de structure est issue de la constitution des mathématiques en tant que science des « relations entre objets », point de vue relationnel abstrait qui domine les mathématiques contemporaines depuis l'élaboration des axiomatiques formelles par Hilbert notamment. La pensée structuraliste se caractérise par une méthodologie et un style spécifique, qui font école à Göttingen autour de Noether dans les années 1920. Cette école change la manière de prouver en privilégiant les preuves générales limitant les calculs et en mettant en avant les concepts. Définir des concepts a pour objectif de reconstruire un domaine sur une nouvelle base, sur la base de concepts plus fondamentaux, plus généraux et plus « simples » :

Il faut s'appliquer à réduire un domaine mathématique à ses concepts fondamentaux les plus généraux, donc les plus simples, puis à construire et à reconstruire à l'aide de ces seuls concepts (Hasse 1930).

Il s'agit donc d'une refondation mathématique, portée par un projet qui relève également du didactique (permettre l'intelligibilité d'un contenu structuré). Cette reconstruction apporte une vision nouvelle de la matière mathématique et ouvre la voie à des constructions inédites, de nouveaux objets. La citation de Hasse est également intéressante par la mise en avant de la généralité, posée en principe (recherche de la généralité maximale), car vecteur de simplifications. Ce point de vue, s'il est emblématique du projet structuraliste, ne manquera pas d'être débattu au sein de la communauté mathématique ; ainsi, pour Mandelbrojt :

Il y a un moment où l'ensemble d'objets, auxquels il s'applique, explique le sens même du théorème. [...] C'est ainsi qu'on obtient la généralité explicative. Personnellement, je sens qu'il y a un optimum à cette généralité. (Mandelbrojt 1952, p. 426-427)

Les *raisons d'être* des concepts sont ainsi à trouver dans l'examen des preuves, en les faisant apparaître comme des « ressorts » (phase d'analyse). On en fait ensuite une théorie déductive, en posant ces principes isolément puis en les combinant, de façon à produire des systèmes axiomatiques fertiles (comme celui définissant un groupe) tels que les théorèmes sur les objets considérés apparaissent comme des conséquences logiques de ces derniers systèmes (phase de synthèse). Ce processus de constitution des concepts structuralistes est bien décrit dans le Manifeste (Bourbaki 1948) rédigé par un groupe de mathématiciens, Bourbaki, qui en fut l'un des grands promoteurs et qui s'est donné comme projet de diffuser à l'ensemble des mathématiques la méthodologie développée par les algébristes allemands.

Bourbaki évoque la portée applicative de la méthode structuraliste, appliquée sur des problèmes nouveaux, dans un paragraphe intitulé « La standardisation de l'outillage mathématique » :

Son trait le plus saillant, d'après ce qui précède, est de réaliser une économie de pensée considérable. Les structures sont des outils pour le mathématicien ; une fois qu'il a

discerné, entre les éléments qu'il étudie, des relations satisfaisant aux axiomes d'une structure d'un type connu, il dispose aussitôt de tout l'arsenal des théorèmes généraux relatifs aux structures de ce type, là où, auparavant, il devait péniblement se forger lui-même des moyens d'attaque dont la puissance dépendait de son talent personnel, et qui s'encombraient souvent d'hypothèses inutilement restrictives, provenant des particularités du problème étudié. On pourrait donc dire que la méthode axiomatique n'est autre que le « système Taylor » des mathématiques. (Bourbaki 1948, p. 42).

Le discours de Bourbaki est clairement de nature technologique : tous les ingrédients sont présents pour définir des praxéologies. Cependant, afin de traduire l'esprit de la méthode, les types de tâches ne doivent pas rester au niveau théorique (montrer une inférence dans le jeu axiomatique abstrait) mais attraper des objets concrets et montrer le gain du point de vue conceptuel. Comme nous allons le constater bientôt, nous nous heurtons pour cela à une difficulté liée à la présence de méthodes élémentaires qui agissent comme un pôle attractif vis-à-vis de l'apprenant comparativement au pôle conceptuel abstrait souvent répulsif.

LES PRAXÉOLOGIES STRUCTURALISTES DANS LE CAS DE L'ARITHMÉTIQUE DES ANNEAUX ABSTRAITS

L'arithmétique de \mathbf{Z} et des anneaux des entiers des corps de nombres, dans le cadre de généralisations successives, a été l'une des branches prépondérantes dans le cheminement historique, progressif, de la construction de la théorie des anneaux, avec la mise en évidence des grandes classes d'anneaux : anneaux euclidiens, principaux, factoriels. La problématique est la suivante : quel est l'analogue de \mathbf{Z} si l'on remplace \mathbf{Q} par une extension finie de \mathbf{Q} (i.e. un corps de nombres) ? On veut que le théorème fondamental de l'arithmétique soit vérifié (existence et unicité de la décomposition en facteurs premiers). Or ce n'est pas toujours le cas, même pour la bonne notion d'anneau des entiers ; ce résultat négatif conduit Dedekind à introduire les idéaux : dans les anneaux des entiers, tout idéal se décompose comme un produit d'idéaux premiers.

Une autre branche qui a contribué historiquement est l'arithmétique des polynômes, développée par Hilbert bien que ce dernier n'ait pas fait le lien avec les corps de nombres : il a fallu attendre Noether vers les années 1920 pour que l'on comprenne bien l'intérêt des structures comme *principe organisateur*. Le rapprochement est alors effectué entre la théorie des groupes et celle des anneaux, à travers les théorèmes d'isomorphismes par exemple (Hausberger 2013), ce qui met en évidence l'intérêt d'un traitement unifié des structures et conduit à la réécriture des deux théories selon la méthode structuraliste.

Les exemples $\mathbf{Z}[i]$ (anneau des entiers de Gauss) et $\mathbf{K}[X]$ apparaissent ainsi comme les deux grands exemples paradigmatiques à unifier. Certains livres (par exemple Guin 2013) préfèrent $\mathbf{K}[X]$ en vertu de la propriété universelle d'un tel anneau (point de vue théorie des catégories). D'un autre point de vue, $\mathbf{Z}[i]$ engendre toute une classe

d'exemples (les anneaux des entiers) que l'on peut traiter avec des techniques similaires, donc génère des *types de tâches*.

Les notions d'anneau, d'homomorphisme et d'idéal constituent un « fond théorique » dans lequel s'inscrit cette théorie des anneaux factoriels, principaux et euclidiens. De nombreux auteurs (par exemple Guin 2013, Chap. I) font ainsi précéder ce chapitre d'un chapitre « généralités sur les anneaux » très abstrait (sans problématique sur des objets), en s'appuyant sur les analogies formelles entre les groupes et les anneaux dans la structuration de la théorie : par exemple, les sous-groupes distingués et les idéaux sont les noyaux des homomorphismes. D'autres auteurs (Colmez 2011) vont plus loin et posent, au sein d'un même chapitre « structures algébriques », en début d'ouvrage, la totalité des structures usuelles en algèbre ainsi que les notions structuralistes « usuelles » (« sous-truc », morphisme, noyau et image, produits et sommes de « trucs », construction de quotients). Les concepts et résultats mis en avant dans de tels chapitres constituent typiquement, dans notre approche, les *techniques et technologies structuralistes* à dégager et mettre en relation avec les tâches en algèbres abstraites engageant des objets.

Nous donnons ci-dessous une liste de tâches qui nous apparaissent comme récurrentes dans les photocopiés et manuels que nous avons consultés (Guin 2013, Perrin 1996 par exemple) :

- T_1^+ (resp. T_2^+ , T_3^+ , T_4^+) montrer qu'un anneau est intègre (resp. factoriel, principal, euclidien) et T_1^- (resp. T_2^- , T_3^- , T_4^-) : non intègre (resp. non factoriel, non principal, non euclidien)
- T_5 : déterminer le groupe des inversibles A^* d'un anneau A
- T_6 : déterminer les irréductibles, ou simplement montrer qu'un élément donné est irréductible, ou encore décomposer un élément en produit d'irréductibles
- T_7 : calculer le pgcd de 2 éléments

Il est important de noter que le domaine mathématique qui nous concerne est caractérisé par une *structure écologique en « poupées russes »* (la chaîne d'inclusions des différentes classes d'anneaux : factoriel, principal, euclidien), qui impacte les types de tâches. Se situer par rapport à des classes d'objets et en tirer parti, délimiter ces classes donc l'extension des concepts, occupent une partie importante des tâches dévolues à l'étudiant.

La réalisation des tâches nécessite également le passage à un mode de pensée ensembliste et la mise en relation de propriétés et d'opérations sur les éléments avec des propriétés et opérations correspondantes sur les ensembles, sous la forme d'un véritable « dictionnaire » : par exemple, a est un élément premier si et seulement (a) est un idéal premier non réduit à (0) , le pgcd est lié à la somme des idéaux, un quotient A/I est intègre si et seulement si l'idéal I est premier. Cet héritage de Dedekind et Noether représente un saut conceptuel conséquent. Dans Hausberger (2013), nous mettons en relation le problème de transition généré par l'accès à

l'algèbre structuraliste à l'Université avec la transition épistémologique vers ce *mode de pensée en termes de sous-ensembles distingués et d'homomorphismes* qui engendre les praxéologies structuralistes.

L'exemple de la tâche T_1^+ (montrer qu'un anneau est intègre)

Cette tâche peut être traitée à différents niveaux, selon l'importance de la *dimension structuraliste*. Prenons l'exemple de l'anneau $\mathbf{Z}[i]$:

- Au premier niveau (niveau 1), il s'agit de démontrer que la définition de l'intégrité est satisfaite, autrement dit qu'« un produit est nul si et seulement si un des facteurs est nul ». On écrit pour cela $(a+ib)(c+id)=0$, ce qui conduit à un système un peu pénible à résoudre dans les entiers, d'où probablement une impasse pour un grand nombre d'étudiants. Par contre, lorsque l'on réalise que $\mathbf{Z}[i]$ est inclus dans l'ensemble \mathbf{C} des nombres complexes et qu'un élément non nul est inversible donc simplifiable, la preuve devient « triviale » : $zz'=0$ avec z non nul, donne, en multipliant par z^{-1} , la nullité de z' .
- Au niveau 2, on utilise toujours l'inclusion de $\mathbf{Z}[i]$ dans \mathbf{C} mais on invoque le résultat général que le sous-anneau d'un corps est intègre. Le bloc technologico-théorique, réduit dans le niveau 1 aux propriétés des nombres complexes, intègre maintenant des résultats abstraits généraux, des structures. C'est ce type d'organisation mathématique qui est visée, et non l'organisation mathématique restant au niveau de la théorie des objets.
- Au niveau 3, on invoque qu'un corps est intègre et que l'intégrité est une propriété stable par sous-anneau. C'est le même bloc théorique que le niveau 2 en apparence mais la formulation de la réponse et son mode d'obtention sont différents : on n'applique plus un théorème du cours, mais le mode de pensée structuraliste par rapport à la question posée : elle concerne l'intégrité ; on raisonne alors en termes de classes d'objets, de relation entre ces classes (anneau-corps) et de conservation de la propriété (intégrité) vis-à-vis des opérations structuralistes sur ces classes (passage à un sous-anneau).

L'exemple de la tâche T_3^+ (montrer qu'un anneau est principal)

Dans le cas de l'anneau des entiers relatifs \mathbf{Z} , au niveau 1, on se donne un idéal I de \mathbf{Z} non réduit à (0) et considère $a = \inf(I \cap \mathbf{N})$. On montre alors que $I = a\mathbf{Z}$ en utilisant la division euclidienne. C'est la même preuve que pour les sous-groupes de \mathbf{Z} . Le niveau 2 consiste à remarquer que la preuve précédente fonctionne dès que l'on a une généralisation de la division euclidienne : cela revient à effectuer T_3^+ via la tâche T_4^+ .

Ci-dessous, il sera question de la tâche T_3^+ dans le cas de l'anneau \mathbf{D} des nombres décimaux. La première idée est de généraliser la preuve de \mathbf{Z} (niveau 1). On se rend alors compte que c'est la relation de \mathbf{D} à \mathbf{Q} qui fait fonctionner la preuve : tout sous-anneau de \mathbf{Q} est principal (niveau 2'). Un apprenant ayant accompli ce chemin et assimilé le nouveau contrat structuraliste retiendra par la suite cet argument plus synthétique. Il est également possible de montrer que \mathbf{D} est euclidien (donc effectuer

T_3^+ via T_4^+), en généralisant à \mathbf{D} la division euclidienne de \mathbf{Z} , en appui sur l'écriture d'un décimal sous la forme $a/10^n$.

Définition des praxéologies structuralistes

En définitive, chaque type de tâche présente une *dialectique entre le particulier et le général*, où l'on essaie soit de généraliser/adapter des preuves connues, soit de généraliser l'énoncé à démontrer en conjecturant que le nouvel énoncé est vrai et porteur de simplification. La *pensée structuraliste* se caractérise par des raisonnements *en termes de classes d'objets, de relations entre ces classes et de stabilité de propriétés par des opérations sur les structures*. Sur les exemples précédents, dès que le niveau 2 est atteint, nous pouvons parler de praxéologie structuraliste. De façon générale, une telle praxéologie va viser la réalisation de la tâche en se plaçant à un niveau de généralité qui soit porteur de simplification, en appui sur les concepts et sur l'*outillage technologique structuraliste* (combinatoire des structures, théorèmes d'isomorphismes, théorèmes de structures, etc.). La méthodologie structuraliste vise ainsi à remplacer une praxéologie $(T, *, *, *)$ par une praxéologie structuraliste $(T^s, \tau, \theta, \Theta)$, où T^s est une généralisation de T qui permette l'usage de techniques structuralistes. Nous allons observer ce phénomène plus en détail à travers le travail réalisé par un collectif d'apprenants en situation adidactique.

ETUDE DES PRAXÉOLOGIES STRUCTURALISTES DÉVELOPPÉES PAR UN COLLECTIF D'APPRENANTS SUR UN FORUM

Le fil de discussion qui nous concerne, intitulé « les nombres décimaux », est visible à l'adresse suivante : <http://www.les-mathematiques.net/phorum/read.php?3,318936,page=1> Les échanges ont eu lieu probablement pendant un temps assez court, en 2007. L'intervention initiatrice du fil est le fait d'un forumeur, *Mic*, lequel met avant deux assertions et deux questions : A_1 (\mathbf{D} est un sous-anneau de \mathbf{Q}), A_2 (Tout sous-anneau de \mathbf{Q} est principal), Q_1 : Comment le démontrer ?), Q_2 (Comment définit-on le pgcd de deux décimaux ?).

D'emblée, nous remarquons que les assertions A_1 et A_2 sont les deux prémisses d'un syllogisme dont la conclusion est « \mathbf{D} est principal », assertion notée A_0 et qui est probablement visée par *Mic*. L'assertion A_2 est une généralisation de A_0 (nous notons $A_2 = A_0^g$), dans l'esprit de la méthode structuraliste : la preuve recherchée se place au niveau de généralité supérieur (A_0^g), reflétant la pratique experte des mathématiciens qui d'une part postulent que cette généralisation est porteuse de simplification, d'autre part considèrent qu'elle est éclairante quant aux « raisons profondes » à l'origine du phénomène (la principalité de \mathbf{D}). La question Q_2 lui est également liée : tant l'existence du pgcd que les diverses définitions (ou propriétés) du pgcd que l'on peut énoncer dépendent du type d'anneau dans lequel on se place.

L'investigation de ces questions va conduire un autre forumeur, *bs*, à porter la question à un niveau de généralité encore supérieur et formuler Q_1^g (tout sous-anneau d'un anneau principal est-il principal ?). Le forumeur *barbu rasé* y répond ensuite à travers une généralisation Q_1^{gg} de la question : il donne une classe de contre-exemples

à l'assertion « toute propriété remarquable des anneaux (euclidien, principal, factoriel, noethérien, de Bezout) est stable par sous-anneau ». Un autre participant, Toto le zéro, énonce de son côté l'assertion A_3 ($\mathbf{Z}[X]$ n'est pas principal), destiné à fournir également un contre-exemple à la question Q_1^g qui porte sur une assertion universelle. Le forumier Olivier G complète l'argument en affirmant A_4 (l'idéal $(2, X)$ de $\mathbf{Z}[X]$ n'est pas principal). L'assertion A_3 fait en fait l'objet d'une pluralité de preuves, données de façon incomplète sur le forum, et qui laissent apparaître une gradation au niveau de leur *dimension structuraliste*.

- Une preuve élémentaire, via l'assertion A_4 , consiste à raisonner par l'absurde et écrire $(2, X) = (P)$. Ceci signifie que l'idéal (P) contient 2 et X , donc que P divise 2 et X . En raisonnant sur le degré, on montre que P est une constante puis que P est une unité de l'anneau \mathbf{Z} , donc ± 1 . Or une écriture $1 = 2U + XV$ est impossible dans $\mathbf{Z}[X]$, comme on le voit en l'évaluant en 0 pour des raisons de parité. La preuve utilise la propriété $\deg(PQ) = \deg(P) + \deg(Q)$, valable dans $A[X]$ pour tout anneau intègre A , qui est une propriété structurale liant une structure multiplicative et additive. Cependant, la dimension conceptuelle de la preuve reste peu visible et ne met pas en jeu les techniques structuralistes au-delà des *raisonnements logiques et ensemblistes à partir des définitions*.
- Une lecture plus *conceptuelle* de la preuve, qui sert aussi d'*heuristique*, est de questionner les *propriétés des éléments* 2 et X : on montre, toujours avec le degré, que ce sont des éléments irréductibles de l'anneau. Comme les unités de $\mathbf{Z}[X]$ coïncident avec ceux de \mathbf{Z} (propriété valable pour tout anneau $A[X]$, A intègre), donc avec ± 1 , on voit que ces éléments ne sont *pas associés*, donc ils sont *premiers entre eux*. C'est pour cette raison que l'idéal $(2, X)$ est considéré comme un bon candidat pour fournir un idéal non principal. Enfin, si $\mathbf{Z}[X]$ était principal, l'idéal $(2, X)$ serait engendré par un pgcd. On conclut comme précédemment.
- Une dernière preuve consiste à remarquer que l'idéal (X) est inclus strictement dans l'idéal $(2, X)$. Ce fait est alors relié au *théorème structuraliste* bien connu en théorie des anneaux : dans un anneau principal, tout idéal premier est maximal. Il suffit donc de montrer rigoureusement que (X) est un idéal premier non maximal. Une *technique* structuraliste consiste à raisonner sur le quotient $\mathbf{Z}[X]/(X)$ et montrer que ce dernier est un anneau intègre qui n'est pas un corps. La *technologie* repose sur les liens entre les propriétés de l'idéal et celles du quotient. Enfin, ce quotient s'interprète comme l'ajout de la relation $X=0$ à l'anneau $\mathbf{Z}[X]$: cet anneau n'est autre que \mathbf{Z} , à isomorphisme près, ce que l'on démontre en utilisant le « premier théorème d'isomorphisme », selon sa dénomination officielle, qui fait partie de l'*outillage structuraliste* standard. Les propriétés visées étant *stables par isomorphisme*, le résultat est démontré.

L'étude de ce fil de discussion montre ainsi le fonctionnement de deux dialectiques fondamentales en algèbre abstraite :

- *Dialectique particulier-général*. La reformulation du problème avec un niveau de généralité supérieur (passage de A à A^g) apparaît comme une démarche employée à plusieurs reprises par certains membres du collectif. Ceci reflète les démarches expertes des mathématiciens en algèbre abstraite et participe du développement de praxéologies structuralistes.
- *Dialectique objets-structures*. L'examen de la structure des objets, des généralisations éventuelles des énoncés et des preuves, de l'insertion de ces dernières dans la théorie constituée en tissu axiomatique fait des structures axiomatiques un point de vue conceptuel généralisateur-simplificateur pour démontrer des propriétés sur les objets. Réciproquement, un contrôle sémantique sur les énoncés axiomatiques s'exerce en les mettant à l'épreuve des exemples connus, donc des objets. En ce sens, la dialectique objets-structures s'apparente à une dialectique syntaxe-sémantique.

CONCLUSION

La pratique de l'algèbre abstraite, qui engage des objets et des structures dans un rapport dialectique, se caractérise par un ensemble de techniques spécifiques que nous avons appelées structuralistes. En termes praxéologistes, une tâche se décompose ainsi en sous-tâches abstraites telles que : démontrer qu'un idéal premier est maximal (en utilisant les quotients), ou « simplifier » un quotient (en utilisant les théorèmes d'isomorphisme) dont la technologie mobilise les théorèmes généraux sur les structures concernées. Ce travail de description et d'analyse des praxéologies est à poursuivre et à raffiner encore. Au stade actuel de nos recherches, l'analyse des échanges d'un collectif hétérogène sur le forum a montré le développement de praxéologies structuralistes, du fait de praticiens expérimentés. En milieu didactique, la division de la tâche en sous-tâches est souvent réalisée par l'enseignant. Nous faisons l'hypothèse qu'un manque de recul des étudiants vis-à-vis de la méthode structuraliste, qui est également une heuristique, est un réel obstacle lorsqu'il s'agit de comprendre les raisons d'être de cette division et reconstituer le fil d'une démonstration présentant une dimension structuraliste importante. Un travail sur les preuves et l'usage de l'épistémologie en tant que levier méta (Hausberger 2012) nous apparaît comme un angle d'attaque pertinent pour faire face à cet obstacle.

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A commognitive analysis of mathematics undergraduates' responses to a commutativity verification Group Theory task

Marios Ioannou

University of the West of England, Alexander College,
Department of Education, Cyprus, mioannou@alexander.ac.cy

The introduction of the notion of group is an important milestone in the study of mathematics at the university level. This study focuses on how students respond to this notion, especially during the process of verifying commutativity. Commognitive Theoretical Framework has been used in order to identify, from a participationist perspective, how students use the metalevel rules to prove the given task. Results suggest that although the overall response to the given task is satisfactory, there have been identified two types of commognitive conflicts. The first is related to the proof that commutativity holds, indicating problematic object level understanding of the particular axiom. The second is related to the process of proof per se indicating inherited problematic engagement with metarules.

Keywords: Group Theory, Abelian Group, Commognition, Proof.

INTRODUCTION

Research focusing on the learning of Group Theory has attested on the university students' predicament to cope with this module, in their first engagement. Group Theory has proved to be a particularly demanding module for mathematics students, since they are required to successfully cope with its abstract and rigorous nature and invent new learning approaches. It is the first module in which students must go "beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems)" (Dubinsky et al, 1994, p268). Aim of this study is to elaborate further on the undergraduate mathematics students' reaction to the notion of abelian group, focusing in particular on its ontological characteristics, namely its structure and the axioms that should satisfy, i.e. associativity, existence of identity element, existence of inverses and, most particularly, commutativity. For the purposes of this study, there has been used the Commognitive Theoretical Framework (Sfard, 2008), due to its great potential to investigate learning in both object and meta-discursive levels (Presmeg, 2016).

THEORETICAL FRAMEWORK

Commognitive Theoretical Framework (CTF) is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis (Yackel, 2009). It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal

communication and cognitive processes, with commognition's five properties *reasoning, abstracting, objectifying, subjectifying* and *consciousness* (Sfard, 2008).

In mathematical discourse, unlike other scientific discourses, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, i.e. "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). CTF defines discursive characteristics of mathematics as the *word use, visual mediators, narratives, and routines* with their associated metarules, namely the *how* and the *when* of the routine. In addition, it involves the various objects of mathematical discourse such as the *signifiers, realisation trees, realisations, primary objects* and *discursive objects*. It also involves the constructs of *object-level* and *metadiscursive level* (or metalevel) *rules*, along with their characteristics *variability, tacitness, normativeness, flexibility* and *contingency*.

Thinking "is an individualized version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches, participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al. 2014; Sfard, 2012). Interpersonal communication processes and cognitive processes are (different) manifestations of the same phenomenon, and therefore Sfard (2008) combines the terms cognition and communication producing the new terms *commognition* and *commognitive*.

Sfard (2008) identifies the commognitive capacities that depend on the human ability to rise to higher commognitive levels and involve an "incessant interplay between utterances and utterances-on-former utterances" (Sfard, 2008, p110). These capacities fall into two distinct categories: those related to *commognitive objects* (i.e. reasoning, abstracting and objectifying), and those who consider the thinkers or speakers, namely the *commognitive subjects* (i.e. subjectifying and consciousness).

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, groups are considered compound d-objects.

The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166) The *realization tree*

is a “hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth” (Sfard, 2008, p300). Realisation trees and consequently mathematical objects are personal constructs, although they emerge from public discourses that support certain types of such trees. Additionally, realisation trees offer valuable information regarding the given individual’s discourse. Moving with dexterity from one realisation to another is the essence of mathematical problem solving. Realisation trees are a personal construction, which may be exceptionally ‘situated’ and easily influenced by external influences such as the interlocutors. Finally, signifiers can be realised by different interlocutors in different ways, according to their own specific needs (Sfard, 2008).

The epistemological tenet of CTF described in the last sentence is cardinal in its development as theoretical framework. Due to this tenet Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel discourse learning*. Moreover, according to Sfard (2008, p253), “object-level learning [...] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse”. In addition, “metalevel learning, which involves changes in metarules of the discourse [...] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses” (Sfard, 2008, p254).

CTF has proved particularly appropriate for the purposes of this study, since, as Presmeg (2016, p423) suggests, it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of mathematics per se, but to investigate “the entire fabric of human development and what it means to be human.”

LITERATURE REVIEW

Research in the learning of Group Theory is relatively scarce compared to other university mathematics fields, such as Calculus, Linear Algebra or Analysis. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al. 2014). In the context of this research strand, Ioannou (2012) has, among other issues, focused on the intertwined nature of object-level and meta-level learning in Group Theory and the commognitive conflicts that emerge.

The first reports on the learning of Group Theory appeared in the early 1990’s. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students’ cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic concepts.

The construction of the newly introduced d-object of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students' difficulty with the construction of the Group Theory concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the 'fundamental concepts' of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry (Carspecken, 1996). This chasm of ontological and historical development proves to be of significant importance in the metalevel development of the group-theoretic discourse for novice students.

From a more participationist perspective, CTF can prove an appropriate and valuable tool in our understanding the learning of Group Theory due both to the ontological characteristics of Group Theory, as well as the epistemological tenets of CTF. Group Theory can be considered as a metalevel development of the theory of permutations and symmetries. Moreover, CTF allows us to consider the historical and ontological development of a rather 'historically decontextualized' modern presentation of this Theory.

Research suggests that students' understanding of the d-object of group proves often primitive at the beginning, predominantly based on their conception of a set. An important step in the development of the understanding of the concept of group is when the student "singles out the binary operation and focuses on its function aspect" (Dubinsky et al, 1994, p292). Students often have the tendency to consider group as a 'special set', ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group. These last conclusions were based on students' encounter with groups presented in the form of group tables. In fact, students when using group tables adopt various methods for reducing the level of abstraction, by retreating to familiar mathematical structure, by using canonical procedure, and by adopting a local perspective (Hazzan, 2001).

An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g . This is partly based on student inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Group Theory, and problematic conception of the symbols used probably causes confusion in other instances.

A distinctive characteristic of advanced mathematics in the university level is the production of rigorous and consistent proofs. Proof production is far from a straightforward task to analyse and identify the difficulties students face. These difficulties have been extensively investigated for various levels of student expertise. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically misuse it.

METHODOLOGY

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Group Theory. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year.

The Abstract Algebra (Group Theory and Ring Theory) module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians.

The lectures consisted largely of exposition by the lecturer, a very experienced pure mathematician, and there was not much interaction between the lecturer and the students. During the lecture wrote self-contained notes on the blackboard, while commenting orally at the same time. Usually, he wrote on the blackboard without looking at his handwritten notes. In the seminars, the students were supposed to work on problem sheets, which were usually distributed to the students a week before the seminars. The students had the opportunity to ask the seminar leaders and assistants about anything they had a problem with and to receive help. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data includes the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Finally, all emerging ethical issues during the

data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

DATA ANALYSIS

In the first piece of coursework, students needed to prove the following task, which was solely focused on the ontology of the concept of group: *Suppose (G, \circ) is a group with the property that $g^2 = e$, for all $g \in G$. Prove that for all $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$ (that is, G is abelian).*

In general, students' encounter with the d-object of group was satisfactory, and their performance was generally in agreement with their impression as this has been revealed in their interviews. Their understanding was generally quite explicit and their mathematical narratives show good use of the mathematical vocabulary and notation, as well as the ability to specifically demonstrate their reasoning in the specific routine. Yet there have occurred **two types of commognitive conflicts** concerning the proof that the group is Abelian: commognitive conflicts in the process of *proving that the group is abelian*; and assumption of *what needs to be proved*;

Problematic proof that a group is Abelian was mostly related to the use of group axioms, and in particular commutativity. These inaccuracies are possibly linked with the incomplete object-level understanding of the definition of group and the involved object-level rules. As the excerpt below suggests, student A assumes that $(g_1 g_2)^2 = e$, but she rather takes it for granted. As seen below, she applies all the necessary manipulations of $g \in G$, for instance $g_1^{-1} = g_1$ and $g_1^2 = e$ as well as associativity, but with no further explanation. In addition, she does not clearly state that since $g_1 g_2 = g_2 g_1$ therefore G is Abelian. This indicates an incomplete object-level understanding of the property of commutativity, or an inaccurate application of the governing metarules, resulting deficient presentation of her reasoning.

The image shows a handwritten mathematical proof on lined paper. The proof starts with the given property $g^2 = e \Rightarrow g = g^{-1} \forall g$. It then proceeds to assume $(g_1 g_2)^2 = e$ and uses various group axioms like associativity and the fact that $g^{-1} = g$ to manipulate the expression $(g_1 g_2)^2 = e$ into $g_1 g_2 = g_2 g_1$. The final conclusion is G is abelian. The proof contains several logical errors, such as assuming $(g_1 g_2)^2 = e$ without justification and incorrectly applying the inverse property.

Excerpt 1: Solution of Student A

Another inaccuracy is related to the actual proof of the expression $g_2g_1 = g_1g_2$. Student B, as the excerpt below indicates, shows good object-level understanding of the definition of group, and demonstrates the ability to use the group axioms and apply the object-level rules for proving that the group is Abelian, yet he does not always justify his steps. He correctly multiplies both sides of the expression g_1g_2 by g_2^2 and g_1^2 respectively and correctly uses associativity. The main problem with his solution appears at the end of the exercise, where a problematic understanding of the notion of commutativity becomes apparent. Instead of completely demonstrating that $g_1g_2 = g_2g_1$, he erases the second part g_2g_1 , which puts in doubt his solutions' endeavour as well as his understanding of the definition of Abelian group. This commognitive conflict indicates problematic application of the object-level rules related to commutativity and the manipulation of the group elements.

g_1g_2
 $= g_2g_2g_1g_2g_1g_1$
 (as $g^2 = e$ and since is a group $g = ege$)
 so $(g_2g_1)^2 = e$
 Using the ass. law:
 $g_1g_2 = g_2(g_2g_1)(g_2g_1)g_1$
 $g_2g_1 \in G$ so $(g_2g_1)^2 = e$
 Therefore:
 $g_1g_2 = g_2^2 e g_1 \neq g_2g_1$
 but we know $g_2^2 = g_2g_1$
 we want to show $g_2g_1 = g_1g_2$
 I think you
 you have just not
 carried this forward.

Excerpt 2: Solution of Student B

A second type of commognitive conflict was grounded on the assumption of *what was supposed to be proved*. This commognitive conflict was usually part of an overall satisfactory attempt that would suggest an explicit understanding of the object-level rules of the d-object of group, but would highlight a problematic encounter with the metalevel rules and the ‘how’ of proving, even during the very first step of the module. This kind of commognitive conflicts is a typical misapplication of the metalevel rules, since it is directly linked with the ‘norms’ of proving and not of the d-objects as such.

Problematic application of metarules does not require problematic object-level understanding of the d-objects under study. In the following excerpt Student C’s writing style is very analytical with very clear mathematical narratives, good presentation and explicit use of symbolisation, in all her written mathematical narratives. Although a high performer, in this task she assumed what was supposed to be proved at the beginning of the solution i.e. $g_1g_2 = g_2g_1$. This indicates an unawareness of how to approach a proof of this kind and the required course of action, and consequently leads to a problematic encounter with this type of routine

and the amenable metarules. In general, a significant obstacle in the application of metarules, as the following excerpt suggests, is the distinction between the different proving techniques and how the amenable metarules should be used. For instance, assuming that a certain mathematical narrative is valid and used within the proof is only applied in proof by contradiction, which is not the case in this exercise.

(*) $g_1 g_2 = g_2 g_1$ \downarrow multiplying both sides by g_1

$$g_1 (g_1 g_2) = g_1 (g_2 g_1)$$

\downarrow by g_1

$$(g_1 g_1) g_2 = g_1 (g_2 g_1)$$

\downarrow $g_1^2 = e$

$$e g_2 = g_1 (g_2 g_1)$$

\downarrow $e g_2 = g_2$ by property of neutral element

$$g_2 = g_1 (g_2 g_1)$$

\downarrow multiplying both sides by g_2

$$g_2 g_2 = g_2 g_1 (g_2 g_1)$$

\downarrow $g_2^2 = e$

$$e = (g_2 g_1) (g_2 g_1)$$

\downarrow $(g_2 g_1)^2 = e$

$$e = e$$

Hence $g_1 g_2 = g_2 g_1$ if $g_1^2 = e$ & $g_2^2 = e$ \square

(*) Problem here is that you are assuming what you want to prove.
 we want to show $g_1 g_2 = g_2 g_1$, so unless we want to show a contradiction we cannot use this. You cannot assume what we need to prove.

Excerpt 3: Solution of Student C

Another example of using what is supposed to be proved within the proof occurred in Student D's solution. He assumes that what he is trying to prove is valid and uses it during the proof, which indicates problems in applying a fundamental metadiscursive rule regarding the role of the 'to-be-proved' mathematical narrative. As in the other two cases, this is possibly directly linked with the metalevel understanding and indicates ignorance of the governing metarules. At the same time, the step indicated by *, reveals incomplete object-level understanding regarding the manipulation of group theoretical expression $(g_2 g_1)^{-1}$. Although he has proven that $(g_2 g_1)^{-1} = g_1 g_2$, at a later stage, instead of writing $(g_2 g_1)^{-1} = g_1^{-1} g_2^{-1}$ he has written $(g_2 g_1)^{-1} = g_2^{-1} g_1^{-1}$, which indicates problematic object-level understanding of the notion of inverse. The following excerpt reinforces the last assumption.

Yeah, I just weren't sure whether I'd done it right, whether I was allowed to do it that way, cos I think I – I did something with like timesing it by, like both of them and then timesing g_1 them by g_2 and getting like the identity and stuff, but – I weren't sure whether it was the right way to do it? But I came out with like, the answer, but...

so $g_1 g_1 = e$
 $g_1 e g_1 = e$
 $g_1 g_2 g_2 g_1 = e \checkmark \Rightarrow (g_1 g_2)^{-1} = g_2 g_1$
 or $(g_2 g_1)^{-1} = g_1 g_2$
 $(g_1 g_2)(g_2 g_1)(g_2 g_1)^{-1} = (g_2 g_1)^{-1}$
 $g_1 g_2 = (g_2 g_1)^{-1} = g_2^{-1} g_1^{-1} = g_2 g_1$
 so $g_1 g_2 = g_2 g_1$ but why is $(g_2 g_1)^{-1} = g_2^{-1} g_1^{-1}$?
 surely above you have $(g_2 g_1)^{-1} = g_1 g_2$
 and this is general result $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$
 so we get $g_1 g_2 = g_1 g_2$ above.
 So you seem to have assumed in $(*)$ what we
 want to prove. See Solution

Excerpt 4: Part of the solution of Student D

The fact that some students use mathematical narrative that needs to be proved, as a datum that should be used during the proof, possibly indicates a problematic engagement with the ‘how’ of the routine. Namely, they have not a stabilised strategy about the ‘course of action’ for the given mathematical task. These particular students, probably, have not yet a clear idea of how to approach a proof, what the role of the final statement is and how to use it in order to achieve rigorous and clear proof. Consequently, the closure conditions, the set of metarules that define circumstances interpreted as signalling a successful completion of the proof, are not clear to these students. This is suggested by the fact that these students are generally not aware that the final narrative should not be used in the middle of the proof, albeit the fact that in the interviews students occasionally express their uncertainty regarding their solution.

CONCLUSION

This study’s aim was to investigate from a commognitive perspective undergraduate mathematics students’ reaction to the notion of Abelian group, focusing in particular on its structure and its axioms. As the above analysis suggests, there have emerged two types of student behaviour. The first one is related to the proof that commutativity holds and therefore the group is Abelian. This commognitive conflict is a result of problematic object-level understanding of the axiom of commutativity. The second commognitive conflict occurred when students used the mathematical statement that was supposed to be proved during proof. This behaviour is related to problematic metalevel learning and application of the metarules that govern the particular routine. Moreover, the above discussion indicates that, at the early stages of the module, students’ object-level understanding is better compared to the metalevel understanding and the application of the required metarules. For some students, metalevel understanding appears to be problematic from the very beginning.

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Should university students know about formal logic: an example of nonroutine problem

Sarah Mathieu-Soucy

Concordia University, sarah.msoucy@gmail.com

The goal of the study presented in this paper is to discuss how knowledge of formal logic changes the way students produce and validate proofs in the context of undergraduate mathematics. With that in mind, we asked 8 students with varied levels of knowledge and different academic background in formal logic to produce and validate proofs through a task based interview and we analyzed their work. In particular, a nonroutine task was proposed and showed interesting work from the students. Our empirical results suggest that a course in logic changes the way students do mathematical work in many ways. For example, it creates alertness to logical characteristics and a need to rely on the context.

Keywords: formal logic, university mathematics, undergraduate students, nonroutine problem.

CONTEXT

This paper reports on a project as part of a master's thesis (Mathieu-Soucy, 2015) where the practical role and the contribution of formal logic in mathematics were investigated. In the literature, this role and contribution is not clear. Some, for example Poincaré (1905), consider that logic is essential to mathematics and others, for example Dieudonné (1987), consider that logic is not useful to mathematics. Mathematicians Thurston (1994) and Thom (1967) claim that their basic (intuitive and theoretical) knowledge of logic is sufficient for their work and that they use different techniques instead that come, at least in part, from their experience doing mathematics. When it comes to university mathematics students, who don't have as much experience, where do they get the knowledge necessary to do mathematics without making any logical error? Selden & Selden (1999) noted that concepts studied in most beginner courses in formal logic, like Venn diagrams or truth tables, aren't that useful in the everyday mathematics students have to perform. Also, complex logical statements can often be written in multiple simple statements so that the person manipulating them doesn't need to control all the more complex aspects of formal logic. In the same line of thought, Cheng & al. (1986) showed that a course in logic does not prevent students from making logical mistakes when doing mathematics. However, among students, gaps in knowledge of formal logic are one of the causes of difficulties in validating and producing proofs (Epp, 2003; Selden & Selden, 1995). In sum, assessing the usefulness and the necessity of logic in the production and validation of proofs is quite difficult. Hence, it appears worthwhile to address this question: how does knowledge of formal logic, or a course in formal logic, changes the way undergraduate mathematics students produce and validate

proofs? This question will be addressed considering the concepts and characteristics presented in the conceptual framework below.

CONCEPTUAL FRAMEWORK

To approach this question, we examine different aspects of mathematics that could help us characterize mathematical work, proofs in our case. First, we usually agree that in order to do mathematics, we need to combine intuition and rigour (which includes logic). But what is intuition? In our work, intuition is a feeling that imposes itself to an individual without being able to explain why. This knowledge arises subjectively to an individual as being true (Fischbein, 1982, 1987). Also, it comes from the experiences of each individual and it can be mathematically incorrect. Finally, regarding the use of logic in mathematical work, we recognize that logical considerations are absent or nearly so from the discourse of educators and textbooks at the beginning of university and consequently from the work of students (Durand-Guerrier & Arsac, 2003). Such considerations are replaced by contextualized reasoning rules and contextualized knowledge, specific to a certain field of mathematics. Their use seems to be directed by the mathematical knowledge of the individual or his mathematical expertise.

METHODOLOGY

Considering that there could be contextualized knowledge as a result of a course in logic, that a course in logic is not the only source of knowledge in logic (mathematical experience, for example) and with the possible characterizations mentioned above, we developed a methodology in two phases involving eight university students from Quebec, Canada in the second half of a 3-year mathematics program (20-21 years old). First, we evaluated their level of knowledge in formal logic with a written test. Questions in this setting were strictly formal. There was no proof to be done in this test, only direct questions on logical ideas (finding the negation of a statement or defining a *modus ponens*). Then, considering those results (ranging from 0 to 4), and their academic background in logic (if they did or did not take a course in logic), we formed 4 different teams of 2 students to move to the second phase (see below). The four teams were as follows: Anna and Michel formed the only heterogeneous team, meaning Anna did not take a course in logic and was the weakest student on the test (0/4) while Michel did take a course in logic and he was one of the strongest on the test (3/4). The second team, Jeanne and Lucie, were considered as having the same profile as Michel (3/4 and a course in logic). The third team consisted of Éléanore and Paul, who did not take a course in logic and showed a slightly less knowledge in logic in the test (2/4). Finally, Julie-Ann and Robert formed the fourth team. They were the students who showed the most knowledge in formal logic in the test (4/4) and they did a course in logic. We should note that the five students who have academic background in logic did the exact same course at the same time.

The second phase of the methodology consisted of audio-recorded task based interviews (Goldin, 1997). We asked each of the four teams to answer four questions (see Appendix 1 for Questions 1, 2 and 4, Question 3 below). They needed to reach a consensus on the solution at the end of the resolution. The task that this paper is mostly interested in is the third one, which is our adaptation of the first three Hilbert's incidence axioms (Arsac, 1996). This was the only task with an unfamiliar context. While still answering to regular mathematical rules, this question does contain an additional difficulty for the students, other than the logical difficulties. Indeed, conceptualizing new objects (dogs, robots and a friendship relationship) in a known mathematical framework is rarely done in a student's life.

Question 3
Here is a question asked to a university student :

With the following axioms :

- *AX1* : For every robot L and for every robot J different from L , there exists an only dog W which is friend with L and with J .
- *AX2* : For every dog W , there exists at least two distincts robots which are friends with W .
- *AX3* : There exists three distincts robots such as no dog is friend with those three robots simultaneously.

Show that for every robot, there exists at least one dog which is not his friend.

Here is his answer :

Let's consider a generic robot R .
 Let's consider $L1$, $L2$ and $L3$ being the three distinct robots such as no dog is friend with those three robots simultaneously (*AX3*).

If R is the same robot as Li , for $i=1$ or 2 or 3 , the dog is easy to find : we take the only dog which is friend with the two other robots, its existence is guaranteed by *AX1*. This dog cannot be friend with Li without contradicting *AX3*.

We can then suppose that R is different from $L1$, $L2$ and $L3$.

Complete the proof.

Figure 1: Translation from French of Question 3

The three other tasks involved familiar contexts, on material from first-year university courses. A posteriori, questions 1, 2 and 4 were considered too easy to get any useful data to give an insightful answer to our question in a familiar context. However, the students work on those tasks was still useful as a basis of comparison, and per respect to other aspects mentioned in the results section.

RESULTS

Our analysis of the participants' work suggests that a course in logic changes the way students produce and validate proofs. Our results were, however, inconclusive regarding any difference between students with significantly different results on the pre-test on formal logic.

Alertness and Uneasiness

An academic background in logic seems to: increase the alertness to logical characteristics, promote an unconscious notice of logical specifications for students and increase their ability to “unpack” (Selden & Selden, 1995) logical characteristics of symbolic and discursive statements. We hypothesize that it is due to intuitions and contextualized knowledge resulting from a course in logic.

To be more precise, this alertness seemed to be very useful as the students who did a course in logic noticed quickly the logical considerations contained in all four questions, as if it was jumping out of the page. Indeed, they were aware of every logical detail of the questions very quickly and could reflect accordingly from the beginning, while others might reread the same statements many times and still miss some implications or quantification and lose time reflecting on a slightly different problem.

On the other hand, students who showed alertness also showed uneasiness to engage into the mathematical work, especially in Question 3. Indeed, teams formed by two students who showed increased alertness took a significant amount of time to solve the question compared to the other teams (48 minutes for Jeanne and Lucie, 38 minutes for Julie-Ann and Robert, compared to 19 minutes for Anna and Michel and 22 minutes for Éléanore and Paul). It was surprising to us at first that students who did the best on the test and the ones that did the course in logic struggled the most and took the most time. Indeed, we expected that the necessary knowledge for the nonroutine task was closer to pure logic than for the other questions. When looking closely at the sessions, the extra time results from moments where students discussed the problem without engaging in the process of producing a proof. For example, they discussed the axioms and their role (by giving examples of settings with different relationships) or how to address the problem (by contradiction or by using the second axiom first).

The uneasiness was particularly flagrant when looking at the heterogeneous team, formed by Anna, who did not do a course in logic, and Michel, who did. Indeed, Michel was the leader of his team when resolving questions 1, 2 and 4. He was much quicker than Anna in every way. However, in Question 3, the opposite happened. Michel, as the other students who did a course in logic, was hesitant to enter the task while Anna jumped right in, trying many different approaches until she found the right one. Michel was just along for the ride. He was hesitant to try any idea he

would get while Anna was not. In both cases, we do not think that Anna or Michel lacked knowledge to solve any of the tasks.

We think that multiple things can cause the uneasiness. For example, the large amount of logical considerations present in this task could be a cause. Indeed, with their increased alertness, the students could have been blinded by the logical structure in a way that slows their progress in the task. Also, since they are more aware of the intricate logical structure, they are aware that the risk of error might very well be greater than usual and think they should refrain from going forward in the resolution until they have a better control over the problem, over the context.

Control Over the Context

When solving the dog and robots' task, the same students who did a course in logic seemed to seek the control they normally have over the context (*what makes sense in a particular situation*). A great example of this idea is when Jeanne became frustrated because Lucie and herself struggled to finish the proof:

Jeanne: Why can't I find the solution?

Lucie: It's only because it's about robots and dogs.

Here we see that for Lucie, it was obvious that the absence of semantic ground on which to rely on as they were solving this task was the root of the whole problem, as opposed to the intricate logical structure or any other characteristic of the problem. It is indeed true that having a control over the context can help in a resolution. For example, let's look at the axioms in their original form: the first three incidence axioms of Hilbert, as formulated by Arsac (1996). It would become:

- Through 2 distinct given points, one and only one line is incident to both
- Given a line, there exists at least two distinct points incident to this line
- There exist three non-collinear points¹

In this case, students would probably be able to assess the righteousness of their reasoning relying on the context. For example, if at some point in the resolution, the students would infer the following statement: *Given a line, there exists a unique point incident to this line*. It would be easy for the students to discard this statement about points and lines since it is obvious that there is more than one point incident to a line according to our knowledge of points and line. However, considering the equivalent statement *Given a dog, there exists a unique robot who is friends with this dog*, it is not shocking that a dog would have an only robot friend! Hence, we hypothesize that many students were resistant to engage into the task in the

¹ My translation of: Par deux points distincts donnés, il passe une droite et une seule; Étant donnée une droite, il existe au moins deux points distincts sur cette droite; Il existe trois points non alignés.

beginning, until they were “familiar” enough with the relationship between dogs and robots to be more confident. For example, Julie-Ann and Robert discussed the three axioms a lot before doing a step toward the proof they were asked to produce, giving examples of what kind of friendship could exist. At this occasion, Robert questioned the second axiom extensively to finally exclaim himself:

Robert : Oh! I was wondering what is the use of AX2: there is no useless dog.

At that point, Robert was more comfortable with the task because he knew that no dog was friendless and it gave him insight on this new “world” populated by robots and dogs. Also, since the second axiom was not used in the beginning of the proof given to the students (see figure 1), it was even more important for them to find its role, what it really meant in the context.

Symbols

Some students transformed the axioms into symbols to increase their understanding and control over the logic involved. For example, in some of Jeanne and Lucie’s work (see figure 2), they let \sim be the symbol for friendship, R_0 the set of all robots and Ch the set of all dogs, they symbolized the axioms:

- AX1 : $\forall L, J \in R_0$ such as $L \neq J, \exists! W \in Ch$ such as $L \sim W$ et $J \sim W$.
- AX2 : $\forall W \in Ch, \exists L, J \in R_0$ such as $L \neq J$ et $L \sim W$ et $J \sim W$.
- AX3 : $\exists L_i \neq L_j \neq L_k \in R_0$ such as $\forall W \in Ch, W \sim L_i$ et $W \sim L_j \Rightarrow W \nrightarrow L_k$.

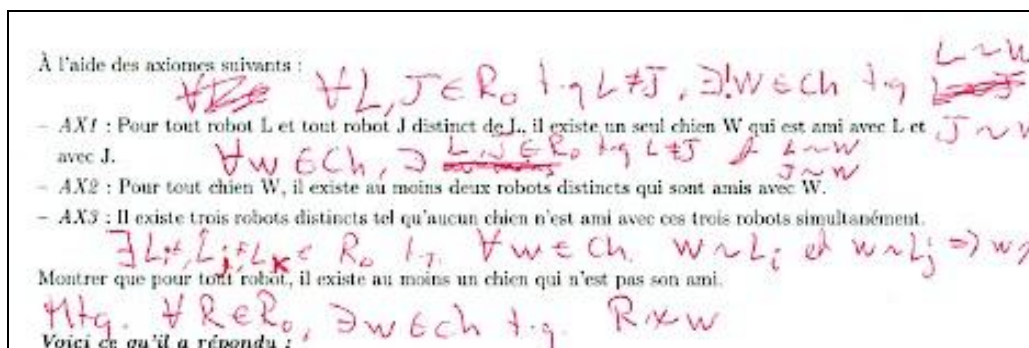


Figure 2: example of the symbolization of the three axioms presented above

The symbolization proposed by this team involved all the right logical specifications (while it is not the most rigorous, we still consider it conveys the right ideas). Jeanne and Lucie mentioned explicitly that this symbolization was necessary for them to grasp the meaning of the axioms. However, later on in the resolution, both Jeanne and Lucie mentioned that they were only looking at the discursive statement and not at the symbolize version to remind themselves and reflect on the axioms during the actual proving process. The symbols helped them grasp the ideas underlying each statement with a more synthesized and compact version, but the symbols became useless for them in the middle of the proof process, as if the symbols were opaque

compared to the words. Also, if reading the discursive version of the axiom did not help them, Jeanne and Lucie would symbolize the axiom they were interested in again from the beginning, on another piece of paper. This shows how much the work of transforming into symbols, and not only the symbols themselves, is helpful to grasp and convey ideas in mathematics.

Does alertness and relying on the context prevent making mistakes?

It would be a mistake to think that being alert (as a result of doing a course in logic) and grasping the meaning of the axioms in the context necessarily means understanding concepts and being able to use them properly. For example, Jeanne and Lucie, who put into symbols the axioms of the nonroutine problem as shows figure 2, considered every logical specification when rewriting the axioms but they thought there was a one-to-one correspondence between $AX1$ and $AX2$. We can easily see that it is not the case since $AX1$ specifies the existence of a unique dog while $AX2$ specifies that there exists at least one robot, eventually many. Hence, even if they saw the difference in the quantification and knew what the quantification meant, they still made a mistake and were not able to resolve the task properly. Similarly, the students who did miss some logical specification in the statements (those students coincide in our study with students who did not do a course in logic), and were consequently making mistakes by omitting information, often showed in the test prior to the interview that they had the formal knowledge associated. For example, after working for a couple minutes to solve the task, Paul said: “‘there exists an only’. I read it, I wrote it [$\exists!$] but I did not take it into consideration. This is why it took us time to conclude.” In this case, Paul read many times the axioms and reflected on them without “absorbing” the special quantification associated with the dog in $AX1$. While it took his team longer to conclude because of this omission, he did not make any mistake and knew exactly what it meant and how to use it, once he actually noticed.

The influence of the type of context (formal, familiar, unfamiliar)

Our data suggests that the type of context influences the students’ ability to perform manipulations associated with formal logic. Every student was confronted with three different contexts during the experiment. In the written test, we consider the context as being strictly formal. During the task based interview, there were three tasks that presented a familiar context: the questions involved mathematical objects and concepts that the students saw many times in many classes. Finally, the third task involved an unfamiliar context: the object and relations were *a priori* unknown by the students but still answered to the same general mathematical rules.

Six of the eight students struggled to find the negation of “ $A \Rightarrow B$ ” (in the test prior to the interview) but they were all perfectly able to negate “the double of any irrational number is irrational” (Question 1). Hence, it seemed important for them to reflect in a familiar context, namely what does it mean for the statement to be false,

according to their knowledge of the concept. Similarly, Julie-Ann struggled to negate “For every robot, there exists at least one dog which is not his friend”, but was perfectly able to negate “ $\forall f \exists a [(\forall u(F(u,a) \Rightarrow G(f,u,a)) \Rightarrow H(f,a)]$ ” (Durand-Guerrier & Njomgang Ngansop, 2009). So for her, negating in a formal context was easier than negating in an unfamiliar context. In other words, control over formal statements is not a necessary and sufficient condition to the control over statements in context and, thereby, on their semantics.

CONCLUSION

Our results suggest that a course in logic did change the way student worked, especially in the case of a nonroutine task. It increased their alertness to logical characteristics while creating uneasiness to progress in a resolution. In both cases of vigilant and less vigilant students, their vigilance was the same when confronted with logical characteristic they could work on and understand or not. This implies what Cheng & al. (1986) already mentioned, namely that a course in logic does not eliminate the risk or errors when working on logical considerations in a mathematical context and also that the production and validation of proof is not necessarily improved by such a class. Also, the students less vigilant would benefit from looking for the logical characteristics more explicitly.

Going back to the title: should university students know about formal logic? While answering this question was not the prior goal of the master’s thesis associated to this work, the results still bring some pieces of answer to this question. This paper shows once again that extensive knowledge of formal logic is not necessary to do mathematics. However, what this research brings is the whole idea of alertness to logical characteristics, which is an interesting asset for mathematics students. What this research also reminds us is that noticing is useless without a strong hold on the notions. This brings us to expand our reflection to the teaching of logic: what kind of knowledge should be taught and in what way to promote students’ understanding and diminish logical mistakes, in order to make logic courses as efficient as possible?

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APPENDIX 1

Translation from French of the tasks proposed in the task based interview

Question 1

A university student suggested this proof. Address a comment to him.

Theorem : The double of any irrational number is irrational

Proof : Suppose it is not. That is, suppose the double of every irrational number is rational. But we previously proved that $\sqrt{2}$ is irrational and also that $2\sqrt{2}$ is irrational. These results contradict our supposition. Hence the theorem is true.

(Epp, 1997)

Question 2

Study the following conjecture : *The composition of two surjective functions is surjective.*

(Epp, 1999)

Question 4

A university student suggested this proof. Address a comment to him.

Proposition : Let's consider E and F two sets and f an application from E to F . Whatever parts A and B from E , we have $f(A \cap B) = f(A) \cap f(B)$.

Proof :

Lets prove first that $f(A \cap B) \subset f(A) \cap f(B)$: if $f(x) \in f(A \cap B)$, then $x \in A \cap B$; since $x \in A$, $f(x) \in f(A)$; and since $x \in B$, $f(x) \in f(B)$ and then $f(x) \in f(A) \cap f(B)$.

Let's now prove that $f(A) \cap f(B) \subset f(A \cap B)$: if $f(x) \in f(A) \cap f(B)$, $f(x) \in f(A)$ then $x \in A$; also $f(x) \in f(B)$ then $x \in B$; since $x \in A$ and $x \in B$, $x \in A \cap B$ and then $f(x) \in f(A \cap B)$.

(Durand-Guerrier, Barrier, Chellougui & Kouki, 2012)

Learning Linear Transformations using models

María Trigueros¹, Barbara Biahchini²

¹Instituto Tecnológico Autónomo de México, ²Pontificia Universidade Católica de São Paulo

Using Action-Process-Object-Schema (APOS) Theory students' strategies while solving a linear transformations modelling problem in a Linear Algebra course are studied. Modelling cycles were complemented by conceptual activities designed with a previously developed genetic decomposition for this concept. The work of students during the modelling process in the classroom is described in terms of questions and knowledge emerging from their own strategies, and in terms of the difficulties they faced. Results show some affordances of the modelling situation and the use of activities, and the difficulties faced by students.

Keywords: APOS, linear algebra, linear transformations, modelling, functions.

INTRODUCTION

Research on the teaching and learning of Linear Algebra has been the focus of attention of several research groups in the last ten years. Researchers coincide that in spite of its many applications, this is a difficult subject for students and many of the difficulties students' face have been underlined and explained in terms of different theoretical frameworks (for example, Dorier, Robert, Robinet & Rogalski, 2000; Sierpiska, 2000). In the last years, an interest on the use of different teaching methodologies in the teaching of Linear Algebra has developed with the aim of helping students in developing their understanding of abstract concepts starting from real or realistic modeling situations (Martin, Loch, Cooley, Dexter & Vidakovic, 2010; Wawro, Larson, Zandieh & Rasmussen, 2012; Trigueros & Possani, 2013) or by the use of technology to help them to relate different representations in order to give meaning to concepts (Maracci, 2008; Gueudet, 2004; Klasa, Oktaç & Soto, 2006; Romero & Oktaç, 2015).

It is in the context of this growing research area that a research project was developed in Mexico. The project has a double aim. On the one hand, investigating the way students construct different Linear Algebra concepts, and, on the other hand, studying the results of the use of modeling situations in the introduction of those concepts, together with activities based on APOS theoretical framework in the classroom (Oktaç & Trigueros, 2010).

In this paper we present part of this work related to the teaching and learning of Linear Transformations. In particular we focus on the design and use of a teaching sequence designed to foster students' understanding of this concept by relating geometrical and algebraic representations through the use of a modeling situation in a geometrical context.

SOME ANTECEDENTS

Linear transformations have received a lot of attention of researchers because of their importance in applications and the difficulties students face when learning them. Some results obtained follow. Students struggle when asked to find a linear transformation in a geometric context starting from the images of basis vectors; they have difficulties using systemic reasoning and using visualization to determine the transformations; students' show a tendency to use intuitive models when working geometrically and conceptualizing transformations as functions (Uicab & Oktaç, 2006; Ellis, Henderson, Rasmussen, & Zandieh, 2012). Roa-Fuentes and Oktaç (2010) developed two genetic decompositions for the concept of linear transformation and used them to investigate the way students may construct this concept. After analyzing students' responses in an interview, they found evidence supporting one of them where construction starts by using specific examples of linear transformations. In a study about the teaching and learning of linear transformation using Cabri- Géomètre, it was found that the use of that tool helped students to find relations between the geometric and the matrix associated to a linear transformations (Karrer & Jahn, 2008). When studying the role of change of basis and matrix representation of a linear transformation Montiel and Batthi (2010) described with care the role that semantics and gestures play in classroom interactions, while Bagley, Rasmussen and Zandieh (2012) discussed that under specific conditions students are able to relate matrices and linear transformations and that they are capable to work with matrices but not to relate the concept of function with that of matrix. Wawro, Larson, Zandieh & Rasmussen (2012) designed a hypothetical collective progression (HCP) to support students' understanding of linear transformations defined in terms of matrix multiplication. Their results show the proposed HCP fostered students reasoning in productive ways and helped students to coordinate local and global views of linear transformations as functions and as matrix multiplication for particular geometric mappings.

THEORETICAL BACKGROUND

APOS Theory is based on Piaget's concept of reflective abstraction. (Arnon et al., 2013). Its main constructs can be defined as follows. An Action is defined as a transformation of a mathematical object memorized by the individual or perceived as driven by external stimuli. After reflecting on Actions, they can be interiorized into a Process; Actions are no longer perceived as external and the individual can use them omitting steps and anticipating the results without having to perform the process. A Process may be coordinated with other Processes, or be reverted as needed in a problem situation. When an individual can see a Process as a totality, and needs to apply Actions on it, the Process can be encapsulated into an Object and new Actions can be applied to it. A Schema for a mathematical topic is considered as a coherent collection of Actions, Processes, Objects, and previously constructed Schemas related to the mathematical topic.

Research using APOS Theory starts by designing a model that intends to predict the mental constructions involved in the construction of the studied concept. This model is called a genetic decomposition (GD). It specifies the mental constructions in terms of the constructs of the theory needed in the understanding of that concept. A GD, as a model, is not unique, different models may be proposed, but it is important that a GD can be supported by experimental data from students. Usually, this is not the case, some of the predicted constructions are not found in students' work, while students show other constructions not predicted by the model. The GD is then refined. This process can be repeated many times until a model predicting students' constructions is found. The GD is also used to design activities to guide students' constructions of the concepts of interest. Students work collaboratively in teams with these activities and whole group discussions are organized in order to promote students' reflection on what they had done.

Although modelling is not included in APOS theoretical framework it is consistent with APOS structures (Trigueros, 2008): When students face a modelling problem, they use the mathematical Schemas and Schemas constructed in other disciplines or in their daily life to approach the problem they face. They take elements of those Schemas to choose variables, and to formulate some hypothesis about the behavior of the expected solution. Through Actions and Processes on some of the components of the Schema, and through coordination of Processes, a mathematical model emerges. This mathematical model is encapsulated into an Object, and new Actions, Processes, coordinations and relations are applied on it to determine its properties and to respond the questions posed by the modelling problem or to pose new questions.

METHOD

We first present the GD used for this study. We did not propose a new GD, but used a slightly modified version of the refined GD proposed by Roa-Fuentes and Okaç (2010) which is schematically described in the Figure 1. It was used in the design of the conceptual activities designed for the course and the research instruments.

A design context based on an illustration shown in a textbook (Nakos & Joyner, 1999) was selected to present a problem situation designed by the authors:

A cartoonist needs to show the figure of a man on a bicycle, he has drawn, in motion and in different positions to appear in a film (Figure 2). He has contacted you to help him by making the necessary calculations to program the drawings on the computer. He asks you to send the calculations together with each figure, so that he is able to write a program.

The problem was used in two occasions accompanied by activities designed with the GD. These activities guided students' constructions emerging from discussion on the problem towards the construction of Linear Transformations. The first experience took place in Brazil with 8 students who were finishing their studies and had taken a

Linear Algebra course. They volunteered to participate in a 4 sessions of 4 hours each

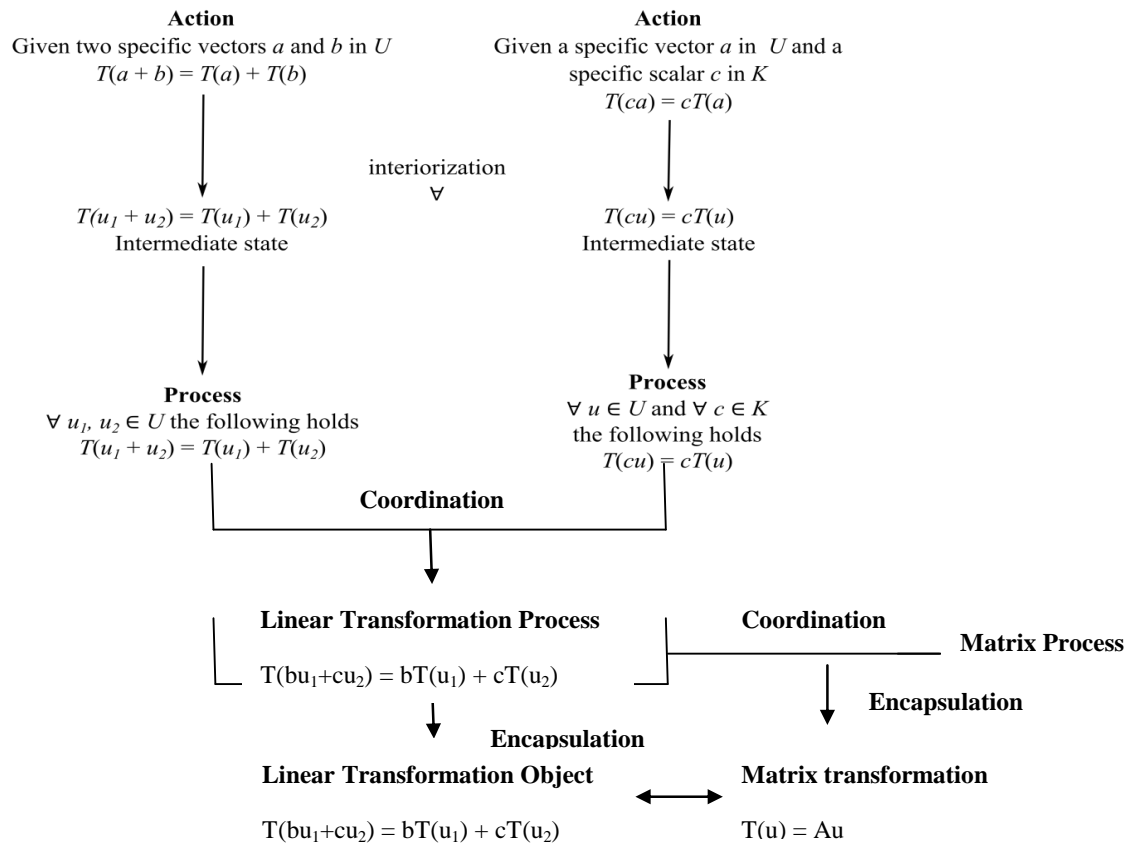


Figure 1: Genetic Decomposition for Linear Transformations

modeling workshop, 3 of them worked in teams during the four sessions while 5 worked individually. The aim of this experience was to test the modeling situation and the conceptual activities. All the work of the students was kept and their dialogues were recorded. The second experience took place during five two hours sessions at the classroom in a Mexican university during an Introduction to Linear

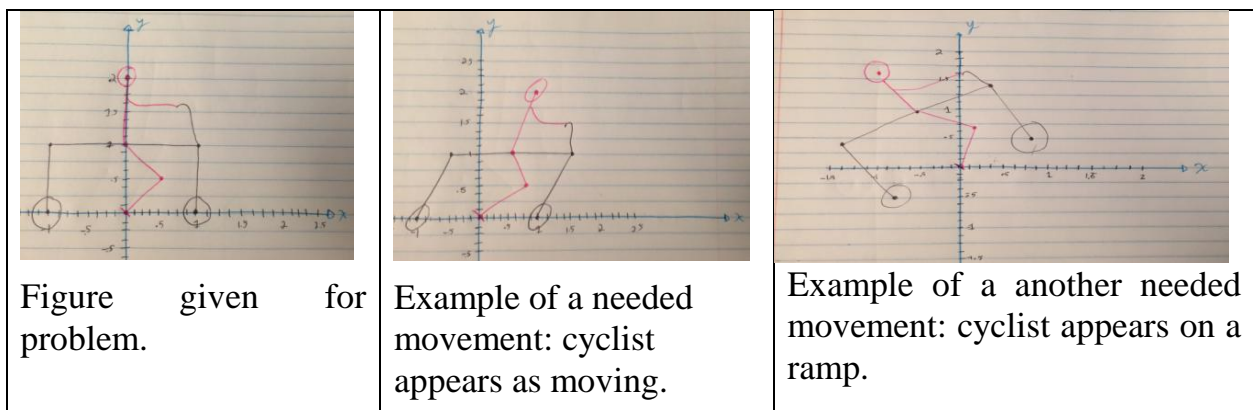


Figure 2: Examples of drawings needed in the problem situation presented

Algebra course attended by 23 engineering and applied mathematics students. Students worked in cycles including collaborative work in teams of three students, followed by whole group discussion. All sessions were audio recorded and all the work of students was kept. Homework problems related to both types of activities were used. After two weeks of the final session 8 students from different teams were invited to a semi-structured interview. Data obtained was analyzed using the GD by both researchers and results were negotiated between them. In this paper we present only results from the work done in the classroom. In excerpts and examples, Brazilian students will be labeled as S1, S2... and Mexican students with a letter describing their team and a number identifying a student.

RESULTS

Students' main strategy during the first cycle, in both experiences, was the use of their knowledge about vectors to explore the given data and making tables to try to find a rule for the transformation. This strategy was successful for them in the case of simple transformations such as shear transformations and translation, but was not easy to apply in the case of more complex transformations, as rotation, where students of both countries showed difficulties.

Most students participating in the first experience, in Brasil, did not recognize, at the beginning of the workshop, that a linear transformation is a function. They did observe some analogy between transformations and functions, but were not clear about their relation. When introduced to the modeling problem they were able to make this relation clear. This was evident when students were surprised when facing situations that were different from those they had encountered before and the conclusions they were making from them:

S1: they are functions, since transformations depend on the initial vectors. The domain in calculus was \mathbb{R} , and now we are studying domain in \mathbb{R}^2 .

S2: In general, in Differential and Integral Calculus courses, domain and codomain of function were real numbers...

The problem made them reflect on what they had studied before and helped them to relate two concepts that had remained compartmentalized in their previous studies; they were able to assimilate transformations into their function's schema. In the Mexican experience, students had not studied transformations before, and students discussions showed that most students were thinking about transformations as functions with domain and range in \mathbb{R}^2 when working with the problem. This was evidenced in comments such as C3: "So it is a function but instead of real numbers you have vectors in the domain and range" or "Yes, and the rule is $x+2y$ for the first component and the second is always the same, y , so you only apply it". They talked about functions in terms of input-output: H1: "...you can check here, if you use this point [writing $(-1, 1)$] with this rule, you get this [writing $(-1/2, 1)$] ... and doing twice the first comma the second, it works for all the points." They evidenced, in

general, a process conception of transformation as a function. They were able to relate the function process they had constructed through algebra and calculus to consider a new type of functions that work in the same way but in \mathbb{R}^2 .

In both countries students presented difficulties related to the use of mathematical language. For example, a Brazilian student wrote:

S7 a) $T(x,y)=(x,0,y) \neq \mathbb{R}^3 \implies T$ is not surjective. $\therefore T$ is not an isomorphism.

Although we consider that the appropriation of mathematical language occurs gradually, it was surprising that students in the last semester of a Mathematics program could still compare a specific vector to a vector space. This issue was taken into account in the Mexican experience by paying a lot of attention to students' productions and pointing out mistakes to be corrected.

Working on the modeling situation made students of both countries reflect on transformations' properties. For example, after doing two transformations, shear transformation and translation, students in two teams, C and E did action to compare the transformations in terms of their properties:

E2: "This one changes the form of the bicycle (*shear transformation*), but this one (*translation*) does not, it only changes its position in space"

E1: Yes, one is like a deformation, it doesn't change the position of the cyclist, but this other only moves it around".

E2: When this changes this form the points move, but the origin stays in the same place. If you move it around, all the points change, including the origin. How can we say this?

E3: What I see is that in both all the points go to a new position, in the first the origin does not change and the points are in different relative positions, while in the other all the points change and all change by the same amount... but, well they are different transformations.

Later on, when trying to do the rotation one student of the same group notices:

E3: This transformation does not change the form, it does not change the origin either...so, some transformations don't move the origin, other leave the drawing as it was.

During whole group discussion the teacher in Mexico recovered this discussion and asked the group about differences they could find about transformations. The teacher defined isometries. She asked students to verify which transformations were isometries. Then gave them activities based on the GD where linear transformations were introduced. Students worked without difficulty with those activities. Students demonstrated they had constructed Linear Transformations as processes.

Students in Brazil had no difficulty while solving activities related to decide if a transformation was or not linear, we can say that all of them had constructed a process conception of linear transformation. They made comments such as:

S3 If we multiply a scalar by a vector or a scalar multiplying by its transformation, the result will be the same.

Or, when discussing if a linear transformation is an isomorphism, they wrote:

S4 and S5: $T(0,0,0) = (0,0)$; $T(1,0,1)=(1,0)$; $T(1,0,2)=(1,0)$. As $T(1,0,1)=T(1,0,2)=(1,0) \implies T$ is not injective. T is not an isomorphism.

Most students in both experiences showed difficulties when facing rotations since the rule was not easy to find from the picture or from a table of values. Most students in Brazil had not been introduced to the matrix form of linear transformations, when the teacher introduced it, they considered it as a novelty and used it without problems, showing again encapsulation of Linear Transformations. Mexican students struggled with the rule using trigonometric functions; they had not been introduced to the matrix representation of the transformation. After some time, students in team A found a possible rule (Figure 3a), and only students in group C realized those equations could be written as the product of a matrix and a vector (Figure 3b). These students also showed encapsulation when they realized that a composition of transformations was needed:

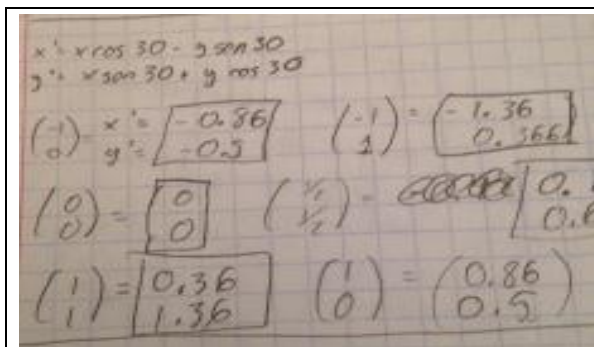


Figure 3a Work on rotation

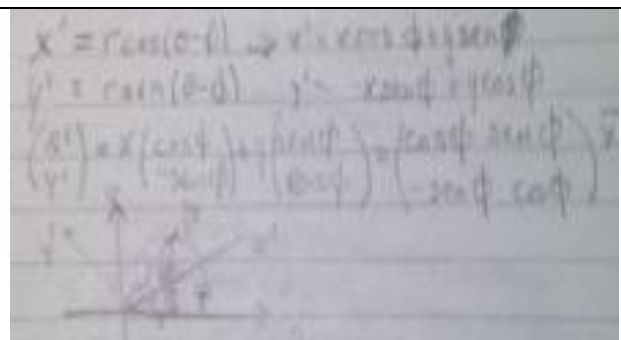


Figure 3b Finding a matrix

C2: If we do this rotation, it is weird; the tires would be underground, because, look the origin is here at the cyclist feet. So I think we need to do first the rotation but then a translation so that the tires are on the ground. It is a composition of transformations.

The second cycle was devoted to work with the composition of transformations and activities related to proving if different transformations given in different representations were or not linear. Students in both countries worked directly, without difficulties, with composition of transformations, demonstrating, again, that they had included the concept of transformation into their function schema, and encapsulation of linear transformations as shown by the following discussion of students in Mexico:

A1: It has to look as if he is going uphill and moving.... so we should use the rotation first and then do the shear transformation, is that OK?

A2: Yes, I guess it is like the composition of functions. We do one and then we apply the second to the vector obtained. OK... but since we have the matrices we can do the matrix product, I think.

Students, in general worked well doing the proofs in different representations and in finding associated matrices to transformations or images of transformations, and doing composition of transformations. At times, work with the modeling problem or with the activities was difficult for students and the teacher had to help. This happened more frequently in Mexico than in Brazil. For example some students in Brazil struggled with an activity involving the kernel of a linear transformation; also they had problems remembering the conditions that made the existence of the inverse of a transformation possible. Although these students had studied these topics, they showed they had constructed an action conception of transformation; work on the activities helped them reflect on their constructions and possibly to interiorize them into a process. Mexican students also showed difficulties with these activities and some of them struggled with finding matrices associated to transformations because they had not constructed a process conception for the concept of basis of a vector space.

DISCUSSION AND CONCLUSION

Results of this study show students' constructions during the modeling process and the work with activities designed with the genetic decomposition. This tool proved to be effective in the analysis of students' constructions during the whole experience in both countries; it can be said that it describes the basic constructions involved in the construction of the concept.

Students in both countries found the modeling experience interesting and important in the understanding of the linear transformation concept. They showed their interest by being involved in their work and even Brazilian students who had already studied this topic found novelties in it which made them re-think about this concept.

The use of the modeling situation mobilized students previously constructed schemas and their development in ways that were not predicted by the researchers. It also favored the emergence of some ideas that previous literature showed to be difficult for students. One important result of this study is students' reconstruction of their function schema and their reflection on properties of different transformations. The problem elicited a group of mathematical models that were used by students as objects to explore what they expected from their predictions, to combine them through compositions and to find their properties. The activities designed with the genetic decomposition played an important role in introducing the notion of linear transformations and in guiding students' exploration towards its construction. They

were also important in focusing students' attention in some aspects of the model that they had not previously taken into account.

The analysis of students' productions and discussions made the recognition of those constructions that seem to play a fundamental role in the learning of linear transformations such as making sense of what linearity means in the context of Linear Algebra and the relation of linear transformation with functions, matrices and with the concept of basis, although not all of this has been described in this paper. The concept of transformation emerged quite easily from work with the modeling problem. However, linear transformations had to be introduced in the activities so that students were able to conclude that translations are not linear transformations. It is important to underline, how, while working on the model, students were able to develop on their own powerful conceptual tools, such as a way to determine the difference between rigid and non rigid transformations; and a relation to the concept of matrix. The emergence of these ideas gives evidence that the use of modeling situations in the classroom promotes the construction of knowledge. The complementary use of activities designed with the genetic decomposition played an important role in the development of students' schema for function and for linearity.

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Fifteenth Conference on Research in Undergraduate Mathematics Education,
Portland, OR.

An Inquiry-Oriented Task Sequence for Eigen theory and Diagonalization in Linear Algebra

Megan Wawro¹ and Michelle Zandieh²

¹Virginia Tech of Blacksburg, Virginia, USA, mwawro@vt.edu; Arizona State University of Tempe, Arizona, USA

We present an innovative task sequence for an introductory linear algebra course that supports students' reinvention of eigen theory and diagonalization. Grounded in the instructional design theory of Realistic Mathematics Education, the task sequence builds from students' experience with linear transformations in \mathbb{R}^2 to introduce the idea of stretch factors and stretch directions. This is leveraged towards defining eigenvalues and eigenvectors, reinventing methods to determine them, and connecting them to change of basis and diagonalization. In the poster, we discuss the development of the task sequence and analyses of students' work on the tasks, specifically on characterizing their approaches for developing eigen theory methods.

Keywords: Linear algebra, task design, eigen theory, inquiry-oriented instruction.

INTRODUCTION

The work presented in this poster stems from a research program focused on student reasoning about linear algebra. It is grounded in the design-based research paradigm of classroom-based teaching experiments (Cobb, 2000), which involves a cyclical process of (a) investigating student reasoning about specific mathematical concepts and (b) designing and refining tasks that honor and leverage students' ideas towards the desired learning goals (Gravemeijer, 1994; Wawro, Rasmussen, Zandieh, & Larson, 2013). One product of this research is the Inquiry-Oriented Linear Algebra (IOLA) curricular material, designed for a first course in linear algebra at the university level. In this poster, we detail the eigen theory and diagonalization IOLA task sequence and present analysis of student work, specifically on Tasks 3-4.

THEORETICAL FRAMEWORK AND METHODS

Our theoretical framework for designing instructional materials draws on Realistic Mathematics Education (Freudenthal, 1991). Briefly stated, task sequences should be based on experientially real starting points; classroom activity should support student development of models of their mathematical activity that can be used as models for subsequent mathematical activity; and student activity, with instructor guidance, should evolve toward the reinvention of formal notions and ways of reasoning about the mathematics initially investigated.

We operationalize the notion of inquiry both in terms of what students do and what instructors do in relation to student activity. Students learn mathematics through inquiry as they work on challenging problems that engage them in authentic mathematical practices such as symbolizing, algorithmatizing, and theoremizing (Rasmussen, Wawro, & Zandieh, 2015). Instructors engage in inquiry by listening to

student ideas, responding to student thinking, and using student thinking to advance the mathematical agenda of the classroom community (Rasmussen & Kwon, 2007).

The data presented in our poster come from classroom teaching experiments in two sections of a first course in linear algebra during Fall 2014 at a large US university. The data sources were classroom videos that captured small-group work and whole-class discussion, students' written work from class, and photos of student work on classroom whiteboards.

RESULTS

In the IOLA task sequence on eigentheory and diagonalization, Task 1 builds from students' experience with linear transformations in \mathbb{R}^2 to introduce the idea of stretch factors and stretch directions and how these create a non-standard coordinate system for \mathbb{R}^2 . In Task 2, students create matrices that convert between the coordinate systems and coordinate with the transformation of Task 1 to reinvent the equation $A\vec{x} = PDP^{-1}\vec{x}$. Task 3 builds from students' experience with stretch factors and directions to reinvent methods to determine eigenvalues and eigenvectors. Finally, in Task 4 students work in \mathbb{R}^3 to develop the characteristic equation as a solution technique and connect eigentheory to their work with diagonalization in Tasks 1-2.

The poster highlights student work on Tasks 3-4. For instance, prior to defining eigenvector, eigenvalue, or characteristic equation, students find the "stretch factors" (i.e., eigenvalues) and "stretch directions" (i.e., eigenvectors) of a given 2x2 matrix. Common student approaches include finding eigenvectors first or eigenvalues first through solving a system of equations, and manipulating $A = PDP^{-1}$.

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TWG4: Teachers' practices and institutions

A bridge between inquiry and transmission: the study and research paths at university level

Berta Barquero¹, Lúdia Serrano², and Noemí Ruiz-Munzón³

¹Faculty of Education, Universitat de Barcelona, Spain, bbarquero@ub.edu,

²INS Vinyet, Barcelona, Spain, ³Escola Superior de Ciències Socials i de l'Empresa-Tecnocampus, Universitat Pompeu Fabra, Spain

This paper focuses on the notion of 'Study and Research Path' (SRP) proposed in the frame of the Anthropological Theory of the Didactic, as it was designed and implemented in first-year courses at university level of business and administration degrees. First, we show how SRPs can 'live' at university level, describing the conditions and constraints under which they take place in two different university institutions. Secondly, we focus on how SRPs can promote the interaction between different teaching approaches: those derived from inquiry-based models and those based on transmission pedagogies. Finally, we indicate why this interaction is essential as it enriches the students' milieu and enables the dialectics of questions and answers, both crucial elements in the evolution of the study process.

Keywords: study and research paths, research and study activities, university level, mathematical modelling, ecology.

INTRODUCTION

Educational researchers and practitioners, at all school levels and across different countries, agree on the basic principle that teaching should not just transmit knowledge to students but should also provide them with tools to question and inquire about reality. It is thus important to promote a change of the pedagogical and school paradigms, with new roles and responsibilities assigned to teachers and students, as well as new functionalities assigned to disciplinary knowledge and, in particular, to mathematics (Chevallard, 2015). In the case of mathematics, approaches like problem-based or project-based learning (PBL), inquiry-based mathematics education (IBME), have appeared with increasing frequency over the last decades in relation to mathematics and science education (Artigue & Blomhøj, 2013), supported by policy makers and curricula guidelines. However, despite the consensus on the importance of this change of paradigm, it is also apparent that any new proposal has to survive in a set of conditions and constraints that does not ensure its long-term survival, and many of said proposals end up disappearing from daily classroom activities. Therefore, to support and analyse any kind of alternative teaching proposal, researchers need reference models that allow them to describe and evaluate the impact that these innovative teaching practices have on the school system, and their relations to institutionalised practices and knowledge (within one or more disciplines). In the face of these needs, we propose the use of the epistemological and didactic model proposed by the anthropological theory of the didactic (ATD) through the notion of study and research paths (Chevallard, 2006 and 2015) in accordance

with the didactic engineering research initiated by the theory of didactic situations (Barquero & Bosch, 2015). On some occasions, SRPs have been wrongly identified as ‘inquiry-based proposals’, as if the transmission of knowledge was not related to their internal functioning. It is therefore important to refocus the meaning of the notions ‘study’ also of the ‘research’ in the SRP proposal, and explain how the SRP arises to dialectically combine ‘inquiry’ with ‘transmission’. This dialectics is what might ensure mathematical instruction moving towards a change of pedagogical paradigm, what Chevallard (2015) designates as the move from ‘visiting works’ to ‘questioning the world’.

According to Winsløw, Matheron and Mercier (2013), an SRP emphasises the dialectics between ‘research’ (inquiry, problem solving, problem posing, etc.) and ‘study’ (consulting existing knowledge, attending lectures where the teacher acts as the main means to provide mathematical knowledge, etc.) that is in fact characteristic of any learning activity, even though the proportion and quality of the two elements may vary. The term ‘path’ emphasises the openness of the possible routes or trajectories to be followed in an effective experimentation of the SRP. That is, the starting point of an SRP should be a ‘lively’ question of genuine interest for the community of study, what we call a *generating question* and refer to as Q_0 , that the group of students wants (or has) to answer with the help of the group of teachers. The study of Q_0 evolves and opens many other *derived questions* that appear as the starting point of new SRPs or new branches of the initial one. Elaborating answers to Q_0 has to become the main purpose of the study and an end in itself. As a result, the study of Q_0 and its derived questions Q_i leads to successive temporary answers A_i tracing out the *possible paths* to be followed in the experimentation of the SRP. In this paper we focus on two cases of SRPs that have been designed and implemented in first-year courses at university level of business administration degrees. The first research questions we aim at answering are: *under which conditions and constraints can SRPs be integrated in regular courses of mathematics at university level? How can SRPs be connected to the traditional university teaching devices?*

Besides the design and implementation of SRPs themselves, the second research question we focus on this paper is: *how to make SRPs progress in a teaching and learning situation? Why do SRPs need the interaction of more inquiry-based teaching devices others that are more based on the transmission of knowledge?* We will use two particular cases of experienced SRPs to analyse the dialectics between inquiry and transmission, emphasizing how the interaction of different didactic devices is crucial to the survival and evolution of the SRPs. Amongst other possible interactions, we assume the following premises about possible forms of integrations of SRPs:

1. When a particular mathematical organisation or praxeology has previously been introduced to students in a more ‘transmissive’ way (in a traditional university lecture for instance). In the ATD, the teaching of a pre-established mathematical praxeology can be described in terms of a *study and research activity* (SRA) (see Barquero &

Bosch, 2015), which is different from an SRP where the objective is not defined in advance. Thus, if the SRP starts from questioning the rationale, the necessity and use of this specific mathematical organisation, that is questioning an SRA, we could characterize this first interaction as the *generation of an SRP from the questioning of a previously developed SRA*.

2. Along the SRP development, some derived questions may appear calling for the introduction of certain mathematical pieces of knowledge (or praxeology). In this case, the starting point of the teaching and learning (sub)process is a fixed praxeology that a group of students should learn under the guidance of the teacher(s). The didactic process can be described as an *SRA originated by the questions that emerge in the SRP*.

3. A basic gesture in an SRP is to invite students, and teachers, to look for possible tools and answers outside, in the external media, which can be helpful in our study (in the sense that they contribute to provide answers). This gesture (closer to inquiry) needs to be followed by an accurate study about how to decompose and build up these external answers to be incorporated in the SRP dynamics. In this case, the main focus of this type of SRA here generated is not a fixed piece of knowledge, but the *search, de- and re-construction* of external answers and objects according to the new SRP needs. This is the most complete interaction between SRPs and SRAs.

AN SRP ABOUT THE EVOLUTION OF A SOCIAL NETWORK

General conditions and research methodology for the testing of the SRP

We focus on an SRP that was designed and implemented in 2010/11 (followed by a second implementation in 2013/14) with first-year university students of a business administration degree at the IQS-School of Management of the Universitat Ramon Llull in Barcelona (Spain). From 2006 to 2015, our research group carried out SRPs in this degree during the subject of Mathematics. On this occasion, the SRP focused on the generating question Q_0 about the *evolution of the number of users of a social network* called *Lunatic World* (Serrano & Bosch, 2011). Q_0 was divided into three sub-questions, based on the necessary tools for their resolution that were approached in each of the terms constituting the course of Mathematics. For instance, Q_0 was partially approached using discrete models and assuming independent generations of users during the first term. The second branch, developed during the second term, was then approached using functional models, so as to fit continuous function to real data. We will refer to both cases in the following section. The a priori mathematical design of this SRP is similar to the one described in Barquero, Bosch and Gascón (2013) in the case of population dynamics.

A special device called the ‘mathematical modelling workshop’ was introduced in the general organisation of the course. It consisted of 90-minute weekly sessions representing one third of the students’ classes, and more than half of their personal work outside of the classroom. Attendance was mandatory for the students. Evaluation of the workshop was the forty per cent of the final grade of the course.

This ran parallel to the three-hour weekly lecture sessions, which included some theory sessions and problem-solving activities. The lecturer of the course was also the person responsible for the workshop, and was accompanied by the authors of the paper who acted as observers. Attendance was mandatory for the students. In the general organisation of the workshop, students worked in teams of 3 or 4 members. Once the initial question was presented, two kinds of workshop sessions were combined every week: teamwork and presentations. In the teamwork sessions, each team had to look for temporary answers to partial questions derived from Q_0 and prepare a partial report with their answers. The reports were then defended orally in the subsequent sessions by some selected working teams. A discussion followed to state what progress had been made, and to agree on how to continue the study process. During the presentation sessions, one member of the class (named the ‘secretary’) prepared a report containing the main points of the discussion and the new questions proposed to follow with. At the end of the term, each student had to individually write a final report on the entire study (evolution of problematic questions, work on and with different models, relationship between them, etc.). The empirical data that were collected, upon which the *analysis a posteriori* of the SRP rested, comprised the students’ team and individual reports, the teacher’s written description of the work carried out during each session, the worksheets given to the students and a brief questionnaire given to the students at the end of each term.

The SRA and SRP: connecting university teaching devices

The generating questions Q_0 of the SRP we focus on are, *how does the population of users of a social network evolve over time? How can we fit models to real data and use them to forecast their future evolution?* With the SRP implementation, we verified how the sequence of questions arising from Q_0 led the students and the teacher to consider most of the main contents of the entire mathematics course (see Barquero, Serrano & Serrano, 2013). In each term, various types of mathematical models were analysed: forecasting the number of users in the short and long term, considering time as a discrete variable (first-order sequences models, 1st term), the same forecast considering time as a continuous variable (differential equations, 2nd term), and the forecast in discrete time distinguishing three user groups with different privileges (models based on matrix algebra, 3rd term). However, during the SRP, these contents appeared in a very different structure from the university’s traditional organization. Instead of the classical ‘logic of the mathematical concepts’, the workshop was more guided by the progressive appearance of the ‘dynamics of questions and answers’ derived from Q_0 (see Figure 1 for an SRP representation in terms of questions and answers).

To answer these questions, new media and milieu were necessary. To facilitate the necessary *enrichment of the students’ milieu* along the progress of the SRP, the interplay between the lecture sessions and the workshop was crucial. The ‘theory-problem’ sessions had their program defined in advance. The first term was devoted to one-variable calculus (functions, their properties, derivatives, etc.), the second term

$Q_{2.1}$: Assuming that $r(t) = p'(t)/p(t)$ is constant, how will the network users evolve over time
 $A_{2.1}$: Construction of the Malthusian continuous model [...]

$Q_{1vs2.2}$: Do we obtain the same conclusions from the discrete and the continuous logistic model? Do the coefficients (K and α) have the same meaning and effect?

During the SRP development, there was a certain moment when questions appearing in the workshop asked for the introduction of certain mathematical tools, and the subsequent enrichment of the students' milieu to be able to follow with the study. In this case, the 'theory-problem' sessions intervened, stopping its regular running to develop a particular study and research activity (SRA) with a clear aim (the construction of a certain praxeology). More specifically, in the case of the first term (10 weeks long), three questions appeared: $Q_{1.1}$, $Q_{1.2}$ and $Q_{1.3}$ (see above) that had to carry out three particular SRA (see Figure 2, SRA 1, 2 and 4), which had not been planned in the regular course. For this reason, some of the lectures and problem sessions had to be devoted to implement the SRA to build up the necessary mathematical praxeologies about: definition of recurrent sequences and their numerical simulation with Excel (SRA1 resulting from $Q_{1.1}$, 3 hours), sequence convergence and velocity of their convergence (SRA2 resulting from $Q_{1.2}$, 1.5h) and methods of graphical numerical simulation (SRA3 resulting from $Q_{1.3}$, 2h). In these cases, the lecturer of the course acted as the main means for students, stopping the regular course and guiding the theoretical and practical activities in accordance with the SRA aims. On the other hand, some of the questions appearing in the workshop achieved to show the functionality and rationale of some contents previously introduced in the regular course. For instance, some contents of the regular course, like the study of C^1 -functions, their representation, graphical techniques to solve equalities or inequalities, reappeared in the workshop, now as tools to provide answers to certain questions derived from Q_0 (as the case of $Q_{1.3}$ - $A_{1.3}$, SRA3 and SRA4 in Figure 2).

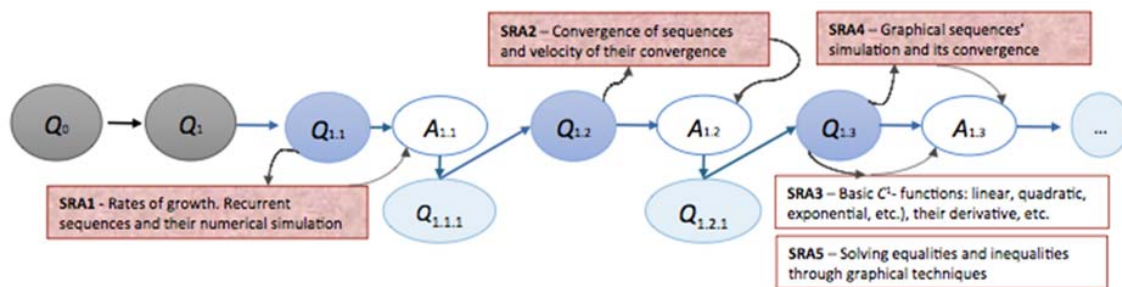


Figure 2: Interplay between the SRP and the necessary SRA during the 1st term

In the second term (also 10 weeks), the 'theory-problem' sessions were devoted to the study of multi-variable functions. In the workshop the 2nd branch of the SRP was implemented, focused on Q_2 about what continuous models can be used to fit data and to provide forecasts about the social network evolution. Students had overcome the initial resistances and progressively accepted a lot of new responsibilities they

were asked to take on: defending their reports, posing new questions, looking for available answers and work outside the classroom reality, etc. Thanks to this and to the parallel structure between the 1st and 2nd branch of the SRP (see Figure 1), the students' autonomy increased significantly. When the necessity of an SRA appeared, students were first asked to search in different media (books, Internet resources, etc.) and to look for available answers to questions, getting involved in a more inquiry nature activity (see for instance Figure 3, with SRA 1 and 2 about: What is a differential equation? How can the discrete Malthusian or the logistic model be reformulated in the continuous world? Under which assumptions?). Then, some of the workshops and/or lecture sessions were used to discuss their findings according to their usefulness in the SRP.

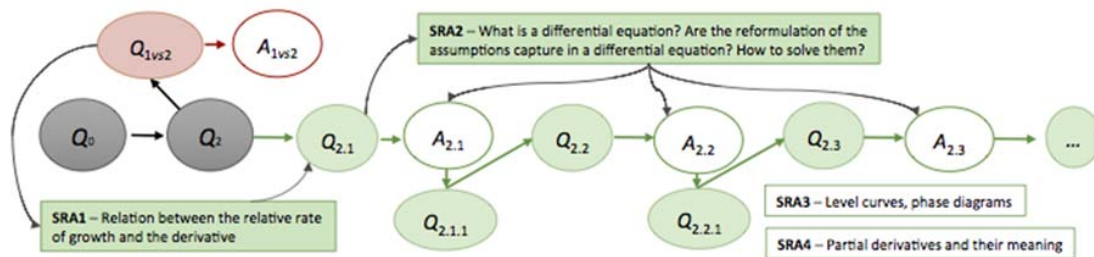


Figure 3: Interplay between the SRP and SRA during the 2nd term

AN SRP ABOUT COMPARING FORECASTS AGAINST REALITY

General conditions and the role of the SRP inside the course

The second case we want to focus on is the case of the SRP on *comparing forecasts against reality in the case of Facebook users' evolution*. On this occasion, we only refer to the SRP a priori mathematical and didactic design, as its implementation is planned to take place in the second term of the current academic year (from January to March 2016) with first-year students of the Business Administration degree and the Marketing and Digital Communities degree at the UPF university. The design of this SRP has been carried out by the authors of this paper in the frame of the European MCSquared¹ project (<http://www.mc2-project.eu>).

The SRP, linked to the first-year course of Mathematics, like in the previous case, is integrated into a new teaching device called the 'modelling workshop', created for this implementation, which is offered to students as a voluntary activity outside the regular schedule of the course, adding an extra point to the final grade of the subject (if they do not fail). The workshop will run in seven sessions of 1h30 each throughout the second term, and includes a certain amount of work the students need to do outside the classroom. The workshop is planned to begin in January this year, after having worked with some important mathematical tools for this SRP: families of basic (polynomial, irrational, logarithmic and exponential) functions, notions of differential calculus and their meaning for the study of one-variable functions in the first term.

The a priori design of the SRP about the evolution of Facebook users and its integration in the institutional conditions of the university

The initial situation begins by presenting some selected journal news about a research project developed by Princeton University, which anticipated that Facebook would lose 80% of their user's before 2017 (see Figure 4). According to the forecast proposed by the Princeton research, the generating question Q_0 is presented to students as follows: *Can these forecasts be true? How can we model real data about the evolution of Facebook users and forecast the short- and long-term evolution of the social network? How can we validate Princeton conclusions?*

La polémica sobre FACEBOOK



Figure 4: Introducing the initial question Q_0 (phase 1)

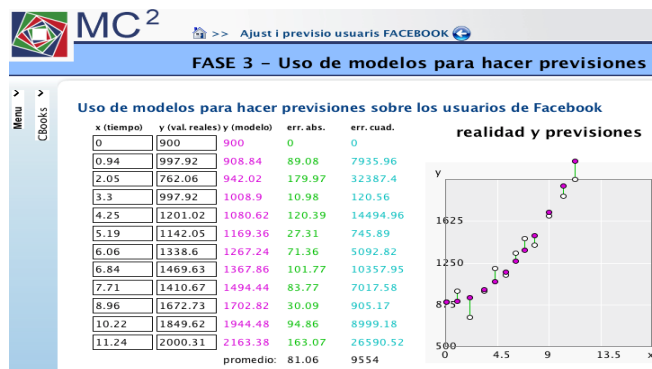


Figure 5: Use of models for the forecast (phase 3)

The workshop is composed of three interconnected phases, built up from the questions derived from Q_0 (see Figure 6). In the first phase (main focus on Q_1), the students are asked to explore and search real data about the evolution of Facebook users (from 2007 to the end of 2013, as Princeton did) and to begin with a descriptive statistical analysis of these sets of real data, their growth and tendency. In the second phase, the main question Q_2 discusses the use of models based on elementary (polynomial, exponential, logarithmic, etc.) functions to fit real data, bringing up the problem of how to better estimate the coefficients' value that define these models. Students are asked to finish this phase by proposing and justifying three mathematical models based on elementary functions. Finally, the third phase aims to use the models to forecast the short-, medium- and long-term evolution of Facebook users (Q_3) and to build up criteria to compare reality vs forecasts (as can be the linear or quadratic error, see Figure 5) and to describe the validity of the long-term forecasts (returning to the starting question, Q_0). Students, working in 'inquiry teams' of 3-4 people, must prepare a final report in answer to the initial question and to the derived questions that guide the three phases structuring the SRP. At certain moments of the study, the teams should prepare a partial report as a summary of their work dealing with certain questions. The discussion and debate moments, planned in advance and corresponding to the end of each phase, are crucial to ask students to defend their proposals, to decide on the new questions to face, to share new resources or answers

found in the external media, etc. For instance, at the end of phase 1, a poster-presentation format was designed to help students institutionalize the real data they chose to work with and their first descriptive analysis of real Facebook data and their tendency. Discussions and debates are also to share possible complementarities among teams with respect to the real data they analyse or to the methods they propose. Moreover, it is planned that, in phase 3, students exchange their partial report at the end of phase 2 and act as reviewers and validators of another team's work.

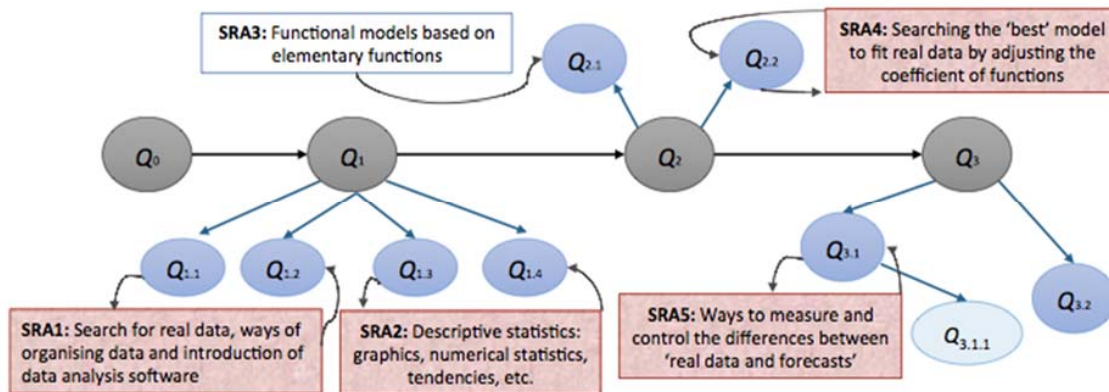


Figure 6: Question-gramme of the SRP with the necessary SRA

Concerning the necessity and interaction between the SRP progress with the SRA (see Figure 6), on the one hand, the workshop is planned after the first term when students have been introduced to one-variable functions (on a more transmission teaching model). In the SRP, however, in relation to Q_2 , students will have to use functions as models to fit data, providing a new use and rationale to their previous introduction in the lectures (SRA3). On the other hand, some particular SRA will be necessary (SRA4 and SRA 5) where the lecturer participates by dedicating some regular classroom time to the introduction of these new mathematical tools. Moreover, although the workshop is mainly linked to the course of mathematics, some important contributions together with other courses have been planned. In particular, some knowledge about statistics concerning SRA1 and 2 will be necessary, running to the statistics course. A course called 'Introduction to digital communities' (starting in the 2nd term) can also provide a general sense and functionality to Q_0 .

CONCLUSIONS

The notions of SRP and SRA provide a productive framework to analyse the necessary connections between 'study' activities consisting of making available a given pieces of knowledge and 'research' activities that consist of raising questions and searching, de-constructing and re-constructing answers. The two particular SRP cases presented in this paper illustrate possible ways of integrating SRA into SRP, thus linking transmissive teaching devices, like lectures or problem sessions, to more inquiry-based ones. The implementation of the second SRP will certainly shed more

light on the real conditions needed and, especially, the constraints found to make these connections existent. The empirical results obtained during this experimentation will be presented at the conference.

NOTES

1. The research leading to these results has received funding from the Spanish R & D project: EDU2012-39312-C03-01 and from the (FP7/2007-2013) Seventh Framework Programme of the European Union under grant agreement n° 610467 - “M C Squared” project, <http://mc2-project.eu>.

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Difficulties of the profession and dissemination as catalysers for the construction of professional praxeologies

Jean-Pierre Bourgade

Aix-Marseille Université, EA 4671 ADEF, ENS de Lyon (Aix-Marseille, France)jean-pierre.bourgade@univ-amu.fr

The introduction of didactic devices such as investigation workshops at university level reveals some difficulties related to missing professional praxeologies. Indeed, the paradigm of the visit of works is dominant in those institutions where a 'pedagogy of teachers' is widespread: investigation workshops require the mastery of a 'pedagogy of investigation', which is based on different professional praxeologies. The aim of this work is to present a didactic analysis of the exploration of a professional type of tasks related to the design of generating questions for such a workshop. We chiefly underline the parts played by the encounter with difficulties of the profession and the processes of dissemination of praxeologies: these two types of situations prove to be catalysers of the construction of praxeologies, in particular in the elaboration of the logos of professional praxeologies.

Keywords: Anthropological Theory of the Didactic, Generating Questions, Investigation Workshops, Problems of the profession, Professional Praxeologies.

INTRODUCTION

Investigation workshops have been introduced by Yves Chevallard (2011, see also Marietti 2009) at the Collège du Vieux-Port, in Marseille, at high school level (students aged 13) and were based on previous works in the frame of the Anthropological Theory of the Didactic (ATD). This was not only a new type of didactic device but also a laboratory for investigations on the professional praxeologies required when a teacher works in the frame of a new didactic paradigm (Chevallard 2013) of *questioning the world*. This paradigm is opposed to the paradigm of the *visit of works*: currently at school in France, students are required to study barely motivated (mathematical) works, which they *visit* just as tourists visit works of art or monuments, that is artificially, since it is not their own questioning that led to this visit but rather a formerly and firmly established program of visit (established by the tour operator in one case, by the ministry of education in the other case). On the other hand, a new paradigm of study emerges with difficulty, in which what is under consideration at school no more is a collection of yet established works, that is answers without questions, but rather *questions needing answers*. The aim of the study is then to elaborate collectively an answer to a question, without prejudging which works might be crossed in the course. In this sense, an investigation workshop leads to the development of a study and research path (SRP), which differs from *praxeologically finalised SRP* (Chevallard 2011), and the management of such a *praxeologically open SRP* raises different professional difficulties, especially in relation with the fact that the 'teacher' no longer knows up to what point, nor in

which direction, the investigation has to be led: he does not play the role of a ‘director of the study’, rather that of an ‘aid to the study’.

The ‘Investigations on the Internet’ workshops at the Collège du Vieux-Port were based on the study of short or medium range questions (four questions were studied in 18 hours the first year). Investigation workshops have also been introduced at University level (students of engineering sciences, aged 18) at *la prépa des INP* in Toulouse. This institution is a preparatory class that leads to engineering schools after two years of intense training in sciences and humanities. Teaching is generally organised following the visit of works paradigm and pedagogy is on the whole *apedagogy of teacher*. As Marietti (2009) mentions it, “today, the transition from the paradigm of the visit of works and from a pedagogy of teacher to a ‘questioning’ paradigm served by an adequate pedagogy of investigation constitutes an open problem” and an important challenge not only is to train teachers in pedagogy of investigation, but already to provide the profession with the identification of praxeological needs and the construction of professional praxeologies to answer these needs.

In this communication, we address the following question: to what extent is the elaboration of a professional praxeological equipment dependent on professional difficulties on the one hand, and on the dissemination of this equipment on the other hand? The author of these lines initiated and supported during two years the implementation of an investigation workshop at *la prépa des INP*. In an institution where no investigation workshop had ever existed previously, the author had to realise a certain number of professional types of tasks related to the design and implementation of the workshop; then, after changing of institution, he had to pass onto his colleagues the praxeological equipment needed for the realisation of these types of tasks. In this communication we proceed to give a didactic analysis of the construction of part of this praxeological equipment; we mainly focus on a specific type of tasks related to the formulation of a generating question and try to identify, in the history of the workshop, the realisation of several didactic moments (first encounter, exploration, building of the technological-theoretical block). In a general fashion, the construction of a professional technique by the subject of an institution does not necessitate the production of a very elaborate *logos* regarding this technique. While, in classrooms, the teacher designs didactic situations where an epistemological obstacle makes it necessary to elaborate further the technique and, consequently, to discuss it, thereby enriching the *logos* on the technique, in a professional context, only a-didactic situations are expected to generate elaborate *logos*. The analysis in terms of didactic moments enables us to identify two catalysers of the *logos* production in the building process of a professional praxeology: professional difficulties and dissemination issues. Finally, the construction of a praxeology also benefits from the *a posteriori* analysis such as the analysis presented in this communication, which can be read as an evaluation and development of the praxeological equipment built up to now.

HOW TO ASK A QUESTION? FIRST ENCOUNTER AND EXPLORATION

We briefly remind the reader with the praxeological model (Chevallard 2007) introduced in the ATD, according to which any human action can be modelled as the realisation in a given institution of a certain type of tasks T , using a technique τ that can be justified in the institution under consideration by means of a certain discourse on the technique, a technology θ , which can in turn be grounded on a theory Θ . The quadruplet $[T/\tau/\theta/\Theta]$ is called a *praxeology*. The study of a given praxeology can be described by the identification of at most six didactic moments (Artaud 2011): the moment of first encounter with the type of tasks and of its identification, the exploratory moment (or moment of the emergence of the technique), the technological-theoretical moment, the praxeological-work moment, the moment of institutionalisation and the moment of the evaluation of the praxeology. In this communication, we will seek to identify both the praxeology under construction for the realisation of a certain type of tasks, but also to make the analysis of the process of study of this praxeology in terms of didactic moments. This will enable us to clarify the parts played by professional difficulties and dissemination issues in the elaboration of a praxeology, especially in the construction of its technological-theoretical block.

The introduction of an investigation workshop at *la prépa des INP* was a lengthy process for the author of this communication (hereinafter designated by the letter y); at first y consider that one of the aims of the workshop was to¹ “promote the emergence of a mathematical knowledge in a functional context of application” (FRR). By that time, y had an indirect acquaintance with the design and implementation of study and research paths (SRP), mainly by reading scientific papers dedicated to these devices. While the management of praxeologically finalised SRP has been considered with accuracy (see Bosch 2010, Chevallard 2011, Ruiz-Munzón 2010 for instance), a guess is that the praxeological equipment related to the pedagogy of investigation also requires components for the *design* of a SRP. In particular, the formulation of a generating question is a problem of the profession, which is generally barely addressed, at least as such and in an explicit way. In the case of praxeologically finalised SRP, a praxeological model of reference (PMR) is built, which indicates the possible paths that can be followed in the study of a given mathematical work (see *e.g.* Ruiz-Munzón 2010 for algebra as a model for arithmetics). As such, it also partially governs the design of the generating question and we can assume that the design of the question is performed in a dialectical process with the construction of the PMR: the design of the question is generally related to the mathematical praxeologies at stake.

In an investigation workshop, the students are asked a question, but the ‘teacher’ has no clue regarding the works that will be encountered in the process of the

¹ Unless otherwise stated, the quotations are drawn from a funding request report (FRR) and from the author’s logbook (LB).

investigation. This can seem to conflict with the idea of aiming at the ‘emergence of a [given] mathematical knowledge’: as a matter of fact, the questions were first selected in order to reassure *y* in an institutional context where the paradigm of the visit of works is dominant. Under these constraints, *y* had to design a generating question for a SRP that would (hopefully) lead to the encounter with mathematical works but would, at the same time, remain praxeologically open. This is a realisation of the moment of first encounter and of identification of the type of tasks:

T_Q: “Design a generating question for an investigation workshop”

The moment of exploration of this type of tasks was realised in collaboration with a high-school teacher at first, then with a didactician, and progressively led to following technical component²: to design a generating question, you have to choose a question which ensures the encounter with mathematical works; though, you must not study the question yourself. Technological components were very limited at that stage since *y* did not have to justify to anyone but himself –and this personal justification boiled down to the fact that studying the question is forbidden in order to avoid the selection of the works that will be studied (which would mean designing a praxeologically finalised SRP).

Using this technique, *y* had selected two questions: the first question was provided by the high-school mathematics teacher who had asked it after reading an article in a mathematics journal for teachers; *y* had elaborated a second question after an investigation based on the reading of a biographical article on Leonhard Euler. In the opinion of *y* at this time, both questions had the advantage of ensuring the encounter with some specific mathematical praxeologies (linear spaces, matrices, eigenvalues, etc.).

A few days before the initiation of the workshop, the two selected questions were:

Q₁: “Some photo editing software can sharpen blurry photos. How do they do it?”

and “There are numerous constraints on the building of a bridge. In particular, the bridge is required to support heavy loads. How is it possible to foresee the maximum weight a bridge can withstand?” (LB) Nevertheless, after a working session with a didactician, the second question was abandoned in favour of the following:

Q₂: “Some mobile phones do not enter into standby until the user stops looking at them. How is it possible?”

Obviously, the didactician had something in mind while proposing this question to replace the question about bridges: *y* understood it as a way of proposing a sharper question, which would facilitate the starting of the workshop by focusing on the

² As our task here is to analyse the influence of professional difficulties and dissemination processes on the emergence of the praxeological equipment of *y*, we do not proceed to evaluate this equipment; in particular, we do not enter into a discussion of the validity of the emerging techniques.

students' interests. Also, this new question was not designed in order to insure the emergence of mathematical praxeologies –and even less, *chosen* mathematical praxeologies. This episode was a further realisation of the exploratory moment and led to a modification of the technique: the question had to be chosen 'sharp' and independently of any *a priori* knowledge regarding the sort of mathematical works that would be encountered in the study of the question. The technology was still very limited since it only included the fact that a sharp question would enhance the students' motivation to work.

PROFESSIONAL DIFFICULTIES AS A DRIVING FORCE FOR PRAXEOLOGICAL CONSTRUCTION

Surprisingly enough for y , the exploration of this type of tasks was constantly renewed over two years. Indeed, y expected T_Q to have a paramount importance at the start of the workshop, but not necessarily once the workshop was launched.

Managing without leading

At the beginning, students were interested or even *seduced* by the workshop: the freedom given them to investigate in any direction and in a large autonomy was appealing. Nevertheless, several students soon confronted y with a reluctance to accept parts of the didactic contract of the workshop (*e.g.* "Continue your investigations as far as you may"); some claimed to have found a satisfactory answer at a very early stage (after one session): "Five minutes after the beginning of the [second] session, two students come to y and claim that their team has 'found the answer'." To deal with this problem, y asked several questions with the effect that the students concluded that their answer was actually "crappy" (LB). This early incident was only the first of a long series that reached its climax on the eve of Christmas holidays when a team refused to work during all the session, only to end with a provocative speech directed to y , blaming him for not giving help and refusing to give precisions regarding the sort of answers that were expected. During the second session (out of 15 one-and-a-half-hour sessions), a girl had already complained: "What is it about? Do we have to program a phone? Do we have to understand engineers' programs?" (LB). A boy of the same team was upset by the fact that "we will never know whether the answer is satisfactory" (LB). We can model the previous incidents as follows. The workshop requires that y realises the following type of tasks:

T_{MW} : "Manage a group of students in the frame of an investigation workshop"

By the answers he gives to a reluctant team, y gives a hint of the kind of technique he has elaborated: not to give a direct answer (yes or no) to the question "is our answer satisfactory?", rather ask questions about the elements of the submitted answer in order to allow the students to identify weaknesses in their proposition. Though it seemingly leaves an important *topos* to the students, this technique, however, raises a problem: letting the team set their own stopping criterion gives no means to tackle the problem raised by teams that believe they have a satisfactory answer and that their

criterion for satisfactoriness is satisfactory. Letting the students choose the stopping criterion can lead them to minimal criterions. There remained to seek a solution, which would not rely on a return towards the ‘pedagogy of teacher’ by allowing y to impose his own criterion since this would conflict with the idea of an *open* SRP and, more specifically, with the goals of the investigation workshop: this workshop was indeed designed as a device for the diffusion of didactic praxeologies such as investigation techniques, etc. Therefore, choosing a stopping criterion can be considered as an essential part of the praxeological equipment that students have to elaborate and make theirs in the course of the workshop.

In search of a third way

A first analysis of the situation led y to conclude that there were actually two ways of managing the workshop: either the study aid y imposes a stopping criterion, or he leaves it to the students to select their own stopping criterion. The first way was soon rejected by y for the above-mentioned reasons. The second way of dealing with the stopping criterion difficulty (to leave it to the student) was first considered ideal by y , but he had no idea how to avoid the production of minimal criterions or, on the contrary, the production of ‘satisfactory’ (from the ‘teacher’s’ viewpoint) criteria for non satisfactory (from y ’s viewpoint) reasons: benevolent students could follow implicit stopping criteria matched to the didactic contract (‘in *classes préparatoires*, it is expected that the student goes as far as possible’, etc.) – though maximal, such a criterion would not emerge for functional reasons, but rather for ecological reasons, which y considered was a flaw. During the first year, y did not really change his vision of this difficulty of the profession. No sooner than in the month of June did he note that: “asking a question that would lead to a *production* [would] avoid the difficulty due to the absence of a stopping criterion” (LB). This is the first sign of the existence of a possible third way to address the difficulty met in realising T_{MW} .

A new *raison d’être* for T_Q

It must be stressed that up to that moment, y had understood the difficulty met in the management of the workshop as relative only to his management praxeologies. Only at that time does he consider the possibility that the design of the generating question may have an influence on the stopping criterion issue. This is a realisation of both the exploratory moment and the moment of the construction of the technological-theoretical block: while the technique slightly evolves (“asking a question that would lead to a *production*”), the technology now includes a new *raison d’être* for T_Q : the design of a generating question must take into account the fact that the question itself may be helpful to tackle the difficulty of the stopping criterion³.

³The realisation of the type of tasks T_Q is subject to many other constraints; we choose to focus here on the relation it has with T_{MW} for several reasons, among which is the fact that the exploration of T_Q described herein is closely articulated with the confrontation with a difficulty in the realisation of T_{MW} .

Enhancing the technological-theoretical block

In an attempt to explore further T_Q and improve the technique under construction, y came across a distinction between two types of questions, which had been made by Chevallard in his *Séminaire* (Chevallard 2010): technical and technological questions. The difference lies in the use of distinct interrogative pronouns –or on the possibility to reformulate questions using one of the two pronouns, *how* and *why*. A *how*-question is a technical question in which it is expected that the person describe a technique commonly used in a given institution to realise the task referred to in the question. A *why*-question leads to an explanation (technology, in the sense of the ATD) of the use in a certain institution of the technique referred to in the question. Questions Q_1 and Q_2 were first analysed by y as being technological questions: though apparently *how*-questions, they pulled the students towards the necessity to explain *why* such or such technique was used, or *why* such or such device actually worked. To put it another way, y thought at first that the problem met in the managing of the group was originated in the fact that the questions asked for *explanations* (technologies) and that the students were provided with no *a priori* criterion for the kind of admissible explanations. Indeed, many teams proposed explanations of a divulgation type –leaving all technicalities unstated.

One could argue, though, that both questions are technical: Q_1 asks “how do they do it?”, while Q_2 asks “how is it possible?”. Answering the first question is to give a description of a technique used “to do it” in a given institution. To answer the second question, it is necessary to explain *why* a certain technique actually works (“how is it possible?” read as “why is it possible?”); yet, to explain *why* something works, it is first necessary to show *how* it works, unless the “how” be given in the question – which was not the case here. The interpretation by y of his difficulties was therefore not entirely satisfactory. Nevertheless, it is a milestone in the process of exploration of T_Q : the identification of the link between T_Q and T_{MW} indicates a certain direction for the elaboration of a technique for the realisation of T_Q , while the previous explanation (the questions were “technological”), though incorrect, is a technological element of the praxeology under construction. Finally, let us point that this thinking on the relations between T_Q and T_{MW} also gave a hint about a part of a technique to realise T_{MW} : managing an investigation workshop can be difficult, but a good realisation of T_Q can make the job easier.

DISSEMINATION AS A DRIVING FORCE FOR LOGOS CONSTRUCTION

After two years, y left *la prépa des INP* and a new team of teachers took the responsibility of the investigation workshop: a teacher of English (y_1), a mathematics teacher (y_2) and a physics teacher (y_3). None of them was acquainted with didactics of mathematics, with the ATD or with pedagogy of investigation –though the three of them had already had an important thinking on their professional (pedagogical) techniques. We will now shortly report on the process of formulating two questions for the workshop as it can be observed in y 's logbook and in the e-mails exchanged with the y_i 's.

Exploration of T_Q by the y_i 's...

First, expectedly, y_2 had designed questions related to mathematics (or that would rapidly reach mathematical problems): y has consequently swept aside these first questions by clarifying the aims of the workshop to y_2 (we find here the first elements of the technique elaborated by y in the first year of the workshop). After some days, a new question arises:

“How to detect counterfeit artworks?”

Comments by y_2 : “Problem: can the question asked to the students result in a catalogue of existing techniques [...]? [...] Up to what point should we investigate to make sure that the question provides a field of investigation neither too wide nor too closed [...], without investigating for them?” (Common logbook of $y_{1,2,3}$, 9/13/2015)

The comments made by y_2 show that part of the technology for T_Q has been acquired by the y_i 's since they recognise the potential influence of the generating question on the ways the students might answer it. Here is a comment formulated by y :

“I think we should find a wording that would allow the students to enter into an investigation that would not finish rapidly in a catalogue of existing answers. [...] The question [...] should be converted to a ‘could you do...’-question.” (e-mail to y_i 's, 9/11/2015).

Here appears the following technological component: “the generating question must be designed in such a way that the investigation will not result in a catalogue of existing answers”. The technique is based on this element: designing a “could you do”-question is, at that time, supposed to avoid the encyclopaedic menace. However, this technology does not seem to convince the y_i 's who propose to yet another generating question:

“To meet energy needs of humanity, how can we use human beings themselves to produce daily a useable energy?” (e-mail, $y_{1,2,3}$ to y , 10/1/2015)

The question is ‘sharp’ in its reference to the energetic problem, and also independent of chosen mathematical praxeologies: it matches with the first requirements identified by y for a “correct” realisation of T_Q . Nonetheless, it is *not* a “could you do”-question – though we observe an attempt to “make technical” the questions by introducing interrogative pronoun “how”; in response, y proposes the following wording:

“To meet energy needs of humanity, it can be contemplated to use energy produced by human beings themselves. Could you suggest a device that would allow covering the needs in energy of the amphitheatre of *la prépa* using only (or mainly) the energy produced by its users?” (e-mail to $y_{1,2,3}$)

...enhancement of the technological-theoretical block by y

Here, y produces a “could you do”-question. Nevertheless, the technology of this technique is not well shared with y_i 's since, in his message, y only gives the question and provides no rationales for the modifications he made. The technology of the

technique proposed by y at this stage was essentially this: if the question is asked at the level of the teaching institution, students will *have* to study it until an effective result is obtained (a more efficient heating of the amphitheatre, *e.g.*). This modification of the technology can be understood as the effect of the reaction of the *milieu* constituted by the y_i 's: the inadequacy of their propositions (though they formally match the requirement of producing 'technical' questions, that is *how*-questions in the sense of Chevallard 2010) forced y to question further the reasons why these propositions did not satisfy him. In the following weeks, y proceeded to read again (Chevallard 2010) and came across the following comment:

“When [the institution] is elided [in the question], it is as if it was unique and as if there also existed a technique, also unique and therefore implicitly universal, which would give an answer to the initial question. This is a language effect that represses and hides the institutional relativity of praxeologies.”

The evocation of the institution '*la prépa des INP*' in the rewriting of the generating question by y can be read as an attempt to make explicit the institution in which the answer must be produced or, to put it more precisely, in which the constructed praxeologies will be used (and, therefore, have to be usable). This new technological component could only be elaborated in the confrontation with competitive rationales produced by the y_i 's, such as: “Thanks for the [question], I feel [it is] indeed *more precise* with your modifications” (e-mail from y_1 to y , 10/8/2015, my emphasis). The justification by the 'precision' of the question did not match with the intentions of y , who had to elaborate further on his own justifications for his techniques (another realisation of the technological-theoretical moment).

CONCLUSION

As he or she takes the responsibility of the realisation of a new (for him or her, or for the institution) type of tasks in a given institution, a subject of this institution generally first encounters the type of tasks and goes on to elaborate a technique to deal with it. The justification of the technique is usually left at a low level of clarification, unless a peculiar *difficulty* makes it necessary to further explore the type of tasks and to analyse the reasons why the previous technique failed to work. However, it is only when the subject of the institution has to disseminate the praxeology under construction towards other subjects of this institution that the construction of a *logos* reaches its highest level: tackling with their difficulties necessitates the explication of previously semi-unconscious choices, etc. Notably, this collective process is the illustration of the conversion of a *difficulty* of the profession into a genuine *problem* of the profession in the sense that an instance takes the responsibility to study it and formulate a (partial) answer (Chevallard et Cirade, 2010).

In the case under consideration in this communication, one step further was taken in the process of submission of the paper since the referees comments led y to relate the praxeology evoked in this paper with other techniques used *e.g.* by Ruiz-Munzón

(2010) in her realisation of T_Q : in her work, the choice was made to consider the class as a *consultancy service* with the objective to answer (real or imaginary) companies' demands. The common point with the praxeology presented here probably lies in the inclusion in the question of a precise institution –which amounts to include in the question some clues regarding the ‘stopping criterion’, living it to the students to build after the situation presented in the question a satisfactory criterion. The choice made by ywas characterised by a close relation between the demanding institution (*la prépa des INP*) and the answering institution (an investigation workshop at *la prépa des INP*), which can be read as an autarkic version of the ‘consultancy service’ fiction.

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The place of computer programming in (undergraduate) mathematical practices

Laura Broley

Concordia University in Montréal, Department of Mathematics and Statistics,
Canada, l_brole@live.concordia.ca

A recent survey of Canadian mathematicians found that while 43% of the participants use computer programming in their research, only 18% integrate this activity into their teaching. The first statistic highlights the significant place that programming may have in professional mathematics practice. The second suggests that such significance may not yet be acknowledged in undergraduate curricula across Canada. Our exploratory study involving 14 Canadian mathematicians sought to gain a deeper understanding of the place of programming in both contexts and therefore describe the gap from a more qualitative perspective. The views of our participants highlight some important issues that may require attention in order to bridge the identified gaps, should that be deemed the favourable direction to take.

Keywords: Mathematical practices, undergraduate teaching and learning, computer programming, institutional constraints.

INTRODUCTION

Several mathematicians and researchers in math education have reported on the disconnection between undergraduate curricula and professional practice. A recent quantitative survey of 302 Canadian mathematicians points to one possible aspect of the gap: while 43% of the participants reported using computer programming in their research, only 18% indicated that they rely on this activity in their teaching (Buteau, Jarvis, & Lavicza, 2014). Furthermore, when compared to the other technologies surveyed, programming was the only one for which such a gap was observed.

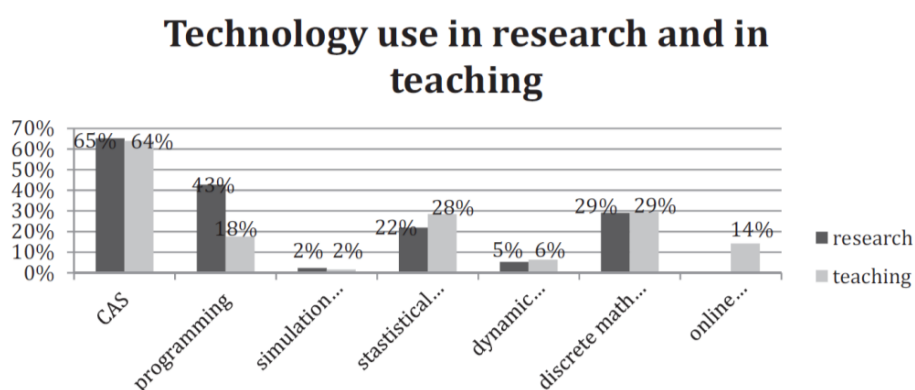


Figure 1: An intriguing gap for programming identified by Buteau et al., 2014.

The first statistic highlights the potential that programming may have for doing mathematics and the possible relevance of integrating it into university courses. The second inspires further research: why would such a gap exist? Buteau et al. (2014)

predict that the learning curve for programming is greater than for other tools such as computer algebra systems (e.g., *Mathematica*). They also mention the logistical obstacles of curriculum-wide integration. We were unaware of studies that verified these hypotheses or explored other possible explanations, while providing an in-depth look at how the 43% and 18% of mathematicians might be using programming in their research and teaching respectively. We hence wondered: what is the place of programming in mathematical research and undergraduate math education?

A FRAMEWORK FOR CAPTURING AND COMPARING PRACTICES

Our research question, posed as is, raises a subtle issue: since education contains two distinct perspectives, the teacher's and the student's, which should we take? After speaking with mathematicians, it appeared that many of them do not see fundamental similarities between the acts of teaching and researching. In contrast, we might assume that student experience should reflect mathematicians' work. We therefore decided that a comparison of research practices with *learning* practices (as opposed to *teaching* practices) could not only be more interesting, but also more important.

To be able to capture and compare such practices, we turned to Chevallard's (1998) Anthropological Theory of the Didactic (ATD), which provides a model for apprehending the various elements of mathematical activity that occur in any context (in this case, professional and educational). According to the ATD, an individual's "practices" are understood through the notion of "praxeology", which takes into account the know-how (the *praxis*) and the discourses (the *logos*) that describe, legitimise, explain, and produce the praxis. The praxis contains two components: *tasks* (things to accomplish) and *techniques* (methods used to accomplish the tasks). Similarly, the logos can be divided into several levels: Chevallard (1998) distinguishes between *technologies* (justifications for the techniques) and *theories* (foundations underlying the technologies). Following the example of Artigue (2002), we have chosen to avoid the ambiguity of the word "technology" and label any *logos* as a "justification". We will also classify different types of justifications, such as those that are *pragmatic* (concerning the productive potential of a technique) and those that are *epistemic* (concerning the potential of a technique to contribute to the understanding of the objects involved). As Artigue (2002) suggests, the place of certain techniques within an institution could depend on such justification types.

This leads us to another crucial aspect of the ATD: individuals' practices are framed by the social institutions where they live. Under the influence of institutional constraints, certain practices are normalized. Conversely, an increased acceptance of new practices can cause a reevaluation of constraints. To describe the variable status of practices within institutions, we use Morrisette's (2011) characterization of

1. *Shared Practices*, which are intimately tied to a profession and remain unquestioned;
2. *Admitted Practices*, which are not shared by everyone in a profession, but are accepted because they have been shown to be effective by innovative practitioners; and

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3. *Contested Practices*, which are not accepted by everyone and are therefore situated at the boundaries of the professional culture.

METHODOLOGICAL CONSIDERATIONS

To gain a deeper understanding of how programming is integrated into the practices of mathematicians and their students, we carried out an exploratory study involving individual semi-directed interviews with 14 mathematicians: 3 women and 11 men of various ages, languages (French or English), and research domains, working within 10 universities in 3 Canadian provinces (British Columbia, Ontario, and Québec).

The structure of our interviews, captured in a written guide, was largely inspired by Vermersch's (2006) *entretien d'explicitation*. This model considers the actions of an individual as the main source of reliable information regarding the reasoning involved in those actions (different from the reasoning adopted outside of the actions). Hence, the interviewer is mainly a guide who tries to lead the interviewee into a state of descriptive verbalisation, where they "re-live" specific experiences. Our participants were encouraged to re-live moments throughout their research and teaching, in relation to computer programming. Some chose to share resources they had developed (e.g., computer programs and activity outlines), which enhanced their descriptions. Nevertheless, some general reflections on mathematics, programming, and institutional constraints were also solicited from the participants.

Interviews were recorded and transcribed. We then performed a categorical analysis (Van der Maren, 1996), using a mixed coding approach to identify, classify, and compare the main ideas. Examples of programming use underwent a supplementary characterisation using Chevallard's (1998) framework in order to extract the types of tasks, techniques, and justifications that define research and learning practices.

CLARIFICATION OF A DEFINITION

At the beginning of our project, we were surprised at how difficult it was to develop a definition of "computer programming". First of all, there was no clear agreement on what constitutes "programming" within the literature we had read. Secondly, we could not decide which kinds of activities to consider in our definition. It was clear that numerically solving a system of differential equations by developing a new method, writing some code, and ensuring the correct functioning of the resulting program should be seen as involving "programming"; but which steps exactly? Additionally, how should we classify activities such as the use of a computer algebra system to calculate a particularly difficult integral? Previous studies (e.g., Buteau et al., 2014) have distinguished between computer algebra systems and programming languages; but we can just as easily write a program in *Mathematica* as in *C++*.

In hopes that our participants could help us circumscribe "programming" in the context of mathematics, we left the interpretation of this word up to them. From the perceptions that emerged, we identify programming as an activity that aims to

construct a computer tool (a program) by way of three nonlinear tasks of varying importance: the development of an algorithm, the coding of the algorithm, and the verification and validation of the program. Still, there was no unanimity on the sorts of activities that correspond to the completion of these tasks. While some questioned whether or not using library routines in *Mathematica* really was “programming”, others proudly described examples of this type that allowed them to make major advances in their research. At times interviewees would take an even larger perspective, wondering, for example, whether or not geometric constructions in *Geometer’s Sketchpad* could be classified as “programming”. Other times, they were more restrictive, reducing programming to the task of writing code. While most participants recognized a mathematical character in programming, comparing it to solving a problem or constructing a proof, they also tended to restrict the whole activity to the status of a technique for accomplishing more important research tasks.

THE PLACE OF PROGRAMMING IN MATHEMATICAL PRACTICES

Based on the experiences described by our participants, we characterized several research and learning practices that may involve programming. What was particularly interesting was the varying degree to which programming (as we have just defined it) could be involved. Our analysis led us to identify six levels on which mathematicians or students may interact with programming: they may

- L0: Strictly observe the results of a computer program;
- L1: Manipulate the interface of a program;
- L2: Observe the code of a program;
- L3: Modify the code of a program;
- L4: Construct the code of a program, with elements (e.g., the algorithm) specified; or
- L5: Develop a program, including algorithm development, coding, and verification.

The higher the level, the more the programming activity becomes visible and the more the mathematician or student becomes active in it. The identification of these levels allowed us to make important comparisons between the practices of a given mathematician and those intended to be developed by his/her students. In what follows, I restrict myself to an insightful selection of these practices, organized into two naturally-arising categories: “pure” and “applied” problem solving.

“Pure” problem solving and the cases of Omar and Paul

This category refers to when a mathematician or mathematics student seeks to develop abstract mathematical knowledge, the former contributing directly to the discipline and the latter supporting the personal education of the student. In both contexts, a main type of task where programming may be involved is the discovery of concepts, properties, or theories, typically realized through an *Exploration Cycle* including the observation of mathematical objects and the formulation/verification of

conjectures. Our study shows that the place of programming within the associated praxeologies may differ significantly for mathematicians and their students.

	Task Type: Discover mathematical concepts, properties, or theories	
	Technique	Justification
Omar	L5 + <i>Exploration Cycle</i> until “sufficient” evidence collected	(1) <i>Epistemic</i> : Gain insight to formulate conjectures and confidence to proceed to proof (2) <i>Pragmatic</i> : Organization, precision, speed, adaptability, and reusability of programs
Omar’s Students	L0 + <i>Guided/ Limited Exploration Cycle</i> in class	<i>Epistemic</i> : Have intuition challenged and abstract theory rendered concrete, exciting, and memorable <i>Pragmatic</i> as above, but from professor’s perspective

Table 1: Omar’s research praxeology vs. the praxeology proposed to his students.

Like several of the pure mathematicians I interviewed, Omar often develops his own computer tools (L5) to collect evidence about the behaviour of the abstract objects he studies. He proceeds through an *Exploration Cycle* to first gain the insight necessary to formulate plausible conjectures and then build confidence in their truth. He justifies his technique principally in an *epistemic* fashion, exclaiming, for example, “Before starting to prove something, you’d better know it’s true beyond a doubt!” Nonetheless, the *pragmatic* character of his programming is undeniable: he is free to control every aspect of his exploration, extend it to any number of otherwise tedious or impossible examples, and adapt his tool for completing future research tasks.

Given his familiarity with creating programs to assist in his own discovery of mathematics, it is not surprising that Omar, like many of our participants, also develops tools within the context of his teaching to support his students’ understanding of challenging notions (e.g., spanning sets and linear independence). From the student’s point of view, however, the proposed praxeology is quite different from their professor’s: Omar’s linear algebra students are invited solely to observe the dynamic images produced by their professor (L0), are prompted to make conjectures, and are provided images that verify or refute their voiced conjectures. Their exploration is heavily guided and limited to the time available in class, and the programming activity remains completely inaccessible to them. The fact that Omar does not share his programs with his students parallels the way he (and other pure mathematicians) communicate their research results: once they have arrived at a theorem and its proof, they typically see no need to discuss the programming that assisted their exploratory work. Similarly in teaching, Omar sees no need to encourage further exploration with a program once he believes the main goal of student discovery has been achieved. Indeed, he emphasizes the *epistemic* quality of the proposed technique, claiming that observing carefully chosen computer-generated results enables students to have their intuitions challenged and the abstract

theories they're learning rendered more concrete, interesting, and memorable. When he attributes a *pragmatic* value to the technique, it is in relation to himself: the examples he generates would be difficult, if not impossible, to reproduce on a board, and he would not have the same flexibility of re-executing programs in response to the needs of his students. A summary of Omar's research praxeology and the counterpart praxeology he proposes to his students is given in Table 1.

Paul is a probability professor whose table of praxeologies would differ from Omar's table in terms of techniques and justifications. In his research projects that require the use of computer tools, he always remains at L0 or L1; the programming is done by a collaborator. And yet, he encourages his students to write and use their own programs (L4 or L5). Like Omar, Paul brings forth principally *epistemic* justifications; the difference is his claim that

It's much better if the students can program it for themselves. If they're sitting in front of the screen and they can play with it and they can adjust parameters, it becomes a kind of a game and it's more interactive for them. And it's better than me just showing them a picture at the front of the classroom.

We will return to this idea that higher-level interactions may have greater *epistemic* value. For now, this claim naturally raises the question: so, why doesn't Omar invite his students to develop their own programs?

Ironically, it is Paul who provides an enlightening story for framing the response to this question. It turns out that the probability course described above is geared towards science students; for math majors, computer technology is completely absent. Of course, there are many ways to "Discover mathematical concepts, properties, or theories", and Paul's pure math students are encouraged to adopt more traditional techniques. Upon reflecting on the addition of programming, Paul concluded: "I think it's the right way to go actually. I think that we're missing an opportunity here." So, why not take the opportunity? At first, Paul discussed curricular constraints: the pure math students may not have the prerequisite knowledge needed for programming and it would take a great deal of time to develop and integrate new activities into an already jam-packed well-defined curriculum. Omar also explained that "There's so much material in [his] course that it seems like it would be an exaggeration to ask them to program as well [...] But, if we had more time, well it would be nice." Since Paul already overcame curricular obstacles during the transformation of the probability course for science students, it would seem like something deeper is at play. Indeed, he eventually revealed that "Academia is a very conservative place. And there's a huge amount of inertia. And there's also a huge amount of independence among the different instructors." He added: "I don't hear a lot of people talking about this being a great idea." Omar elaborated on similar constraints imposed within his institution: "I realize that my department is very abstract [...] And the students in pure math love that. But I believe it limits some of their abilities that are absolutely essential if they want to become researchers."

As reflected in this quote, the pure mathematicians in our study view programming (L5) as *admitted* amongst them and their colleagues. Phillippe summarizes their perspectives when he says that “Programming is really one tool among many others to do mathematics [...] that is not necessary, but that is useful.” Why then is programming still deemed by some departments as *contested* for pure math students? In the past, some mathematicians (e.g., Bailey & Borwein, 2005) have implied that computer-based techniques were *contested* within the pure math community. Our participants point out that many Canadian universities are still anchored in this traditional culture that favors the chalk-and-talk paradigm and by-hand exercises.

“Applied” problem solving and the cases of Barbara and Ben

The institutionally-driven introduction of programming in Paul’s probability course for science students is likely related to the importance attributed to programming in the “applied” math community. As Barbara suggests, “It’s absolutely indispensable for applied mathematicians.” When it comes to solving “real-world” problems, programming is part of the techniques *shared* by all of our participants, allowing them to analyse data (to develop/validate mathematical models), calculate parameters (to specify such models to particular situations), and understand mathematical models (either for validation purposes or to describe, explain, or predict real-world phenomena). As above, I elaborate on the praxeologies for one (the last) type of task.

According to our applied participants, whenever they must explore the behaviour of the mathematical models they develop and/or study, programming (L5) is simply a necessity for *pragmatic* reasons: not only does it create tools capable of performing a massive number of calculations and varying parameters to consider different scenarios, but first and foremost it permits the simulation of models that lack analytic solutions. As Alice explains, “It’s highly unlikely that a mathematical model will give you the quadratic formula in the end. It would be nice, but that doesn’t happen. And so, computer programming is essential.” Though it was not emphasized by the applied group, the underlying *epistemic* character of programming is also clear: it is the visual and dynamic output generated by computer programs that enables the recognition of patterns leading to descriptions, explanations, or predictions.

Given their *pragmatic* justifications, it is not surprising that all the applied researchers engage in programming at the highest level: L5. It may also not come as a surprise that we observed the least dramatic differences between the place of programming in research and in learning within this category. Still, there were some notable differences and interesting debates. Barbara’s students, for example, are not asked to develop their own programs. Instead, in addition to observing some results shown by their professor in class (L0), they are invited to receive explanations of her code (L2), manipulate her programs at the interface level (L1), and modify them (L3) to analyse different models. Barbara explains that “[she] want[s] [students] to see that programming isn’t that bad. You can do lots of interesting stuff with just a few lines of code.” Through the proposed techniques, her students may learn more about

programming itself (e.g., syntax and structure), and may come to appreciate the computer as a powerful tool. Having some insight into the code may also support their understanding of the corresponding output and models. Nevertheless, Barbara justifies their mid-level interactions by saying things like, “It wasn't so much how to program a vector field, it was how to use a vector field to understand the model.” Her ultimate goal is for students to understand models, not necessarily programming.

Task Type: Understand the behaviour of a mathematical model		
	Technique	Justification
Ben	L5 + <i>Experimentation</i> (i.e., variation of parameters to observe different output)	(1) <i>Pragmatic</i> : Otherwise impossible due to lack of analytic solutions and number of calculations/scenarios to consider (2) <i>Epistemic</i> : Visual/dynamic output for various parameter values enables descriptions, explanations, and predictions
Ben's Students		(1) <i>Pragmatic</i> and (2) <i>Epistemic</i> as above, plus: L5 leads to deep understanding and control of the tool, output, and model

Table 2: Ben's research praxeology vs. the praxeology proposed to his students.

In comparison, Ben believes that inviting students to do the programming (L5) might have a higher *epistemic* value. On the one hand, he suggests that “It's very hard to write a program and not understand what it's doing. You know, it's a different level of comprehension.” On the other, he reflects on his experience asking students to manipulate a pre-developed program (L3): “It was an exercise in typing. They really didn't know what it was doing or why it was doing it.” In Ben's view, if students write their own programs, it is more likely that they will deeply understand the tool, enabling them to modify it according to their needs, more effectively interpret the results, and, by extension, better understand the models. Other mathematicians add that while constructing a program, students may come to better grasp the concepts, processes, and methods that they must structure into an algorithm and transpose into a programming language. Then, having created their own tool, students may feel a sense of empowerment and excitement that may further enhance their engagement and understanding. And finally, students may also gain more insight into elements of programming itself (e.g., algorithms, data structures, code efficiency) that could not only allow them to better understand and use existing software (previously “black boxes”), but also provide them with the knowledge required to develop their own tools in the future. After all, the more the power of programming is shifted into the hands of students, the more they may be convinced of the *pragmatic* value of such techniques. In sum, many mathematicians agree with Paul that “It's much better if the students can program it for themselves.”

Once again, we may wonder: why doesn't Barbara ask her students to develop their own programs? Throughout her interview, the professor complained that her university lacks a mandatory training in programming for math students and that the

activity is not widely implemented by her colleagues; some of her students are even afraid of programming! In contrast, learning and using programming is integrated throughout the curricula for all math students in Ben's department. But, as Ben explains, this systematic institutionalization of programming is not necessarily easy:

There's a lot of inertia in Universities. [...] You don't just introduce something and it happens. [...] You introduce it one year, and everybody talks about it, and it's a no. And then there's lots of conversations about it [...] because you want people to have something that they truly need, and that has to evolve through discussion.

Moreover, even after all the discussion, the institutional context may impose serious constraints. Alice, for example, works at a university where programming-based techniques are completely normalized. Yet, she feels she must settle for encouraging lower-level interactions (L4 or L3) because she does not have enough human resources to adequately grade students' code; and in her view, "If it's not assessed in detail, the requirement is shallow." In the end, while programming (L5) may be part of the *shared* practices of applied mathematicians, institutions may render it only *admitted* within the community of students in applied math courses.

SUMMARY AND CONCLUSIONS

In 2014, Buteau et al. reported an intriguing gap: Of 302 mathematicians, 43% claimed to use programming in their research and only 18% said they use the activity in their teaching. In this paper, I presented some results of a qualitative study that sought to gain a deeper understanding of this situation. Our analysis of interviews with 14 mathematicians shows that "using programming" may be interpreted in significantly different ways. The word "programming" itself does not have the same meaning for every mathematician and future research could benefit from clarifying the boundaries of this activity. But in addition to this, when professors "use programming", their students may actually interact with the activity on various levels, from strictly observing the results of programs (L0) to independently developing their own computer tools (L5). The identification of six levels led us to note important differences between the practices of individual mathematicians and those they propose to their students, suggesting that the gap highlighted by Buteau et al. (2014) may actually be greater and more complex than expected. Even if programming (L5) may be *shared* or *admitted* within applied or pure research communities, respectively, it may be *admitted* or *contested* within applied or pure learning communities. Adding to previous predictions as to why such gaps might occur, our participants spoke of different kinds of institutional constraints: curricular (objectives, prerequisites, time), departmental (academic freedom vs. coordination, class size vs. human resources), and cultural (deep-rooted traditions in mathematics). And yet, they spoke equally of the potential benefits of bridging the gaps. Not only might it encourage techniques of high *epistemic* value or make students aware of the *pragmatic* character of programming, but it may also be important for *social/cultural*

reasons: Programming may widen students' vision and appreciation of all mathematical activity, while also encouraging the development of mathematician-like practices that could diversify their options beyond their undergraduate degree. Of course there is the question of the actual experiences of students, which we have not yet addressed: what benefits (and obstacles) do students *actually* experience while programming? Or, more critically, what are the benefits (and obstacles) of each programming level? After all, any level might be required in doing mathematics, whether due to collaborations (recall Paul's research practices) or the development and sharing of tools, which constitutes another category of practices we identified. Indeed, our participants mentioned observing their colleagues' results (L0), using other programs (L1), making sense of existing code (L2), or even reworking such code (L3). I hope to pursue a deeper analysis of these different levels in future research. In the meantime, it is important to note that all 14 of our participants believe that while programming should not constitute the essential element of undergraduate mathematics, it should receive more attention than it does in current Canadian curricula. A path towards change may not yet be clear, nor may it be easy; but the shared perspectives of our participants lead us to conclude that it exists!

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Envelopes in a computerized environment: the transition from 2D to 3D

Thierry Dana-Picard¹ and Nurit Zehavi²

¹Jerusalem College of Technology, Jerusalem, Israel, ndp@jct.ac.il;

²Weizmann Institute of Science, Rehovot, Israel, nurit.zehavi10@gmail.com

Abstract: The usage of a Computer Algebra System (CAS) allows new approaches to classical topics. We describe how envelopes of parameterized families of curves and surfaces can be studied using technology for automatic solutions of equations, dynamical representations and experimentation. Work is based on the knowledge acquired in Calculus and Advanced Calculus. The study of envelopes enables also to make connections with other mathematical domains, such as Abstract Algebra in order to have access to other algorithms implemented in CAS. The transition from 2D to 3D is non-trivial, as technological features available for 2D may either not exist or be harder to use in a 3D setting.

Keywords: Differential Geometry; Envelopes; Computer Algebra Systems; dynamical representations.

INTRODUCTION

The intertwining of experimental approaches and theoretical thinking is the core of discovery in every mathematical field. The usage of Computer Algebra Systems (CAS), packages for Dynamical Geometry Systems (DGS) and other technologies add important tools to the experimental part of the work. Arzarello et al. (2012) mention that the latest developments in these technologies have “rejuvenated mathematics and mathematics education”, as they provide numerous new computational and symbolic tools, and revitalize experimental mathematics and visual tools.

Classical topics in Differential Geometry like the study of 1-parameter families of curves in 2D space and of surfaces in 3D space and their envelopes have been abandoned in the past for various reasons (Thom, 1962). We may mention only the fact that the theory has few powerful theorems and numerous special cases. Visualization is also a crucial issue, in particular in 3D. Moreover, the computations may be quite heavy and unilluminating.

Nevertheless, this topic has a great interest in mathematics and in applied science. Envelopes have numerous applications in science and engineering: caustics and wave fronts (geometrical optics, theory of singularities), robotics (rigid motion in 2-space and in 3-space, collision avoidance), diffusion of chemicals in the soil, etc.

In this paper, we show how the envelopes can be revived and studied making a joint usage of paper-and-pencil together with both CAS and DGS. This is an approach recommended in (Roulier & Böhm, 2015) and (Wegscheider, Himmelbauer, & Böhm, 2015). The examples are based on experimental and computational work.

The transition from 1-parameter families of plane curves to 1-parameter families of surfaces in 3D space presents a double trend: the algebraic part follows a similar pathway in both settings, but visualization is different. The algebraic components of CAS are well developed, and actually the same algorithms are at work. The availability in the software of dynamical features, in particular a slider bar, as in GeoGebra, is a core issue for building intuition and understanding in 2D: tracing changes yields a construction of the desired objects (see Section 2). Direct manipulation of the figures using the mouse is still rare in 3D software.

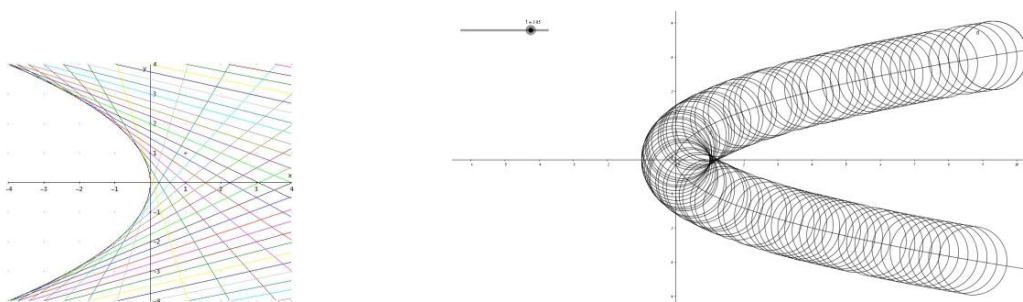
The topic has been taught during last years in sessions for both pre-service and in-service teachers.

ENVELOPES OF 1-PARAMETER FAMILIES OF PLANE CURVES

Let a family of plane curves be given by the equation $f(x, y, t) = 0$, where t is a real parameter. An envelope of the family, if it exists, is a curve tangent to every curve in the family (Struik, 1950). It can be shown that this envelope is the solution set of the following system of equations (op. cit., p. 167):

$$(1) \quad \begin{cases} f(x, y, t) = 0 \\ \frac{\partial f}{\partial t}(x, y, t) = 0 \end{cases}$$

Figure 1a shows the envelope of the family of lines given by the equation $x + ty = t^2$ (it is the parabola whose equation is $x = -y^2/4$). Figure 1b shows the envelope of the family of unit circles centred on the parabola whose equation is $x = y^2$.



(a) A family of lines

(b) A family of circles centred on a parabola

Figure 1: Two examples of envelopes

The usage of a slider bar enables to build envelopes experimentally. The experimental study may enhance the understanding of various properties, among them:

- a. Possibility of non-existence of an envelope.
- b. Possibility of non-unicity of an envelope.
- c. If an envelope exists, the existence of singular points.

The envelope of the family of lines in Figure 1a has one component and no singular point. In Figure 1b, the envelope of the family of circles has two components; the exterior one seems to be smooth, the interior one seems to have singular points, two cusps and one double point. This last property needs to be checked; this may be done using the algebraic presentation of the technology.

A general point on the given parabola has coordinates (t^2, t) , whence a general equation for the circles: $(x - t^2)^2 + (y - t)^2 - 1 = 0$. Denote by $f(x, y, t)$ the left-hand side of this equation. Equations (1) defining the possible envelope of the family read here, after simplification:

$$\begin{cases} x^2 - 2xt^2 + y^2 - 2ty + t^4 + t^2 - 1 = 0 \\ 2tx + y - 2t^3 + t = 0 \end{cases} .$$

Using the **solve** command of the CAS, we obtain:

$$(x, y) = \left(t^2 + \frac{1}{\sqrt{4t^2 + 1}}, \frac{-2t}{\sqrt{4t^2 + 1}} + t \right) \text{ or } (x, y) = \left(t^2 - \frac{1}{\sqrt{4t^2 + 1}}, \frac{2t}{\sqrt{4t^2 + 1}} + t \right),$$

showing the existence of two components. Figure 2a shows the two components and Figure 2b shows the parabola, a few circles and the envelope. These figures have been obtained with parametric plot.

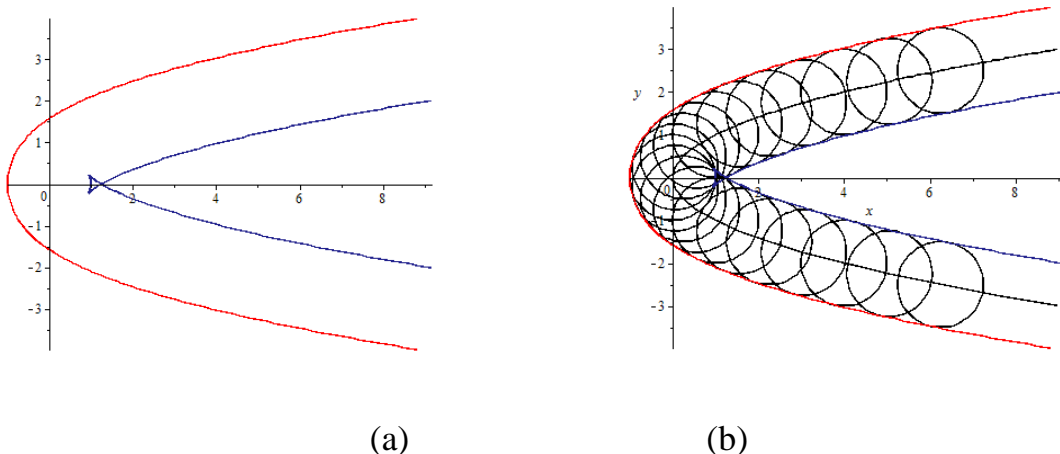


Figure 2: Envelope of a family of unit circles centred on a parabola

Elimination of the parameter in order to obtain an implicit equation requires the usage of a specific algebraic package of the CAS; we present this issue later.

ENVELOPES OF 1-PARAMETER FAMILIES OF SURFACES

The transition to parametrized families of curves and surfaces in 3D space rely on the same techniques and a generalization of Equations (1). New issues have to be addressed: general visualization problems, availability of appropriate features in the software, etc. Let a family of surfaces be given by an equation of the form $F(x, y, z, t) = 0$, where t is a real parameter. If an envelope exists, it is determined by the solutions of the system of equations

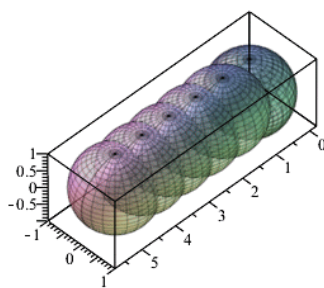
$$(2) \quad \begin{cases} F(x, y, z, t) = 0 \\ \frac{\partial F}{\partial t}(x, y, z, t) = 0 \end{cases}$$

A family of spheres

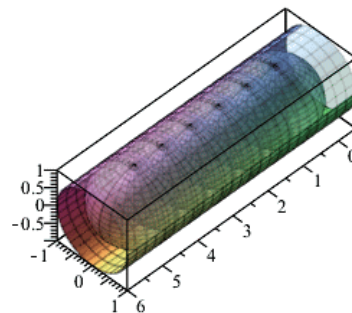
Let be given the family of spheres of radius 1 centred on the x -axis. A general equation for the spheres is $(x-t)^2 + y^2 + z^2 - 1 = 0$. If an envelope exists (Figure 2), it is

defined by the system of equations $\begin{cases} (x-t)^2 + y^2 + z^2 - 1 = 0 \\ 2x = 0 \end{cases}$ whence $y^2 + z^2 = 1$. This is

the equation of the cylinder whose axis is the x -axis and whose basis is the unit circle centred at the origin in the yz -plane (see Figure 3b).



(a) The spheres



(b) The envelope

Figure 3: Envelope of a family of spheres

Intuition and visualization have been enhanced in Figure by using three steps: (a) some spheres of the family, (b) a partial plot of the cylinder and (c) a plot with the cylinder. In more complicated cases, this way of displaying may help, as transparency is not always enough to fully understand the drawing.

A family of planes

A family of planes is given by a general equation $u(t)x + v(t)y + w(t)z + 1 = 0$. In this case, Equations (1) read as follows:

$$(10) \quad \begin{cases} u(t)x + v(t)y + w(t)z - 1 = 0 \\ u'(t)x + v'(t)y + w'(t)z = 0 \end{cases} .$$

This system has either no solution (for example when u, v, w are affine functions of the parameter t) or the solution determines a line in space. This last case means that an envelope exists and is generated by lines, i.e. the envelope is a *ruled surface*. Moreover a) this surface presents a cuspidal edge, i.e. a curve whose points are singular points, b) the tangents to this curve are generators of the envelope, c) this curve is an envelope of the family of generators (Ferreol, 2009).

Consider the family given by the equation $tx + y + t^2z = 2t^3$ where t is a real parameter. The given family has an envelope, displayed in Figure 4 (obtained with Maple). This envelope is determined by the system of equations $\begin{cases} tx + y + t^2z - 2t^3 = 0 \\ x + 2tz - 6t^2 = 0 \end{cases} .$

Solutions of the system are given parametrically by

$$(11) \quad (x, y, z) = (-2st - 6t^2, st^2 + 4t^3, s).$$

Eliminating the parameter yields an implicit equation:

$$(12) \quad x^2z^2 - 4yz^3 - 8x^3 + 36xyz - 108y^2 = 0.$$

The cuspidal edge, called *edge of regression*, appears. Students learning Advanced Calculus know how to find singular points of a surface given parametrically (the first partial derivatives vanish at these points). The mesh chosen by the software to plot the envelope is built on the tangents to the edge of regression.

In order to understand the surface visually, the dynamics of the software are important. Generally, the software enables to rotate the drawing using the mouse, but a slider bar enabling to manipulate directly the plots and to trace the different plots is not available in 3D. In Figure 4, two still pictures have been chosen in order to enable a better understanding, in particular the edge of regression and its tangents. Note that the structure of the envelope as a ruled surface can be revealed by technology.

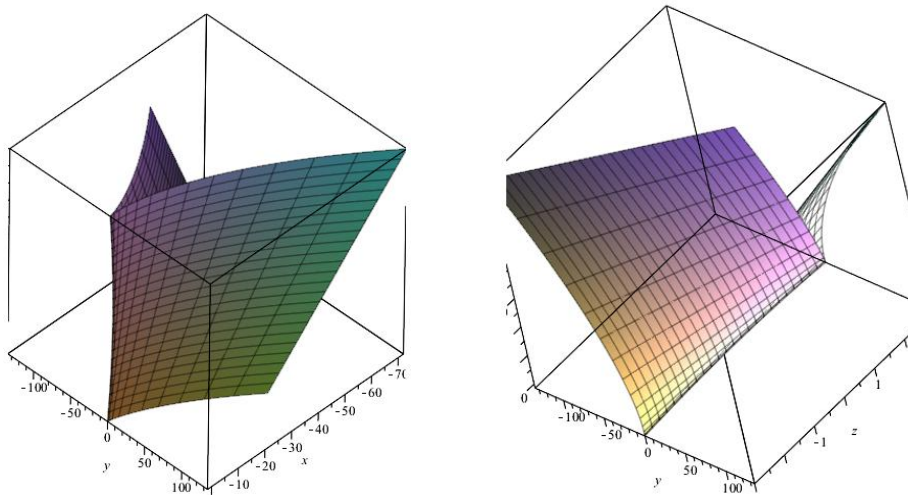


Figure 4: Envelope of a family of planes: two different views

IMPLICITIZATION - ELIMINATION OF THE PARAMETER

Both in 2D and in 3D, the solution of Equations (1) is obtained in parametric form. It happens that an implicit equation is requested. This may be done by elimination of the parameter, an operation often made possible within a specific package.

The algebraic engine to solve polynomial equations and systems of polynomial equations in different CAS is often based on computations of Gröbner bases, a domain belonging to abstract algebra (ring theory). Adams and Loustaunau (1994) explain how this may be viewed as a generalization of the Gauss algorithm for solving systems of linear equations. They provide a nice scaffolding to learn the topic, first leading hand-made computations, and then enabling to understand the algorithms implemented in software. The implementation enables to solve systems of equations, yielding a parametric representation of the solutions. Some packages provide also a tool for implicitization of this parametric representation. The Gröbner bases algorithms have been widely used by Pech (2007) for a new point of view on classical problems in geometry¹.

In the first examples presented here, the parameter appears in polynomial expressions only. Therefore the algorithms in use are based on computations of Gröbner bases. Sometimes, a non-polynomial parametrization can be replaced by a rational one, for example sine and cosine may be given by $\cos t = \frac{2t}{1-t^2}$, $\sin t = \frac{1+t^2}{1-t^2}$.

Then multiplying by the common denominator of the expressions, polynomial

¹ Pech uses a software specifically designed for Gröbner bases computations, the CoCoA System, freely downloadable from cocoa.dima.unige.it

equations are obtained, and from this point the Gröbner bases algorithms may be run. One of the authors taught Gröbner bases to 3rd year pre-service teachers.

If no rational parametrization is available for the family of curves (resp. surfaces), then the same **solve** command of the CAS is used to solve the system of equations defining the envelope. In this case the pattern recognition of the software leads to other algorithms; the output may also be harder to grasp.

WHAT CAN WE LEARN? REFLECTION ON OUR STUDY OF ENVELOPES

CAS literacy for understanding the algorithms

The **solve** command may hide algorithms unknown to the user and the usage of this command may look like a *black box*: the user enters data and receives output, but has no idea of what happened inside. According to the mathematical background of the students and to their CAS literacy, the teacher may consider this as an opportunity to enhance more profound understanding of the mathematics implemented into the CAS command.

Such a situation, with its pedagogical implications, has been described in another mathematical domain by Steiner and Dana-Picard (2004), namely for the computation of either definite or improper integrals. A theorem is presented, which is generally not given in textbooks and is not implemented in CAS. The theorem reads as follows: if the function f is integrable on the interval $[a,b]$ with $a \leq b$, then $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. A proof of this theorem is given by Dana-Picard (2005).

Using this theorem, the computation of the parametric integral $I_r = \int_0^{\pi/2} \frac{1}{1 + \tan^r x} dx$ is an easy task. The Derive software computes the integral I_r for the general parameter r , not only for specific values, and shows that for any value (even a non-integer value), the integral is equal to $\frac{\pi}{4}$.

The fact that Derive is able to compute this integral and other packages are not, was the incentive to understand which mathematical theorem is implemented in the software. The clue is given by the step-by-step feature of Derive.

Technology may be used as a black box, but more important is to catch opportunities to have insight into the mathematics implemented in the software. In the study of envelopes, we meet a similar situation. Using rational parametrization for the curves or the surfaces in the given family, Equations (1) or (2) can be translated into systems of polynomial equations. These systems are solved by the CAS using algorithms based on computations of Gröbner bases of polynomial rings. Students having learnt a first course in Ring Theory may understand these algorithms.

If the parametrization is not rational, the solution of Equations (1) or (2) may involve more complicated expressions, even special functions. This opens a new field of exploration for an undergraduate student.

Interaction between mathematical domains

The interdisciplinary aspect of differential geometry is renewed by technology. Techniques from analysis, abstract algebra and geometry are used within a single session. The title of textbooks for the study of curves and surfaces read as “applications of analysis to geometry” (Cagnac, Ramis, & Commeau, 1966).

Because of this interdisciplinary aspect, the study of envelopes of parametrized families of curves and surfaces provides an opportunity to discover new topics beyond the scope of the regular curriculum, sometimes together with applications to practical situations. Connections between mathematical domains may be discovered and explored. New computational skills with technology may be developed, in particular for the experimental aspect of the work (conjecturing the existence of an envelope and its properties, exploring the existence of cusps, as in Figure 1b). For this, the availability of a slider bar, together with a **Trace on** option, is a central issue.

A great advantage of a CAS is its multi-purpose structure. It enables switches from one register of representation to another. Switching from the algebraic register to the graphic one is standard, but not the reversed direction. GeoGebra has the reversed switching, even when the slider bar is in use. The joint usage of a CAS and an external grapher does not provide such a switch.

There exist freely downloadable graphers², which enable direct manipulation of plots. These graphers are useful, but they can only be used side-by-side with a CAS and they do not collaborate and do not communicate with the CAS. Moreover, the ability to switch between different registers of representation may be improved, within mathematical domains (parametric vs implicit) and with the computer (algebraic, graphical, etc.).

Finally, Duval (2006) pointed out that there is a basic difference between mathematics and the other domains of scientific knowledge. In contrast to phenomena of astronomy, physics, chemistry, biology, etc., mathematics used not to be accessible by perception or by instruments (microscopes, telescopes, and measurement apparatus). The usage of technology has changed the situation: mathematics became accessible to experimentation.

² For example, MathStudio. It can be used online at <http://mathstud.io/welcome/>

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Moments d'exposition des connaissances à l'université : le cas de la notion de limite

Nicolas Grenier-Boley¹, Stéphanie Bridoux² et Christophe Hache³

¹²³LDAR, EA 4434 ; ¹Université de Rouen, France, Nicolas.Grenier-Boley@univ-rouen.fr ; ²Université de Mons, Belgique, stephanie.bridoux@umons.ac.be ;

³Université Paris Diderot, France, christophe.hache@univ-paris-diderot.fr

Dans cette contribution, nous abordons la question générale de l'utilité pour les étudiants des cours magistraux à l'université. Pour cela, nous comparons deux types de cours de première année d'université scientifique consacrés à la première rencontre des étudiants avec la définition formelle de limite de suite ou de fonction : le cours d'un manuel et un cours en amphitheâtre [1] filmé. Pour chaque type de cours, nous caractérisons en termes de proximités-en-acte discursives (Robert & Vandebrouck, 2014) les tentatives de rapprochement de l'enseignant entre nouvelles et anciennes connaissances des étudiants. Il semble que le cours en amphitheâtre soit plus à même de donner des occasions possibles de développer des liens avec les connaissances anciennes des étudiants et d'éclairer ce qui est nouveau, notamment certains éléments logiques de ces définitions, favorisant ainsi des rapprochements propices aux apprentissages.

Mots-clés: cours magistral, limite d'une suite, limite d'une fonction, proximité-en-acte, formalisme.

INTRODUCTION

Nous abordons dans ce texte la question de savoir à quoi peut servir le cours magistral donné dans l'enseignement supérieur, pour les étudiants. Par « cours », nous entendons un moment où l'enseignant expose à l'écrit ou à l'oral des connaissances (savoirs) [2] aux étudiants. Ce questionnement s'intègre dans une recherche plus globale qui se donne comme objectif d'apprécier ce que peut apporter l'exposition des connaissances aux élèves ou aux étudiants au lycée et à l'université (Bridoux et al., 2015). Ces moments de cours ne sont pas ceux durant lesquels les élèves sont les plus actifs. Ils le sont peut-être encore moins dans le supérieur où il y a en général peu d'interaction entre les étudiants et le professeur. Dans l'enseignement supérieur, les étudiants ont également à leur disposition de nombreuses ressources qui leur donnent directement accès au texte du savoir, comme des livres, des photocopies, des vidéos de cours en ligne... et ceci justifie encore plus le questionnement. L'objectif de ce texte est de mettre en regard un cours issu d'un manuel et un cours filmé donné à l'université, en France, de manière à mieux comprendre certaines spécificités du cours « classique » et son impact possible sur les apprentissages des étudiants.

Nous avons ciblé la notion de limite (suite et fonction) qui est une notion clé de l'enseignement de l'Analyse dans la première année du supérieur. Cette notion, qui est également enseignée dans de nombreux pays en première année d'université, est

aussi source de nombreuses difficultés chez les élèves et les étudiants : on en trouvera une synthèse assez complète dans l'article d'Oktaç et Vivier (à paraître). Dans la suite, et sauf mention contraire, nous entendrons par « notion de limite » de suite ou de fonction la « notion mathématisée formelle quantifiée de limite » (en epsilon-N pour les suites et en epsilon-alpha pour les fonctions).

PROBLEMATIQUE

La notion de limite (de suite, de fonction) est une notion formalisatrice, unificatrice et généralisatrice (notion « FUG », Robert, 1998), c'est-à-dire portée par un formalisme nouveau et complexe, généralisateur, unifiant des connaissances antérieures des étudiants dont elle est nécessairement éloignée. Pour cette notion, il paraît donc difficile de trouver un problème initial pour lequel elle serait un outil de résolution optimal et à la suite duquel les étudiants parviendraient à écrire en autonomie la définition formalisée attendue puisque cette notion n'a pas ou peu été travaillée dans l'enseignement secondaire, du moins en France.

Pour ces raisons, ce type de notions est délicat à introduire et nous faisons l'hypothèse que l'enseignant joue un rôle important au moment de leur introduction, notamment pour amener les étudiants à donner du sens à la notion et à l'utiliser correctement. Des travaux antérieurs montrent que l'enseignant peut recourir à certains leviers tels que les commentaires méta (Dorier, 1997), des formalisations intermédiaires (Bridoux, 2011) ou encore des changements de cadres et de registres (Robert, 1983) pour favoriser la prise de sens et l'appropriation de certaines notions FUG (respectivement espace vectoriel, notions de topologie, limite d'une suite).

Dans cette contribution, nous nous focalisons sur les questions suivantes : qu'est-ce qui est explicité par l'enseignant lorsqu'il introduit la définition formalisée de limite ? En particulier, reformule-t-il la définition pour rester « proche » des connaissances des étudiants ? Donne-t-il des commentaires méta sur ce que la définition formelle traduit, la manière dont elle est construite ?

Pour mieux comprendre le rôle des échanges entre l'enseignant et les élèves dans la classe, il nous a semblé pertinent d'étudier un média, ici le manuel, où le professeur est absent. Dans cette situation, le savoir n'est pas présenté oralement et il n'y a pas non plus d'échanges verbaux comme dans un amphi entre l'enseignant et les étudiants. Le lecteur est seul face au livre. Se pose donc la question de la transformation des « connexions » réalisées entre le lecteur et le texte du savoir en connaissances. En ce sens, nous pensons pouvoir dégager de la lecture du manuel des occasions de proximités qui pourraient se créer avec le lecteur. Ces proximités, que nous qualifions de « potentielles » ou « tentées », pourraient être développées par l'enseignant, c'est un élément à étudier dans les cours filmés en amphi. L'étude de manuels que nous poursuivons encore actuellement, nous sert également de référence mathématique et curriculaire pour mieux apprécier ce qui est présenté par l'enseignant dans un cours.

OUTILS THEORIQUES

Un moment de cours est l'occasion de présenter aux étudiants des concepts (du moins perçus comme tels par l'enseignant) avec des mots, des formules,... avec un certain environnement éventuellement (histoire, commentaires, questions,...) mais en sachant que ce ne sont pas encore des concepts pour les étudiants. L'enjeu de ce type de moment est que cela participe aux acquisitions visées. Nous faisons l'hypothèse qu'une des manières développées par les enseignants pour y arriver est de rester aussi « proche » que possible des connaissances des étudiants, notamment grâce à l'activation de connexions entre ces mots, ces formules,... et ce qu'ils savent déjà. On reconnaît ici une « opérationnalisation » de la notion de ZPD chez Vygostky. Pour caractériser cette proximité, Robert et Vandebrouck (2014) introduisent la notion de *proximité-en-acte* pour qualifier ce qui, dans les discours ou dans les décisions des enseignants pendant les déroulements des séances, peut être interprété par les chercheurs comme une tentative de rapprochement avec les élèves. Les proximités-en-acte traduisent ainsi une activité de l'enseignant (discursive ou autre) visant à provoquer et/ou à exploiter une proximité avec les réflexions ou les activités ou les connaissances des élèves. Cette activité de l'enseignant peut plus ou moins être consciente et voulue. Ces proximités peuvent être d'ordre cognitif ou non, et concernent ou non tous les élèves. Dans ce travail, notre attention se porte sur les proximités discursives de nature cognitive.

Nous suggérons qu'il y a, dans les cours, plusieurs grands types de ces proximités-en-acte particulières. Elles se déclinent en relation avec les contenus précis. Elles sont soit le fait de l'enseignant, soit se jouent dans des échanges questions/réponses avec les étudiants. Nous pensons qu'elles peuvent avoir des finalités différentes, mais qu'elles n'atteignent leurs objectifs qu'à certaines conditions, liées à la nature du savoir non contextualisé en jeu, c'est-à-dire le savoir général à retenir et à réutiliser, et notamment à son degré de généralité, et aussi à la nature des exemples et activités proposés ainsi qu'au travail que les élèves ont eu ou ont encore à faire avec ces contenus. C'est leur place par rapport aux moments d'exposition des connaissances et les liens qu'elles supposent entre contextualisé et hors-contexte qui déterminent ces types. Soulignons qu'un passage du cours ou une activité introductive ou illustrative peut activer plusieurs types de proximités successivement.

Les *proximités inductives ou ascendantes* se placent entre ce qu'ont déjà pu faire les étudiants et du nouveau (mots, définitions, propriétés) – il y a généralisation, décontextualisation ou énoncés hors-contexte, soit d'un caractère outil qui donne naissance à un « nouvel » objet ou à un « nouvel » outil, soit directement d'un nouvel objet, définition ou propriété. Ce type de proximité peut par exemple se retrouver dans beaucoup d'ingénieries ou de problèmes développant une dialectique outil-objet (Brousseau, 1998 ; Douady 1987 ; Butlen & Pézard 2003) où en s'appuyant sur le travail des élèves, l'enseignant dégage le savoir en le « sortant » du contexte dans lequel les élèves ont travaillé.

Les *proximités déductives ou descendantes* se placent entre ce qui a été exposé et des exemples ou exercices à faire ensuite avec ou par les élèves (contextualisation, voire re-contextualisation différente des anciennes), par exemple des illustrations faites par l'enseignant. Les élèves peuvent être associés plus ou moins directement. L'enjeu est d'explicitier ce qui est à contextualiser, la manière d'inscrire le cas particulier traité dans l'invariant général, de substituer les données aux bonnes variables (à repérer).

Les *proximités horizontales* ne font pas changer de niveau de discours (restant sur du contextualisé ou du général). Cela peut porter sur le cours en train de se faire, y compris sur les expressions formelles (demandes de compléments, de réponses) ou sur la structuration du cours (« on en est où ? ») ou même sur des discours plus méta, mises en évidence de relations, d'analogies, statuts des éléments en jeu, voire questions sur le savoir concerné, selon ce qui est en jeu. C'est l'enseignant qui les gère, même s'il peut poser des questions.

Des exemples illustrant la présence ou non de ces proximités sont maintenant donnés pour chaque type de média (un manuel et un cours en amphi).

PRESENTATION DES ANALYSES

Méthodologie

Comme nous l'avons précédemment suggéré, l'efficacité des cours dépendrait notamment des occasions et de la qualité de l'activation de ces proximités-en-acte, et donc du discours de l'enseignant, que ce soit par des reprises, des questions, des exemples, des explications, des corrections,... avec des diversités selon les étudiants. Nous faisons donc le choix, ici, d'étudier dans le manuel la présence de proximités « potentielles » de manière à anticiper ce qui, dans le discours de l'enseignant, pourrait donner lieu à des proximités « tentées » par celui-ci. Ces occasions de proximités dépendent aussi des notions étudiées. Compte tenu des spécificités de la notion de limite (de suite et de fonction) décrites précédemment, notamment la distance importante entre les connaissances anciennes des étudiants et la nouvelle notion à introduire ainsi que la structure logique complexe de la définition, nous nous intéressons à la présence ou non de reformulations de la définition avec d'autres mots pour rester « proche » de ce que peuvent entendre les étudiants, associé notamment à la structure logique globale de la (nouvelle) définition. Cela s'appuie sur la prise en compte des connaissances anciennes (ou actuelles) des étudiants. Les premiers exemples proposés par l'enseignant, puis les premiers résultats et leurs démonstrations ont eux aussi été étudiés (Bridoux et al., 2015).

Étude d'un manuel et occasions de proximités

Nous regardons ici le manuel *Mathématiques Tout-en-Un* (Gautier et al., 2007). Ce manuel couvre le programme de mathématiques de la première année des classes préparatoires économiques et commerciales et concerne principalement la filière scientifique. La section qui traite des suites convergentes démarre par la définition suivante et fixe ensuite les notations (p.322):

Une suite réelle $(u_n)_{n \in \mathbb{N}}$ est dite convergente s'il existe un réel l tel que

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \in \mathbb{N} (n \geq n_0 \Rightarrow |u_n - l| \leq \varepsilon)$$

On dit que l est la limite de la suite $(u_n)_{n \in \mathbb{N}}$ ou que la suite $(u_n)_{n \in \mathbb{N}}$ converge vers l .

On note $l = \lim u$ ou $l = \lim_{n \rightarrow +\infty} u_n$. Une suite qui ne converge pas est dite divergente.

La caractérisation choisie pour définir la notion de convergence vers un réel n'utilise que des symboles mathématiques. Elle n'est accompagnée d'aucun commentaire explicatif que ce soit sur les quantifications, sur la présence de la valeur absolue ou encore sur le fait que cette caractérisation traduise effectivement l'idée de « converger vers ». Aucun dessin ou graphique n'est présenté dans cette partie. Telle qu'elle est présentée, il nous semble difficile d'établir ici un rapprochement avec les connaissances « déjà-là » d'un étudiant en première année universitaire qui tenterait de s'approprier seul la notion en lisant ce manuel, d'autant que le choix de définition suggéré dans les programmes du lycée repose sur une caractérisation en termes d'intervalles centrés en la limite contenant tous les termes de la suite à partir d'un certain rang. Celui-ci aurait en effet complètement à sa charge de donner du sens au formalisme utilisé dans la définition.

Deux remarques suivent la définition dans le manuel. La première consiste justement à reformuler la définition précédente en termes d'intervalles (comme le préconisent les programmes du lycée) :

La définition signifie que, pour tout $\varepsilon > 0$, tous les termes de la suite à partir d'un certain rang, ou encore tous les termes de la suite sauf un nombre fini, sont dans l'intervalle $[l - \varepsilon; l + \varepsilon]$.

Cette reformulation est une occasion de donner un certain sens à la définition précédente en lien avec les connaissances antérieures des étudiants. Cependant, aucune explication n'est donnée sur le passage des symboles présents dans la définition aux expressions utilisées ici, comme « tous les termes de la suite à partir d'un certain rang », ni sur le passage de l'inégalité à la notion d'intervalle. De nouveau, il n'y a ici aucune trace d'une tentative de connexion avec les connaissances du lecteur.

La deuxième remarque concerne le fait que la caractérisation en termes d'intervalles est aussi vérifiée pour des intervalles ouverts, avec une tentative de justification :

Comme tout intervalle ouvert contenant l contient un intervalle de la forme $[l - \varepsilon; l + \varepsilon]$, la propriété est vérifiée pour tout intervalle ouvert contenant l : la suite $(u_n)_{n \in \mathbb{N}}$ converge vers l si, pour tout intervalle ouvert I contenant l tous les termes de la suite à partir d'un certain rang sont dans I .

Remarquons que la justification ne porte que sur la transformation des intervalles, la question de savoir si le même entier n_0 convient selon qu'on travaille avec un intervalle ouvert ou un intervalle fermé n'est pas abordée. Elle est de nouveau laissée à la charge du lecteur.

Dans ce manuel, la notion de convergence d'une suite numérique est présentée sans motivation, la nécessité de la nouvelle définition n'apparaît pas. Dans un cours magistral, l'enseignant pourrait commenter, même brièvement, l'introduction d'une nouvelle notion, même si la motivation est justifiée seulement ici par l'évocation du savoir mathématique. En anticipant sur ce qui va suivre, on peut déjà se dire que dans un amphi, en cours magistral, l'enseignant qui introduirait cette définition l'exprimerait sans doute à voix haute tout en l'écrivant. L'étudiant aurait alors au moins l'occasion d'entendre comment le professeur prononce cette phrase mathématique uniquement constituée de symboles, en ajoutant ne serait-ce que des petits mots. Dans un cours effectif, le passage de la définition à la caractérisation en termes d'intervalles pourrait aussi être accompagné de commentaires explicatifs de l'enseignant, notamment de rappels sur la manipulation d'inégalités avec valeur absolue et serait ainsi l'occasion d'une proximité horizontale. Ainsi, ce moment de première rencontre avec la définition formelle de convergence offre sans doute des occasions au moins de proximités horizontales à travers les reformulations qui sont présentées, les symboles utilisés et les notations introduites alors que le manuel ne donne aucun commentaire explicatif sur ces différents passages.

Étude d'un cours magistral

Pour mesurer l'importance du rôle de l'enseignant lors de l'introduction de la notion de limite, nous étudions maintenant une vidéo de cours magistral en première année d'université. Le cours étudié est une séance de 1h30 donnée au second semestre devant 200 étudiants pour lesquels il constitue la première rencontre avec la définition formalisée de la notion de limite de fonction.

Notre objectif est toujours de mettre en valeur les occasions que l'enseignant a ou pas de créer des proximités avec les connaissances des étudiants au cours des éventuelles reformulations qu'il fait de la définition de limite. Pour cela nous analysons tout ce qui, dans l'activité discursive de l'enseignant – orale ou écrite –, peut être vu comme un rapprochement potentiel avec les connaissances des étudiants : en particulier, nous prenons en compte le fait que dans un cours magistral le texte du savoir est accompagné du discours oral de l'enseignant, de ses gestes et d'interactions éventuelles avec certains étudiants qui peuvent lui permettre de donner une certaine cohérence au savoir et de tisser des liens.

Pour analyser ce moment d'enseignement, nous avons procédé à la transcription intégrale des écrits au tableau et du discours de l'enseignant puis nous les avons mis en regard avant de procéder à un découpage du déroulement en phases qui suit celui effectué par l'enseignant sur le plan de son cours. Nous insistons ici seulement sur les phases (1, 3, 6) qui ont trait à la définition de limite de fonction en un point.

Lors de la phase 1, l'enseignant tente de faire une introduction intuitive. Après avoir demandé et obtenu d'un étudiant visiblement avancé la définition de limite en un point, il la reformule de deux façons. Dans un premier temps, il dit « *OK, $f(x)$ doit se rapprocher autant que l'on veut de l mais quand x se rapproche de x_0* ». Dans un

second temps, il traduit sur un exemple graphique l'expression « se rapprocher de » par des flèches et des gestes avant d'aboutir à une seconde reformulation : « $f(x)$ est aussi proche que l'on veut de l si x est suffisamment proche de x_0 ». Cette phase témoigne de la volonté de l'enseignant de tenter de partir des représentations intuitives des étudiants pour en arriver à deux reformulations contenant déjà en germe certains éléments logiques et formels de la définition : ce faisant, il tente de créer a priori un début de proximité horizontale avec la définition de limite. Cependant, cet avis est à tempérer puisque la seconde reformulation reste encore éloignée de la définition formelle quantifiée : non prise en compte de la nature des quantificateurs, inversion possible de leur ordre. En outre, on peut se demander si cette tentative relaie des besoins réels des étudiants.

Lors des phases 3 et 6, l'enseignant en vient aux premières définitions formelles de la notion de limite en un point. Lors de la phase 3, il tente d'écrire cette définition progressivement en respectant une chronologie qui part de la seconde reformulation obtenue lors de la phase 1. Lors de la phase 6, l'enseignant écrit au tableau la définition formelle et une troisième reformulation :

Définition : soit $x_0 \in \mathbb{R}$, f une fonction définie sur un voisinage de x_0 . f a pour limite $l \in \mathbb{R}$ en x_0 si et seulement si $\forall \varepsilon > 0, \exists \alpha > 0$ tel que $\forall x \in Df, |x - x_0| < \alpha \Rightarrow |f(x) - l| < \varepsilon$. Autrement dit, aussi petit que soit ε on peut trouver un intervalle suffisamment petit autour de x_0 sur lequel la distance de $f(x)$ à l est inférieure à ε .

Il fait ensuite oralement le parallèle entre les éléments de la définition formelle et cette reformulation : « D'accord, le « quel que soit » c'est « aussi petit que soit ε », « $\exists \alpha > 0$ tel que $\forall x \in Df, |x - x_0| < \alpha$ » ça c'est un intervalle en fait donc il existe un intervalle suffisamment petit autour de x_0 sur lequel la distance de $f(x)$ à l est inférieure à ε ». Il interroge ensuite les étudiants pour savoir s'il est clair que : $|x - x_0| < \alpha \Leftrightarrow x \in]x_0 - \alpha; x_0 + \alpha[$.

Pendant ces deux phases, le travail de l'enseignant témoigne d'une volonté d'introduire la définition de limite en un point en travaillant à la fois sur son formalisme et sa structure logique par le biais d'un réseau de proximités horizontales appuyées sur de nombreuses reformulations de certaines parties de la définition [3]: en mots (oral et écrit), avec un graphique (oral et écrit), en termes de distance (oral), d'inégalités strictes impliquant des valeurs absolues (écrit), en termes d'intervalles et de voisinages [4] (oral et écrit). En guise de bilan, nous pouvons dire au travers de l'étude des premiers moments en lien avec la définition de limite qu'il y a des différences sensibles entre ce cours magistral et le manuel. Si le caractère FUG de la notion de limite semble interdire à l'enseignant des liens effectifs avec les connaissances antérieures des étudiants lors de l'introduction à la notion de limite de fonction, de nombreux indicateurs au sein de son activité discursive montrent d'abord des essais de proximités horizontales avec ces connaissances. Les proximités potentielles pointées dans la section consacrée au manuel sont effectivement présentes dans ce cours magistral et se trouvent même enrichies. Pour cela,

l'enseignant se base sur un formalisme progressif des diverses reformulations utilisées (mots, graphique, valeur absolue, distance, voisinage) qu'il commente abondamment. Cependant ces tentatives de rapprochement sont à relativiser en ce qu'elles sont pour la plupart orales et non écrites et que la distance dit/écrit chez cet enseignant est grande : on peut donc se demander ce que les étudiants en retiennent. D'autre part, même si l'enseignant prend en charge une part de la structure logique de la définition, il laisse de côté des difficultés au sujet de la nature et de l'ordre d'apparition des quantificateurs, voire même l'inversion partielle de l'ordre suivi dans la définition et l'imbrication des déterminants par rapport à ceux suivis dans l'expression antérieure familière « f se rapproche de l quand x tend vers a ». Par exemple, l'introduction de « $\exists \alpha > 0$ » dans cette phrase logique semble retardée puis forcée par l'enseignant, étant donné qu'aucun élément de ses différentes reformulations ne permet de la motiver réellement. Pour mesurer la portée de son travail sur la définition et sur les proximités repérées, sans doute faudra-t-il attendre les cours ou travaux dirigés suivants.

Dans ce cours, nous n'avons pas repéré de proximités ascendantes : selon nous, cette absence viendrait de ce que les notions introduites ont été identifiées comme FUG. En revanche, notre étude (Bridoux et al., 2015) a montré des essais de proximités descendantes avec les connaissances des étudiants lors d'autres phases. Dans la phase 12 pour laquelle il s'agit de montrer que $\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$, l'enseignant met en valeur la logique locale et le formalisme de la définition [5] qu'il reformule en utilisant des connaissances antérieures des étudiants sur les valeurs absolues et la fonction inverse. Ce faisant, il tente de créer une proximité descendante vis-à-vis de la définition formelle sous-jacente. En fin de démonstration, il revient explicitement mais à nouveau juste oralement sur une méthode générale et féconde pour prouver ce type de résultat : en quelque sorte, il tente de rendre la définition « procédurale », il faut se donner un epsilon et réaliser quelque chose ensuite qui est de l'ordre de la recherche d'une condition suffisante d'une forme souhaitée. En d'autres termes, il a introduit auparavant la définition formelle de limite par le biais de différentes reformulations (proximités horizontales) qu'il utilise et la preuve d'un tel résultat demande une démonstration formelle qu'il tente de rapprocher (proximité descendante) d'une utilisation procédurale de la définition au sens où les étudiants peuvent traduire ce qu'il faut réaliser en termes de procédures associées à des connaissances qu'ils sont supposés maîtriser.

CONCLUSION

Une nouvelle notion, et particulièrement lorsqu'elle s'accompagne d'un nouveau formalisme comme c'est le cas de la notion de limite, embarque avec elle un certain nombre de connaissances, y compris anciennes, et de nouveautés, notamment logiques, – invisibles directement à partir des seuls mots ou des premières formules. Ainsi, l'étude des proximités discursives « potentielles » ou « tentées » permet selon

nous de repérer comment l'enseignant s'appuie, dans les « moments de cours », sur les connaissances antérieures des étudiants et d'éclairer ce qui est visé.

Nous avons montré que, globalement, le manuel étudié ne donne même pas d'occasion de proximités horizontales au moment de l'introduction de la définition de limite alors que le cours en amphitheâtre est sans doute un lieu où il est davantage possible de développer ces proximités. Dans le cours étudié, l'enseignant tente effectivement d'établir des proximités soit horizontales en se basant sur un formalisme progressif des diverses reformulations utilisées (mots, graphique, distances et valeurs absolues, voisinages...), soit descendantes, notamment lorsqu'il traite les premiers exemples d'utilisation des définitions formelles, en mettant en valeur certaines connaissances en logique (quantificateurs, méthodes de raisonnement, condition suffisante...) : en outre, ces reformulations et mises en valeur sont en général accompagnées de commentaires explicatifs de l'enseignant, contrairement au manuel où certains passages étaient laissés à la charge du lecteur.

Cette étude a évidemment ses limites. Il manque à la fois une étude de davantage de cours portant sur les suites et les fonctions, et une mise en relation directe avec des effets sur les élèves – ou au moins des hypothèses à ce sujet. Des recherches plus précises seraient à mener, en faisant varier les contenus étudiés, les élèves, les enseignants, en interrogeant les élèves à la sortie d'un cours, en comparant des productions d'élèves et leur cours...

NOTES

1. L'amphitheâtre est la salle où se donne le cours magistral.
2. La distinction entre connaissances et savoirs est peu présente ici: les savoirs sont associés à ce qui est adopté par les mathématiciens alors que les connaissances relèvent plutôt de ce que les étudiants retiennent et utilisent.
3. Voir Bridoux et al. (2015) pour plus de précisions.
4. Les voisinages (pointés) ont été définis par l'enseignant lors des phases 4 et 5 du déroulement global.
5. La définition de limite finie en $\pm\infty$ a été donnée par l'enseignant lors de la phase précédente et reformulée de manière similaire à celle de limite finie en un réel.

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Addressing large cohorts of first year mathematics students in lectures

Georgia Petropoulou¹, Barbara Jaworski², Despina Potari^{1,3} and Theodossios Zachariades¹

¹Athens University, Greece; ²Loughborough University, UK; ³Linnaeus University, Sweden

We investigate university teaching practices in the context of lectures to identify how students' learning needs are conceptualized and addressed in this context. In this paper we focus on one lecturer's goals for teaching and the associated teaching practices. His teaching to a large cohort of mathematics students in a Calculus course is analysed by using grounded techniques and the Teaching Triad construct (Jaworski, 1994). The analysis suggests that this lecturer's main goal is to help students start their university studies smoothly. In his practice he tries to support students with the advanced mathematical content to be learned and to introduce them to aspects of advanced mathematical thinking. The Triad brings to our insight that Sensitivity to Students could be central in teaching, even in the lecture context.

Keywords: teaching practices, lectures, Sensitivity to Students.

INTRODUCTION

Lectures have been widely criticized as a method of teaching but remain the common element of teaching mathematics at the university level with the potential to contribute significantly to learning (Pritchard, 2015). Despite of being the predominant format of teaching at university level, the lecture format has attracted very few studies possibly because the lecture is taken as a description of how teaching practice looks like at this level (Speer, Smith and Horvath, 2010). However, existed research studies in mathematics education have shown that teaching in lectures may vary and needs studying (e.g. Weber, 2004). Studying teaching practices in lectures, especially those practices that afford learning potentials to students, could be an important source of insights into the processes and practices of university mathematics teaching. Such studying could contribute to researchers' awareness about potentials of university mathematics lecturing. It could also contribute to university lecturers' reflective thinking on their own practices towards the development of enriched learning opportunities for mathematics students.

A general question that we try to address is how teaching at this level, and in the particular context of lectures to large cohorts of students, takes students into account. There is a body of research seeking to characterize elements of teaching practice that takes students into account largely at school level (e.g. Stein, Engle, Smith and Hughes, 2008). However, university students, like the students at the other levels, have also learning needs particularly in the first year of their studies. For example, they struggle with the abstraction and formalism of university mathematics (Nardi 1996) and they experience difficulties related to the secondary-tertiary transition (Pritchard, 2015). It is essential to know how students' learning needs are

conceptualized by university teachers forming goals for teaching and how these conceptualizations are enacted with specific teaching practices. Thus, we seek to address the scarcity of empirical research and to gain better understandings of the mathematics teaching at this level drawing on direct observations of teaching practice. In this paper we investigate the teaching practice of one lecturer who teaches Calculus in a mathematics department. He is a lecturer whom students seem to consider of great help and who has very high rates of students' success in the course's examinations. In particular, here: a) we identify this lecturer's goal-directed teaching practices related to students and b) we interpret the identified teaching practices in the particular context of lectures.

THE THEORETICAL BACKGROUND

Our theoretical perspective of teaching is that it is an activity which: “first, it aims to bring about learning, second, it takes account of where the learner is at, and, third it has regard for the nature of what has to be learnt” (Pring, 2000; p. 23). We employ the language of Activity Theory in relation to teaching actions and goals (Leontiev, 1978). Our perspective towards teaching practice is sociocultural; within this perspective we analyze our data and we interpret our findings in the social setting of a university mathematics amphitheatre and in the culture of mathematics. We agree with Morgan (2014), that the study of a university teacher's conscious goal-directed teaching actions makes more sense when these actions are interpreted in the light of the broader context within which this individual teacher is situated. The students in this context have, like every other student affective, social and cognitive ‘needs’ (this term is elaborated in Hannula, 2006). In fact, they move from the school culture which is organised around the mastery of rather familiar tasks to a culture where the routinization of practices is much more difficult (Artigue, Batanero and Kent, 2007). This ‘move’ could be eased in lectures according to Pritchard (2015) who argues that lecturers can help first year students deal with transition related challenges by paying attention to students' technical difficulties; by demonstrating how mathematicians think and how real mathematics are; and by giving mathematics a human face.

We investigate how students are taken into account in lectures responding to the calls for attention of “how and why teaching happens in certain ways” at university level (Speer Smith & Horvath, 2010). We adopt Speer et al.'s (2010) distinction between instructional activities and teaching practice. According to this distinction the lecture, the context of our study, is an instructional activity while teaching practice concerns what teachers do when they are planning, teaching and reflecting on their lesson. By teaching practice we mean the lecturer's teaching actions (what he does intentionally) and the rationale behind these actions.

We draw on studies that characterized teaching approaches through observations of practice at both secondary and university level. At the university level for example, Weber (2004) studied the teaching of one mathematician in a proof-oriented course and described his actions which influenced the way that his students attempted to learn the material. Mali, Biza and Jaworski (2014) identified characteristics of

university mathematics teaching in the tutorial setting. At the secondary level, Lobato, Clarke and Ellis (2005) examined the processes of teachers' 'telling' and pointed out that telling is instructionally important since students cannot be expected to reinvent entire bodies of mathematics. Identifying levels of 'scaffolding' teaching practices that can enhance mathematics learning, Anghileri (2006) considered "explaining the ideas to be learned" as a central practice even if it is not so responsive to the learner. 'Explaining' is a code that we also use in our analysis. Moreover, Baxter and Williams (2010) addressed the "dilemma of telling" students what they need to know and facilitating their mathematical understandings at the same time while Grandi and Rowland (2013) pointed out the importance of the context in the management of the same dilemma. Drageset (2014) characterized in detail elements of teaching practice such as teachers' comments. The above studies focused on the teaching of one or a very small number of teachers and used qualitative approaches to categorize teaching actions and teaching approaches. Their findings informed the coding process in our attempt to identify the lecturer's actions and practices.

Our research tool in the endeavour to interpret the identified teaching practices in the context of the lectures is Jaworski's (1994) Teaching Triad (TT). TT is an analytic framework that emerged from an ethnographic study at secondary level. Its main goal was to capture essential elements of the complexity of mathematics teaching. Jaworski describes that the Triad consists of three "domains" of activity in which teachers engage: management of learning (ML), sensitivity to students (SS) and mathematical challenge (MC). ML describes how the teacher organizes the classroom learning environment. SS describes teacher's knowledge of students needs. SS has been shown to relate to both affective (e.g., offering praise) (SSA) and cognitive (e.g., inviting explanation) (SSC) domains. MC describes the challenges offered to students to engender mathematical thinking. The above elements are closely interrelated. Jaworski and Potari (2009) further pointed out to a need for a broader appreciation from the side of the teacher of what is possible for the students or how much help they might need to achieve teaching objectives; an appreciation which is not specifically related to particular students. They used the term "social sensitivity" to describe this dimension. The Triad has also been used in studying interactions in university mathematics tutorials (Nardi, Jaworski and Hegedus, 2005) but it has not been used in studying lecturing so far. It is a question for example, what is the potential meaning that Sensitivity to Students could gain in this setting.

METHODOLOGICAL ISSUES: DATA AND ANALYSIS

This paper is a part of an ongoing study with aim to investigate first year's university teaching in Greek mathematics departments. The topic in focus is Calculus, a compulsory first year course, is taught exclusively in a lecture format. Calculus is a topic also taught in high school (age 17). The main difference between Calculus taught in school and Calculus taught in university is in emphasis given to the concepts. In mathematics departments, Calculus courses have a more theoretical focus while in high school the emphasis is on computations and methods. The

participant in the study presented here is a very active research mathematician and an experienced university teacher. In his department, the Calculus course is taught in two parts. The first part includes sequences of real numbers, functions and derivatives. The second part, from which is our data here, is taught during the spring semester and includes series of real numbers, integrals, sequences of functions and power series. The course is taught for 13 weeks, 6 hours per week (4 hours for theory and 2 hours for exercises), to large cohorts of students. While the Calculus course is compulsory, the attendance of the lectures is not. This means that a student can participate in the final exams even if she has not attended the lectures. The course is taught in parallel in three classes from three lecturers. There is an indicative alphabetical allocation of students in these three classes, which is proposed by the department, but, in practice, each student can attend whomever of the lecturers she chooses. Interestingly, the vast majority of students (200+) choose and attend this lecturer's class. Notably, first year Calculus is one of the most difficult courses for the students in this department and many students fail in the final exams. This failure leads students to take their degree in 6.5 years on average instead of 4 years which is the formal duration of studies for a mathematics degree in this department. The lecturer is aware of and interested in this problem. He keeps statistical information about students' success in Calculus courses. The rate of success of students in his course is very high. The course is supported by an accessible to all students web site (e-class) which includes general information about the course, notes and questions from past exams. Data for this lecturer were collected during two years (2012-2013) through lectures' observations (19 hours of lectures); field notes; and interviews right after some lectures discussing issues from teaching (7 interviews, conducted by the first author). In addition to interviews, informal short discussions with students during the time of collecting observational data were also conducted. All lectures and interviews were audio-recorded and transcribed.

In data analysis, grounded approaches (Charmaz, 2006) and the Teaching Triad (Jaworski, 1994) were used. The analysis was conducted in three layers. In the first one, each lecture was divided into episodes typically including a section where one theorem or one proposition was taught. Since the course was proof-based, the episodes included the largest part of the lectures. In each episode, teaching actions were coded. Grounded coding of lecturers' typical teaching actions (mostly observations from the lectures) as well as codes from the literature (e.g. Anghileri, 2006; Drageset, 2014) used to characterize teaching practices. In the second layer of analysis, the rationale of the teaching had been investigated through the analysis of the interviews (also divided in themes). Considering successively and thoroughly the outcomes of each of the first two phases of analysis resulted to the identification of the lecturer's teaching practices (repeated teaching actions and the rationale behind these actions). In the third layer of analysis, the TT was used as an analytical frame to gain insights into the nature of the identified goals and teaching practices. In this way we explored potentials of TT's elements at this level.

RESULTS

The lecturer seems to take into account the broader context into which teaching is situated. In particular, he considers that university newcomers face difficulties in their transition to university. Some of these difficulties relate to the advanced mathematical subject per se while some others relate to the new setting, for instance to the “enormous number of students” in lectures. The lecturer considers that these difficulties may lead some students to fail in the final examination and have a delay in their studies. He values that “it is important for students not to waste time in getting their degree. I know that they get lost in their first year studies”. Taking into account all these difficulties that students may have was judged as an expression of lecturer’s sensitivity to students (SS). As it is emerged from his first interview, his main goal is to help students to overcome these difficulties that is, as he says, to start their university studies “smoothly”.

“We want as many students as possible to start their studies smoothly. Given the enormous number of students, general adaption difficulties of first year students and the difficulty of the subject, ideally the average student could pass all the compulsory courses in a time period of three years instead of two which is expected. I believe it is possible.”

He thinks that a good organization of the course is important for his goal.

“Which are the needs for the course...? Students want to know what exactly is expected for them. Even these organizational things about the intermediate assessment etc... They want to know. This is one need for them: to be organized”.

He also takes into account that there are students who do not attend the lectures perhaps because they have to work in parallel with their studies for financial reasons. This may be an expression of lecturer’s sensitivity to students’ social background, a social sensitivity to students (designated as SSS from now on). SSS was identified also in this lecturer’s actual teaching (exemplified bellow).

“When you believe that students attend the lectures, then you ignore all these students who do not attend and study and... You have also to think about a crowd of people who do not come here so, you have to take this into account.”

To support students who cannot attend the lectures, he assigns to a student to keep notes from the lectures, he corrects these notes once a week and he uploads the notes on e-class. Organizing the course and using the e-class was judged as managerial of students’ learning (ML) but also as an indication of social sensitivity (SSS).

In class, his main goal is carried out with specific teaching practices. The format of teaching is mainly the traditional one. The lecturer stands at the board and does most of the “telling” with rare interaction with students. Interestingly, this rare interaction is an expression of lecturer’s affective sensitivity (SSA) to large cohorts of students:

“In an audience of 200 students, if you discuss with 2 – 3 of them, these probably will be the strongest students and the other will feel bad. ... And finally nothing will remain on

the board... Here, we talk about masses of students and how to achieve a practical result for them. That's the point!"

In elementary or in secondary classrooms, interaction is a key part of current visions of effective mathematics teaching (Stein, 2008). But how realistic could be to expect interaction in a university amphitheatre stuffed with 200 students? The lecturer here cares about students' who "will feel bad" if he interacts with 2-3 of their colleagues. At the same time he points to the effectiveness of teaching for 'masses' of students as practically opposite to interaction probably due to the time the last requires. His perspective could contribute to a discussion of what sensitivity to large cohorts of students could mean and thus could be of help to reassess this element of the Triad.

The analysis of a teaching episode that is typical of this lecturer's teaching follows (Table 1). In this episode, the proposition "if a series converges then the sequence of the series is a null sequence" is taught. The concept of series and the definition of a convergent series had been introduced before. Also, the harmonic series had been given as an example of a non convergent series, still written on the board.

Episode	Teaching practices
[1] L: Now, there is a basic question: Given the sequence a_k , we want to see if we can add them i.e. <i>[he writes]</i> if the series of a_k , converges or not....	Posing a problem (MC)
[2] I shall show a proposition.... They have given me a sequence a_k , ok? If the series of a_k converges, then necessarily the sequence a_k must tend to 0 <i>[he writes]</i> ...	Formulating the proposition (ML)
[3] Of course, because you are advanced now, you will ask if the inverse holds. If the inverse held, just taking a look at a_k and seeing that it tends to 0... would be enough. I would say that the series converges. If it didn't tend to 0, I would say that the series does not converge, and that would be all!	Connecting the proposition with the initial problem (SSC)
[4] However, I have already written an example for you [the harmonic series]This series tends to infinity.	Justifying (SSC)
[5] Attention here! This point [non – convergence of the harmonic series] remains up to the final assessment.	Highlighting (ML)
[6] The quick way [of proof] and I will show the slow way as well. Ok? ...	Evaluating (ML)
[7] I repeat again. We should not forget that a_k are the terms of the sequence. S_n is the sum of the first n terms of the sequence. If a series converges to $s \in \mathbb{R}$, then S_n also converges to s . <i>[he writes]</i>	Repeating (SSC)
[8] Now, I define a second sequence t_n as follows – I am going to write down for you the terms of this sequence. First, I set ... let's say t_1 , to be equal to 0. Then I set the 2 nd term of t_n to be equal to S_1 , 3 rd to S_2 ... Ok? 4 th to S_3 etc. Namely, I set t_1 to be 0 - you can set everything you want. Let t_n to be S_{n-1} ; t_n is S_{n-1} if $n \geq 2$. <i>[he writes]</i>	Explaining a process formally (ML)
[9] I want to define the sequence clearly. The books just write "consider the sequence S_{n-1} ", but what is the S_{n-1} if $n=1$? Is it S_0 ? It is not defined. Ok?	Evaluating (ML)

[10] If you want to define this sequence S_{n-1} , you set a sequence t_n ; you set the first term and then you transfer the terms. Namely the 2 nd term of S_{n-1} is the 1 st term of S_n ; the 3 rd is the 2 nd etc. Fine. ... So, you crack the first term and then you get all the other terms of a sequence which tends to s .	Explaining a process informally (SSC)
[11] So the sequence you get tends to s . So t_n goes to s , too. So the difference of the two sequences goes to 0. [<i>he writes: "Then $t_n \rightarrow s$. So, $S_n - t_n \rightarrow s - s = 0$"</i>]	Inferring (ML)
[12] But what is the difference of the two sequences? For $n \geq 2$, the difference $S_n - t_n$ is the following: S_n is the sum of the first n terms of a_k and t_n is the sum of the first $n-1$ terms. Ok? Because $t_n = S_{n-1}$, so $t_n = a_1 + a_2 + \dots + a_{n-1}$. [<i>he writes</i>]. Thus the difference $S_n - t_n$ is... a_n , it is the only term left.	Explaining a process formally (ML)
[13] I have done all these analytically because the book writes "the difference $S_n - S_{n-1}$ is equal to a_n " and nothing more. This is how we calculate this difference!	Evaluating (ML)
[14] Now, if you don't like this way, you can prove the proposition using ϵ . Second proof – we will not set the sequence t_n . [<i>he continues with the second proof</i>] ...	Giving an alternative method (SSC)
[15] So, you prove a very useful proposition: you keep on hoping to add the terms a_k if they tend to 0. If a_k does not tend to 0 then you directly say "it is over".	Connecting with the initial problem (SSC)
[16] This proposition is very useful as a non - convergence criterion. Ok?	Evaluating (ML)
[17] For example: Someone gives you the sequence $a_i = \frac{k-1}{k+1}$ and asks you if the series of a_k converges or not. .. Then, he does not ask you anything!	Applying (ML)
[18] The first thing you have to do is to look at a_k . You say to him that a_k tends to 1 and not to 0, so the series doesn't converge. Ok? I.e. the first thing you look at is if the k -term inside the series tends to 0. [<i>he writes</i>]	Providing a solution method (ML)
[19] Therefore, the only interesting question about a series can be formulated in the case that the sequence of the series tends to 0. All the other series do not converge!	Refining the initial problem (ML)

Table 1: A teaching episode and its analysis (Translated from Greek)

In the above episode we see an example of how the lecturer attempts to help students to start their studies smoothly in practice. Students were used to more method-oriented teaching practices at school. Here, methods are also provided (e.g. in [18]) but in the context of a more global perspective: a problem is posed at the beginning [1] and it is refined on the basis of what has been proved at the end [19]. The same global perspective is identified in the other episodes, too. Also, technical processes are clarified and explicated [8]; what students need to know is repeated [7]; gaps found in textbooks are fulfilled [9], [13]; and all the explanations are written the board neatly arranged (e.g. in [1], [2] etc). This explaining of the mathematical content clearly and systematically seems to support students' learning. In fact, several students told the observer during the lectures' breaks that they consider this lecturer's way of explaining very "analytic" and the notes they keep from the board during the

lectures very helpful for their studying. Further, the lecturer demonstrates aspects of advanced mathematical thinking [3], [4], [6], [11], [14], [15], [16], [17]. He uses verbal representations to describe a mathematical process [10] and the familiar to students natural language (e.g. in [15] “keep on hoping to add the terms”) to further clarify a process. He uses the pronouns “I” and “me” (e.g. in [2], [8]) giving to mathematics a human face. At the same time, he generates an air of relief (Pritchard, 2015) inside the amphitheatre (e.g. in [14] “if you don’t like this way, you can prove the proposition using...”) and retains an atmosphere of interpersonal conduct with each student using “you” (singular, e.g. in [17]). In his words:

“It is as though I have a particular student in front of me ... right here, and ... you say to him ‘be careful! Here, I try to do this’ – but I do it for all students together.”

In terms of the TT, in this episode, we mainly see lecturer’s management of students’ learning (ML) in his teaching practices. However, this ML stems from Sensitivity to Students (SS). For example, the lecturer takes into account that some students cannot attend the lectures for personal reasons. Taking into account the broader macro context into which the students study is judged as an expression of his social sensitivity to students (SSS); based on his perception on what students may need to study he organizes the course with an e-class (ML) where all the students, even those who cannot attend the lectures, have access. Moreover, accommodating students’ possible difficulties with gaps found in the textbooks is judged as an expression of lecturer’s cognitive sensitivity (SSC) but it is instantiated for example by explaining analytically a new sequence [8] (ML) to bridge that gap. Also, taking into account students’ feelings in the case of interaction with others is judged as an expression of his affective sensitivity (SSA) even if it leads to a teaching closer to ‘showing and telling’ (ML). Thus SS is judged to be central for his practice. Possibly the large number of students that attend his lectures is also an impact of this sensitivity.

CONCLUSION - DISCUSSION

In this paper, we studied a lecturer’s Calculus teaching to a large cohort of first year mathematics students in a mathematics department. This lecturer is an exemplary case in terms of the large number of students who choose to attend his classes. His main goal was to help first year students to ‘start their university studies smoothly’ namely to overcome difficulties they might have in their move from school to university mathematics culture. He carried out this goal by supporting students’ learning of the advanced mathematical content and by introducing students to aspects of advanced mathematical thinking. In particular, teaching practices such as explaining; highlighting subtle aspects; repeating and providing steps and methods were intended to help students to overcome possible difficulties with the mathematical content and thus to support their learning. Teaching practices such as posing a problem and refining it; using alternative methods; relating mathematical ideas; representing and justifying were intended to introduce students into aspects of advanced mathematical thinking. Further, the organization of the course, mainly by

using electronic sources (e-class), was intended to support students' learning, especially with regard to students who could not attend the lectures.

We interpreted the identified teaching practices in terms of TT's Sensitivity to Students and Management of students' Learning in an attempt to substantiate what could be meant by "taking students into account" in teaching large cohorts of first year mathematics students. This interpretation was simultaneously a process of reassessing TT's elements by identifying new possibilities and relations in the particular context. We found that, in a lecture context, Sensitivity to Students can be central in teaching practice and that Management of students' Learning, which is expected to be predominant in this context and is closer to 'showing and telling', can stem from this sensitivity. We also found that, at the particular context, the interaction between the lecturer and the students can be questioned by the lecturer's Affective Sensitivity to large groups of students and that Sensitivity to Students in the social setting of an amphitheatre can receive a social dimension. In the particular case of the lecturer we presented, this dimension of sensitivity seemed to create a positive learning atmosphere in the amphitheatre. Sensitivity to Students in the social domain has possibly a particular meaning in the context of large groups' university lecturing which may deserve a further exploration. In this study, lecturer's Sensitivity to Students was central in his teaching practice and affected the Management of students' Learning.

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Study and research paths in online teacher professional development

Avenilde Romo¹, Berta Barquero², and Marianna Bosch³

¹Instituto Politécnico Nacional, CICATA, Mexico, aromov@ipn.mx;

²Universitat de Barcelona, Spain; ³Univ. Ramon Llull, Barcelona, Spain

We discuss a course on mathematical modelling implemented in a Master's Programme for in-service secondary mathematics teachers using an online modality. The course adopts the SRP-TE methodology proposed by Ruíz-Olarría (2015) and satisfies the principles of online participation suggested by Scott (2010). Our goal is to provide teachers with tools for creating, adapting and managing mathematical modelling activities based on the epistemological, didactic and ecological analysis proposed by the Anthropological Theory of the Didactic.

Keywords: teacher professional development, study and research paths, online education, anthropological theory of the didactic.

AN ONLINE MATHEMATICS TEACHER EDUCATION PROGRAMME

The Latin American society requires better-prepared and more professional mathematics teachers, equipped to adapt their teaching practice to the reforms and updating of new education plans. This situation is especially complex in Mexico, where no established education programme for high school or university mathematics teachers exists, so that engineers, architects and economists can directly become mathematics teachers. All these specialists turned into mathematical instructors feel a pressing need for professional development, for learning theoretical and methodological tools that will enable them to control their classrooms activities, as well as to incorporate information technologies, promote innovative forms of learning, integrate the focus on competences, etc.

While other Latin American countries have specific programs for pre-service teacher education, in-service teachers also feel the need for professional development and sometimes find it hard to obtain. Given this situation, in 2000 the *Instituto Politécnico Nacional* of Mexico created an online distance learning Master of Science Program in Mathematics Education for in-service mathematics teachers. This program, offered by the *Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada* (hereinafter, CICATA-IPN) has spawned teaching units whose main objective is to disseminate the most important results of mathematics education research among the teachers. The online modality has great potential, but at the same time supposes big challenges, as shown by many investigations in general and in mathematics education. For example, Barberá and Fuentes (2012) point out that the design of teaching units must go hand in hand with the use of multimedia tools like videos, forums, *wikis*, and online correction questionnaires, among others. Costa (2009) stressed the importance of second-generation tools (e.g. web 2.0) in helping students develop their own learning paths –autonomously and in community– in

more dynamic settings. Online education appears as an interesting means for teachers to transfer all the potential of information technologies into the classroom. In this respect, Scott (2010) presented empirical evidence of the efficacy of online teacher educational programs and indicates six basic principles that other authors have echoed (Castañeda & Adell, 2011):

1. Adopt a problem-solving orientation
2. Incorporate opportunities for cooperative work by teachers and with experts teachers to work together along with experts
3. Facilitate contact with innovations in knowledge, teaching practice and supporting technologies
4. Prepare teachers to test new teaching strategies and skills
5. Promote resource creation and sharing
6. Enable ongoing, purposeful reflections and discussion

In the context of mathematics teacher education, a review from Goldsmith, Doerr and Lewis of 106 studies on in-service teachers' learning from 1985 to 2008 highlight the need "to know more about the specific content of activities and, for facilitated professional learning experiences, the nature of facilitation" (Goldsmith, Doerr & Lewis, 2014, p. 22). Following these authors, we here present a research on teacher professional development working on the Anthropological Theory of the Didactic (ATD) which consists in an adaptation of the methodology of 'Study and Research Path' for teacher education (SRP-TE), initially developed by Sierra (2006) and Ruiz-Olarría (2015) for pre-service teacher education, to this online modality.

STUDY AND RESEARCH PATHS FOR TEACHER EDUCATION

The ATD stresses the need to base teacher professional development on problematic questions related to their daily practice and to particularly focus on the difficulties related to an insufficient development of the 'mathematics for teaching' (Cirade 2006). This includes problems related to the design and management of teaching and learning processes but especially to *questioning* current school mathematical organisations and the scholar knowledge they refer to. The SRP-TE methodology developed by Ruiz-Olarría (2015) takes as a starting point of the teacher educational process a teaching problematic question Q_0 proposed by the teacher-students or the educators (such as "How to teach [a given topic]?") and propose to approach it in five broad phases. The first one consists in searching for available answers Q_0 in teacher and mathematics education literature; the second one in experiencing one of these answers generated by didactics research as mathematical students (the educators acting as teachers). The third phase proposes to analyse the experience from a didactics perspective, including both an epistemological and didactic modelling of the activities carried out, which includes the introduction of analytic tools elaborated in the field of didactics. Finally, the fourth and fifth phase consist in

the design, implementation (if possible) and development of a class activity adapted to the real school conditions. The SRP-TE methodology has been implemented in pre-service elementary and secondary school teacher education starting with questions related to different mathematical domains (number system, proportionality, algebra, functional modelling) and constitutes an open domain of research.

The SRP-TE methodology seems to have adapted well to the online distance learning course modality through the integration of multimedia tools like forums, videos, the *Moodle* platform, and asynchronous work conditions. Similarly, the fact that participants are in-service mathematics teachers in high schools or universities primarily in Mexico and other Latin American countries (Argentina, Chile, Colombia, Paraguay, Uruguay) offers the possibility to experiment in their classrooms or with small groups of volunteer students. This is why we decided to adapt this methodology to the design of a course initially entitled *Processes of Institutionalization of School Mathematics* (hereinafter PISM).

SRP-TE IN AN ONLINE COURSE

The main objective of the research here presented is to implement a PISM course enabling teachers to connect knowledge produced by research in mathematical education with the difficulties encountered in their teaching practice, adapting the methodology of SRP-TE to in-service teacher mathematical education. The PISM course was designed for implementation in the Masters of Science Program in Mathematical Education at the CICATA-IPN, a 2-year program that offers twelve, 4-week courses in the first three semesters. The final semester is entirely devoted to research concerning the study of a problematic issue related to the teaching practice, starting in the second semester and running parallel to the course.

Our design of the PISM course is based on four main phases: 1) experiencing an innovative teaching proposal as mathematics students; 2) analysing and adapting it for implementation in the classroom through the elaboration of a *lesson plan*; 3) putting it into practice with secondary school students; 4) identifying the institutional constraints revealed by the experimentation and subsequently redesigning the activity. These four phases are associated with the four stages of didactic analysis; *i.e.*, epistemological analysis of the content at stake; *a priori* didactic analysis; experimentation and *in vivo* didactic analysis; *a posteriori* analysis of the didactic ecology (Barquero & Bosch, 2015).

The PISM course was implemented with the CICATA *Moodle* platform and proposed four activities (one for each phase) explained in a document that also contains the program's objectives, calendar and evaluation system (see Figure 1). The online modality requires informing potential students of the entire contents from the beginning to ensure that they have a clear understanding of the nature of the theoretical and experimental activities proposed, and the requirements associated with their development. The problematic questions involved in the teaching

profession that guided the design and development of this course are related to the teaching of mathematical modelling: how to analyse, adapt, develop and integrate a didactic process related to mathematical modelling into teaching practice; how to disseminate a didactic process based on modelling and ensure its long-term institutionalization; which are the difficulties to overcome; what didactic tools can help teachers to make this possible; what new questions emerge in the implementation and how to best deal with them. To approach these questions, we chose a case study on sales forecasting that was carried out and analysed with first year students of mathematics for business and management (Serrano, Bosch & Gascón, 2011). The organisation and results obtained in the four activities of the course are discussed in the following paragraphs.



Figure 1: Description of the PISM course on Moodle

Activity 1. Sales forecast for *Desigual*

Activity 1 presents a problem sales forecasting from *Desigual*. In this activity, students were given 11 variables representing weekly product sales in different stores in Barcelona. For example, variables 9 and 10 show the sales of one-print t-shirts in the stores situated on *Ramblas* and *Rambla de Catalunya*, respectively.

9. Evolution of weekly sales of one-print t-shirts from February 15 2010 in the *Ramblas 136* store (Barcelona):

Week	1	2	3	4	5	6	7	8	9	10	11
One-print t-shirts sold	233	112	118	130	116	151	159	173	175	230	253

10. Evolution of weekly sales of one-print t-shirts from March 1 2010 in the *Rambla de Catalunya 140* store (Barcelona):

Week	1	2	3	4	5	6	7	8	9	10
One-print t-shirts sold	100	101	107	115	125	140	140	164	194	210

In this activity, the group of teachers was divided into teams of three, and each team had to make short- and long-term forecasts for two assigned variables. They had to play the role of mathematicians working at a consulting firm –Pi&SA– hired by *Desigual* to forecast their sales. The instructors took on the role of senior consultants. The aim of adopting these roles is for the teachers to ‘live’ the modelling activity as mathematicians and experience the study of an open question to which they must provide an answer in a written report. In developing this activity we implemented a forum where participants discussed the solutions sharing files with lineal, polynomial, exponential and logarithmic models, and raised questions such as how to determine the short and long terms or how to choose the better model. They were asked to generate two reports, a partial and a final one. The objective of the partial report was to discuss their first forecasting techniques, the models used, and the means of validating them with another team. For example, team 5 presented three models –a quadratic model, an exponential one and a third one based on a modified Gompertz curve (see Figure 2)– the last one being accompanied by this analysis:

In order to increase the fit, we considered the modified Gompertz curve, which in short periods of time enables an adequate simulation of data, and in long time periods it predicts a stationary value for the number of items sold. The disadvantage may be that it predicts a larger number of items sold in the short term than other models.

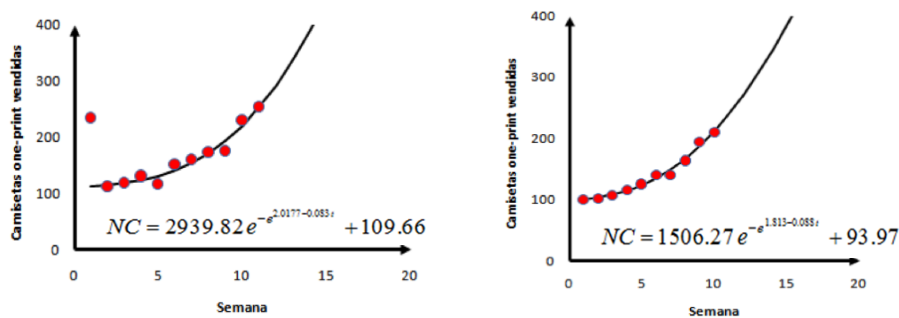


Figure 2: Modified Gompertz curve

Nº semana	Nº camisetas vendidas	Ajuste cuadrático	Ajuste exponencial	Error cuadrático	Error cuadrático al cuadrado	Error exponencial	Error exponencial al cuadrado
x_i	y_i	yc_i	ye_i	$yc_i - y_i$	$(yc_i - y_i)^2$	$ye_i - y_i$	$(ye_i - y_i)^2$
1	100	99,7857	95,9343207	-0,2143	0,04592449	-4,0656793	16,5297482
2	101	101,6428	102,725851	0,6428	0,41319184	1,72585089	2,9785613
3	107	106,5713	109,998178	-0,4287	0,18378369	2,99817755	8,98906863
4	115	114,5712	117,785338	-0,4288	0,18386944	2,78533796	7,75810755
5	125	125,6425	126,123779	0,6425	0,41280625	1,12377902	1,26287929
6	140	139,7852	135,052528	-0,2148	0,04613904	4,94747215	24,4774807
7	140	156,9993	144,613374	16,9993	288,9762	4,61337441	21,2832235
8	164	177,2848	154,851067	13,2848	176,485911	9,14893284	83,7029722
9	194	200,6417	165,813522	6,6417	44,1121789	28,1864776	794,477519
10	210	227,07	177,552049	17,07	291,3849	32,4479513	1052,86954
					802,244905		2014,32911

Figure 3: Sum of quadratic errors

In their final report, the team discussed the same three models but, inspired by the discussion with another team, added an analysis to determine which one was the most adequate. The technique they used was to select only certain data –particularly the first six– and contrast them to those provided by the company. Subsequently, they calculated the sum of the quadratic errors (see Figure 3). Once this table was obtained, they chose the quadratic model for variable 9 and the Gompertz curve for variable 10. The team argued that a larger amount of data would enable more precise forecasting. The other two teams performed activity 1 in the same way, and they all found it difficult to determine the most suitable model. Feedback from the instructors was needed to highlight the fact that the choice of one model over others had to be based on criteria that needed to be made explicit.

Activity 2. Elaborating the lesson plan

Activity 2 consisted in producing a lesson plan (or didactic guide) indicating how to implement the material on the sales forecasts for *Desigual* in a mathematics class. The aim was to let teachers adapt the open activity to the commonly assumed institutional constraints of school mathematics, so that the adaptation obtained acted as the starting point of the analyses of activities 3 and 4. Two phases were proposed to elaborate the lesson plan. The first phase was individual: participants were told to imagine that they decided to implement this activity in the classroom, but that some unforeseen circumstance obliged them to ask a colleague to take their place. They thus had to write a detailed lesson plan explaining how to conduct the activity. In the second phase, the team members discussed the individual lesson plans and had to agree on producing a collective guide. We expected that the discussions would explicitly state the institutional constraints assumed by the teachers in the design of the teaching activity. In fact, these constraints appeared even before the discussions as some teachers adopted the institutional format of the lesson plan (competences, goals, class activity, etc.) and most of them proposed a guided activity, divided into small questions or exercises where students were told on a regular basis what to do in the next step. Some of them also tried to ‘open’ the activity, fairly successfully. For instance, Daniela, from team 5, proposed an implementation that reflected the spirit of the modelling activity that she experienced:

In the next sessions you’ll play the role of a company that must make short- and long-term forecasts for variables that appears in the *Problem Desigual Sales.pdf* file.

Activity:

1) Working individually, read the information from *Desigual* carefully and answer the following questions:

a) What are the variables on which *Desigual* based the information provided? What is the nature of those variables?

b) How is the variation of each variable presented?

2) Find the following information and write it down in your notebook:

a) What garments do *Desigual* stores sell?

b) What is the logo of *Desigual* stores?

c) Where are *Desigual* stores found in Spain... and outside Spain?

3) Work in teams of 3 or 4 (the instructor assigns the participants to teams)

Tasks assigned to each team:

To analyse the series assigned to it and choose the model that best represents – according to its criteria– the evolution of the variable that appears in each series. Participants are specifically asked to record in writing all their thoughts, all the decisions taken and all the agreements and disagreements experienced. They can use an *Excel* or *GeoGebra* spreadsheet to conduct their analyses.

One member of Daniela’s team proposed a lesson plan in which the responsibility of the study process fell entirely on the professor, telling the students everything they needed to know in advance. In the lesson plan of the last team, it was Daniela’s proposal that was adopted, with comments such as “When students solve the activities, the teacher should intervene as little as possible. His or her role should be reduced to accompanying the different teams, attentively following what they do and, if asked, pose questions that let them progress in the activity”. This comment is certainly too general to be really useful...

Activity 3. Implementation of the lesson plan

Activity 3 was designed to implement the lesson plan with groups of students in ordinary classes, or with volunteer groups. Participants were asked to write a report on the implementation describing the experimental conditions, the work expected of students, an *a priori* analysis based on the team-generated lesson plan, and a description of what the students actually did. In general, participants relied on lineal, polynomial and exponential models using primarily two computer programs: *Excel* and *Geogebra*.

The teacher in charge of the implementation included this reflection in her report:

My participation consisted in orienting the work with interventions made basically to focus the team’s activity. In their eagerness to find precise mathematical responses they often chose the model that coincided with the data, refusing to consider any option that strayed from the information provided because they considered that the forecast would be unreliable. I urged them to reflect on the importance of considering models that approach reality, even if they were not so close to the company’s data.

It is interesting to note how the modelling experience from activity 1 led the professors to recognize their nature and then to lead their students to experience it as well. In this case, the teacher insisted on the difficulties for her students to carry out

a good ‘model’ activity where the conditions of the real context could be as important as the degree of precision obtained. Based on the analysis of this experimentation (phase 3 of SRP-TE) the teacher was able to identify institutional constraints associated with the traditional pedagogical paradigm, especially the passive role and lack of responsibility/ or: minimum amount of/ responsibility of the students in their learning process.

With respect to students, the main obstacle I see is that they are not used to this dynamic. They have learned math through a traditional focus where their participation is minimal compared to the professor’s activity. This difficulty can be overcome gradually as they learn to model certain problems in real contexts.

Activity 4. Joint analysis and final review of the didactic guide

The objective of the fourth and last activity was to generate a global analysis of the sales forecasting activity. To this end, each team participated in a forum where they presented and analysed the reports from the individual experimentations. They then reviewed and modified the team-generated lesson plans from activity 2 through an explicitly mathematical and didactic analysis of the activity. To elaborate this new guide, participants were given a theoretical document –produced by the educators– that presented tools for didactic analysis to describe the mathematical work developed and hence stimulate and regulate open modelling and research activities that contrast more traditional modes of transmitting knowledge. Finally, participants were asked to add a special section to compare the previously established conditions for conducting experimentations and the (individual or shared) restrictions that emerged during the implementation. Through interaction with the instructors (videos, feedback, forums), special emphasis was placed on identifying the institutional restrictions that the teachers had not foreseen, but that inevitably emerged during the experimentations.

RESULTS AND CONCLUSIONS

We have described the process of developing a course on mathematical modelling for teacher professional development based on the SRP-TE methodology. The aim of our research is twofold. On the one hand, to develop the design of SRP-TE as proposed by Ruiz-Olarría (2015); on the other hand, to adapt this methodology to online distance-learning courses for in-service teachers. With regard to the SRP-TE methodology, our research confirms the pertinence of promoting the questioning of both the mathematical school organisation of knowledge and the prevailing didactic model related to it. The teachers who carried out an open modelling activity and the way some of them ‘closed’ the initial problem to adapt it to the specific conditions of their teaching institution appear to be a rich *milieu* for the educational process. It helps teachers be aware of the institutional constraints affecting the implementation of new pedagogical and mathematical forms of activity. The theoretical and methodological tools of the ATD are introduced as a way to identify these

constraints and to start thinking about the possible ways to overcome them (even if most of them are related to schools and pedagogical organisations of learning and are beyond the teachers' scope). We can confirm that the SRP-TE methodology can be adapted to in-service teacher education and highlight the productivity of two new devices introduced. The first one is the role-play that defines the four activities of the course (acting as a mathematician, then as a teacher-designer, then as a teacher and finally as a didactic analyst) helping teachers 'detach' themselves from their current role to look at their activity from an outsider's perspective. The second device is the lesson plan that is first prepared from the perspective of an 'ordinary teacher' in total agreement with the school institution, then implemented and finally developed after taking into account the institutional constraints assumed in its first version.

In what concerns the adaptation of the SRP-TE to the online distance-learning pedagogy, it seems to satisfy the six principles stated by Scott (2010) concerning the effectiveness of online teacher education courses. However, the online modality also presents important difficulties. The first one is related to the online mathematical activity carried out in the different phases. The teachers' interactions show a difficulty to 'speak' about the mathematical activity and, more specifically, to include parts of it in the forum. The participants' productions were always added to the forums as attached files and, thus, the exchanges could include some references to the work done, but not the pointing at some of its parts to include more specific comments ("In this calculation I used...", etc.). We do not know if any of the teams overcame this limitation by using Skype. In any case, this first analysis of the implementation of the course reveals that it is necessary to incorporate other information technology tools to improve the teachers' interactions, such as chats and, above all, those that allow direct mathematical work.

Another difficulty found was the organisation of the interactions between teachers and educators, especially with respect to the interventions of the latter in the teachers' forums. Structuring the forums threads and the most effective moments to make contributions is not clear for the educators, especially when the discussions in the teams generate several pages. This difficulty is associated with the didactic tools that should be made available to the teachers and the most appropriate moment to do so. Finally, and coming back to the initial proposal of SRP-TE by Ruiz-Olarría (2015), the PISM course does not incorporate the first phase of searching information and teaching resources available related to mathematical modelling. The way to incorporate them to the course seems difficult given the fact that the participants tend to find it too demanding in terms of work. Obviously, each new edition of the course will be an opportunity to include new developments based on didactic analyses such as the ones here presented.

NOTES

1. The research leading to these results have received funding from the Spanish R & D projects: EDU2012-39312-C03-01 and The "Obra Social La Caixa - Universitat Ramon Llull".

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A commognitive analysis of closed-book examination tasks and lecturers' perspectives

Athina Thoma and Elena Nardi

University of East Anglia, United Kingdom, a.thoma@uea.ac.uk

In this paper, we investigate the discourse of the closed-book examinations using a commognitive perspective. We analyse the routines of the discourse aiming to describe lecturers' expectations about students' engagement with mathematical discourse. Our data consists of the examination tasks of a year one course from a mathematics department in the United Kingdom and interviews with the lecturers of the course. In our analysis we identify the routines of the assessment discourse. The analysis reveals routines focusing largely on: directions given on the how of the mathematical routines; the gradual structure of tasks; students' enculturation; and that the majority of the mathematical routines the students are expected to engage with are substantiation and recall.

Keywords: Closed-book examinations, commognition, routines, assessment discourse.

INTRODUCTION

The dominant method of assessment in mathematics departments in the United Kingdom is the closed-book [1] examinations (Iannone and Simpson, 2011). There is a wealth of frameworks analysing the tasks used in closed-book examinations (e.g. Bergqvist, 2007). The focus of these frameworks is on the range of skills, knowledge and reasoning assessed in the tasks. Taking a discursive approach when analysing the closed-book examinations provides us with a wider understanding of assessment practices: it allows us to characterise the mathematical discourse the students are expected to engage with and provides insight into the lecturers' rationale for the way they pose examination tasks as well as into their expectations from student responses.

This paper focuses on a closed-book examination from a year one course in a mathematics department in the United Kingdom. The course consists of two parts: Sets, Numbers and Proofs taught in the autumn semester; and, Probability taught in the spring semester. The examination tasks are analysed using Sfard's (2008) theory of commognition. We focus on the routines of the discourse of the closed-book examination tasks and we complement the analysis of the tasks with data from interviews with the lecturers who designed the examinations.

In what follows, we first present the commognitive framework and review the literature on lecturers' perspectives on examination tasks. We then introduce the context of our study and analyse two of the examination tasks. We conclude with a discussion of the discursive characteristics of these closed-book examination tasks.

COMMUNICATIVE FRAMEWORK AND RELEVANT LITERATURE

There is a wealth of studies in mathematics education using discursive approaches (Ryve, 2011), with Sfard's (2008) theory of commognition rapidly becoming a quite widely used one (Tabach & Nachlieli, 2016). Sfard (2008) defines discourse as “different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors” (p.93). Mathematics is seen as a discourse and doing mathematics is seen as engaging with mathematical discourse. The rules followed by the participants of the discourse are distinguished in *object-level* rules (“narratives about regularities in the behavior of objects of the discourse” (p. 201)) and *metarules* which “define patterns in the activity of the discursants trying to produce and substantiate object-level narratives” (p.201). Discourses are described in terms of four characteristics: *word use*, *visual mediators*, *endorsed narratives* and *routines*. More specifically, *word use* refers to the use of words specific to the discourse or everyday words (colloquial discourse) which may have different meaning when used in this discourse. In the mathematical discourse *word use* includes mathematical terminology (i.e. integers) and some words with special meaning in mathematics (i.e. disjoint sets). The *visual mediators* are objects and artifacts used to describe objects of the discourse. Some examples of visual mediators in the mathematical discourse are symbols and diagrams (i.e. Venn diagrams). *Endorsed narratives* are “sequence(s) of utterances, spoken or written, framed as a description of objects of relations between objects, or of activities with or by objects” (p.223). In the mathematical discourse an example of an endorsed narrative is a definition or a theorem. Finally, *routines* are a set of metarules describing patterns in the activity of the discursants. Some examples of routines in the mathematical discourse are the routines of proving and defining.

Sfard defines the *how* of a routine as “a set of metarules that determine, or just constrain, the course of the patterned discursive performance” (p.208) and the *when* of a routine as “a collection of metarules that determine, or just constrain those situations in which the discursant would deem this performance as appropriate” (p.208). She categorises routines in: *deeds* (effecting change on objects), *rituals* (“creating and sustaining a bond with other people”, p.241) and *explorations* (producing or substantiating an endorsed narrative, p.224). The exploration routines are further distinguished in: *construction* (resulting in new endorsable narratives); *substantiation* (aiming to decide whether to endorse previously constructed narratives); and *recall* routines (aiming to remember endorsed narratives).

The theory of commognition is being used increasingly in studies in mathematics education at university level (Nardi *et al.*, 2014). Viirman's (2014, 2015) work on the routines of the discourse is of particular relevance to this study. His study analysed in detail the routines of the teaching practices of university mathematics teachers when giving lectures. The participants of the study were teaching year one

mathematics courses in three Swedish universities. The analysis of the discourse of mathematics teaching resulted in a categorisation of the mathematical routines (Viirman, 2014) and the didactical routines (Viirman, 2015). In our study we extend this focus on routines related to assessment, aiming to describe the lecturers' expectations about the students' engagement with mathematical discourse in the context of closed-book examination.

Lecturers' perspectives on mathematical tasks is not a widely researched area. Schoenfeld and Herrmann (1982) investigated the way that students and lecturers classified mathematical tasks and showed that the lecturers sorted the tasks according to the mathematical principles necessary for the solution of the task (e.g. solution by induction). The students however in the same study classified the tasks according to the items described in the problems (e.g. roots of polynomials). In a commognitive sense we could argue that in Schoenfeld and Herrmann's study the lecturers classified the tasks according to the rules of the discourse and the students categorised the tasks according to the objects of the discourse.

The lecturers' perspectives on mathematical tasks is also the focus of Tallman and Carlson (2012). These researchers produced a classification of Calculus examination tasks based on orientation, representation and format of the task. Furthermore, they investigated the lecturers' intended and implemented practices examining their views regarding the focus of the task on a mathematical concept or a procedure and whether the tasks ask students to provide explanation for their answers. The findings from the analysis of the tasks are in stark contrast with the findings from the analysis of the lecturers' questionnaire responses. Specifically, the lecturers claim that they usually require their students to explain their thinking and also believe that the proportion of tasks focusing on demonstrating the understanding of mathematical concepts was the same as the tasks focusing on procedures. However, the results of the task analysis did not substantiate those claims, showing a difference between intended and implemented assessment practices.

Similarly, focusing again on Calculus examinations, Bergqvist (2012) examines the lecturers' views on the reasoning expected from the students during the examinations. The results of the study show that the reasoning required in the exams is imitative and not creative. The lecturers, commenting on their implemented assessment practices, argue that otherwise the examinations would be too difficult for the students and this would lead to low passing rates. Also, reporting on the factors they take into account when designing examination tasks, the lecturers include student proficiency, prior knowledge, course content, perceived degree of difficulty and students' familiarity with the task. Our study, building on Bergqvist's (2012) study of intended assessment practices, seeks to provide insight into these practices by taking a discursive perspective and characterising the discourse of closed-book examinations.

METHODOLOGY

This paper reports part of a larger study which aims to analyse the assessment discourse in mathematical courses at university level using a commognitive perspective. Our focus in this paper is to characterise the routines of the closed-book examination discourse. As described in the previous section an example of a mathematical routine is proving, whereas an example of a routine of the assessment discourse evidenced in an examination task is whether, and if so to what extent, students are provided with hints regarding the *how* of the mathematical routine.

Data collection took place in a well-regarded mathematics department in the United Kingdom during the spring semester of the academic year 2014-2015. The data analysed for the purposes of this paper consists of: the tasks from a closed-book examination of a year one compulsory course; and, semi-structured interviews with the two lecturers, L1 and L2, each teaching one part of the course (Sets, Numbers and Proofs and Probability respectively). The duration of the two interviews was 110 minutes with L1 and 83 minutes with L2. The interview discussion was focused on the examination tasks set for the final examination of that academic year.

This year one course focuses on Sets, Numbers and Proofs in the autumn semester and Probability in the spring semester. The final examination has six tasks with the first two compulsory and the rest optional. One of the compulsory and two of the optional tasks are from the Sets, Numbers and Proofs part of the module and the rest are from the Probability part. The duration of the examination was two hours and the students had to respond to both of the compulsory tasks and choose three from the optional tasks. Non-programmable calculators were permitted and the statistical tables were provided to the students. The total grade of the examination was one hundred marks, with the pass mark set at forty marks. In the following we analyse the two compulsory tasks also with reference to the interviews with the lecturers.

ANALYSIS

The compulsory task from Numbers, Sets and Proofs (figure 1) consists of two sub-tasks. In (i) the students are expected to engage in a substantiation routine as they are

asked to prove that the given equality stands for all natural numbers. They are directed regarding the *how* of

(i) Prove by induction that for all natural numbers n ,

$$2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2.$$

(ii) (a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a . Using this, prove that if d is a divisor of a and d is a divisor of b , then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

Figure 1: Compulsory task on Sets, Numbers and Proofs

the substantiation routine with the phrase “Prove by induction”.

At the start of sub-task (ii) the students are expected to engage with a recall routine. They have to recall an endorsed narrative; the definition of a divisor. Then, they are asked to engage in a substantiation routine, specifically in a direct proof. The students are instructed regarding the *how* of the substantiation routine as illustrated by the phrase “using this, prove that if...then...”. They are asked to use the endorsed narrative they recalled in the previous part three times: twice, to express the relationship between a and d , and b and d ; and, once, to show that d is a divisor of the linear combination of a and b . In part (b), the students are asked to engage in a ritual, using the Euclidean algorithm, in order to compute the common divisor of 123 and 45. They are again instructed regarding the *how* of the routine, as they are explicitly asked to use the endorsed narrative of the Euclidean algorithm. After finding the greatest common divisor they are asked to find the integers for which the equality is true. This is a substantiation routine as they are asked to identify the integers which substantiate the narrative. In this part they are instructed implicitly to use the work they did with the Euclidean algorithm or use another procedure as indicated by the “Hence, (or otherwise)”. During the interview L1, comments on how the phrase “(or otherwise)” allows him to give full marks to students that use a different *how* for this substantiation routine and thus reward also those who take an alternative approach. In doing so, he talks about the creativity in the *how* of the mathematical routines and how this assessment routine allows students to be creative. This creativity in the *how* of the mathematical routines is common practice, this is also highlighted in the way L1 talks about it in the excerpt below:

L1: “in mathematics generally, solving some mathematical problem usually there is not a unique way to do that, and that is a good thing, that is a nice thing about mathematics. So a very bright student might be able to solve some mathematical problem in some, in some completely interesting different way that you don't expect and that sometimes happens and it is really fantastic when it happens and they should get credit for it”

In the final part of the task the students have to combine the endorsed narratives they substantiated in the previous parts in order to decide whether the narrative describing the relationship between 7 and the linear combination of 123 and 45 can be endorsed. The students are expected to engage in a substantiation routine as they are asked to consider whether a linear combination which is a multiple of 3 is equal to 7. Additionally, they are instructed by the phrase “Explain your answer carefully” to provide justification for their response. L1 justifies this choice of words as follows:

L1: “the danger would be that the student would write yes or no and then write nothing else (...) I guess it's to remind them that I want them to explain why they are saying what they are saying”

We also observe that the structure of sub-task (ii) is gradually leading the students in answering (c). It starts with the definition of the divisor and the relationship between

the linear combination of two numbers that have the same divisor. Next, an example of two specific numbers, 123 and 45, is given and the students are expected to find the greatest common divisor. Then, the students are asked to express this number in the form of a linear combination of 123 and 45. Finally, they are asked to examine whether 7 could be expressed in a linear combination of these two numbers. This structure of sub-task (ii) assists the students with engaging in routines that lead to endorsing or rejecting the final narrative. We also note that parts (b) and (c) are dependent on each other: in order to answer (c) the students need to have identified the greatest common divisor of 123 and 45.

Next, we analyse the compulsory task from the Probability part of the course (figure 2). The task starts with a small introduction “In the framework of the modern probability...” situating the students in the historical context set out in L2’s lecture notes (which include an account of Probability as a subject, starting from the 16th century until the modern axiomatic definition of probability given by Kolmogorov). Then the students are expected to engage in a recall routine as they are asked to

remember and state some endorsed narratives. More specifically, they have to recall the definition of

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov’s axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq B$, prove that $P(A) \leq P(B)$.

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p .

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

disjoint events and

Figure 2: Compulsory task in Probability.

Kolmogorov's axioms. In the following excerpt, L2 illustrates how he poses the specific task aiming to guide the students in constructing complete definitions. L2 seems aware of students’ previous engagement with the mathematical discourse and, more specifically, with the routine of defining. Consequently he assists students in providing a complete definition aiming to shift their discursive practices.

L2: “Usually students, especially in the first year they are not used to give proper definitions or if they give the definitions or they write a part of the theorems they don't specify what are the objects they are talking about. Okay? So in other words is like if you are speaking a language but you, you are speaking to somebody which is able to understand you. But what I want the students is to make an effort to try to explain something in a most complete way and that is why I sometimes try to guide them in giving the right definition or writing the right axioms. Because in the third axiom they need to speak about disjoint event, pairwise disjoint event, I've asked them to give first the

definition of disjoint even just to see if they really know what is a disjoint event and they are able to explain it in the axiom.”

The prompt “then use them to demonstrate” illustrates the *how* of the substantiation routines the students have to engage in while proving the two propositions. Both of these substantiation routines are direct proofs. Also, L2 gives to the students another narrative, namely Proposition 2, which can be endorsed without substantiation.

The structure of sub-task (ii) assists the students in solving it. At first, the students are asked to engage in a substantiation routine in order to substantiate that the probability of the intersection is $\frac{1}{4}$. In engaging with this substantiation routine, the students have to recall the multiplication rule or the definition of conditional probability. For sub-task (b) the students are expected to engage in a ritual based on recalling a proposition to calculate $P(B)$. Then, using Kolmogorov's axioms, they need to determine the range of the possible values for p . Finally, in (c) the students are asked to calculate two probabilities. In order to do that they have to engage in a ritual, recalling a proposition and the multiplication rule. First, they are asked to calculate the probability of the complement of B given A , which assists in calculating the probability of A intersecting the complement of B . The parts (a) and (b) are dependent, meaning that the value from (a) is needed to engage with the ritual in (b). However, the lecturer, knowing this and aiming to make all the tasks independent, he decided to phrase the task as a “Show that” task providing the value of the probability and thus assisting the students with achieving the desired response. This is evident in the following excerpt.

L2: “(...) they need to use this value in the second part. So, I don't want them to penalise if they are not able to get the first solution (...) while for part c they don't need this value to solve any other task, any other part of that task. So, I can ask them find. Of course it would be better to ask them to find everything but it's just to again help them in order to do to let's say separate all subsections of exercise so that they can do it. They can do them separately without the need of any other values.”

In this task, we note that the structure of the sub-tasks indicates helpful ways to solve the sub-tasks. In (i) the students are asked first to recall the endorsed narratives which are needed to engage in the substantiation routines in (a) and (b). Similarly, for sub-task (ii) the relation in (a) is needed in order to engage with (b) and (c).

DISCUSSION

In this paper we deployed a commognitive perspective in order to describe the assessment discourse at university level. Sfarid's theory of commognition, alongside Systemic Functional Linguistics (Halliday, 1978), has been the basis of a framework [2] introduced to examine changes in the nature of students' participation in the mathematical discourse over the years 1987-2011. The aim of this framework is to identify changes in the form of the tasks and the expectations from students’

responses focusing on the public examinations (GCSE – General Certificate of Secondary Education) in the United Kingdom taken at age 16. Our analysis is in the spirit of this framework: it focuses on closed-book examination tasks at university level, as well as the interviews with the lecturers who designed the tasks, and aims to offer a characterisation of the assessment discourse and examine the choices the lecturers make when designing the tasks. The analysis highlights discursive routines some aimed at the mathematical discourse the students are asked to engage with and others aimed at assisting the students in their engagement.

Regarding the mathematical routines the students are expected to engage with, we observed engagement with rituals and explorations. The analysis of the explorations in the compulsory and optional tasks showed that the students are asked to engage mostly in substantiation and recall routines, with only two instances being construction routines. Examining further the substantiation routines from the whole examination paper (we note that here we elaborated only on two of the six tasks in the paper), the *how* of these routines can be distinguished as follows: there are sub-tasks using proof by induction, direct proof and proof by counterexample. Examining the recall routines, the students are asked mostly for definitions of mathematical objects. However, we note that in engaging in the rituals and the substantiation routines, the students have to recall endorsed narratives and thus also engage in recall routines. For example, in 1(ii)b from the Sets, Numbers and Proofs task (figure 1), the students have to recall the Euclidean algorithm in order to be able to engage with the ritual.

One assessment routine aimed at assisting students' engagement, is the gradual structure of the task. This step-by-step structure assists the students in gradually recalling or substantiating endorsed narratives needed in subsequent sub-tasks. Furthermore, the sub-tasks are independent or dependent, allowing the students to engage with further tasks or requiring them to find answers that will allow them to proceed further with the task.

Another assessment routine concerns the different degrees of guidance provided to the students in terms of the *how* of the routine or the endorsed narratives that would be needed in order to substantiate a narrative. Similarly, there are instances where students are guided to justify their response – thus assisting the students in providing arguments for the claims they make and helping them to understand this quintessential characteristic of the university mathematics discourse. These directions are aimed at helping students to shift their mathematical discourse towards what is required at university level. Also we observe that, to assist this shift, the lecturers illustrate some of the routines of the new, for the students, discourse. For example, L1 uses the directions for justification and L2 assists students to explain the term disjoint sets in the third axiom and thus provide a complete narrative of Kolmogorov's axioms. With these requests in the examination tasks the lecturers are encouraging students to demonstrate enculturation into the practices of university

mathematics. Sfard (2014) comments on some of the characteristics of the university mathematical discourse as follows when she observes

“first, this discourse’s extreme objectification; secondly, its reliance on rules of endorsement that privilege analytic thinking and leave little space for empirical evidence; and thirdly, the unprecedented level of rigour that is to be attained by following a set of well-defined formal rules.” (p.200).

In our analysis we focus on the lecturers’ expectations regarding students’ engagement with the mathematical discourse. We examine the choices the lecturers made while posing the tasks and how they justified those choices. Their choices seem to be guided by their experience of where the students might face problems. The gradual structure, the guidance in terms of the *how* of the routines and the explicit directions regarding justification aim to assist students in achieving a correct and complete response. These choices though could potentially foster a somewhat limited image of mathematics as they suggest mathematics as a predominantly step by step, highly directed activity. However, we have to consider also that the lecturers take into account the context for which these tasks are designed for. In the examination the students have limited time – and they are stressed – and lecturers calibrate their examination task design accordingly.

Our analysis resonates with the analysis of others in the field such as Schoenfeld and Herrmann (1982) and Bergqvist (2012). However, the results of our analysis slightly digress from the results of Tallman and Carlson (2012) regarding the directions for justification. They noted an inconsistency between the lecturers’ claims regarding justification and the results of the task analysis. Whereas, we observed that the students are explicitly directed regarding the justification of their response. Of course we recognize that this is based on a small set of data, one examination from a year one course and interviews with the two lecturers. More examination tasks and interviews with the lecturers are needed to provide a richer characterisation of assessment routines.

Finally, we note that in this paper we sampled from our analysis of the examination tasks in terms of routines, one of several aspects of discourse that our analysis is focussing on. In addition to other aspects of said discourse, we are now analysing the students’ written responses to the same tasks in order to examine their actual engagement with the mathematical discourse and whether there are differences between what the lecturers intended and what the students actually did.

ENDNOTE

1. Closed-book examination means that the students are not allowed to use textbooks or notes during the examination.
2. An ESRC funded project: “The evolution of school mathematics discourse as seen through the lens of GCSE examinations” (<http://gtr.rcuk.ac.uk/project/D23BF129-B7CC-4BEA-83E2-8EB9D0EDBF17> accessed on 28/04/2016)

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A training concept for supervising self-directed problem-solving in the STEM disciplines

Martin Bracke¹, Detlev Friedewold² and Jörn Schnieder³

¹University of Kaiserslautern, Department of Mathematics, Germany, bracke@mathematik.uni-kl.de, ²Curriculum Institute, Hamburg, Germany,

³University of Lübeck, Institute of Mathematics, Germany

How can school and university tutors support pupils and students in self-directed exploration and open-ended problem-solving? What are helpful didactic methods and pedagogical approaches, and how can teachers and tutors learn which approach they should use? In our poster we first define what we mean by open-ended problem-solving, using an example problem as illustration. We then detail some of the issues that can be encountered by tutors during the supervision of such problem-solving and outline our proposed method for addressing these issues.

Keywords: Mathematical modelling, problem-solving, research tutors, workshop concept.

WHY MODELLING?

Undertaking mathematical modelling tasks, in the form of mini research projects, has been shown to be highly educationally beneficial. Our understanding of a mathematical modelling project is one:

1. where there is no fixed method for obtaining a solution, so that it can be approached by students with different levels of mathematical knowledge and ability; and
2. that is open-ended.

Engaging in applied mathematical exercises promotes the development of skills similar to those used in research, such as self-directed study, problem-solving and explorative ways of thinking (Bracke & Humenberger, 2012; Greefrath et al., 2013). In our case we wish to focus on cases where the problem being set has a relationship one or more of the other STEM (science, technology, engineering and mathematics) disciplines, thereby strengthening both the mathematical understanding of the student as well as their competence in the related fields.

Mathematical modelling is a goal-oriented process, and the success of the activity depends, in part, on whether the predefined goals given by a (fictive) client are achieved or not. In our case the client is the problem setter, in most cases the school teacher or university tutor supervising the project.

ROLE AND RESPONSIBILITIES OF THE MODELLING TUTOR

To date there is no scientific evidence regarding didactics and methods for appropriate advanced training of both research tutors and problem-solving tutors (Kunter et al., 2011; Link, 2011). Our training concept is based on the fundamental assumption that the effectiveness of a tutor depends on his or her attitude towards the learners and students as subjects of their learning. The idea is for the tutor to have confidence in the learners' strengths and potentials; to believe in the inherent resources and strengths of the students' as well as their ability to independently find appropriate help in their environment. The role of the tutors is not only to support the learners on the subject-related objective level but also to perceive and understand the attempts the students make and the steps they take to find solutions.

Our training concept leads the tutor through three separate roles they play over the course of a modelling project:

- in the role of the (fictive) client;
- as a specialist in his or her subject field; and
- as an empathetic and understanding listener.

Through playing these three separate roles we believe that the tutor can support the student with the subject-specific contents and at the emotional and motivational level, while also helping them to find a good strategy and progress with their problem.

We strongly believe that our innovative training concept will give school teachers and university tutors the theoretical background with which to successfully supervise modelling projects and inquiry based learning processes in the STEM subjects.

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MAIN CHALLENGES IN TEACHING/LEARNING OF MATHEMATICS FOR CYBER-SECURITY

¹M.V. Carriegos, ²Noemí DeCastro-García and ³J.F. García-Sierra

¹Universidad de León, Algebra, School of Engineering, Spain, miguel.carriegos@unileon.es ; ²Universidad de León, Didactic of Mathematics, School of Engineering, Spain; ³Universidad de León, RIASC (Research Institute Cyber-Security), Spain.

In this poster the didactics of a specific matter, Cryptography in Master degree studies of Cyber-Security, is studied. Some concrete weakness are found by performing an assessment survey and observation. This weakness is related with some kind of applicationism. Some improvements are proposed by taking into account the design of real-world experiences perhaps by using old-fashioned data.

Keywords: Master Degree, Cryptography, Applicationism

RESEARCH TOPIC

This poster deals with teaching and learning challenges, weakness and possible solutions of matter of Mathematics for Cyber-security at Master degree studies. Some problems are detected and pointed-out. Moreover, some conclusions are given based on results of a concrete questionnaire of assessment. Finally some improvements are proposed.

THEORETICAL FRAMEWORK

Following Barquero, Bosch and Gascón (2014), the “Applicationism” or “Aplicacionismo” is a epistemology based, roughly speaking, on stressing theoretical matters of mathematical concepts devaluating concrete applications and scientific scenarios or real-world situations. Applicationism is detected in the framework of cryptography matter. To be concise, the topic is described from its theoretical roots and thus applications are proposed to the students. However our results show that it is necessary a more active approach to the matter maybe by process of teaching and learning based on projects; or giving an historical approach to the subject.

DESCRIPTION OF THE RESEARCH

The study was carried out by at Master Degree on Cybersecurity of Universidad de León by the Research Institute of Cyber-Security of the University. The participants were all the 20 students of the Master; 18 of them (90%) were Graduate Computer Engineers, 1 of them (5%) was Mathematician and 1 of them (5%) was Graduate Engineer in Electronics.

Our research is based in both observation and an assessment survey involving questions about learning, satisfaction, teachers, facilities and didactical material given in the subject of Cryptography. The objective of the quiz was to assess the quality of Master studies. This quiz was composed following SEEQ standard (Perry & Smart, 1997) and performed for all the 20 students of the Master.

RESEARCH RESULTS

A first overview of the results shows that our students were motivated and skilled and consequently overall scores were high: In fact means of about 4,5/5 in every matter except Cryptography. In this specific matter the score was 4/5; more than 10% less. This deviation was analyzed in a concrete session together with the students and several answers focused on “too much theory”, “too much mathematics”, “too few applications”, “we want to put our hands on concrete problems in real scenarios”, and so on.

IMPLICATIONS AND CONCLUSIONS

Cyber-security is a novel matter in university studies curricula; it is not clear yet its concrete competences, skills or contents. Moreover it is still not defined as a scientific field because its dynamic aspect, hence the contents are unstable. Therefore conclusions of any study must be restricted to the concrete environment of the study, and hence we won't try to generalize our conclusions to the whole list of mathematics matters at university level or even to mathematics for cyber-security. Maybe this subject could be researched by means of a wide data collection across the whole university system when available. We do not have such data at this moment.

However we think we're able to state some conclusions related to the field: detecting weakness and stating acting guidelines to avoid difficulties and improve the studies. In particular it is necessary to create active situations of teaching and learning of Cryptography where students can model real problems. But real data is hard to obtain, an alternative deals with use of old-fashioned data of concrete cyber-security problems like the Enron email Corpus (Kessler, 2010), which is a public database obtained by the authorities after Enron bankrupt; thus this real old-fashioned data show criminal corporative behaviors or structures, data-flows, &c.

ACKNOWLEDGEMENTS

This research was partially supported by Instituto de Ciberseguridad (Spanish Ministry of Industry, Tourism and Communications) under the contract X43 with Universidad de León.

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Consistency, specificity, reification of pedagogical and mathematical discourses in student teacher narratives on the challenges of their school placement experience

Irene Biza and Elena Nardi

University of East Anglia, Norwich, UK, i.biza@uea.ac.uk

Mathematics student teachers often express commendable pedagogical and didactical aspirations the application of which may prove challenging when they enter the profession. The potential mismatch between aspirations and institutional reality is the focus of the study we present in this poster. Our interest is to investigate how the mathematical and pedagogical content addressed in the university-based component of a mathematics teacher education programme interacts with student teachers' experiences in school placements and also with their reflection on these experiences. For example, how do student teachers transform mathematical knowledge into teaching practice? How do they transform commendable pedagogical intention into teaching strategies? What mathematical discourses characterise their reflections on the mathematical content of their teaching? And what pedagogical discourses characterise their reflections on their teaching experiences?

In our on-going research programme on mathematics teacher knowledge and beliefs which this study draws on we invite pre- and in- service teachers to reflect on fictional but realistic and research grounded classroom scenarios that include a mathematical problem and a reaction by one or more students (and a teacher) to this problem (e.g. Nardi, Biza & Zachariades, 2012). Teachers' responses to these tasks and interviews reveal a complex set of considerations (e.g. personal, professional, institutional) that teachers take into account when they determine their actions (ibid.). A recent elaboration of this work suggests a typology of four interrelated characteristics of teachers' responses (Biza, Nardi & Zachariades, submitted):

- *consistency*: how *consistent* student teachers' stated beliefs and their intended (or reflected upon) practice are;
- *specificity*: how *contextualized and specific* student teachers' reflection is to the classroom situations under consideration;
- *reification of pedagogical discourse*: how *reified* the *pedagogical discourses* of student teachers are in order to describe the pedagogical and didactical issues of the classroom situations under consideration; and,
- *reification of mathematical discourse*: how *reified* the *mathematical discourses* of student teachers are in order to identify the underpinning mathematical content of the classroom situations under consideration and the transformation of this mathematical content into their intended (or reflected upon) pedagogical practice.

We use these characteristics in the analysis of how student teachers experience and reflect on their school placement. To this aim we conducted this study in the context of the mathematics Initial Teacher Education (ITE) programme in our institution that has an expectation of a minimum 24 weeks in two schools with a further 12 weeks study at university. The university based component aims to provide students with the academic, professional and personal skills and competencies needed for teaching. In the phase of the study reported in this poster, we invited mathematics student teachers to compose a scenario (approximately one A4 page) drawing on their classroom experience soon after their first block of school placement. We specified that this scenario could be fictional but based on a real situation they had experienced. We suggested that they include: (a) a brief description of the context, (b) a story, possibly in a dialogic format, telling the incident their scenario is about, and (c) their reflection on this scenario (e.g. the reasons they selected it, their concerns, their questions, etc.). The participants were familiar with this type of situation specific scenarios as a trigger for mathematics teacher reflection and professional development through participation in other phases of our research programme earlier in the academic year. However, this was the first time they were asked to produce their own scenarios.

We collected twelve scenarios and grouped them thematically into four groups that covered topics such as student disengagement with mathematics, relational/instrumental understanding in the secondary mathematics classroom, students' (mis)conceptions and the secondary mathematics classroom management. Then the 17 participants discussed the scenarios in small groups first and then in a plenary discussion. All discussions were audio-recorded and transcribed.

We are now analysing the scenarios and subsequent discussions with aforementioned typology. In the poster presentation we will show outcomes from this analysis. We will focus specifically on the mathematical aspects of the scenarios and the ensuing discussions – and the challenges student teachers face when they deal with this content in the secondary mathematics classroom. We will conclude with discussing implications for university-based mathematics teacher education programmes.

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Stimulating and facilitating Norwegian RUME

Simon Goodchild

University of Agder, Norway, simon.goodchild@uia.no

MatRIC – Centre for Research, Innovation and Coordination of Mathematics Teaching is a Norwegian centre of excellence in higher education. The centre focuses on innovation in university level mathematics teaching. We foster research in teaching and learning mathematics, especially but not exclusively in the context of innovative practice. MatRIC also seeks to facilitate the networking of university level mathematics teachers within Norway and with the international community. This poster focuses on MatRIC's activity aimed at the development of research in university mathematics education. The poster sets out strands of activity within MatRIC's research programme, types of inquiry pursued, and categories of purpose.

Keywords: Norway, Research programme, Centre of Excellence.

STRANDS OF ACTIVITY WITHIN MATRIC'S RESEARCH PROGRAMME

There are four strands of activity: i. Researcher education and development (masters, PhD and post-doctoral research); ii. Small research grants; iii. Research embedded within MatRIC's networks for innovation; iv. Externally funded research projects.

i. Researcher education and development spans project and dissertation work in a masters programme in mathematics education and with PhD fellowships and a post-doctoral researcher. To date one master's dissertation has been completed; this explores engineering students' response to interactive software designed to visualize and simulate mathematical concepts. Five PhD fellowships (3 started and 2 to begin August 2016) are focused on teaching and learning undergraduate mathematics in the context of innovative practice, such as use of video and flipped classroom approaches. The post-doctoral researcher contributes to research embedded with the networks (strand iii) and pursues research that builds on his PhD that explored university mathematics teaching from a discourse perspective.

ii. MatRIC makes available small research grants (up to about €6000). These are open to researchers in higher education institutions throughout Norway who propose research into teaching and learning mathematics at university level. Proposals are peer-reviewed before funding is made available. In the last two years 10 such awards have been made, about half of these to researchers outside the University of Agder. Reports from these small scale studies are published on the MatRIC web pages (www.matric.no).

iii. MatRIC organizes and supports networks of university level mathematics teachers who are using and developing innovative approaches in their practice: video, computer aided assessment, digital visualization, and mathematical modelling. There is also a teacher education network. Research is embedded within these

networks, including that conducted by PhD fellows affiliated with different networks. Teachers within networks also pursue forms of developmental research to explore the consequences of the innovative approaches they introduce to teaching and learning mathematics.

iv. External research funds are sought from agencies and programmes that sponsor educational research in Norway. MatRIC's networks and activities provide a base for launching larger scale research project proposals. At the time of writing one project related to mathematical modelling and based on collaboration with Brno University of Technology is supported by EEA Grants (European Economic Area/Norway Grants). Other proposals are under review.

TYPES OF INQUIRY PURSUED AND SUPPORTED BY MATRIC

Research sponsored and pursued by MatRIC is aligned with the so called 'Pasteur's quadrant' in Donald Stokes's depiction of scientific research (Stokes, 1997), that is MatRIC's research is focused on understanding the field better to inform practice within the field. The studies pursued fall into two types. Exploratory studies, which seek better understanding of teaching and learning in different learning environments to inform teaching. Second, as noted above, developmental research which pursues theoretical and principle-driven innovation in practice and researches the process to inform theory and principles in addition to developing practice.

CATEGORIES OF PURPOSE IN MATRIC'S RESEARCH

Developmental research is pursued to inform innovation in teaching and learning, this seeks to explore and develop as outlined above. Developmental research is also pursued in what might be characterized as traditional higher education teaching-learning situations such as lectures and seminars. There is no assumption that 'innovation means better' or 'traditional' approaches cannot be better understood or improved.

Hermeneutic research that seeks to explore and understand better teaching and learning in both innovative and traditional settings is also pursued. The outcomes from this research will be of value in future developmental activity.

THE POSTER

MatRIC's research is presented in the form of an annotated matrix, rows and columns represent types of inquiry and strands within the research programme respectively. Individual projects, represented by elements within the matrix, are colour coded to indicate the category of purpose.

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Using the interactive visualization tool Simreal+ to teach mathematics at university level: an instrumental approach

Said Hadjerrouit and Harald Hoven Gautestad

University of Agder, Norway

This work focuses on mathematics education at the university level using the interactive visualization tool SimReal+. To capture the complexity of teaching and learning processes in a digital environment, this work uses the instrumental approach to explore teachers' orchestrations in classroom. In addition, this study proposes a set of criteria to analyse students' perceptions of teachers' use of SimReal+ to orchestrate the learning process. The findings indicate that SimReal+ is globally a useful tool in mathematics education at the university level.

Keywords: Instrumental orchestration, SimReal+, usefulness, visualization.

THE INTERACTIVE VISUALIZATION TOOL SIMREAL+

SimReal+ is an interactive visualization tool for teaching and learning mathematics. SimReal+ uses a graphic calculator, video lectures, video streaming, and interactive simulations to teach mathematics. The basic idea of SimReal+ is that visualizations are powerful mechanisms for learning mathematics and explaining difficult topics. According to Arcavi (2003), visualization is the ability to use and reflect upon pictures, graphs, animations, images, and diagrams on paper or with digital tools with the purpose of communicating information, thinking about and advancing understandings. There is a huge interest in visualization in mathematics education. However, there is little empirical support for the use of visualizations in mathematics education (Macnab, Phillips, and Norris, 2012).

THEORETICAL FRAMEWORK

This work uses the instrumental approach to analyse teachers' use of SimReal+ to orchestrate the students' learning of mathematics. The instrumental approach to mathematics education is rooted in a number of research studies (Drivjers, 2010; Trouche, 2004; Verillon and Rabardel, 1995). These studies highlight the role of instrumental orchestration as external steering of students' instrumental genesis, that is to say the process by which a tool becomes an instrument while students carry out mathematical tasks. Instrumental orchestration is more than a one-way process directed from the teacher to the students. It includes a socio-cultural aspect as the digital tool serves as a medium between teacher and students in teaching and learning processes. Starting from a set of possible orchestrations as defined by Drivjers (2010), new types of orchestrations may emerge as a consequence of digital tool use. This work also uses technical, mathematical, and pedagogical criteria to analyse the usefulness of SimReal+ in mathematics education at the university level.

RESULTS

A total of 13 different orchestrations have been observed, eight in lectures and five in exercises. The results show that traditional teaching with white board/projector and/without SimReal+ is the most used teacher-orchestration (41.8%), followed by Individual Guide-and-explain (10.60%), Individual technical support (7.55%), Individual Discuss-the-screen (6.97%) and technical-demo (6.31%). Regarding the technical usefulness, the findings show that it is difficult to navigate through the menus of SimReal+. This indicates that the orchestration Technical-demo should play a greater role in teaching. From the mathematical point of view, the majority of the students believed that SimReal+ is useful in learning mathematical topics such as differentiation and integration. Students' responses to interviews also indicate that SimReal+ can provide a deeper understanding of differentiation and integration. From the pedagogical point of view, a majority of the students believed that they are more motivated to use SimReal+ due to the varied methods provided by the tool.

CONCLUSION

The findings indicate that SimReal+ is a potentially useful digital tool for teaching and learning mathematics at the university level. Still, SimReal+ needs to be improved to make the user interface more intuitive and easy to use. The large number of participants ($N=500$) may be sufficient to adequately support the generalization of the results. However, new cycles of experimentations and evaluations of SimReal+ in higher education are needed to ensure more validity and reliability of the results.

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The challenge of being a mathematics teacher

Verónica Martín-Molina

*Department of Didactics of Mathematics, Faculty of Educational Sciences,
Universidad de Sevilla, Seville, Spain, veronicamartin@us.es*

One of the things that teachers at the university level have to learn by experience is how to adapt their lectures and classroom activities depending on their intended audience. For instance, it is understood that teaching mathematics to mathematicians is different than teaching them to pre-service elementary school teachers. However, how to carry out this adaptation or what exactly should be changed is not often discussed. On the basis of my personal experience in the matter, here we use the Atherton model to reflect about my own practice and try to extract some conclusions about the difficulties of the above-mentioned adaptation.

Keywords: teaching at university level, reflective practice model, Atherton model.

INTRODUCTION

There are many challenges that young researchers face when they finish their Ph.D. and want to embark on a career as teachers at the university level. Firstly, doctoral students often receive no mentoring on how to teach. Indeed, most Spanish universities do not have induction or mentoring programs at this level. Secondly, most young teachers have temporary jobs in different universities for the first years before acquiring a permanent job. This impacts negatively on their training as teachers. Moreover, as Zucker (1996) points out, student evaluations are important to keeping a position and being promoted. He states that young assistant professors sometimes put getting good ratings above all, even if that means lowering the level of what they teach. Zucker speaks of his first years of teaching experience, in which he determined that a fundamental problem that many teachers face is that most high school students graduate without knowing how to learn, something that may be remedied by proper orientation at the beginning of the students' first university year.

Finally, mathematics university teachers face another challenge: a widely diverse array of students, from future scientists or engineers to elementary school teachers.

In this context, I try to answer the following question: How do young teachers adapt the content of their classes and their teaching style to the intended audience, beyond following the syllabus and general advice given by colleagues?

THEORETICAL FRAMEWORK

Schön (1983) and Dreyfus and Dreyfus (1986) have proposed reflective practice and skills models, in which future professionals must cultivate either their capacity to reflect in and on action, or their ability to do things automatically. Other authors like Atherton (2013) think that there are problems with both models and proposes to

concentrate on exploring the components of expertise. His model has the form of a pyramid, with the following levels: Competence, Contextualisation, Contingency and Creativity. Therefore, this last model provides us with a framework in which to reflect about the adaptations that a teacher at each level is capable of making.

METHODOLOGY

I have taught different Mathematics subjects in the undergraduate degrees of Mathematics, Chemistry, Agricultural/Industrial Management Engineering and Primary Education. However, my way of teaching was different in each of them. In this work, I focus on three degrees (Chemistry, Engineering and Primary Education) and a common topic (Statistics) and use my lesson plans for all of them as a source of data in order to analyze the adaptations made in each degree.

SOME RESULTS

The first time I had to teach Statistics was to Chemistry first-year students. It was my first year teaching and I had received no training whatsoever in how to do so. Therefore, I simply explained the material that was given to me by the more senior teachers, in order to acquire a competence similar to them. Later, I had to teach Statistics to future engineers. I was given both a syllabus and a textbook that I had to follow, but both my experience and the familiarity with the subject allowed me to reflect on what I was doing, and to discuss with my colleagues what to teach and in what order. In other words, I tried to contextualise my knowledge to the situation.

From last year, I am teaching future elementary school teachers and one of the topics that we teach them is Statistics. However, the perspective is different, since I no longer have to concentrate on the applications of what I teach (what most interests chemists and engineers) but rather on how to teach it. This has forced me to adapt the level of formalism and the goal of my explanations. Perhaps, the adaptation to future events or circumstances in such a way approaches the contingency level.

In general, the reflection about this transition of levels has improved my way of teaching and advanced my acquisition of professional skills.

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TWG5 : Students' practices

Anxiety and Personality Factors Influencing the Completion Rates of Developmental Mathematics Students

Edgar Fuller¹, Jessica Deshler, Marjorie Darrah, Marcela Trujillo and Xiangming Wu

Department of Mathematics, West Virginia University

¹ef@math.wvu.edu

Students lacking core mathematical skills in algebra and arithmetic are traditionally placed into developmental mathematics courses at colleges and universities. These courses attempt to bridge the gap between students' existing skill sets and mastery levels needed to be successful at the level of college algebra, precalculus and calculus. In this paper we describe the interaction of anxiety and personality traits with course content completion for 404 students enrolled in a developmental mathematics course at a large research university in the United States.

Keywords: mathematics anxiety, personality, developmental, success.

INTRODUCTION

Students in developmental mathematics courses typically have major deficits in their ability to complete foundational arithmetic and algebraic manipulations. These deficits can lead to anxiety associated with mathematics courses, mathematics exams and/or numerical operations, creating a complex set of interactions between their underlying mathematical abilities, their existing personality traits, and the levels of anxiety they experience during a course. Connections between cognitive states and mathematical performance have been studied previously (Schoenfeld, 1983), and others have identified relationships between anxiety and performance (Ma and Xu, 2004) at the secondary level. In this project we seek to understand these relationships at the undergraduate level and to develop heuristics that can be used as indicators of detrimental internal states for students with the ultimate goal of building interventions that are adapted to different combinations of student abilities, anxiety levels and personality factors. In the fall semester of 2015 a series of surveys were administered to a cohort of students enrolled in a self-paced developmental mathematics course at our institution, a large research university in the United States, in an attempt to assess their anxiety levels, personality traits and career aspirations. In addition, data representing students' patterns of task completion were collected weekly. This collection of information along with the existing demographics of the student population have been analysed in an effort to identify patterns that facilitate or inhibit success. In the current work, we present several of these analyses and attempt to find indicators in the data that would be useful for future interventions.

The ultimate goal of the larger research project is to develop profiles of students intending to major in science, technology, engineering and mathematics (STEM) fields, are underprepared in mathematics yet have the potential to be successful. We will use longitudinal data collected over two years of following a STEM-intending cohort of students who begin in a developmental mathematics course.

At our institution, students enter mathematics courses based on either existing standardized exam scores or via a locally administered placement exam. Students not meeting minimum requirements for entry to a college level algebra course must enrol in a remedial course intended to develop their arithmetic and basic algebra skills. In this course students progress at their own pace through online modules facilitated by an instructor. In a given year, approximately 30% of 5,000 incoming first-year students place into this course. Of the 823 students enrolled in the course in fall 2015, 404 consented to participate in this study and responded to the surveys administered by the research team.

INSTRUMENTS

Anxiety related to mathematics has been studied extensively due to its impact on student performance. Richardson and Suinn (1972) define mathematics anxiety as ‘feelings of tension’ surrounding different aspects of mathematical thought. They developed the Mathematics Anxiety Rating Scale (MARS) to measure levels of anxiety in individuals by rating 98 items that present behavioural situations. In this way, MARS can gauge the level of mathematics related anxiety present in the emotional state of a student. The items focus on a self-evaluation of respondents’ relationships with mathematical content and activities such as performing basic calculations, taking exams, and using a mathematics textbook. The instrument was validated for internal consistency and for test-retest reliability (Richardson and Suinn, 1972) on a population of university students in Missouri.

We collected data during the fall 2015 semester using the Abbreviated Mathematics Anxiety Rating Scale (AMARS) instrument (Alexander & Martray, 1989). This survey is an abbreviated version of the MARS instrument and consists of 25 items split into three categories focusing on Mathematics Exam Anxiety (EA, 15 items), Numerical Task Anxiety (NTA, 5 items), and Mathematics Course Anxiety (CA, 5 items). This instrument shows high internal consistency and has been shown to reliably measure anxiety levels among students (Peterson, Casillas & Robbins, 2006).

Hembree (1990) and Ashcraft & Krause (2007) have further observed that anxiety interacts strongly with mathematics performance by inhibiting working memory and creating a cycle of difficulty for students. Such a process is particularly damaging to the capabilities of students who are weak mathematically to start with (e.g. remedial students). We seek to monitor anxiety levels in a way that provides insight into the interaction of anxiety with course completion and persistence in STEM majors, especially those who begin their study in developmental coursework.

In order to understand the way that anxiety interacts with underlying student predispositions, we collected data to measure baseline personality traits in this population. To do this we utilized the Integrative Big-Five Trait Taxonomy and the corresponding Big Five Inventory survey (BFI) developed by John and Srivastava (1999). This instrument and its underlying framework identify five core groups of personality traits as a way of characterizing individual behaviour. This work is

founded on the notion that personality traits manifest as verbalizations, and that the five core groups identified in their work can be measured by a collection of test items that identify the levels of these traits present in a person's behaviour by their response to linguistic prompts. Survey respondents rate items using a five-point Likert scale from 'Disagree Strongly' to 'Agree Strongly' over a range of 44 statements such as 'I am someone who is talkative' or 'I am someone who tends to be lazy.' The responses are averaged across a defined group of questions to give a composite score in one of five personality areas including Extraversion (EV), Agreeability (AG), Conscientiousness (CS), Neuroticism (NR) and Openness (OP). This instrument has been validated on multiple populations (John, Naumann & Soto, 2008; John and Srivastava, 1999; Fossati, Borroni, Marchione & Maffei, 2011) and has high reliability for reporting these underlying personality trait levels (John and Srivastava, 1999)

SURVEY AND DEMOGRAPHIC DATA AND RESULTS

The values shown in the second column of Table 1 indicate the average levels of anxiety found in an average student in our population. These anxiety measures provide a way to compare the level of apprehension a student possesses to the average score found in a given group, and to look at the way anxiety levels compare to student performance. Personality trait measures given by the BFI for our cohort are given in the fourth column. For comparison, we note the values for a population of 468 college students surveyed in Peterson, Casillas & Robbins (2006) at other colleges and universities in the last column of Table 1. As noted in John, Naumann & Soto (2008) these values represent the relative level of these personality traits in our population. Several studies have compiled data related to the personality traits found in other populations but this group of students at a university are relevant to the current work.

We examined the breakdown of all these scores along a number of demographic categories including gender, ethnicity, high school grade point average (GPA) and major. Majors of students in the study were designated as STEM or non-STEM using the list of STEM fields maintained by the National Science Foundation cross-listed with major codes at the university and developed from student records. These data are presented in Tables 2 - 4.

The outputs from these analyses indicate that female students exhibit higher levels of Exam Anxiety. In addition, non-STEM intending students tended to maintain higher levels of anxiety. Neither of these was found to differ significantly between populations, however. It was observed and noted below that anxiety and Conscientiousness interact significantly with student success.

Anxiety levels and personality traits showed no discernable association to GPA and also did not differ between racial groups except those for Native American and

Pacific Islander groups. Further analysis is needed to see how these demographic factors interact with anxiety and personality traits.

Anxiety Factor	Average Anxiety N=404	Personality Trait	Average Score	2006 Study Average Score
Exam	52.11	EV	3.35	3.50
Task	10.41	AG	3.93	4.10
Course	11.44	CS	3.64	3.92
		NR	3.09	2.81
		OP	3.43	3.68

Table 1: Averages for anxiety levels and personality traits

	EA	NTA	CA	EV	AG	CS	NR	OP
Total (N = 404)	52.11	10.41	11.44	3.35	3.93	3.64	3.09	3.43
Non-STEM (N = 320)	52.55	10.32	11.60	3.39	3.93	3.65	3.10	3.43
STEM (N = 84)	50.44	10.74	10.83	3.20	3.93	3.61	2.04	3.43
F (N = 253)	54.18	10.34	11.36	3.37	4.01	3.70	3.27	3.40
M (N = 151)	48.64	10.53	11.56	3.33	3.80	3.54	2.78	3.48

Table 2: Average anxiety levels and personality traits for total respondent population, by major and gender

COURSE COMPLETION DATA AND RESULTS

Students complete this course by passing online exams administered in the Pearson MyMathLab system. They must demonstrate 80% mastery on each of seven chapter

exams, and then 70% mastery on a comprehensive final in order to pass the course. The number of students of the 404 survey respondents who had completed each of the seven exams and the final at the beginning of weeks five, six, eleven and twelve are shown in Table 5. Week five was the earliest available data. We chose week six to provide a one-week snapshot of student behaviour, and this observed the data again five weeks later for another set of indicators. Finally, we observed student performance again in week twelve to capture the same one-week change. In subsequent work the rate of progress over one week will be compared at the two different times to develop a 'rate of completion' model for the students. In the current work we focus on correlations with anxiety and personality traits.

High School GPA	EA	NA	CA	Total Anxiety	EV	AG	CS	NR	OP
<2.0	32.67	5.67	8.00	46.33	3.17	3.96	4.22	2.54	3.77
2.0-2.49	51.94	11.00	11.59	74.53	3.20	3.72	3.36	2.91	3.29
2.5-2.99	50.58	10.54	11.33	72.45	3.38	3.83	3.53	3.02	3.45
3.0-3.49	54.13	10.75	12.03	76.91	3.38	3.99	3.69	3.12	3.40
3.5-3.79	52.11	10.68	11.58	74.38	3.45	3.87	3.64	3.10	3.43
>3.8	51.64	9.56	10.39	71.58	3.14	4.15	3.81	3.18	3.36
No data	49.94	10.44	11.25	71.63	3.21	4.14	3.86	2.98	3.76

Table 3: Average anxiety levels and personality traits by high school GPA

Race	EA	NA	CA	Total Anxiety	EV	AG	CS	NV	OP
African American	51.42	12.09	13.42	76.94	3.20	3.85	3.73	2.90	3.46
Native American	58.88	12.75	13.88	85.50	3.23	3.60	3.41	3.45	3.26
Asian	52.45	13.27	13.82	79.55	3.25	3.88	3.44	3.07	3.53
Pacific Islander	48.00	9.50	16.00	73.50	3.88	3.44	3.33	3.75	3.90
Hispanic	54.54	11.54	12.46	78.54	3.57	4.01	3.68	2.84	3.58
No data	49.33	14.50	12.00	75.83	2.85	4.18	3.38	3.03	3.28
White	52.01	10.15	11.15	73.31	3.38	3.94	3.65	3.10	3.42

Table 4: Average anxiety levels and personality traits by ethnicity

Week	Exam							
	1	2	3	4	5	6	7	Final
Five	250	89	57	24	7	1	1	0
Six	288	140	89	36	12	2	1	0
Eleven	343	316	292	149	59	21	12	10
Twelve	346	330	325	265	160	68	33	23

Table 5: Number of students in completing exams by weeks

A graphic representation of these data shows that the overall trend of the population is a logistic response in the sense that over time, a small fraction of the students will complete the later exams followed slowly by the main bulk of the population. Some students will remain ‘stuck’ on earlier exams.

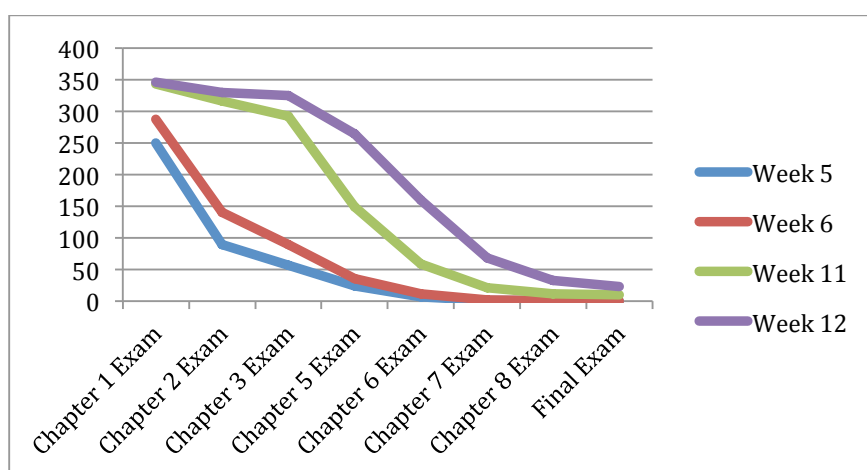


Figure 1: Number of exams completed by students at beginning and end of semester

Anxiety levels, personality traits and other demographic variables provide a method for analysing progress through these exams. First, we give the average number of exams completed at four points for the respondent student population and then for gender and STEM major subsets. In week 5, $N=390$ students were observed to be active in the course. By week 12, this number had dropped to $N=382$.

	Week 5	Week 6	Week 11	Week 12
Total (N=390)	1.09	1.44	3.06	3.97
Non-STEM (N=307)	1.09	1.44	2.99	3.87
STEM (N=83)	1.06	1.47	3.32	4.31
Female (N=244)	1.09	1.43	3.11	4.01
Male (N=146)	1.09	1.46	2.96	3.89

Table 6: Average number of completed exams by major and gender

Splitting the same data for STEM vs. non-STEM majors shows that as expected, STEM intending students are more motivated and attentive to the completion of course requirements. Interestingly, however, this distinction does not become apparent until week 11.

PERSONALITY, ANXIETY AND COMPLETION RATES

Completion rates can also be compared to anxiety levels. We first observe that average anxiety levels correlate negatively with completion. Specifically, for the $N=382$ students who remained active in the course in week twelve, we present data relating anxiety measures and exam completion for each of the eight exams in the semester as noted in Table 7.

Exams Completed	EA	NA	CA
0	54.71	11.47	12.95
1	55.14	11.77	13.17
2	49.93	10.2	11.13
3	51.85	9.77	10.71
4	50.74	11.16	11.28
5	52.46	9.95	11.41
6	51.29	6.43	10.43
7	34	8	7
8	48.56	9.22	12
Total (N=382)	51.96	10.42	11.41

Table 7: Average anxiety levels by number of exams completed in week 12

Bivariate Pearson correlations measures were computed for all anxiety measures and personality trait levels against completion rates in weeks 5, 6, 11, and 12. Table 8 shows significant correlations of Conscientiousness with week eleven and week twelve completion rates, as well as between week twelve completion and both exam and course anxiety.

Linear regression models were constructed for exam completion in the twelfth week (EC_{12}) against both Conscientiousness (CS) and Exam Anxiety (EA). The functions representing these models were

$$EC_{12} = 2.605 + .374CS$$

$$EC_{12} = 4.812 - .016EA$$

Using these functions and hypothesizing that a student would need to have completed four out of eight exams by week twelve we see that students below $CS=3.73$ and above $EA=50.75$ are at risk of not completing the course. These values are close to

the population averages of EA=52.12 and CS=3.62 and provide guidance for identifying students at risk for failure.

		EA	CA	Total Anxiety	CS
Week 11	Pearson				
	Correlation	-0.098	-0.093	-.112*	.106*
	Sig. (2-tailed)	0.056	0.069	0.029	0.038
Week 12	Pearson				
	Correlation	-.103*	-.122*	-.121*	.111*
	Sig. (2-tailed)	0.044	0.017	0.018	0.03
** Correlation is significant at the 0.01 level (2-tailed).					
* Correlation is significant at the 0.05 level (2-tailed).					

Table 8: Completion correlations with anxiety and personality factors

CONCLUSIONS AND DISCUSSION

As a first step to try to predict who will be successful in the self-paced Developmental Mathematics Course, we have analysed two possible contributing factors from the AMARS and the BFI measures. We see that anxiety, especially exam anxiety, seems to be a contributing factor to explain why students do not successfully complete the course requirements. It seems that students with higher exam anxiety may be avoiding taking exams and may run out of time at the end of the semester. It is also no real surprise that conscientiousness plays an important role in successful completion of the requirements for the course. We see that students need a higher than average conscientiousness score to be able to successfully complete the course requirements. These two factors will be further examined to determine if it is possible to provide interventions at the beginning of the semester to effect changes for students who indicate on these two measures that they are below cut-off values. Approximate cut values can be obtained using either t-test analyses or stronger correlation matrices for both anxiety levels and personality trait scores. These cut values will be the focus of future work and interventions. In particular, an analysis of these same data within the STEM intending subgroup would allow for interventions that may support the persistence of STEM identifying students.

Although, the self-paced course structure fits this type of Developmental Mathematics Course very well because students come in with such varied backgrounds, it is clear that the lack of deadlines and lack of strict oversight may be a problem with students who have higher exam anxiety or who have a lower conscientiousness score. In order to ensure success of students that exhibit these indicators, we need to provide interventions (e.g. peer coaches, suggested completion schedules) from the beginning that assist them in overcoming these barriers to success.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the help of their colleagues Betsy Kuhn and Mary Beth Angeline during the administration of these surveys as well as the assistance of the students and instructors of the course. Partial support for this work was provided by the National Science Foundation's Improving Undergraduate STEM Education (IUSE) program under Award No. 1544011. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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Students' work in mathematics and resources mediation at entry to university

Ghislaine Gueudet¹ and Birgit Pepin²

¹CREAD, University of Brest, France, ghislaine.gueudet@espe-bretagne.fr,

²Technische Universiteit Eindhoven, The Netherlands

In this paper we study the use of resources by students in their mathematical work at the beginning of university. The institution offers a variety of resources: lecture notes, books, exercises, websites, to name but a few. Leaning on a theoretical framework by Rabardel, we argue that the university teachers expected an epistemic mediation of these resources, as they supported student learning of (higher level) mathematics. However, analysing two case studies (one in the UK and one in France) we observe that the actual use of resources by novice mathematics students corresponded to a pragmatic mediation, as they searched for worked examples and “reproduction techniques”, all very similar to their use of resources at secondary school.

Keywords: epistemic mediation, pragmatic mediation, resources, secondary-tertiary transition

STUDENTS' WORK WITH RESOURCES AT THE SECONDARY-TERTIARY TRANSITION

The secondary-tertiary transition is a moment of important institutional evolutions: e.g. in terms of the mathematics taught; the mathematical practices expected from students; the institutional support for student learning (Gueudet 2008; Pepin 2014). In this article we claim that the resources available for students' work with mathematics, and the expected use of these resources (expected by the institution) significantly change from secondary school to university mathematics education. When entering university, it is assumed that students develop new ways to use resources, mathematical texts in particular. Previous research (e.g. Rezat 2010) contends that at secondary school even grade 12 students used their mathematics textbook mostly to search for worked examples, in order to learn rules and how to apply these rules to tasks similar to the tasks they worked on with their teachers in class. The same holds true at the beginning of university: Lithner (2003) shows that students' homework with textbooks was mostly oriented towards solving exercises; and that students searched for surface similarities between exercises, in order to choose a procedure. Our hypothesis is that this use of textbooks and other mathematical curriculum material by university students derives from similar practices at secondary school, in particular their use of textbooks. Hence, our research question is the following:

- which use of resources is expected by university mathematics teachers, and how does this compare with the actual use of resources by students?

In the next section we explain the theoretical frame we used, Rabardel's (2002/1995) instrumental approach. Subsequently, we present a case study from the UK TransMaths project¹, considering students' work in mathematics at a general level; and a case study in France concerning students' work in the area of Number Theory.

RESOURCE MEDIATION: THEORETICAL FRAME AND METHODS

In terms of theoretical frame we refer to the instrumental approach by Rabardel (2002/1995). He distinguishes between an artefact - produced by humans, for an aim of human activity; and an instrument - developed by a subject along his/her goal-directed activity with this artefact. The instrument incorporates the artefact (or parts of it) and a scheme of use for this artefact (Vergnaud 1998). Following Vygotsky (1978), Rabardel and Bourmaud (2003) consider that the goal-directed activity of a subject is mediated by instruments, in particular between the subject and the object of his/her activity. This mediation has a *pragmatic* value: the instrument permits to reach the aim of the activity; it contributes to the production of an outcome. At the same time it has an *epistemic* value: it contributes to the development of the subject him/herself and of his/her understanding of the object. This pragmatic/epistemic distinction has been stressed in particular by Artigue (2002) in her work about the use of CAS (Computer Algebra Systems). She identified pragmatic and epistemic values in the instrumented techniques developed by students: these techniques both permitted to reach an aim, for example solve an exercise; and to build new mathematical knowledge, enabling the students to solve further exercises. Drawing on the work of Rabardel, we developed the theoretical frame of the documentational approach (Gueudet, Pepin & Trouche 2012). In this approach, instead of artefacts we considered resources of different kinds, following the definition proposed by Adler (2000): everything that is likely to re-source the activity. Our previous research has mostly been concerned with teacher interaction with resources; our focus here is on students' use of resources. We distinguish between available/proposed resources and resources-in-use, comparing the resources available for students (resources offered by the institution), and the resources actually used. We already analysed the two cases presented here in terms of links between the Didactic Contract (Brousseau 1997) and the use of resources (Gueudet & Pepin 2015). In this paper we extend the epistemic/pragmatic distinction to the mediation realised by various kinds of resources (Gueudet, Pepin & Trouche 2012) intervening in students' work, and investigate these mediations. The resource mediation can be represented by the following figure (figure 1).

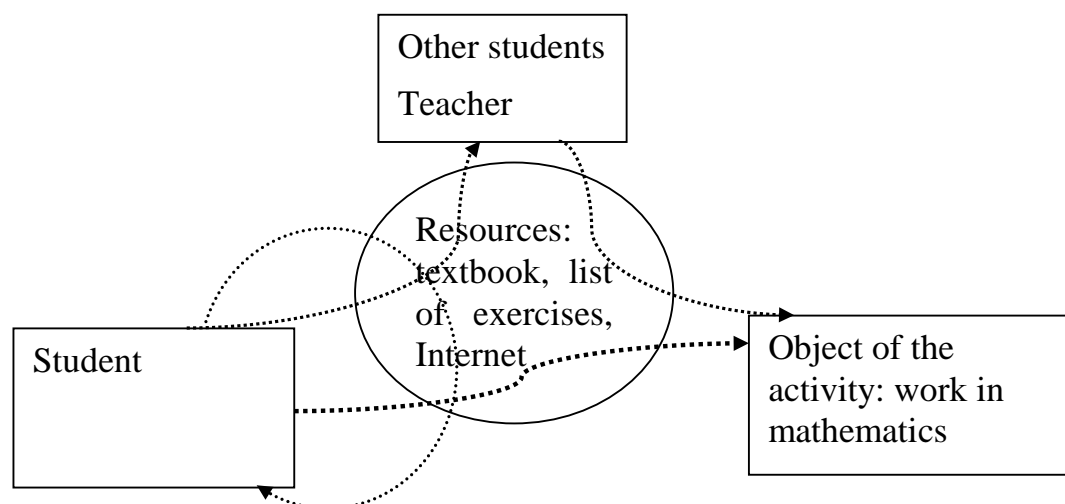


Figure 1. Resource mediations of students' work with mathematics.

Students at university use resources of various kinds: paper resources presenting the text of the lecture; their own notes; lists of exercises; textbooks; online resources found on the Internet; etc. These resources mediate the interaction between the student and the object of his/her activity; it has both a pragmatic and an epistemic value.

Both teachers' expectations and students' actual use of resources can be investigated through interviews with teachers and students, and we conducted such interviews in our two cases in the UK and in France.

In the UK, we explored the use of resources at a general level, at a medium-size university in a large city in the South of England. We mean by 'general level': for all mathematical content areas (taught during the first year), focusing on the general organisation of students' work. Beside conducting student interviews (with selected students), we surveyed all students of that particular mathematics course; we observed lectures, and interviewed lecturers, in addition to other support staff (e.g. teaching assistants at university) and teachers at secondary school. We also collected documents at both institutions (school, university, in the UK). For the work presented here, we selected a case study subject who is an ethnic minority student, Simar and his (ethnic minority) friends, all studying mathematics in the faculty of mathematics at the same university. We followed Simar over approximately two years: starting when he transited from a local upper secondary school into a university mathematics course, and into his second year at university. Over that period observations and interviews were conducted at several data points: (1) in his previous school (with his mathematics teacher/s); (2) at entry to university (with Simar and his friends; and with lecturers and support tutors); (3) towards the end of the first year at university (with Simar and his friends); and (4) during the first semester of his second year at university (as under (2)). For this study, interviews

with Simar (3); focus interviews with his friends (2); and interviews with selected lecturers (5) were analysed using our theoretical frame explained earlier.

The case in France has been conducted with a focus on a particular mathematical content, Number Theory. We collected data in a medium-size university where Number theory was taught in a first year teaching unit. During the academic year 2014-2015, we interviewed the teacher responsible of the course about the use of resources she expected from students. We also proposed an online questionnaire to the 140 students about their use of resources. We collected 85 answers (around 61 %). In 2015-2016 we complemented this study by proposing exercises as homework to a group of students, and interviewing three of them about their use of resources for solving these exercises. The collection of data at different levels, we claim, is likely to give us insights at different phenomena, which (it is hoped) complement each other.

CASE 1: STUDENT RESOURCES AND THEIR MEDIATION AT GENERAL LEVEL

From interviews with Simar (and his friends), who studied at City University, we could identify the main resources used in their first year of study: the lecture and lecture notes; the coursework; notes and tutorial notes made during tutorial and/or with his friends/study group. It was clear that these resources were quite different, in nature and quantity, from what students were used to at school: at school students had one textbook (which was portraying mathematics as something that one can learn by solving “tons of exercises”), whereas at university they were expected to work with many resources: e.g. different textbooks; lecture notes; examination papers; etc.

At City University the main resources were clearly the lecture, usually in halls of up to 300 students, and the lecture notes provided by the lecturer/professor (sometimes supported by a textbook). Different lecturers had different styles of lecture notes. Some lecturers would produce hand-written notes projected onto a screen (and talk students through the content during the lecture) (e.g. calculus). Simar and his friends would copy these notes, most of the time with little or no understanding.

However, Simar talked quite enthusiastically about one of his lecturers, and he pointed to what he (and his peers) would regard as a good lecture and lecture notes.

S: Geometry: the feedback we got from geometry is, basically he’s faultless. He’s brilliant, he’s excellent; the lecture’s engaging, the notes are available- clear notes. You can use the notes for the coursework.

Int: The notes are handwritten?

S: Yeah handwritten notes yeah. And they actually, you can see the kind of proofs- he doesn’t give too much away, but it’s just enough to get you thinking in the coursework’s, which is excellent. ... Because like, what students are finding is that they can go, because the lectures, they’re not gonna walk around with the lecture twenty-four, seven are they? They need something to take away from the lecture and you know, they’re gonna ready at home, they’re gonna read it, and they understand it. ... And they can go to the tutorial, ask

whatever questions and do the questions with confidence, knowing that they've done well like because everything's there, available. They don't need to go anywhere else, and if they do, the tutorial's available or the office hours. So really it's probably one of the best.

Int: So do you think they understand because in the lecture he explains well, or do you think they understand because the, it's so well-prepared and written out?

S: I think mainly it's mostly well-prepared, definitely, and then to accompany that, the lectures are brilliant as well. Yeah it's really, really kind of funny. He catches your interest..." (DP5, Simar)

Another lecturer would provide notes that students had to read in advance of the lecture. These notes had "holes" that needed to be filled in. During the lecture the lecturer would then discuss the content, and subsequently fill in the "holes". In such a way, students were not only obliged to prepare the lecture in advance (in order to be able to understand the notes in the lecture and fill in the missing text), but they also needed to attend the lectures to have "complete" notes.

In addition to lectures, the coursework (provided once a week) was to support student understanding of the lecture, through exercises. Simar and his friends/learning group were clear that unless the coursework was well aligned with the lectures, it did not help their understanding of the subject area (see Calculus as compared to geometry lectures/coursework). Indeed, in some cases students did not know what to ask in tutorial time, or in lectures, so little had they understood of the topic area. Other resources included textbooks (suggested/approved by the lecturers), but these were seen as less helpful than the lecture notes and coursework (provided by lecturers and tutors), in particular as students were often "learning to the test". However, the same resources (e.g. lecture notes) were often evaluated very differently by students, in terms of support for their learning, so much so that Simar (as student representative) had asked for a change in form and practice concerning lecture notes: as students did not want to be presented with "one slide after another".

Interestingly, in terms of lecture notes students distinguished between different types of notes: (1) "understanding notes" were well prepared and developed, apparently useful for understanding and coursework (and tests); (2) "comfort notes" were those that students did not understand but "you've got to have the notes" and "you have gone to the lecture", which in their views helped for knowing what to study for revision and examination purposes; and (3) "motivation notes" were provided on the student web, before the lecture, and which apparently "makes you want to come to the lecture ... because they are different" (DP5, p.4).

At the same time institutional practices, such as lectures, and accompanying resources played a crucial role in the ways that mathematics, and what it meant to "do mathematics" was portrayed. On the basis of video footage of selected lectures and pre- and post-video stimulated recall discussions with lecturers one could identify meanings that were attached to particular practices. Particular lectures reflected the kinds of things that a "rigorous mathematician" may need to learn:

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- ‘reasoning and proof’ based thinking and practices were expected to be developed through Geometry and Linear Algebra;
 - ‘procedural fluency’ (methods) was seen to be developed through Calculus;
 - practical and context relatedness was regarded to be developed through Statistics.

However, it was clear that during lectures student would not learn how to work as a “rigorous mathematician”, neither did students expect this from lectures and lecture notes. What students wanted were “help notes” for doing their course work, and worked examples suitable for studying for examinations.

“The only way I understand to do my work is, when I’m doing my coursework and there are help questions to do your coursework, and this is how I tend to them more and during the tutorials, and I think the tutorials and the courseworks are more helpful than, the lecture. The lecture you just get the notes.” (DP5 Focus group interview)

In terms of mediation of resources, in particular lecture notes, it can be argued that for students they had mainly pragmatic value: Simar and his friends were content, if they were given the “instruments” to do their coursework and examination questions. However, for university lecturers the mediation of (for example) lecture notes had epistemic value: they wanted to develop students into “rigorous mathematicians”, and lecture notes (and lectures) would show them how ‘rigorous mathematicians’ worked. How that could be learnt was not clear, except for alignment with what the lecture notes showed as examples. In fact, at City University one lecturer realized the problematic, and he had started a module on “writing mathematics” which was to provide students with the language they needed to appreciate the epistemic side of the subject.

CASE 2: STUDENT RESOURCES AND THEIR MEDIATION IN NUMBER THEORY

Our investigations took place in a first year teaching of Number Theory spanned over twelve weeks, with four hours each week (two hours of lecture, and two hours of tutorial). The first half of the course concerned logic, sets and combinatorics; the second half more directly number theory, with Euclidean division, Euclid algorithm, prime numbers and congruencies.

Use of resources by students and pragmatic mediation

In 2014-2015, at the beginning of this course the students were provided with a “polycopie”, which included more or less the text of the lecture; and a list of exercises. They could also access complementary resources, on the webpage of the teacher responsible for the teaching: previous exam texts; references of books; links towards online exercises. We proposed an online questionnaire to the 140 students concerned and obtained 85 answers. Only 52% of these 85 students declared that they found the polycopie useful. They considered that the text of the lecture of their

teacher was enough, and used the polycopie only before the final exam (83%). Moreover 90% would have liked to find worked examples in the polycopie; and 44% looked for additional resources on the Internet, in particular worked examples. We contend that these answers evidence that the polycopie mediation remains pragmatic for the students, similar to their use of textbooks at secondary school where number theory is limited to the application of some techniques (Battie 2010). Alike students in the UK, they search for worked examples in order to reproduce techniques, whereas the teacher expects that the polycopie has a strong epistemic value, and is used to work on the course: learn definitions, understand proofs of theorems etc. (declared by the teacher responsible for this teaching).

Mathematical resources for Number Theory

In 2015-2016, following the results of the questionnaire evoked above (a report about the answers was presented to the teachers of the number theory unit), no polycopie was given to the students. A book was recommended instead, together with a website, Braise² proposing exercises associated with different mathematical texts: description of methods, extracts of the course, hints, partial solution etc.

The teacher proposed to the students the following exercise as homework:

Exercice 28

Soit n un entier positif, prouver que $2n + 3$ et $n^2 + 3n + 2$ sont premiers entre eux.

Figure 2. Exercise given as homework (Let n be a positive integer, prove that $2n+3$ and n^2+3n+2 are coprime).

This exercise can be solved by searching for a and b such that $a(2n+3)+b(n^2+3n+2)=1$. There is a specific difficulty in this exercise, since a had to be itself of the form $un+v$, where u and v were constant integers.

A student can solve this exercise without any explicit use of resources; nevertheless, since the homework was given at a stage where such tasks are not yet familiar to students, we consider that this will probably not happen. Amongst the resources provided by the institution, the students can use their course notes, in particular to find the definition of coprime. They can also use the notes taken during the tutorial, where an exercise using a similar method has been done: “Let m and n be integers, such that m divides both $8n+7$ and $6n+5$. Show that $m=\pm 1$ ”(Exercise 8). In this exercise, the students also need to find a linear combination of $8n+7$ and $6n+5$, which does not depend on n . The students can also visit the Braise website; they could find on it a method entitled “Determine if two integers are coprime” which can also be used here as a resource. It does not mean that the students will only need to reproduce the same method: in particular, the presence of n^2 in exercise 28 requires a

significant adaptation of the method. We claim that, in such a case, the resources mediation has an important epistemic value.

16 students did the proposed homework; ten of them proposed a correct solution, and for six of them proposed a wrong solution. We met three of these students for an individual interview about the resources they used to solve the exercise: Brian, and Franck, who did not succeed; and Tom who found a correct solution. Tom used exercise 8 that he found in his tutorial notes, and correctly adapted the method. Franck used exercises that he found on the Internet (but not on BRAISE), which he identified as useful in terms of including the idea of Euclidean division of polynomials, and divides n^2+3n+2 by $2n+3$. This led him to conclude that the gcd is $(-1/4)$, “so is ± 1 up to a constant multiplier”). Brian searched his lecture notes, found the property “if p is prime and n an integer, then p and n are coprime”. He tried to apply it but realised that $2n+3$ is not always prime. Then he searched grade 12 textbooks, found the linear combination method but only with constant coefficients, and thought that the coefficient cannot depend on n . We claim that, while for Tom and Franck epistemic mediations of the resources took place, for Brian the mediation was limited to a pragmatic aspect. He searched for a method that he wanted to apply without any adaptation. Franck took personal initiative searching for resources that were not proposed by the teacher. He tried to build an original method, but was not successful in controlling its correctness. It is noticeable that, while Brian came directly from secondary school to university, Tom and Franck have had previous experiences at university: Tom did a first year of law studies before deciding to study mathematics, while Franck studied two years of “computer science and networks”. We can assume that the influence of secondary school was less important for them, as they had previous experiences at university.

CONCLUSIONS

The changes in resources (for teachers and for students), and in the use of these resources, at the secondary-tertiary transition have been under-researched. We contend that the study of “resource use” is an important theme for research, likely to deepen our understandings of teaching and learning processes (initiated or supported by resources) at the beginning of university. In the two case studies presented here we observe that the institution provides the students with numerous resources, mainly mathematical texts. According to the teachers, the epistemic mediation of these resources should support students transiting from school to university mathematics, both in their ways of learning mathematics (more self-regulated work, more autonomous reading of mathematical texts, see e.g. Farah 2015), and in their ways of “doing” mathematical work, so that it would become similar to the work of a “real mathematician”. However, asking the students about their actual use of resources led to a different picture. The learning at university seemed to be based on listening to the teacher (in lecture), writing down notes, trying similar worked examples, reading the polycopie for exam preparation (in France) – in other words an

alignment based on a kind of apprenticeship learning. We contend that, based on the two cases we studied, the pragmatic mediation of resources took over the epistemic aspect, at least in the first year. Students used worked examples and lecture notes in order to produce the desired results. At the level of mathematical content, in our case number theory, we detected a potential epistemic aspect in the use of worked examples and lecture notes, when a significant adaptation of a given method was needed. From our study we claim that the “enculturation” and “alignment” processes associated with the change from school to university mathematics education take longer than expected (by university staff), and more awareness and didactical flexibility (from the side of university staff) might help students to bridge this gap more successfully.

NOTES

1. TransMaths project, University of Manchester: <http://www.transmaths.org>
2. <http://braise.univ-rennes1.fr/braise.cgi>

Acknowledgement

The second author of this paper recognizes the contribution made by the TransMaths team in the collection of data, the design of instruments and the project, and discussions involving analyses and interpretations of the results. She would also like to acknowledge the support of the ESRC-TLRP award RES-139-25-0241, and continuing support from ESRC-TransMaths award(s) RES-139-25-0241 and RES-000-22-2890.

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Multiple choice questions and peer instruction as pedagogical tools to learn the mathematical language

Axel Hoppenbrock, University of Paderborn, Germany, hoppenbrock@khdn.de

Previous research has shown that the use of clicker questions and Peer Instruction in a lecture can have a positive impact on students' understanding, especially their conceptual understanding. The quality of students' discussions plays a crucial role for increasing the understanding. However, little is known about the role that clicker questions play in triggering high quality collaborative discussions in undergraduate analysis courses. In this case study, I will show how a clicker question, designed to help understand AE and EA expressions, triggered the meaning making process of one group. Different interpretations of the expression were an ideal trigger to a high quality discussion. At the end I set up some hypotheses about the design of good clicker questions.

INTRODUCTION

At the beginning of their studies students have to face many challenges. One major problem that math students face is to learn the mathematical language. The modern symbolic mathematical language developed over centuries and became increasingly dominant from early 19th century onwards (Nardi, 2011, p. 2053). This language is subject to rules that are beyond the rules of ordinary language (Schichl & Steinbauer, 2009, p. 8). These rules have to be learned quickly, because most of the mathematical content - especially in lectures - is presented in this language. Therefore lecturers should ask themselves, how they can support the process of learning this language. One possibility is to integrate multiple choice questions (clicker questions¹) with Peer Instruction (PI) into lectures themselves, as recommended by Mazur (1997): the lecturer presents a clicker question, the students vote for the first time, discuss their vote for a few minutes with their neighbours (this is PI) and then re-vote a second time before the solution and reasoning is explained. Numerous research studies showed that this method can increase conceptual understanding (e.g. Deslauriers, Schelew, & Wieman, 2011; Freeman et al., 2014; Hake, 1998). Moreover Smith et al. (M. K. Smith et al., 2009; M.K. Smith, Wood, Krauter, & Knight, 2011) showed the particular importance of PI for this increase.

Clicker questions for learning the mathematical language

Increasing conceptual understanding alone does not guarantee a better understanding of the rules of mathematical language and of typical mathematical expressions. But there are two reasons why PI can support the understanding. It allows students to deal with course material on their level of understanding and they have to express their

¹ *Multiple Choice question like the presented one here were often named clicker question because many lecturers use clicker devices in their lectures.*

ideas in ordinary language. This kind of verbalisation and reasoning can help to understand mathematical expressions. It “can act as a crucial semiotic mediator between symbolic and visual mathematical expression” (Nardi, 2011, p. 2060).

Research Question

So collaboration during PI is important to improve the understanding. A mathematical discussion becomes collaborative when it is useful for the task at hand, and the students “explore each other’s reasoning and viewpoints while working on a common activity, so that shared understanding evolves simultaneously for all participants” (Goos, Galbright, & Renshaw, 1996, p. 237).

A study (Knight, Wise, & Southard, 2013) that investigated the quality of collaboration during PI found a correlation between the quality of the clicker question and the quality of students’ discussion. In this paper I will analyse the role of clicker questions in triggering the discussion and the construction of new mathematical knowledge of the group. This deep insight might help create high quality clicker questions, especially to support students’ in learning the language of mathematics.

Construction of mathematical knowledge through interaction

The idea of PI is to allow students to create new mathematical knowledge by discussing with their neighbours. Creating new mathematical knowledge cannot be seen as a given product to which further knowledge elements can simply be added. Instead it has to be understood as an extension of the old knowledge by means of new, extensive relations and allow old knowledge to shine in a new light (Steinbring, 2005).

Steinbring supplies a “theoretical basis, where the epistemological conditions of mathematical knowledge are particularly related to interactive constructions of knowledge”. (Steinbring, 2005, p. xii). He combines the epistemological triangle as seen in figure 1 with Luhmann’s concept of communication (Steinbring, 2000).

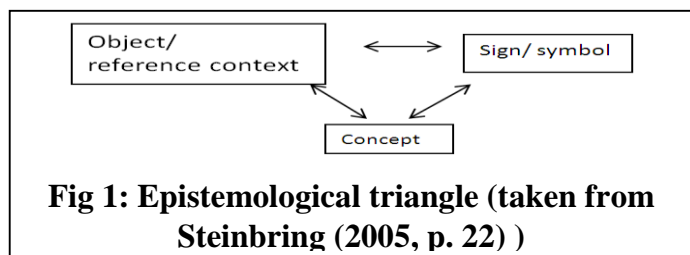


Fig 1: Epistemological triangle (taken from Steinbring (2005, p. 22))

In interaction with others, the students must produce actively reciprocal connections between the “points” of the triangle (Steinbring, 2005). For example while students discuss the concept of functions, they relate the symbol “f” with a diagram as a reference context. But this relation is not fixed; it can be modified during the interaction with others. So the “epistemological triangle reflects the particular status of mathematical knowledge as it has been constructed in the interaction to a certain point of time” (Steinbring, 2005, p. 78)

This view of producing mathematical knowledge through interaction allows us to model “the nature of the (invisible) mathematical knowledge by means of

representing the relations and structures constructed by the learner in the interaction” (Steinbring, 2005, p. 23). Moreover, the learning progress of a student or a group in the form of the development of interpretations can be represented as a sequence of epistemological triangles. “In the ongoing development of mathematical knowledge, the interpretations of the sign systems and the appropriately chosen reference contexts are modified or if necessary further generalized by the student” (Steinbring, 2005, p. 23).

THE STUDY

A case study can provide a rich and significant insight into events and behaviours, provide descriptive details about a particular phenomenon, increase understanding of phenomenon and explore uncharted issues (Yin, 2006).

In this paper I will present a case study on students’ discussions on one clicker question focussing on learning the mathematical language. The results presented here were part of a larger study in an undergraduate analysis course that almost 100 students attended. In the larger study 16 questions were presented and discussed in four theatre style lectures each 90 minutes long. Six or seven discussions were recorded per clicker question. The clicker question of this paper (figure 2) was presented at the beginning of the second lesson.

According to Yins’ (2014) six differentiations of case studies, this case study was an open participant observation. The students knew that their discussions were recorded and the investigator attended the lecturer but only as a passive bystander.

With the given clicker questions the students had about 6 minutes to discuss the solution with their peers. For analysing the Peer Instruction, the students were asked which group was willing to have their discussion recorded. Seven groups volunteered. But only six of these discussions were useful because one group turned off the dictaphone after 17 seconds.

For the validity and reliability of the case study, as postulated by Yin (2014), the audio recordings of the discussions were transcribed using GAT rules (Breidenstein, 2004). Afterwards the transcripts were interpreted by turn-by-turn analyses among members of the study group (investigator triangulation) as described by Krummheuer(2010). Afterwards, in order to uncover the knowledge construction, it was analysed with Steinbring’s epistemology oriented methodology as described above.

The clicker question

One example of the difficulties that undergraduate students face with the understanding typical mathematical

Definition of local maximum

D is a subset for \mathbb{R} and $f: D \rightarrow \mathbb{R}$ with $x_0 \in D$:

Please choose the definition that fits best to your understanding of a local maximum:

f has a local maximum at x_0 if and only if,

A) for all $\varepsilon \in \mathbb{R}^+$ and all $x \in D$ with $|x - x_0| < \varepsilon$ we have: $f(x_0) \geq f(x)$.

B) there exists an $\varepsilon \in \mathbb{R}^+$ such that for all $x \in D$ with $|x - x_0| < \varepsilon$ we have: $f(x_0) \geq f(x)$.

C) for all $\varepsilon \in \mathbb{R}^+$ there exists a $x \in D$ with $|x - x_0| < \varepsilon$ and $f(x_0) \geq f(x)$.

Fig 2: The clicker question

expressions was presented in the paper of Dubinsky and Yiparaki(2000). They presented major problems with the understanding of the interlacing of “for all...there exists” (AE) and “there exist...for all” (EA). “Most students [...] could not distinguish between AE and EA statements in mathematics and did not seem to be aware of the standard mathematical conventions for parsing statements” (Dubinsky & Yiparaki, 2000, p. 239). Based on these findings a clicker question was designed and presented in the lectures to teach such expressions for a specific example.

In this question the correct answer B) is contrasted by the two definitions A) and C). In definition A) the students should realize that “all $\epsilon \in \mathbb{R}$ ” and “all $x \in D$ ” can be shortened to “all $x \in D$ ” hereby defining an absolute (global) maximum. In definition C) the EA statement were switched around to an AE statement with the result that every point of a function fulfil the requirements of definition C.

ANALYSES OF THE GROUP DISCUSSIONS

In this analysis I will examine three parts of the discussion that the three students Susan, Mike and Lucas had, as an example to show how the clicker question triggered the students’ discussion process and influenced the learning process of the group.

Phase 1: Exchanging the decisions on the first vote

At the beginning of the discussion the three students informed each other about their decision in the first vote and justified it:

8.²S:I chose C simply because from our experience in the course, it has always been “for all epsilons”. I don’t know, that was my initial rationale. (1.2)

9.M: hmm (2.0)

10.L: the good old way

11.S: yes but still an explanation.

12.M: are you sure because in principle the idea is (---) that you have the maximum here

13.S: yes

14.M: and principally a kind of curtain that we hang up at the maximum and pull it down (--) and it should be (--) for all epsilon, so that we can create (--) all of them below that distance. That is why I decided on A (3.2)

15.S: hmm (affirmative)

16.L: Well now I am also for definition A. First I decided B to be the correct answer and now I am rejecting this there exist an epsilon because mmm. It doesn’t make any sense if there is only one.

All three students voted for different variants of the clicker question and their approaches differed greatly. Susan voted for C on purely formal reasons. She focused on the expression of “for all $\epsilon \in \mathbb{R}$ there exist a $x \in D$ ” (AE) in definition C and

² *The numbers of the original transcript have been retained*

concluded that her past experience with definitions and theorems in the lectures, there were only AE expressions.

Mike instead tried to connect his mental picture of a local maximum with the formal definition. His statement influenced Lucas. After he had heard the explanation from Mike he discards definition B and favoured definition A instead.

However Mike’s statement had an influence on the group discussion, too. Susan asked for a sketch for a better understanding of Mike’s explanation, and the whole group started to compare their conceptual image of a local maximum with the formal definitions of the clicker question.

Phase 2: The comparison of definition A and C

The group has ruled out definition B with the words “there exists one epsilon that must be a joke” very quickly. Afterwards they started to compare definition A) and C) with a sketch in front of them. Mike, who favoured definition A, started with the words:

- 35. M: here we have our $f(x_0)$ and $(- -) x$ minus x_0 is smaller than epsilon must define the interval where for all x the $f(x)$ must be smaller because if.....
- 36. S: But wouldn’t it be all x
- 37. M: no no that’s true, wait. You’re right (---)
- 38. S: if it’s valid for all x then it would be really big, don’t you think?
- 39. M: no no it says it’s valid for all x within this interval
- 40. L: Yes that is definition A (---) because if it is valid for only one x like in definition C that doesn’t work.
- 41. M: because otherwise there could be something higher next to it.

The group mainly focussed on the two statements in definitions A and C: “there exists a $x \in D$ ” (Def. C) and “for all $x \in D$ ” (Def. A). The students tried to understand the impact of the differences on the meaning of the definitions.

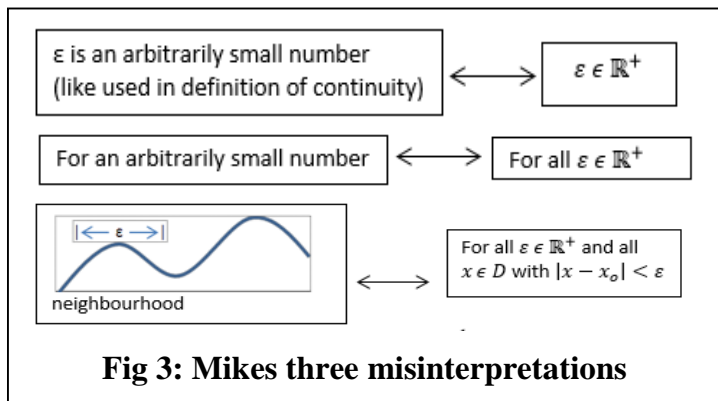


Fig 3: Mikes three misinterpretations

In turn 35 Mike started with his interpretation of definition A by creating a sketch in front of them. His assumption that definition A is correct was based on three misinterpretations he expressed before. From his previous experience in the analysis course, he connected the sign/symbol “ $\epsilon \in \mathbb{R}^+$ ” in definition A with the idea of an arbitrarily small number ϵ . The other misinterpretations were the result out of the first one. The mathematical symbol “for all“ $\epsilon \in \mathbb{R}^+$ ” was connected to the idea (reference

context) of “for arbitrary small numbers” which in return resulted in the interpretation of the sign/symbol “for all $\epsilon \in \mathbb{R}^+$ ” and all $x \in D$ with $|x - x_0| < \epsilon$ ” as an ϵ -neighbourhood of x_0 (illustrated in fig. 3).

Susan’s objections in turn 36 and 38 was based on the focus on the expression of “for all $x \in D$ ” in definition A. At this moment she didn’t see the connection between the two expressions “for all $\epsilon \in \mathbb{R}^+$ ” and “for all $x \in D$.” However, her words helped Mike to focus on the connection between these two statements. When he tried to argue against Susan’s objections he started to recognise his mistakes illustrated in figure 3:

- 54. M: But the problem is....what if here we, if here we ummm(--)
- 55. S: That’s why you have your epsilons here, right?
- 56. L: no that is for every epsilon
- 57. M: but if it’s valid for every epsilon then we aren’t getting any closer here.

Mike’s words in line 54 ends with an eureka moment. Suddenly he realised his misinterpretation and in line 57 he tells his fellow students his new view of definition A. This new view ends up with the realisation that definition A defines a global maximum

M: if it is for all epsilon and you choose this

as the maximum, then you don’t get this one (--) because you say it’s for all epsilons. But for a global maximum it obviously works.

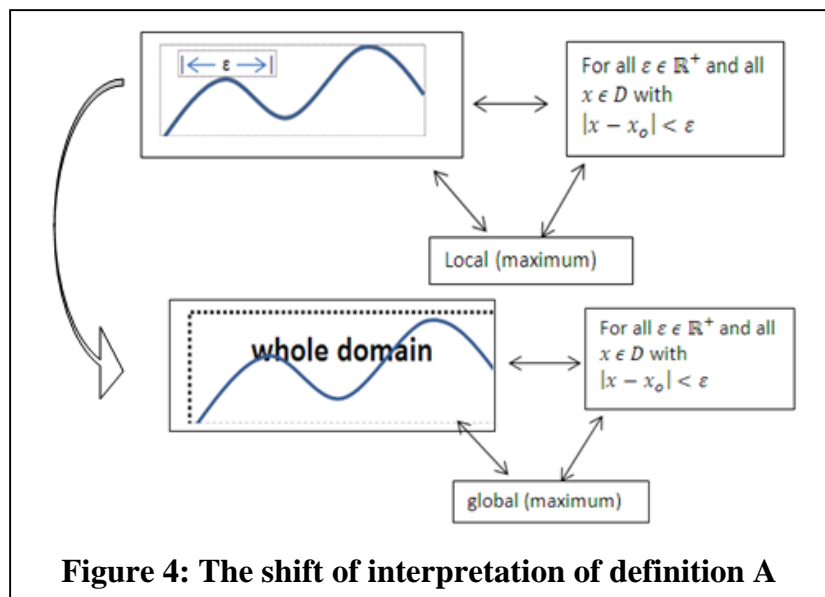


Figure 4: The shift of interpretation of definition A

This shift is illustrated in figure 4 with the epistemological triangle. Now the sign “for all $\epsilon \in \mathbb{R}^+$ ” and all $x \in D$ with $|x - x_0| < \epsilon$ ” was connected with the concept of “global” and the reference context illustrated in figure 4.

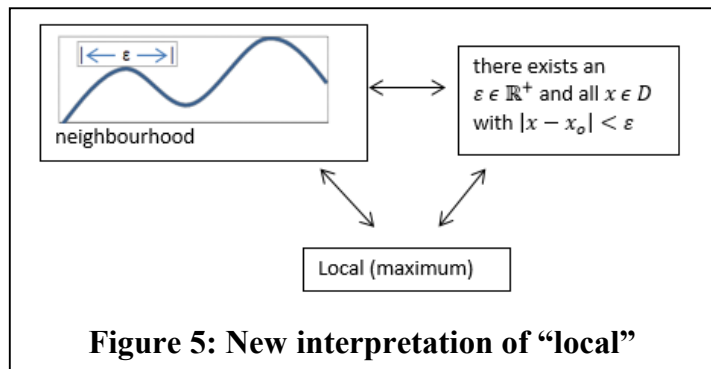
Phase 3: Understanding definition B

After the group recognised definition A as a definition for a global maximum Mike and Susan interpreted definition C:

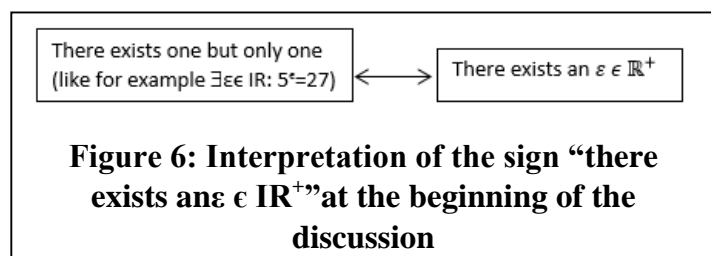
- 77. M: [...] I am almost convinced to say definition C is correct because of the expression there exist one x (--) I think you can find always an x for every small neighbourhood. No matter how close you get to x_0 , you always find an x that is smaller.
- 78. S: Yes [so far as you say

79. M: [and that is for a local maximums
 80. S: yes that makes sense, because the maximum is the highest (--) and at least there must exist a x that is smaller.
 81. M: exactly because the maximum is local (--) I will try the definition with one x now. I would say C now.
 82. S: OK good then I'll stick with C too
 83. L: I think I will go with choice B
 84. S: what is your idea behind B
 85. L: I would explain it this way if at this point you can find any interval that was B
 86. S: yes
 87. L: You find any interval so that they are all smaller, then you have a local maximum and that is exactly what is stated in B: Find an epsilon interval around this and they must all be smaller. That is exactly how it is formulated in B.
 88. S: Then you would have a solid epsilon.
 89. L. Yes, you only have to find one.
 90. M: YES, you're right.

When Mike and Susan talked about definition C, they used their imagination and the sketch of a local maximum in front of them. They tried to figure out if a local maximum fulfils the requirements that are stated in definition C. The result is that the maximum of their sketch meets the necessary requirements of definition C. Thus they decide for C although Sarah does not seem to be completely convinced (line 82).



Then in line 83 Lucas surprisingly proclaimed definition B to be correct. Before he was quiet and didn't argue with the other about definition C. He had used the small break to think individually about the question. He explained his decision for B in line 87. Susan was surprised and Max agreed with the words "Yes, you are right". Eventually Lucas explanation leads to a new interpretation of definition B and the concept of local (see figure 5).



The group had ruled out definition B to be correct because of their misinterpretation of the sign "there exists an $\epsilon \in \mathbb{R}^+$ " (see figure 6) and finally found the correct definition for a local maximum at the end of the discussion.

CONCLUSIONS AND DISCUSSION

The clicker question was designed to help the students to understand the different meanings of the expressions of AE and EA statements in a mathematical context as recommended by Dybinsky and Yiparaki(2000). As seen in the discussion many meaning making situations were triggered by different interpretations of the variants of the multiple choice question (mc question).

The change of the interlacing of “for all...” and “there exist...” statements between the variants of the mc questions had a great influence on the meaning of the three definitions. The students had to work out these differences. During that process misunderstandings and misinterpretations were seen. According to Muller (2008), the addressing of misunderstandings is an important part to overcome such misunderstandings. Definition C started with the AE statement like many other definitions³ in the course before. So definition C was able to unveil Susan’s generalisation that any definition with quantifiers had to start like this.

The “for all $\varepsilon \in \mathbb{R}$ ” statement in definition A was interpreted by Mike as an arbitrarily small number. One explanation for such an interpretation is the common use of ε in the course before, like in the definition of convergence of sequences. This definition starts with the statement “for all $\varepsilon \in \mathbb{R}$ ” but it is just “used” in the way expressed by one student in another group during their discussion:

N: definition A makes most sense for me, it means you approach over all x but let the interval get smaller and smaller. I see a connection to the concept of convergence (.) that you shorten the distance more and more (1.0) nevertheless the $f(x_0)$ is the greatest.

Despite these difficulties, the students were able to find the right answer at the end of the discussion. The key for the construction of new mathematical knowledge was the attempt to find connections between the mathematical symbolic expressions and their conceptual image, as well as to discuss different interpretations. Phase 2 is a particularly good example of this. Both Susan and Mike were making mistakes, but together they influenced each other in a positive way. Susan’s objection and Mike’s counterarguments led to a new view and understanding of the meaning of definition A (figure 4).

According to Goos et al. (1996) three factors influence the quality of collaborative mathematical discussions in school (see figure 6): the task has to be for learning and not for performance, the students should have equal task specific expertise and the task should be challenging for all students (Goos et al., 1996, p. 243). These conditions were met here. None of the students in the group knew the right answer at the beginning of the discussion and solving the task was challenging for all of them. Therefore, this leads to the hypothesis that these factors are influencing the quality of PI at University as well.

³ *Like the definition of convergence of sequences and the definition of continuity*

Moreover the analysis of the discussion shows that the presented clicker question complies with the four demands for tasks to support collaborative learning in some way: meaningful, complex, need for different ability to be solved and aim of level raising (Dekker & Elshout-Mohr, 1998).

This question was complex and difficult enough to encourage the involved students to debate the meaning of the different expressions (complex). But it was not too difficult in comparison to the students' skills and knowledge. The students used their conceptual image of a local maximum that they learnt during high school in order to work out the different meanings. Their different interpretations of the AE and EA expressions helped the group to find the right answer (different ability) and the construction of new mathematical knowledge (level raising).

However, one has to consider whether the quality of the discussion depends on the interplay between the clicker question and the skills, motivations and knowledge of students in the group. The impact on such questions on the quality of the discussion in relation to different group dynamics should be investigated further because lecturers have to find questions that challenge as many students as possible. The analysis of all six recorded discussions is an encouraging sign that these kind of questions can trigger high quality discussions in different kind group compositions as well: one discussion failed totally because one student were afraid to say something wrong but four of the remaining five groups were also able to construe new mathematical knowledge during the discussion.

Clicker questions like the one presented here could be implemented more often during undergraduate analysis courses. For example a question could be designed on the definition of convergence of sequences or continuity in the same way. Then such clicker questions might be one pedagogical tool to learn the mathematical language as desired by Nardi(2011, p. 2056).

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Quelques difficultés d'étudiants universitaires à reconnaître les objets « droites » et « plans » dans l'espace : une étude de cas

Céline Nihoul¹

¹UMONS, Belgique, celine.nihoul@umons.ac.be

Nous proposons ici une étude didactique de quelques difficultés rencontrées par des étudiants de première année universitaire dans un cours de mathématiques générales à propos des droites et des plans dans l'espace. Celles-ci concernent principalement la reconnaissance des objets géométriques à partir de leurs équations et de leurs descriptions en termes d'ensembles de points. À partir d'une analyse du cours et des exercices proposés aux étudiants, nous repérons dans une évaluation quelles sont les difficultés persistantes liées à la reconnaissance des objets.

Mots clefs: droites dans l'espace, plans, registre d'écriture, point de vue, enseignement universitaire.

INTRODUCTION

Ce travail s'inscrit dans le contexte d'un cours de mathématiques générales dans lequel nous intervenons, donné à des étudiants belges en première année universitaire dans les filières mathématique, informatique et physique. Ce cours vise à reprendre des notions abordées au lycée tout en y intégrant des exigences spécifiques à l'enseignement universitaire telles que la rédaction des raisonnements et une utilisation appropriée des connaissances en logique et en théorie des ensembles. Il est organisé avant tout autre cours de mathématiques tels que l'Analyse et l'Algèbre linéaire. Ces objectifs sont guidés par une meilleure prise en compte, des professeurs du département de mathématiques des difficultés des étudiants dans la transition secondaire-université. Une description détaillée des exigences du cours est donnée dans (Bridoux, 2014). Nous nous centrons ici sur le chapitre de géométrie analytique et plus précisément sur les notions de droites et de plans dans l'espace. Cette recherche s'inscrit dans notre travail de thèse qui en est à ses débuts et notre objectif est actuellement de mieux comprendre les spécificités de l'enseignement, en particulier les choix didactiques de l'enseignant. Nous montrons donc tout d'abord comment ces notions sont travaillées dans le cours dont il est question ici. Nous abordons ensuite notre problématique et les outils théoriques qui nous permettent de mieux appréhender les spécificités du cours et des TD. Nous analysons ensuite une question issue d'une évaluation pour confronter certaines difficultés persistantes aux choix didactiques effectués par l'enseignant.

ÉLÉMENTS DE CONTEXTE ET PROBLÉMATIQUE

Dans le cours visé ici, le chapitre sur les droites et les plans dans l'espace vient après l'étude des droites dans le plan. L'enseignant choisit d'aborder le chapitre qui nous

intéresse en partant des connaissances déjà là chez les étudiants sur le chapitre des droites dans le plan à partir du questionnement suivant : l'équation $ax + by = c$ vient d'être étudiée dans le plan et on a en particulier établi que la droite d'équation $ax + by = c$ est l'ensemble des couples (x, y) du plan qui vérifient l'égalité $ax + by = c$. Quel est maintenant l'objet géométrique de l'espace décrit par cette équation ?

Le chapitre sur l'espace à trois dimensions débute alors par l'étude d'équations cartésiennes de plans, comme par exemple le plan d'équation $2x + 3y = 6$, qui sont visuellement proches de celles des droites dans le plan. Le choix d'introduction du professeur consiste donc à initier un travail sur la notion d'« équation » et sur ce que représente une équation de la forme $ax + by = c$ dans l'espace, équations dites incomplètes à cause de l'absence d'une variable. Il décrit alors l'équation $2x + 3y = 6$ comme l'ensemble des triplets (x, y, z) qui vérifient tous l'égalité $2x + 3y = 6$. Ensuite, quelques triplets satisfaisant cette égalité sont déterminés afin de pouvoir construire graphiquement le plan représenté par cette équation. Son objectif est aussi de provoquer chez les étudiants une contradiction et de leur faire prendre conscience que l'objet décrit par une même forme d'équation n'est plus une droite comme dans le plan mais un plan dans l'espace. Ainsi, l'introduction du chapitre sur les droites et les plans dans l'espace ne s'inscrit pas dans le contexte général d'une ingénierie didactique au sens de Brousseau (1998) mais bien dans un contrat ostensif où le professeur montre aux étudiants que des équations cartésiennes a priori similaires ne représentent pas les mêmes objets géométriques selon que l'on se place dans le plan ou l'espace. En ce sens, ce choix semble prendre en compte des travaux de recherches sur ces notions tels que ceux de Schneider & Lebeau (2009) sur les équations incomplètes de plans et ceux de Chevillard (1985) sur la notion d'« équation », qui est selon lui une notion paramathématique car elle n'est pas vraiment définie dans l'enseignement secondaire.

L'enseignant généralise ensuite le travail déjà réalisé sur les équations incomplètes pour caractériser un plan comme un ensemble de triplets satisfaisant tous une équation cartésienne de forme générale $ax + by + cz = d$. Ensuite, l'enseignant définit les droites de l'espace comme des ensembles de points alignés et les caractérise dans un premier temps par une équation paramétrique de la forme $(x, y, z) = (x_A, y_A, z_A) + \lambda(x_v, y_v, z_v)$ où (x_A, y_A, z_A) et (x_v, y_v, z_v) sont respectivement un point et un vecteur directeur de la droite et λ est un réel. Le cheminement est ici identique à celui réalisé dans le chapitre des droites dans le plan. En éliminant le paramètre, l'enseignant montre que les droites sont caractérisées par un système d'équations cartésiennes de la forme

$$\begin{cases} \frac{x - x_A}{x_v} = \frac{y - y_A}{y_v} \\ \frac{y - y_A}{y_v} = \frac{z - z_A}{z_v} \end{cases}$$

où (x_A, y_A, z_A) est un point de la droite et (x_v, y_v, z_v) est un vecteur directeur de la droite tel que chacune de ses coordonnées sont différentes de 0. Il insiste alors sur la description géométrique d'un système d'équations cartésiennes décrivant une droite comme étant l'intersection de deux plans sécants. Le fait que deux plans sécants se coupent suivant une droite est une connaissance étudiée en première au lycée lorsque les positions relatives de plans sont abordées. Nous remarquons encore une tentative de rapprochement de la part de l'enseignant avec les connaissances antérieures des étudiants. De plus, il nous semble que l'enseignant essaie vraiment d'établir des liens entre les objets géométriques et leurs descriptions. En effet, il multiplie les représentations possibles des droites et des plans dans l'espace, soit en termes d'équations, soit en termes d'ensembles de points ou encore avec l'aide d'un dessin.

Dès l'introduction de ce chapitre, des difficultés apparaissent chez les étudiants. Par exemple, face à l'équation $3x + 4y = 12$ dans l'espace, nombreux sont les étudiants qui ne sont pas capables de reconnaître l'objet géométrique décrit par cette équation. La plupart des étudiants considèrent que cette équation décrit une droite dans l'espace tout comme dans le plan à cause de la proximité visuelle entre les équations cartésiennes d'une droite dans le plan et d'un plan dans l'espace. De plus, la majorité des étudiants associe l'objet décrit par l'équation $3x + 4y = 12$ à un ensemble de couples (x, y) vérifiant cette égalité plutôt qu'à un ensemble de triplets, et quand c'est le cas, les étudiants suggèrent souvent seuls les triplets de la forme $(x, y, 0)$ sont solutions de cette équation. De nombreux étudiants considèrent aussi que l'absence de la variable z provient de l'annulation de la cote plutôt que de celle du coefficient devant celle-ci. De ce fait, pour la plupart des étudiants, cette équation décrit encore une fois une droite dans l'espace. Cette difficulté a également été identifiée par Maurel (2001) qui l'interprète comme une confusion dans le statut des différentes lettres, ici la variable et le coefficient. Selon nous, ces constats peuvent probablement être liés à un manque important de questionnement sur la nature des objets représentés par une équation ou par un ensemble de points. En outre, la description ensembliste de ces objets n'apparaît pas comme naturelle chez la majorité des étudiants. Cela peut s'expliquer par le fait que les connaissances liées à la théorie des ensembles, dont la description des objets comme étant des ensembles de points, sont nouvelles pour les étudiants puisqu'elles ne font pas l'objet d'une étude spécifique dans l'enseignement secondaire.

D'autres difficultés sont quant à elles plus spécifiquement liées aux connaissances en théorie des ensembles. Par exemple, si A désigne l'ensemble $\{(x, y, z) \mid 2x + 3y + 3z = 8\}$, nombreux sont les étudiants qui ne sont pas capables de traduire la proposition $\alpha \in A$. Cela entraîne des difficultés à vérifier si par exemple $(1, 1, 1)$ appartient bien à l'ensemble A mais également à déterminer que tout triplet de A est de la forme $(x, y, \frac{8-2x-3y}{3})$ avec x, y des réels quelconques. Dans ce cas également le manque de travail sur les ensembles au lycée amène des difficultés

chez les étudiants à donner du sens à la manière dont on écrit un ensemble et aux éléments qui appartiennent à l'ensemble. Ce constat peut être mis en lien avec une difficulté décrite par Dieudonné, Droniou, Durand-Guerrier, Ray & Theret (2011) qui notent un manque de sens accordé au symbole d'appartenance et aux éléments qui constituent les ensembles. Nous repérons également, lors de la description de l'objet géométrique associé à l'ensemble A que la majorité des étudiants a des difficultés à identifier l'ensemble des éléments de A comme formant un plan. En effet, ils ont besoin de déterminer quelques éléments particuliers de l'ensemble pour identifier le plan et ne considèrent pas tous les éléments de l'ensemble. Dieudonné et al. (2011) ont déjà repéré qu'il est difficile pour les étudiants de concevoir un ensemble de points comme un objet mathématique sur lequel ils peuvent raisonner.

Alors que les notions de droites et de plans ont été étudiées au lycée, leur reprise à l'université dans le cours dont il est question ici a montré que de nombreuses difficultés sont bien présentes. Les étudiants ont en fait des conceptions personnelles (au sens de Hitt, 2006) sur ces notions lorsqu'ils entrent à l'université. Vu les objectifs du cours précédemment décrits, il semble aussi que le savoir personnel construit par la plupart des étudiants soit assez éloigné du savoir institutionnalisé au lycée. Dès lors, nous abordons la problématique suivante : après un travail explicite sur la reconnaissance des objets « droites » et « plans » dans l'espace à partir de diverses descriptions, soit en termes d'équations cartésiennes ou d'équations paramétriques, soit en termes d'ensembles de points satisfaisant tous une même relation, objectifs clés du cours, quelles sont les éventuelles difficultés persistantes chez les étudiants et à quel(s) type(s) de description sont-elles liées ?

Pour tenter d'obtenir des éléments de réponse à ce questionnement, il nous semble nécessaire de compléter cette première étude du cours par une analyse des exercices proposés en TD et du travail attendu des étudiants. C'est l'objet de la section suivante.

SPÉCIFICITÉS DE L'ENSEIGNEMENT ÉTUDIÉ

La reconnaissance des objets géométriques dans l'espace est selon nous liée aux diverses manières de décrire ces objets. Les droites et les plans peuvent en effet être décrits par des équations cartésiennes, par des équations paramétriques, mais aussi être considérés comme des ensembles de points vérifiant tous une même propriété. Puisque les informations dont on a besoin pour déterminer des équations paramétriques, des équations cartésiennes ou décrire des ensembles de points ne sont pas identiques, le travail à réaliser pour chaque type de description n'est pas le même. Rogalski (1995) parle de point de vue sur un objet mathématique pour caractériser la manière de regarder, de faire fonctionner et éventuellement de définir cet objet. Ainsi, au sein du cours dont il est question ici, la description d'un objet peut se faire d'un point de vue paramétrique, d'un point de vue cartésien ou d'un

point de vue ensembliste et cela n'engendre pas le même travail mathématique même s'il est possible de passer facilement d'un point de vue à un autre.

Selon Dorier & Robert (1997), ces notions, et de façon plus générales les notions d'algèbre linéaire, provoquent de nombreuses difficultés chez les étudiants puisque sur un plan épistémologique, différents points de vue sont en jeu dans la manière de les décrire. Notre étude du cours théorique a permis de préciser certaines difficultés relatives aux points de vue présentés. De plus, le passage d'un point de vue à un autre nécessite une certaine flexibilité liée aux changements de cadres (au sens de Douady, 1987) et aux changements de registres (au sens de Duval, 1993) associés à ce passage. Notre problématique nous amène donc maintenant à regarder les exercices proposés et à étudier spécifiquement les difficultés liées à la reconnaissance des objets suivant un point de vue particulier et celles éventuellement en rapport avec l'articulation de différents points de vue.

Les exercices proposés dans le cours étudié ici, visent à faire travailler les étudiants sur la reconnaissance des objets et sollicitent tous au moins un de ces trois points de vue (paramétrique, cartésien ou ensembliste). Nombreux sont les exercices nécessitant une articulation entre ces points de vue. C'est le cas par exemple de la résolution de systèmes linéaires. Dans les exercices, les systèmes linéaires à résoudre sont en général composés de deux ou trois équations cartésiennes. Ainsi, les objets dont on cherche une éventuelle intersection sont donnés dans le registre algébrique et selon le point de vue cartésien. La tâche principale des étudiants est de résoudre le système et d'identifier l'objet décrit par l'ensemble des solutions du système linéaire. Ils sont libres de choisir quelle sera la méthode de résolution du système et à partir de quel point de vue l'objet décrit par l'ensemble des solutions sera identifié. En voici un exemple :

$$\begin{cases} 2x + y + z = 1 \\ x + y + z = 2 \end{cases}$$

La résolution en elle-même s'effectue dans le registre algébrique. L'ensemble des solutions du système peut s'écrire sous la forme $\{(-1, 3 - \alpha, \alpha) | \alpha \in \mathbb{R}\}$. Ici, on peut reconnaître immédiatement l'objet « droite » à partir du point de vue ensembliste en se rendant compte que l'abscisse est fixe et que l'ordonnée et la cote dépendent l'une de l'autre et évoluent de façon linéaire. On peut aussi s'appuyer sur le point de vue paramétrique. En effet, on peut reconstruire une équation paramétrique de l'objet à partir des triplets de cet ensemble. On obtient alors par exemple l'équation paramétrique $(x, y, z) = (-1, 3, 0) + \alpha(0, -1, 1)$ où α est un réel. De cette équation paramétrique, on peut aisément déduire un point et un vecteur directeur de la droite décrite. Il y a donc une articulation entre les points de vue dans cet exercice. L'articulation peut se résumer aux points de vue cartésien et ensembliste si on reconnaît la droite à partir de l'ensemble de points, mais une autre articulation peut s'ajouter entre le point de vue ensembliste et le point de vue paramétrique si on reconnaît et décrit la droite à partir d'une équation paramétrique.

Ce type d'exercice nécessite alors d'être capable de reconnaître les objets dans chaque point de vue mais aussi de les articuler. On voit donc bien ici la nécessité d'avoir une certaine flexibilité entre les différents points de vue pour identifier les objets décrits. Cette flexibilité ne s'apprend pas de manière autonome par les étudiants et un réel travail doit être effectué en phase d'apprentissage (Artigue, Chartier, & Dorier, 2000). Alves-Dias (1998) ajoute que l'articulation entre les points de vue cartésien et paramétrique n'est pas souvent travaillée dans l'enseignement et que cela amène des difficultés chez les étudiants lorsqu'ils abordent l'algèbre linéaire. Ainsi, cette flexibilité se développe selon elle à partir d'un travail important sur l'articulation des points de vue cartésien et paramétrique et ce, à la fois dans le sens cartésien/paramétrique mais aussi dans le sens paramétrique/cartésien. De plus, selon Rogalski (1994), la majorité des étudiants considère que l'algèbre linéaire n'est qu'un catalogue de notions très abstraites. Il souligne qu'une de leurs difficultés concerne les raisonnements de type « ensembliste ». C'est pourquoi, il nous semble également nécessaire d'articuler le point de vue ensembliste, nouvelle manière de regarder les objets pour nos étudiants, avec les points de vue cartésien et paramétrique. Selon Dorier & Robert (1997), la flexibilité et le jeu entre les différents points de vue sont des activités cognitives essentielles pour l'apprentissage des notions d'algèbre linéaire. Nous pensons alors que cette flexibilité entre les points de vue paramétrique, cartésien et ensembliste peut amener les étudiants dans ce chapitre à donner du sens aux notions (Douady, 1987). Rogalski (2000) ajoute que les connaissances liées à la géométrie cartésienne et à une pratique de la géométrie dans l'espace constituent un support important pour le langage et le sens en algèbre linéaire car ces connaissances géométriques peuvent donner des « images » de certains concepts vectoriels tels que les ensembles de solutions de systèmes d'équations linéaires. Harel (2000) va aussi dans ce sens lorsqu'il dit qu'une réflexion géométrique doit être incorporée dans l'enseignement des premières notions d'algèbre linéaire car selon lui, cela contribue de façon significative à la compréhension des étudiants. C'est pourquoi, dans la perspective de préparer nos étudiants à l'algèbre linéaire, nous essayons dans le cours dont il est question ici, de proposer des exercices consistants sur les notions de droites et de plans dans l'espace travaillant et articulant les points de vue.

Comme décrit précédemment, le cours magistral permet selon nous de mettre en fonctionnement de nombreuses adaptations (au sens de Robert, 2008). En effet, des conversions de registres (algébrique et graphique) et des changements de points de vue (cartésien, paramétrique, ensembliste) apparaissent tout au long de la séquence d'enseignement. De plus, les exercices mettent en jeu des connaissances anciennes telle que la résolution de système linéaire et requièrent une certaine flexibilité entre les points de vue. Ainsi, les tâches demandées aux étudiants sont complexes et nécessitent une certaine disponibilité des connaissances. Nous analysons maintenant des copies d'étudiants à une évaluation pour étudier l'impact d'un tel travail sur les apprentissages des étudiants. Notre objectif vise à savoir s'ils sont capables ou non

après un enseignement spécifique sur la reconnaissance des objets, d'identifier ces objets selon différents points de vue. Nous mettons alors les difficultés persistantes en lien avec l'utilisation d'un point de vue précis ou à une articulation précise entre points de vue.

EXPÉRIMENTATION

Nous reprenons dans cette section une question issue des évaluations de l'année 2014 [1]. Nous en présentons tout d'abord une analyse a priori et nous mettons en évidence les difficultés repérées dans les copies des étudiants liées à la reconnaissance des objets géométriques. Nous nous restreignons aux copies des 24 étudiants de la filière mathématique. Nous considérons qu'une question est réussie lorsque l'étudiant obtient au moins la moitié de la note attribuée à la question. Nous analysons la question proposée à la figure 1.

Question 11. Soit l'ensemble

$$S_2 := \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

Décrivez géométriquement l'ensemble et représentez le graphiquement. Expliquez votre démarche.

Figure 1: Question 11, Examen, 2014.

L'ensemble S_2 représente l'ensemble de tous les triplets de la forme $(x, y, 0)$ avec x, y deux réels quelconques. Il s'agit donc du plan Oxy ou encore du plan dont une équation cartésienne est $z = 0$. Le niveau de mise en fonctionnement des connaissances relève du mobilisable (au sens de Robert, 1998). En effet, le travail attendu des étudiants n'est pas nouveau puisque cette question est assez proche de ce qui a été réalisé lors du cours et des TD à la fois dans le plan et dans l'espace. L'objet décrit dans cette question est donné selon un point de vue ensembliste. On attend que l'étudiant retrouve une équation de l'objet et donc qu'il décrive le plan selon un point de vue cartésien ou paramétrique. Nous sommes donc dans le registre algébrique et une articulation entre points de vue est effectuée. Ce changement de points de vue amène une conversion entre le registre algébrique et le registre graphique pour représenter graphiquement le plan Oxy .

Bien que 79 % des étudiants décrivent l'ensemble S_2 comme étant un plan, cette question n'est réussie que par 58 % des étudiants. Cela s'explique par un manque de précision dans leur description. Certains étudiants sont capables de décrire un plan mais pas qu'il s'agit du plan Oxy . 21% des étudiants parviennent à reconnaître le plan et le caractérisent par l'équation cartésienne $z = 0$. Ces étudiants sont donc passés du point de vue ensembliste au point de vue cartésien. 25% des étudiants disent qu'il s'agit du plan Oxy . Il nous semble que cette description du plan est donnée selon le point de vue paramétrique puisque cela représente la donnée d'un point O et des vecteurs directeurs $(1,0,0)$ et $(0,1,0)$. Ces étudiants ont donc articulé le point de vue ensembliste avec le point de vue paramétrique. Nous remarquons qu'il y a presque autant d'étudiants qui décrivent le plan selon les points de vue cartésien et

paramétrique. 16.5 % des étudiants citent les deux descriptions. Il est plus compliqué de nous prononcer sur ce résultat. En effet, lorsqu'un étudiant donne les deux définitions, est-ce parce qu'il pense que les deux descriptions sont équivalentes ou bien si au contraire il pense qu'elles ne décrivent pas le même objet. Pour ces étudiants, nous avons besoin de regarder la représentation graphique du plan.

Nous repérons beaucoup d'erreurs lors de la représentation graphique du plan Oxy . Ces erreurs sont plus ponctuelles. Une difficulté persistante chez 45 % des étudiants est liée à la représentation graphique du plan, comme l'illustre la figure 2.

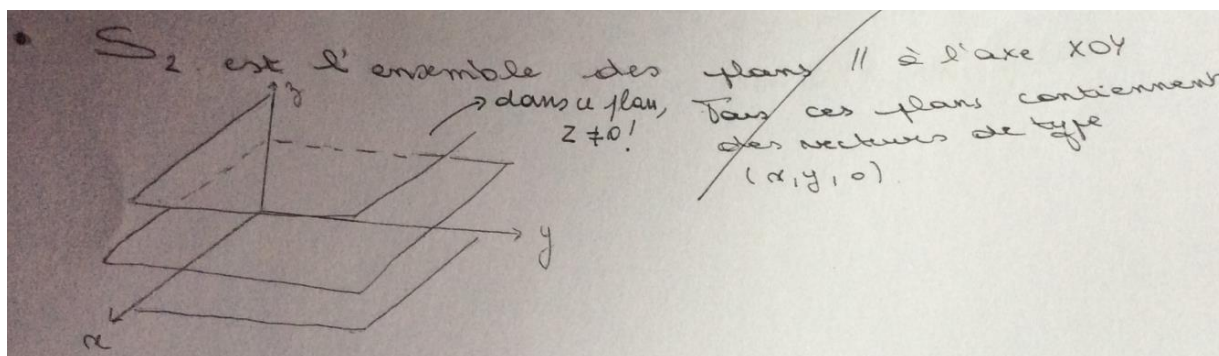


Figure 2: Question 11, Examen, 2014.

Cette copie montre que l'étudiant reconnaît l'objet « plan » mais il ne peut dire lequel. Pour lui, l'ensemble S_2 décrit un ensemble de plans parallèles au plan Oxy (qu'il appelle l'axe Oxy). Le fait que ces quelques étudiants décrivent un plan parallèle à Oxy comme étant l'ensemble des triplets de la forme $(x, y, 0)$ montre que le lien entre la description du plan d'un point de vue ensembliste et d'un point de vue cartésien ou paramétrique est totalement transparent pour eux. De plus, l'unicité du plan décrit n'est apparemment pas claire chez ces étudiants. La reconnaissance des objets est donc encore absente chez certains étudiants. Pourtant, parmi les étudiants n'étant pas capable de représenter le plan, 8% des étudiants ont donné une description correcte en termes d'équations cartésiennes. Selon nous, cela témoigne d'un problème lié à l'articulation des registres algébrique et graphique mais peut-être aussi d'un manque de sens accordé aux équations.

Tous les étudiants ayant décrit le plan comme étant le plan Oxy l'ont tous correctement représenté. Par contre, parmi ceux qui l'ont décrit par l'équation cartésienne $z = 0$ (21%), seulement 12.5 % des étudiants ont réussi à le représenter. Ainsi, le passage du registre algébrique et en particulier de la description des objets selon le point de vue cartésien vers le registre graphique semble amener des difficultés chez les étudiants. Ceux ayant donné une description selon un point de vue cartésien et un point de vue paramétrique ont tous bien représenté le plan. Nous pouvons désormais supposer que ces étudiants ont développé une certaine flexibilité entre les points de vue et sont capables d'articuler les registres.

Cette question travaille la reconnaissance des objets selon un point de vue cartésien ou paramétrique à partir d'une description donnée dans le point de vue ensembliste. L'analyse des copies révèle que majoritairement la reconnaissance de l'objet « plan » et la description du plan par une équation semblent maîtrisées par la majorité des étudiants. L'articulation entre le point de vue ensembliste et les autres points de vue est correctement effectuée par la plupart des étudiants. Leurs conceptions personnelles semblent avoir évolué au fil de leur apprentissage. Toutefois, des difficultés persistent en lien avec la représentation du plan dans le registre graphique. La conversion du registre algébrique vers le registre graphique semble être encore problématique chez certains étudiants. Cette difficulté semble davantage marquée auprès des étudiants ayant utilisé un point de vue cartésien pour décrire le plan. Ainsi, la représentation graphique et la description selon le point de vue cartésien sont deux éléments importants qui doivent être encore approfondis.

CONCLUSION

Nous nous sommes intéressée aux difficultés rencontrées par les étudiants de première année universitaire dans un cours de mathématiques générales pour reconnaître les objets « droites » et « plans » dans l'espace à partir d'une équation ou à partir d'un ensemble de points. Nous avons montré que l'enseignement étudié ici mettait en évidence plusieurs descriptions des objets aussi bien à travers des conversions entre le registre algébrique et le registre graphique, qu'à travers une articulation entre points de vue cartésien, paramétrique et ensembliste. De plus, les exercices proposés dans ce chapitre requièrent une certaine flexibilité en termes de points de vue.

Notre objectif était ensuite de décrire les difficultés persistantes liées à la reconnaissance des objets géométriques et de les mettre en lien avec les points de vue en œuvre dans les exercices. Nous avons alors analysé des copies d'étudiants issues d'une évaluation. Cette analyse montre que la reconnaissance des objets selon un point de vue précis semble être atteinte chez un grand nombre d'étudiants. Cependant, la plupart d'entre eux rencontrent des difficultés à représenter graphiquement les objets, c'est-à-dire quand une conversion de registre est nécessaire. Selon nous, ce constat témoigne notamment d'un manque de sens accordé à l'objet « équation », surtout selon le point de vue cartésien. Dans notre travail de thèse en cours, une étude des difficultés liées à l'utilisation de l'objet « équation » doit encore être réalisée. De plus, une analyse des exercices proposés dans le cours visé ici et des copies des étudiants aux évaluations doit être approfondie pour déterminer plus précisément quelles sont les difficultés en lien avec cette flexibilité attendue chez les étudiants.

NOTES

1. Plusieurs évaluations sont prévues pendant le cours de mathématiques générales. Elles sont organisées de manière hebdomadaire pendant toute la période du cours.

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Making sense of students' sense making through the lens of the structural abstraction framework

Márcia M. F. Pinto¹ and Thorsten Scheiner²

¹Federal University of Rio de Janeiro, Brazil, marcia@im.ufrj.br; ²University of Hamburg, Germany, thorsten.scheiner@uni-hamburg.de

In this paper we use the evolving framework of structural abstraction as a theoretical lens to investigate how mathematics major university students understand the limit concept of a sequence. To this aim the theoretical framework is outlined and previous empirical data on one individual's partial (re-)construction of a convergent sequence is revisited. In doing so, we provide insights in how students, who consider the formal definition of a mathematical concept as one of the components of their concept image, involve it into their overall mathematical discourse when building new knowledge. Deeper analysis also reveals unsettled issues about structural abstraction and provides new directions for advancing our understanding of this kind of abstraction.

Keywords: generic representation, mathematical learning, sense making, structural abstraction, theory development.

INTRODUCTION

There has been a growing interest in revisiting the notion of abstraction in mathematics education. Recent contributions from socio-cultural perspectives on the learning of mathematics have strengthened our theoretical understanding and framed our empirical investigation on abstraction in knowing and learning mathematics, as Hershkowitz, Schwarz, and Dreyfus' (2001) *abstraction in context* approach and Noss and Hoyles' (1996) *situated abstraction* approach indicated. With regard to cognitive approaches on abstraction in mathematics education, Scheiner (2016) observed that the literature demonstrated substantial progress in explicating the significance of Piaget's (1977/2001) *reflective abstraction* in mathematical concept construction, the kind of abstraction that is often described in terms of forming a (structural) concept from an (operational) process (see Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). However, in the past, the literature rarely explored differences in cognitive processes with regard to whether the primary focus is on the actions (abstraction from actions) or on the objects (abstraction from objects). The former takes place on the actions on objects, in particular, individual's reflections on actions on known objects; the latter takes place on the objects themselves, in particular, paying attention to the properties and structures inherent in those objects. However, Piaget considered abstraction from actions as the only form of abstraction for mathematical epistemology; separating it from abstraction from objects. Given these historical origins of our field, it is not surprising that the literature reveals a bias towards abstraction from actions as the dominating form of abstraction in knowing and learning mathematics.

Only recently, abstraction from objects has attracted attention as a form of abstraction that provides an account for the complex cognitive processes compatible with students' sense-making strategy of 'giving meaning' (Scheiner, 2016; Scheiner & Pinto, 2014). An important contribution within abstraction from objects has been provided by Mitchelmore and White (2007) who investigated *empirical abstraction* in learning elementary mathematics drawing on Skemp's (1986) conception of abstraction. Their approach goes beyond Piaget's idea of empirical abstraction, as their understanding of abstraction accounts for the similarities of the underlying structures rather than the superficial (or external) characteristics of objects, as Piaget did. While Mitchelmore and White (2007) considered physical objects, Scheiner (2016) described a framework of a kind of abstraction, namely *structural abstraction*, that takes place on mental objects, and, even more important, considers *complementarity* of diverse features of mathematical objects instead of their similarity. The notion of structural abstraction has been introduced by Tall (see 2013) as a form of long term development in mathematical thinking with a focus on the properties of objects. Scheiner (2016) and Scheiner and Pinto (2014) further elaborated Tall's notion of structural abstraction to draw out the cognitive architecture of this kind of abstraction, accounting for both an objects-structure perspective and a knowledge-structure perspective. The data of a previous study (Pinto, 1998) was revisited, offering in the present paper a context for insights into students' sense-making of formal mathematics through the lens of the structural abstraction framework. Reinterpreting the data resulted in, and still contributes to, an evolving framework that may serve as a potentially useful tool in analyzing cognitive processes in mathematics learning with students' particular sense-making strategies that have not been captured by abstraction-from-actions approaches.

In this paper, we build upon previous research using the evolving framework of structural abstraction in providing insights in students' mathematical concept construction compatible with their sense-making strategy of 'giving meaning' (Scheiner, 2016; Scheiner & Pinto, 2014). Particularly, we take the revision of a case study of a student, called Chris, as a point of reference (Scheiner & Pinto, 2014) – a first-year undergraduate mathematics student, who “consistently understood [the formal concept] by just reconstructing it from the concept image” (Pinto, 1998). The object of consideration in this paper is another student, called Colin, who – similar to Chris – ‘gave meaning’ to the formal content. We begin this paper by sketching the structural abstraction framework and the research methodology of our project. The selected instances from Colin's case do not only highlight the analytical power of the structural abstraction framework but also indicate profitable directions for its advancement. It is important to note that the overall agenda in developing a theoretical framework of structural abstraction is not to challenge or explain ideas presented in an original work or to contrast and compete with recent approaches in mathematics education but to theorize about, to provide deeper meaning to older ideas, and to take them forward in ways not conceived yet.

THEORETICAL BACKGROUND

Structural abstraction is proposed as embedded in a cognitive architecture that takes place both on the objects-structures and on the knowledge-structures. It has a dual nature: (1) *complementarizing* the meaningful aspects and the structure underlying specific objects falling under a particular mathematical concept, and (2) promoting the growth of coherent and complex knowledge structures through *restructuring* of the knowledge system gained through the former process.

From the objects-structure perspective, we assume that the meaning of a concept is almost always contained in a unity of meaningful components of a variety of specific objects that fall under the particular concept. For the (socially constructed) meaning of a mathematical concept we draw on Frege's (1892) observation that the meaning is not directly accessible through the concept itself but through objects that fall under the concept. In this sense, we cannot take as absolute the 'complete construction' of the meaning of the concept. Rather than trying to draw a sharp line between whether an individual has (or has not) constructed the whole meaning of a mathematical concept, or to elaborate stages of objects-structure development, we pay particular attention to partial constructions of the concept that students develop, and how they make use of them in constructing new knowledge. In our view of the structural abstraction framework, a concretizing process is demanded to particularize meaningful components and the underlying structure of an object falling under the mathematical concept. Concretizing may occur through contextualization that is, placing object(s) in different specific contexts. Structural abstraction, then, means (mentally) structuring aspects and the underlying structure of these specific objects. In contrast to an empiricist view whose conceptual unity relies on the commonality of elements, it is the interrelatedness of diverse elements that creates unity. Thus, the core mechanism of structural abstraction is complementarizing rather than seeking for similarity. In addition, we suggest that, in the complementarizing process, a representation may be developed that is used generically for several other instances, and, in doing so, may provide a theoretical structure in constructing the meaningful components of the objects. Here we draw on Yopp and Ely's (2016) insightful contribution indicating that what makes an example generic has not only to do with whether the example is a carrier of the general but also with the actions performed on it – a lesson that Balacheff (1988) tried to teach long ago.

For students who 'give meaning' such 'representations of' are used generically as 'representations for' sense-making in mathematics. This shift from establishing a representation of a concept to using this representation generically for constructing and reconstructing the concept in new contexts, could be described in terms of shifting from a 'model of' to a 'model for' (Streefland, 1985). Models are, in this sense, intermediate in abstractness between 'the abstract' and 'the concrete'. This means that in the beginning of a learning process a model is constituted that supports the 'ascending from the abstract to the concrete' as described by Davydov (see 1972/1990). Davydov's strategy of ascending from the abstract to the concrete draws

the transition from the general to the particular in the sense that learners initially seek out a primary general structure, and, in further progress, deduce multiple particular features of objects using that structure as their mainstay. The crucial aspect in this approach is Ilyenkov's (1982) observation that "the concrete is realized in thinking through the abstract" (p. 37). The key feature within the objects-structure perspective, however, lays in the idea that specific objects falling under a particular concept mutually complement each other, so that the abstractness of each of them, taken separately, is overcome. In this sense, and in line with a dialectical perspective described by Ilyenkov (1982) but different from empiricist approaches, structural abstraction is a movement towards complementarity of diverse aspects that creates conceptual unity among objects.

From the knowledge-structure perspective, we take the view that knowledge is a complex system of many kinds of knowledge elements and structures. Structural abstraction implies a process of restructuring and expanding the knowledge system, consisting of such 'pieces of knowledge' that have been constructed through the processes described above. The cognitive function of structural abstraction is to facilitate the assembly of more complex knowledge structures. The guiding philosophy of this approach is rooted in the assumption that learners acquire mathematical concepts initially on their backgrounds of existing domain-specific conceptual knowledge through progressive integration of previous concept images and/or by the insertion of a new discourse alongside existing concept images.

The reanalysis of empirical data gained from Pinto's (1998) study has shown that students, who give meaning, build a representation of the concept and, at the same time, use it generically for reconstructing the concept in other contexts – such as in verbal recovering the formal definition. The analysis also showed that students generically used representations of the concept to build pieces of knowledge. To put it in other words, the representations are actively taken as representations for producing new knowledge and sense-making of mathematics. This mental shift from 'representations of' to 'representations for' may indicate a degree of awareness of the meaningful components and a level of complexity of the knowledge system (Scheiner & Pinto, 2014). In this paper, we discuss one student's non-linear knowing and learning development of the limit concept of a sequence.

RESEARCH PURPOSE

The purpose of the paper is twofold: (a) refining and extending the theoretical framework through paying particular attention to eventually unsettled issues about structural abstraction, and (b) providing further insights in its potential power for the analysis of an individual's partial construction of the limit concept of a sequence, consistent with his sense-making strategy. In doing so, we focus on those aspects of the learning phenomena that are illuminated by using the structural abstraction framework (and that have not been noticed before). Thus, the framework functions both as a tool for research and as an object of research, a distinction already made by

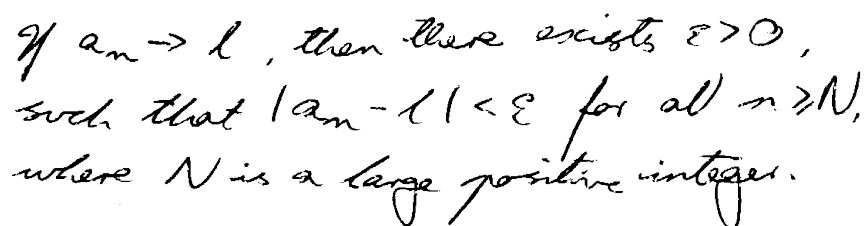
Assude, Boero, Herbst, Lerman, and Radford (2008). Our agenda is driven by re-examining an earlier study (Pinto, 1998) that identified a sense-making strategy of formal mathematics that has not fully been captured by abstraction-from-actions approaches in the literature on knowing and learning mathematics. The original data were collected taking an inductive approach throughout two academic terms during students' first-year at a university in England, through classroom observation field notes and transcriptions of semi-structural individual interviews. Interviews took place every two weeks with eleven students in total. A cross-sectional analysis of three pairs of students resulted in an identification of two prototypical sense-making strategies: 'extracting meaning' and 'giving meaning'.

"Extracting meaning involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

The latter strategy is the object of our study. In this paper, we selected instances from the available data of the case study of a particular student, called Colin.

SELECTED INSTANCES FROM A CASE STUDY

At the beginning of his first course on real analysis, Colin expressed, in his first interview, the formal definition of the limit of a sequence as follows:



If $a_n \rightarrow l$, then there exists $\epsilon > 0$,
such that $|a_n - l| < \epsilon$ for all $n \geq N$,
where N is a large positive integer.

(Pinto, 1998, p. 201)

His partial reconstruction of the formal concept definition of limit is a productive formulation (in a sense that it works in various contexts) of a property of a convergent sequence. His sense-making is coherent with his written definition:

Umm ... it means that the difference between ... umm ... the terms in sequence a_n and the limit gets very very small indeed and it gets below a certain umm pre-determined value. [...]. Err ... yes, after you go far enough out in the sequence. (Colin, first interview)

(Pinto, 1998, p. 203)

A dynamic view of a sequence as a process, implicit in evoked images such as 'you may go far enough out', and of the limit concept in terms of 'getting very small' are both indicated. Images such as 'arbitrarily small quantities', or 'infinitesimals', which are common in secondary school learning settings, are recalled with the use of the dynamic language of 'gets very very small indeed'. This is consistent with results in Martinez-Planell, Gonzalez, DiCristina, and Acevedo (2012) on students' understanding of series. The authors focused on whether students were seeing series

as a process without an end or as a sequence of partial sums, as stated by definition; and referred to Arnon et al.'s (2014) APOS theory to respond how students may construct the notion, by considering a distinction amongst the understandings of the concept of a sequence as a list of numbers or as a function defined in natural numbers (McDonald, Mathews, & Strobel, 2000). Martinez-Planell et al. (2012) concluded that even after formal training, students often think of sequences and series as an infinite unending process, and evoke dynamical aspects, as Colin did.

We approach the phenomena from the perspective of the structural abstraction framework and understand that, in line with his personal concept definition, the result of Colin's contextualizing processes resulted in a representation of the limit of a sequence as that of a descending curve (see Fig.1):

... umm, [I] sort of imagine the curve just coming down like this and dipping below a point which is epsilon... and this would be N . So as soon as they dip below this point then ... the terms bigger than this [pointing from N to the right] tend to a certain limit, if you make this small enough [pointing to the value of epsilon]. (Colin, first interview)

(Pinto, 1998, p. 202).

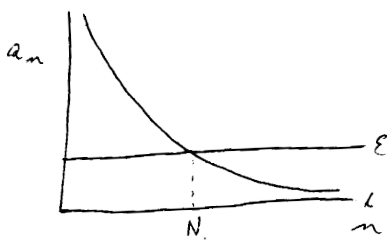


Fig. 1: Colin's first picture

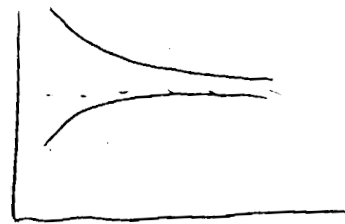


Fig. 2: Colin's second picture

Colin accentuated the image of a decreasing sequence by saying “as soon as they dip below the point then”. The second picture he drew (see Fig. 2) is based on the idea that “convergence could happen from above and below”. In other words, it seems that he evoked images of a convergent sequence identifying it with monotonic ones. He was able to explore his representations dynamically, interpreting and exploring the actions involved in his written definition relying on partial constructions that are specific and productive to some familiar contexts. In this light, it seems that Colin has interpreted (in the sense of Piaget's notion of assimilation) new concepts in terms of his prior knowledge:

Umm ... in A' level we used to ... umm ... plot sequences and generally you might get a sequence like this and ... *it would tend down* to a value or something. ... the little bit I had done at A-level I just sort of settled into it quite well... (Colin, first interview)

(Pinto, 1998, p. 203)

Thus, from the objects-structure perspective, Colin's partial reconstruction of a convergent sequence still related to a descending function or sequence, in a context

where a formal discourse is inserted. From the knowledge-structure perspective, his *representation of* a convergent sequence was generically used as a *representation for* constructing knowledge, as those students who ‘give meaning’ did. On the one hand, his representation of a convergent sequence was productive, as many times he sensed when results and claims were true. On the other hand, many statements became self-evident for Colin while his earlier mathematical discourse still was not recontextualized within the formal experience, as when he was asked to prove:

If $a_n \rightarrow 1$, prove that there exists $N \in \mathbb{N}$ such that $a_n > \frac{3}{4}$ for all $n > N$.

Colin said: ‘It seemed to be a silly question that ... if a_n tends to 1 then if you question when a_n is greater than $\frac{3}{4}$... this is a bound, it seems ... I don’t know why.’ (Colin, third interview)

(Pinto, 1998, p. 221)

Colin’s representation of a convergent sequence and its limit, which was coherent with his sense-making and his written definition, is a potential conflict factor (Tall & Vinner, 1991) concerning its use and the formal discourse. It did not enable him to produce a formal proof. Here, other than seeing the formal content as demanding, it is its complementary aspect that matters. Colin eventually noticed the new discourse introduced by the formalism as increasingly conflicting with his sense-making of the theory. In many occasions he ignored it and simply added it as an information:

There are certain things that ... I think they’re okay and I just learn that, it’s sort of that’s defined to be that ... (Colin, seventh interview)

(Pinto, 1998, p. 205)

In synthesis, in many contexts and situations, students may activate the various partial constructions productively. Such an attitude could be common; but in Colin’s case, various issues related to recontextualization of his concept image seem to miss.

DISCUSSION

In this paper, we presented data showing that a student, called Colin, built partial constructions of a convergent sequence that he used as representations of the concept. Such use was productive to particular contexts, but remained unproductive in others; for instance, to deal with formal mathematics. Colin could perceive that a statement is true, based on the properties of the concept he observed and concretized in a representation that he used generically (see Yopp & Elly, 2016), as a representation for building knowledge. Using his representation of the limit concept as a ‘definition’ Colin was able to evoke formal results, although he was unable to make deductions. Colin’s awareness of the formal requirements in the new context at university was not immediate. His description of the natural flow of his transition from school to university, expressed during his first interview, indicates that he did not perceive that the concretized knowledge he learned at school and the formal context at university were already in conflict. As the course progressed, he was gradually conscious of

conflicting aspects in his understanding; though he added new knowledge as information rather than (re-)structuring the prior mathematical experience. There are students whose sense-making of the mathematics is detached from their learning of the institutional knowledge. They deal with those as if sense-making and institutional knowledge were compartmentalized knowledge structures (see Vinner, 1991, p. 70). What strikes us in the selected instances of Colin's case study was the cohesion in his sense-making and in learning the formal mathematics concept. Coherence amongst students' sense-making and their (re-)construction of the formal content has been proven to be a central characteristic of those students who 'give meaning'.

From the objects-structures perspective of the structural abstraction framework, the aspect of 'complementarizing' meaningful components reflects the idea that whether an individual has 'grasped' the meaning of a concept can only be considered in specific contexts. This makes clear that "the subjective nature of understanding [...] is not [...] an all-or-nothing state" (Skemp, 1986, p. 43). A comparison with Chris' case (Scheiner & Pinto, 2014), another student who 'gave meaning', shows that although Chris did not 'have' all relevant meaningful components at hand, he was able – using his 'generic representation' – to generate some of them at need. The growing complexity of his representation of the convergence of a sequence, gradually constructed in particular settings, served as a representation for reconstructing and recontextualizing the limit concept in the formal context. We argue that Colin's understanding of a convergent sequence must increase in complexity and complementarity, which could be achieved through contextualizing as well as integrating various constructions; the latter may be promoted by the insertion of new mathematical discourses alongside earlier concept images.

From the knowledge-structures perspective, structural abstraction is a process of restructuring the 'pieces of knowledge' constructed through contextualizing and complementarizing. In using the structure of the representation, some meaningful components of the concept may be productively activated in diverse contexts. Such use may even allow to generate new knowledge pieces. In both cases, Chris and Colin, a shift from a representation of (the convergent sequence) to a representation for generating knowledge can be documented. The shift does not result in knowledge restructuring per se, as we could identify in Colin's case. On the other hand, Chris' case suggested that even a 'representation for' may be complementarized by new knowledge elements, and such a process becomes recursive (Scheiner & Pinto, 2014).

CONCLUDING REMARKS

The structural abstraction framework takes the view that knowledge is an evolving, complex, and dynamic system of many kinds of knowledge elements and structures. Abstraction is seen as a movement across levels of complementarity and complexity (Scheiner, 2016). The case study in Scheiner and Pinto (2014) and the one provided in this paper raise directions for advancing our understanding of structural abstraction. Both cases reveal (1) a *cohesion* amongst their sense-making strategy of

giving meaning, and (2) a *generic use* of their constructed representations to reconstruct the limit concept in other and new contexts. Contrasting the two cases shows that the two students differed in the degree of complementarity and complexity of the representations used. Chris' representation could be considered as being generic in terms of being a carrier of the general (Mason & Pimm, 1984) that he used to reconstruct meaningful components at need. Colin's representation did not allow him to do so – maybe due to its degree of complexity and complementarity. Other questions to be addressed are raised by the use of representations in knowledge structuring – as a tool to reconstruct knowledge, as Chris did, or as an object in place of the definitions, maintaining the earlier mathematical discourse, as in Colin's case.

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Une étude de l'autonomie en mathématiques dans la transition secondaire-supérieur

Pierre-Vincent Quéré¹

¹CREAD, UBO, France, pierre-vincent.quere@ac-rennes.fr

Cet article traite du thème de l'autonomie en mathématiques pour des étudiants débutant des études menant à un diplôme d'ingénieur. Quelle forme d'autonomie est réellement attendue ? Et quels sont les moyens proposés, et les moyens effectivement utilisés par les étudiants pour le développement de cette autonomie ? Nous étudions ces questions dans le cadre de l'approche anthropologique. Nous avons proposé un questionnaire et réalisé des entretiens avec des étudiants. L'analyse de ces données montre, tout d'abord, que les ressources ou dispositifs proposés sont variés, ce qui représente pour eux une réelle nouveauté. Néanmoins, dans les faits, les étudiants n'en apprécient et n'en utilisent qu'une partie et développent autour de ces propositions leurs propres stratégies de travail autonome.

Mots-clés: Autonomie, classes préparatoires, ressources, transition secondaire-supérieur.

INTRODUCTION

Dans la formation d'un étudiant après le bac, la transition entre le secondaire et le supérieur représente un réel défi (Gueudet, 2008). À cette occasion en effet, les bouleversements auxquels il est amené à faire face sont de plusieurs ordres, que l'on peut catégoriser grossièrement : social, méthodologique, cognitif. Dans tous ces domaines, l'étudiant devra faire preuve d'une autonomie croissante et l'injonction à l'autonomie à laquelle il est soumis dans cette transition peut provenir de l'institution elle-même en se déclinant encore sous différents aspects : par exemple dans l'organisation du travail, dans l'utilisation de ressources nouvelles, ou encore, à l'échelle d'une discipline (ici en mathématiques) dans la prise d'initiative pour la résolution d'un problème.

Dans notre travail, nous nous sommes donc intéressé à des étudiants en début de parcours universitaire, plus précisément dans un Cycle Préparatoire Intégré (CPI) correspondant aux deux premières années de la formation au sein d'une école d'ingénieurs. Ce cursus est à distinguer des Classes Préparatoires aux Grandes Ecoles (CPGE) qui correspondent au même niveau de formation, mais dans un lycée indépendant de la future école d'ingénieurs qui sera, en cas de succès, intégrée par l'étudiant après concours. L'organisation du travail personnel en CPGE a été l'objet récent de travaux (Farah, 2015) reprenant ceux de Castela au lycée (Castela, 2008). Nous nous situerons dans le prolongement de ceux-ci, en centrant notre questionnement sur l'autonomie, qui est en lien avec le travail personnel mais ne se limite pas à celui-ci.

Pour aborder la problématique soulevée ici, nous commencerons par énoncer, dans le cadre théorique de l'approche anthropologique, la question de recherche à laquelle nous nous proposons d'apporter des éléments de réponses. Nous présenterons ensuite la méthodologie que nous avons utilisée pour obtenir les résultats que nous détaillerons dans notre dernière partie.

QUESTION DE RECHERCHE ET CADRE THEORIQUE

Le point précis sur lequel nous allons étudier le thème de l'autonomie est celui des moyens utilisés pour parvenir à la développer, ce qui nous permet d'énoncer ici la question de recherche suivante : "Quelles sont les ressources, les dispositifs ou les supports à même d'aider le nouvel étudiant à développer ses capacités d'autonomie en mathématiques ?"

L'idée philosophique, politique et pédagogique (voire économique ?) d'autonomie s'est imposée dans le monde de la Formation ces dernières années. Cette apparition est le fruit d'un besoin de modernisation de l'enseignement sous-tendu par une remise en cause importante due à la massification et la diversification des publics à l'université (Albero & Poteaux, 2010). Cette diversification n'a pas épargné la formation d'ingénieurs. Or, derrière cette notion générale d'autonomie, se cachent plusieurs concepts ; il ne faudrait surtout pas réduire cette notion à celle d'un travail autonome de l'étudiant utilisant les Technologies de l'Information et de la Communication (TIC). Il y a aussi plusieurs conceptions possibles des étudiants vis-à-vis de leur propre autonomie (Liebendörfer & Hochmut, 2015). Ici, nous menons une étude du point de vue de la didactique des mathématiques.

Le cadre que nous choisissons pour préciser le sens de cette notion d'autonomie dans le cas des mathématiques est la Théorie Anthropologique du Didactique (TAD) (Chevallard, 2007). Dans cette théorie, nous nous intéressons plus précisément au concept de praxéologie (ou organisation mathématique) qui est un système complet $[T, \tau, \theta, \Theta]$ destiné à modéliser toute activité (vue comme activité d'un sujet d'une certaine institution). Parmi les quatre éléments qui composent cette organisation, on trouve tout d'abord le type de tâche T . Par exemple, si l'on s'intéresse à la résolution d'équations différentielles du second ordre, on peut choisir d'affiner T en précisant que les coefficients sont constants ou que le second membre est d'un type particulier. Le type de tâche T ainsi choisi pour l'observation est alors mis en relation avec des techniques τ , pour définir un *bloc pratico-technique* (identifié couramment à un savoir-faire, ou une pratique). Vient enfin le deuxième bloc, appelé *bloc technologico-théorique* (ou plus couramment théorie) formé des technologies θ (à prendre au sens d'un discours rationnel permettant de justifier la technique τ utilisée, et non pas directement au sens moderne), et de la théorie Θ , laquelle reprend, par rapport à la technologie θ , le rôle que cette dernière tient par rapport à la technique τ . Comme nous l'avons soulevé précédemment, nous admettons que seul un petit nombre de techniques sera reconnu par une institution donnée, ce qui en fait un outil adapté à notre travail.

C'est ainsi également que ce modèle nous permet une étude de la transition secondaire-supérieur en terme d'autonomie. Étudier les praxéologies utilisées par l'institution peut amener à proposer une définition tenant compte de cette perspective institutionnelle : "l'autonomie, c'est être capable de développer par soi-même une technique adéquate pour effectuer un type de tâches, et de tenir un discours technologique cohérent associé". Nous reviendrons en conclusion sur sa pertinence.

La TAD permet ensuite de se placer à différents niveaux appelés "niveaux de codétermination didactiques" : celui d'une question, pour un exercice particulier (ou d'un thème) ; celui de la discipline mathématique (dans ce cas le type de tâches serait par exemple : "apprendre une démonstration", et une technique possible "recopier la démonstration" ou "refaire la démonstration sans la regarder"), mais aussi celui de la pédagogie (le type de tâche étant par exemple "organiser son travail personnel"). Rappelons ici tous les niveaux définis dans cette échelle :

sujet<>thème<>secteur<>domaine<>discipline<>pédagogie<>école<>société

Pour notre part, nous allons utiliser les praxéologies à plusieurs niveaux : celui d'un *sujet*, en observant comment les étudiants résolvent des exercices particuliers ; celui de la *discipline*, en regardant plus généralement comment ils organisent leur travail en mathématiques ; et celui de la *pédagogie*, concernant plus généralement leur travail en CPI. Nous allons tenter d'identifier à chaque niveau des types de tâches présents dans leur travail personnel. Pour cela, nous nous appuyons tout d'abord sur notre connaissance des étudiants, ainsi que sur le travail de Farah sur les CPGE (Farah, 2015) pour chercher à définir *a priori* une liste non exhaustive de types de tâches, répertoriés à partir des phases de travail personnel des étudiants :

- au niveau de la pédagogie, il y a l'organisation du travail personnel avec le choix de ce qu'il faut travailler, du temps à y consacrer, des supports et des aides.
- à un niveau de la discipline, il y a le travail (le type de travail en mathématiques n'est pas le même que, par exemple, en psychologie) entre deux séances, la préparation des évaluations, le travail sur des projets donnés par l'enseignant, la recherche d'aides à la compréhension d'une notion.
- au niveau du sujet, il y a le travail du cours correspondant, le travail d'exercices (traités ou non en TD, donnés en préparation par l'enseignant, ou choisis seuls par l'étudiant dans un livre ou sur internet),

Nous recouperons *a posteriori* cette liste avec nos données et nos observations pour affiner notre description. Nous chercherons si ces types de tâches sont nouveaux en CPI ou s'ils existent aussi au lycée; puis nous tenterons de voir si les étudiants disposent d'une, ou plusieurs techniques pour accomplir ces tâches.

METHODOLOGIE

Ainsi, pour répondre à la question formulée, nous avons mis en place une enquête en deux temps sur le thème général de l'autonomie dans deux classes de CPI (première

et deuxième année). Nous allons ici en détailler les parties qui touchent exclusivement au thème de notre article.

La première partie de notre enquête a consisté à faire réagir l'ensemble des 199 étudiants du CPI concerné sur un questionnaire général sur l'autonomie. Ce questionnaire en ligne, anonyme, comportait une partie "Autonomie et dispositifs institutionnels" qui était principalement axée sur la formation en mathématiques. Il s'agissait d'avoir une première série de données relatives aux moyens les plus utilisés au lycée puis en CPI, à savoir, dans le désordre : logiciels pour l'aide à la résolution de problèmes, devoirs maison, projets ou exposés, ressources pour la préparation ou l'approfondissement de séances (internet, livres, etc.), photocopiés de cours et/ou d'exercices, cours hybride, cours magistral, projection d'un diaporama en cours, questions au professeurs, aux camarades.

Nous avons également posé une question ouverte (« *Avez-vous des commentaires à ajouter au sujet de votre autonomie dans vos études ?*») à laquelle nous avons eu, comme nous le verrons dans la partie sur les résultats, des réponses en lien direct avec l'utilisation de ces dispositifs.

La suite de l'enquête a été de nous entretenir avec des étudiants volontaires (nous le leur avons proposé dans le questionnaire). Dans ces entretiens semi-directifs de type ethnologique, nous avons d'abord cherché à approfondir un ensemble de points toujours en rapport avec l'autonomie. De plus, quelques jours avant l'entretien individuel, nous avons remis aux étudiants une liste de trois exercices de mathématiques dans des domaines que nous avons repérés dans le questionnaire comme étant source de différentes mises en pratique de leur autonomie : la géométrie avec les nombres complexes, les équations différentielles et le calcul matriciel.

RESULTATS

Le questionnaire

Le questionnaire en ligne (cf. annexe) a été rempli par 179 étudiants sur les 199, ce qui représente un très bon taux de représentation. La plupart des résultats dont nous soulignons l'importance ici sont ceux qui sont majoritaires (souvent à plus de 70-75%).

Les deux premières questions sont relatives au thème de l'autonomie et des dispositifs institutionnels ("*Dans votre formation antérieure, vos enseignants de mathématiques vous ont proposé les dispositifs suivants dans le but de développer votre autonomie*", cf. annexe), nous avons pu relever que seule la pratique du "devoir maison" est employée dans la majorité des lycées, et ce, d'après 96% des étudiants interrogés. Ce type de devoir possède la particularité d'être rédigé par les élèves (seuls ou en groupe) en dehors du temps de classe. Le travail rendu est ensuite corrigé par l'enseignant. Aucune des autres propositions faites dans la liste présentée (logiciels, projets, exposé, utilisation d'internet, de livres) n'a reçu une majorité de réponses positives : d'après les étudiants, cette utilisation du "devoir maison" dans le secondaire serait

donc représentative des attentes et des pratiques de l'autonomie pour les enseignants. À la lecture de ces réponses, il semble que l'utilisation de ressources pour préparer ou approfondir une séance, l'utilisation de logiciel ou enfin la préparation de projets ou d'exposés ne soit pas courantes au lycée.

Ensuite, pour les dispositifs ou ressources proposés par l'institution favorisant l'autonomie dans leur formation actuelle, le plus grand nombre d'avis positifs va au photocopie de cours ou d'exercices (81% et 86%). Viennent ensuite les "questions au professeur" avec 72% d'adhésion. Nous avons en effet fait cette proposition car elle représente à nos yeux une capacité relevant pleinement de l'autonomie : outre l'affrontement du regard des autres membres du groupe, poser une question au professeur signifie pour l'étudiant avoir su identifier un problème et l'exprimer sous forme d'une question. La base de données scientifique WOLFRAM ALPHA (qui se présente comme un moteur de recherche classique, auquel il est possible de soumettre une requête, sans que la syntaxe ne respecte une forme mathématique précise), utilisée en accompagnement libre de recherche de solutions pour des exercices arrive également, avec le logiciel institutionnel de calcul formel SAGE (utilisé, lui, pour des TP de mathématiques) en bonne position avec respectivement 69% et 64% d'avis favorables.

Enfin, à l'occasion de la question ouverte qui a été posée dans cette partie du questionnaire, nous avons relevé plusieurs réponses qui prouvent l'existence d'une réflexion au sujet de l'autonomie de la part des nouveaux étudiants. Cette réflexion s'appuie sur leur expérience passée ainsi que sur leur ressenti dû à leur nouveau statut. Nous en présentons ici les plus pertinentes pour notre question de recherche, en terme d'autonomie :

- *en prépa, un travail autonome important s'impose (recherches sur internet, exercices supplémentaires).*
- *par rapport au collège et au lycée, on peut se faire de bons amis avec qui travailler en groupe.*
- *la manière de travailler change complètement par rapport au lycée : refaire les TD, faire des fiches.*
- *au lycée, on est beaucoup plus guidés, on n'a pas de méthode de travail. On devrait nous apprendre à être autonome en arrivant.*
- *il est délicat de faire des exercices sans correction.*
- *l'autonomie nécessite une bonne organisation (notamment pour gérer l'irrégularité des semaines en terme de besoin de travail personnel).*
- *les travaux dirigés aident à l'autonomie, si on cherche par soi-même et qu'on a une correction après.*

Que nous enseignent ces réponses au questionnaire, en termes de praxéologies ? Pour les niveaux de détermination, nous pouvons ici nous situer au niveau général de la *pédagogie*, puis au niveau de la *discipline* mathématique. Nous identifions, dans les

réponses des étudiants, certains types de tâches et techniques associées qui nous semblent pertinents pour une étude de l'autonomie.

Au niveau pédagogique, le type de tâche que nous pouvons faire ressortir serait : "Organiser son travail personnel". Ce type de tâche est certainement moins présent au lycée, les professeurs cadrant plus le travail personnel à effectuer, ce qui se traduit par le fait de se sentir "guidés". Arrivés en CPI, les étudiants développent alors de nouvelles techniques : se donner des moments réguliers pour reprendre le cours ou les TD ainsi que des moments de travail collectif.

Au niveau de la discipline, un type de tâche pourrait être : "Choisir des exercices à faire" en remarquant que la notion d'exercice est centrale en mathématiques, ce qui n'est pas le cas dans d'autres disciplines. Ce type de tâche n'est certainement pas présent au lycée, car il s'agit seulement de faire les exercices donnés par le professeur. En CPI, on constate que les étudiants essaient de travailler de leur propre initiative certains exercices. Et selon les cas, diverses techniques apparaissent dans les réponses au questionnaire : "Refaire les exercices traités en TD" ; "Tenter de faire des exercices de la feuille de TD qui n'ont pas déjà été traités" ; "Chercher des exercices corrigés dans des ressources externes" ;

On peut encore définir un autre type de tâche : "Travailler une démonstration" ou "Apprendre un théorème". Là encore, il semble que ce type de tâche ne soit pas présent au lycée et la technique mise en place alors peut être "Faire des fiches".

Pour aller plus loin dans notre analyse des praxéologies selon notre modèle, nous allons nous servir de données recueillies dans la deuxième phase de notre enquête.

Les entretiens

Pour affiner notre étude et notre analyse, nous avons choisi de nous entretenir avec des étudiants après avoir fait une première extraction des réponses produites lors du questionnaire. Notre objectif était de sélectionner un panel d'étudiants représentatif de l'ensemble, tout en repérant ceux qui semblaient, notamment par leurs réponses à la question ouverte, plus ou moins engagés dans une réflexion sur le thème de l'autonomie.

A propos des ressources utilisées par les étudiants, les retours que nous avons obtenus lors de ces entretiens corroborent les réponses déjà observées lors du questionnaire et nous n'allons pas en proposer de nouvelle analyse. Nous pouvons néanmoins énoncer des précisions qui nous ont été formulées, par exemple, pour les différents usages du photocopié : certains s'en servent comme document de référence *a posteriori*, d'autres recopient son contenu pour apprendre et mémoriser, d'autres encore y voient une manière de percevoir un autre point de vue sur la notion (parfois plus synthétique) que celui présenté en cours.

Les entretiens individuels ont également mis au jour deux ressources qui n'avaient pas été proposées dans le questionnaire : les sujets d'examens passés ; travailler sur ces sujets peut représenter en termes de praxéologies, une technique possible pour un

type de tâche qui serait "Préparer une évaluation". Notons aussi l'ajout de la ressource "des amis de travail" dans la liste de celles fréquemment utilisées. En effet, comme le montraient déjà certaines phrases relevées dans les réponses à la question ouverte, le travail personnel est pour certains étudiants une nouveauté en terme d'organisation. Si l'on ajoute à cela la nouveauté que représente pour la majorité des étudiants la vie éloignée de sa famille, il paraît alors de plus en plus naturel pour plusieurs d'entre eux de partager des moments de travail en commun où certains jouent parfois le rôle de celui qui explique et l'autre celui qui reçoit les explications. Ceci s'intègre parfaitement dans leur vision de l'autonomie.

Intéressons nous pour terminer à la partie des entretiens relative à la résolution d'exercices de mathématiques. L'objectif étant ici de percevoir les ressorts utilisés par les étudiants en terme d'autonomie, dans une situation de résolution de problème, en nous basant sur le modèle des praxéologies présenté plus haut. L'activité que nous décidons d'analyser est la résolution de l'exercice suivant :

"Résoudre l'équation différentielle d'inconnue y suivante :

$$y''(t)-2y'(t)-8y(t)=\sin^2(t) "$$

Pour l'objectif de nos recherches, cet exercice présente plusieurs avantages :

- il doit être résolu par une méthode systématique qui doit faire partie du bagage d'un futur ingénieur (résolution de l'équation homogène associée à l'aide de l'équation caractéristique, puis recherche d'une solution particulière),
- il nécessite l'utilisation de techniques de transformations, à l'initiative de l'étudiant, afin de réécrire le second membre (linéarisation du terme en \sin^2) pour trouver une solution particulière par superposition,
- les calculs sont réalisables "à la main", mais il peut se résoudre sans difficulté à l'aide d'un logiciel de calcul formel.

Chez les huit étudiants rencontrés, nous nous sommes intéressés à la démarche mise en place pour l'activité représentée par cet exercice. Nous pouvons chercher à traduire cette démarche en terme de praxéologies pertinentes pour l'autonomie.

En analysant les réponses on constate qu'après "Identifier le type d'équation" qui peut engager une technique de type "Recherche dans un répertoire", apparaît un autre type de tâche que l'on peut dénommer : "Se souvenir de la méthode générale" (six étudiants). La technique alors mise en place est de se tourner vers le photocopié de cours, ne serait-ce que pour vérifier avant de se lancer dans les calculs. La deuxième possibilité est d'utiliser une de leurs fiches méthodes (rédigée à partir du cours).

Un deuxième type de tâche apparaissant ici est : "Rechercher une solution particulière", et comme nous l'avons vu, c'est l'idée de la linéarisation de $\sin^2(t)$ qui représentait sans doute la plus grande difficulté de cet exercice. Là encore, en tant que technique, beaucoup ont utilisé leur cours qui contient une liste de seconds membres à connaître par cœur. Cependant, pour la formule de linéarisation, qui nécessite une réelle autonomie mathématique, il est opportun de constater que deux

d'entre eux ont utilisé le moteur WOLFRAM ALPHA (qui permet d'obtenir des formules, et pas seulement d'effectuer des calculs). Signalons que deux des étudiants nous ont raconté avoir eu par hasard l'idée de linéariser, parce qu'au même moment en classe, il étaient en train d'étudier le cours sur le calcul de primitives et que cette même méthode de linéarisation y est utilisée. Ceci confirme l'utilisation de ressources non directement prévues, *a priori* par l'institution.

Un troisième type de tâche peut être enfin mis en avant ici, que l'on peut dénommer "Effectuer les calculs". Pour tous les étudiants, la technique principale est de les effectuer à la main. A ce dernier type de tâche, on peut même en ajouter un ultime, celui de "Vérifier ses résultats". Seuls deux étudiants n'ont pas estimé nécessaire de le faire, et la technique la plus commune a été d'utiliser un logiciel ou une calculatrice.

Notons que certains étudiants n'ont pas hésité à se renseigner auprès de leurs camarades pour obtenir de l'aide, ce qui représente une technique mobilisable à tous les niveaux, et qu'ils ont tout fait pour parvenir à résoudre ce problème, en réfléchissant parfois sur plusieurs jours, ce qui représente également une technique possible ("Laisser passer du temps pour la réflexion"). Seuls deux étudiants n'ont pas abouti à la bonne solution de cette équation différentielle.

Avec toutes les observations que nous avons relevées dans cette activité, pour laquelle le type de tâche peut être défini comme "résoudre une équation différentielle du second ordre à coefficient constant", nous pouvons donc conclure que les techniques utilisées se développent à l'intérieur de plusieurs sous-niveaux en relation avec des sous-tâches. Tout d'abord, il leur faut connaître ou retrouver la méthode générale de résolution d'un tel problème. Pour cela une technique est de rechercher dans son cours ou ses fiches, ou bien éventuellement de se diriger vers une source extérieure comme internet ou un camarade.

Notons que la partie concernant la résolution de l'équation homogène (qui représente un type de tâche à part entière), il n'a pas été fait mention de difficulté particulière. Nous ne relèverons donc rien à ce sujet, si ce n'est que les techniques de résolution semblent acquises par le plus grand nombre, une fois la méthode générale retrouvée.

DISCUSSION – CONCLUSION

Le but de ce travail était d'éclairer, notamment à la lumière du cadre de la TAD, différents aspects de l'autonomie du travail des étudiants novices.

Dans un premier temps, nous pouvons dire que nous avons constaté que dans leur formation antérieure de lycéens, les étudiants semblent avoir eu assez peu l'occasion de mettre en action une quelconque forme d'autonomie. D'après notre enquête, la seule expérience en la matière, en dehors du travail quotidien, semble être la pratique du devoir maison. Cela soulève, dans la continuité de l'étude sur travail personnel en CPGE de Farah (2015), une sorte de paradoxe pour les étudiants du Cycle Préparatoire Intégré sur lequel notre étude a porté : leur réussite jusqu'au bac n'est pas nécessairement la conséquence d'un travail régulier, mais plutôt de capacités (dont les

étudiants eux-mêmes ont d'ailleurs souvent conscience) qui ne les ont pas obligés à développer un travail personnel régulier autonome. C'est alors peut-être suite à la rencontre de premières difficultés que les étudiants mettent en place des stratégies et des méthodes d'apprentissage dans leur travail personnel. Ceci se passe à différents niveaux, et bien qu'ils utilisent diverses ressources et divers dispositifs, on peut y déceler différents degrés d'autonomie. En effet, certaines de ces ressources peuvent être mises à leur disposition par l'institution, comme c'est le cas du photocopié de cours qui sert de référence. Mais, certains étudiants l'utilisent peu pour des raisons d'organisation personnelle. Ces ressources peuvent alors être d'une toute autre nature, comme par exemple l'aide précieuse d'un camarade ou d'un logiciel. Ceci nous permet d'entrevoir une sorte de décalage entre les attentes et les propositions institutionnelles en terme de praxéologies, et les réels gestes d'étude (Castela, 2008) des étudiants eux-mêmes : par exemple, l'étape "comment faire pour retrouver la méthode de résolution d'une équation différentielle ?" n'est pas prévue par l'institution. Nos résultats rejoignent ici ceux de Castela (2008), qui identifie des besoins d'apprentissage ignorés par l'institution.

Pour aller plus loin dans ce travail, nous pourrions chercher à identifier, toujours dans le cadre de la TAD, des parties du bloc *technologico-théorique* mis à disposition par l'institution et réellement utilisé par les étudiants dans les différents niveaux de codétermination mathématiques et didactiques. Ceci fournirait sans nul doute un aspect supplémentaire à notre analyse. De plus, nous tenons à souligner que le travail engagé ici ne saurait être considéré comme exhaustif, mais bien comme l'amorce de notre travail en thèse de doctorat.

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Partie 2 - Autonomie et dispositifs institutionnels

8 [21]
 Dans votre formation antérieure (lycée), vos enseignants de mathématiques vous ont proposé les dispositifs suivants, dans le but de développer votre autonomie :

* (01.NAOK == "A1" or 01.NAOK == "A3")

Choisissez la réponse appropriée pour chaque élément :

	Oui	Non
Utilisation de logiciels	<input type="radio"/>	<input type="radio"/>
Travail au calcul ou à la résolution de problèmes	<input type="radio"/>	<input type="radio"/>
Rédaction de devoirs	<input type="radio"/>	<input type="radio"/>
Réalisation de projets ou d'exposés	<input type="radio"/>	<input type="radio"/>
Utilisation de ressources (internet, livres) pour l'approfondissement d'une notion vue en cours ou en TD	<input type="radio"/>	<input type="radio"/>
Utilisation de ressources (internet, livres) pour l'approfondissement d'un thème en cours ou en TD	<input type="radio"/>	<input type="radio"/>

9 [22]
 Dans votre formation antérieure (lycée), avez vous ressenti des attentes de la part de vos enseignants de mathématiques pour que vous travailliez de manière autonome lorsqu'il s'agissait d(e)

* (01.NAOK == "A1" or 01.NAOK == "A3")

Répondre à cette question seulement si les conditions suivantes sont réunies :

Choisissez la réponse appropriée pour chaque élément :

	oui, toujours	oui, souvent	non, rarement	non, jamais
utiliser un (des) logiciel(s) comme aide pour la résolution de problèmes	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
rédigier des devoirs	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
réaliser des projets ou des exposés	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
utiliser des ressources (internet, livres) pour l'approfondissement d'une notion vue en cours ou en TD	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
utiliser des ressources (internet, livres) pour la préparation d'une future séance	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

11 [24] Les dispositifs listés ci-dessous favorisent le développement de votre autonomie :

Choisissez la réponse appropriée pour chaque élément :

	Totalement en désaccord	Plutôt en désaccord	Plutôt en accord	Totalement en accord
Le polycopié du cours	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Le polycopié des exercices	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Le logiciel sage	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
La base de donnée scientifique "Wolfram Alpha"	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Le travail sur des projets	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Un cours magistral	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Un diaporama projeté en cours	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Un cours hybride (en vidéo)	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Poser une question au professeur (en classe ou en dehors de la classe)	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

Activity and performance on a student-centred undergraduate mathematics course

Johanna Rämö¹, Lotta Oinonen¹, and Arto Vihavainen²

University of Helsinki, ¹Department of Mathematics and Statistics / ²Department of Computer Science, Finland, johanna.ramo@helsinki.fi.

This study investigated the connection between students' performance and activity for an introductory mathematics course organised using a student-centred teaching method. A relationship between performance and activity was found: the highest performing students reported that they attended drop-in sessions, worked with their peers, discussed with instructors, and attended lectures more than the weakest performing students. Moreover, students' motivations and learning strategies were related to their behaviour. The results suggest that it is important to provide the students with an accessible learning environment without time constraints, and offer them opportunities to collaborate with peers.

Keywords: student-centred, Extreme Apprenticeship, performance, attendance, Motivated Strategies for Learning.

INTRODUCTION

This study investigates the relationship between students' course performance and activity such as discussing with course instructors and working with peers. It focuses on an introductory mathematics course taught using a student-centred teaching method, Extreme Apprenticeship (Rämö, Oinonen, & Vikberg 2015; Vihavainen, Paksula, & Luukkainen, 2011). The core idea of the Extreme Apprenticeship method (XA) is to support students in becoming experts in their field by having them participate in activities that resemble those carried out by professionals. In mathematics this may mean, for example, reading mathematics, asking questions, discussing mathematical ideas and explaining one's reasoning. The theoretical background of XA is in Cognitive Apprenticeship (Brown, Collins, & Duguid, 1989). In the XA method, students learn skills and gain knowledge by working on tasks that have been divided into smaller and approachable goals, which are then merged together as the students start to master a topic. An important part of the teaching method is one-on-one instruction, and the students are also encouraged to work collaboratively. Bi-directional feedback between the instructor and the student plays a significant role: the students receive continuous feedback from their work, and at the same time the teachers receive feedback from the progress of the students, which can be used to craft materials and assignments that help students understand a topic.

In XA, studying revolves around working on tasks and discussing them with others. This means that the students' activity, such as attendance, help-seeking and collaboration is crucial. Several studies have found class attendance to be one of the

main factors predicting performance (e.g., Credé, Roch, & Kieszczynka, 2010; Moore, 2006; Newman-Ford, Fitzgibbon, Lloyd, & Thomas, 2008). Reasons found for students' absenteeism are, among others, pressures from other courses (Van Blerkom, 1992), and not seeing a studied class as interesting or important (Moore, Armstrong, & Pearson, 2008; Gump, 2006).

In the XA method, help-seeking is a necessity for the student, as discussions with instructors and other students are an important part of teaching. Help-seeking has been studied extensively among younger students in school context, and many of the results have been confirmed in higher education settings. Poor self-esteem or low perceptions of their cognitive competence can make students feel that help-seeking is threatening (Karabenick & Knapp, 1991; Newman, 1990; Ryan & Pintrich, 1997). Also, in classrooms where students felt that the focus is on understanding and mastery, and not as much on competition and proving one's ability, students were less likely to avoid help-seeking (Karabenick, 2004; Ryan, Gheen, & Midgley, 1998). Karabenick and Knapp (1991) have shown that students are more likely to ask for help from their peers than their teachers. Also the social climate of the classroom may have an effect: positive relations have been found to increase the likelihood that students ask for help from the teacher or from their peers (Newman & Schwager, 1993). Symonds, Lawson, and Robinson (2008) have studied university students' help-seeking in mathematics support centres. They noticed that a large portion of students who need support do not seek it. Among the reasons were lack of awareness of the need of support, lack of motivation, time-management issues, and embarrassment.

CONTEXT AND DATA

Context: The course “Introduction to university mathematics”

The participants of this study were students of an introductory mathematics course “Introduction to university mathematics”, which is a compulsory course to mathematics and computer science students. It lasted for 12 weeks, and introduced basic concepts such as sets, functions and relations, and familiarised the students with the concept of proof and various proving techniques. There were 569 students enrolled for the course, 44% of which were computer science students, 34% mathematics students and 6.8% statistics students. The course was taught with the Extreme Apprenticeship method. The teaching consisted of weekly tasks, one-on-one guidance and lectures. There was also a course material written by the teacher responsible for the course.

The students were given approximately 13 tasks per week, two of which were selected for inspection. Students received written feedback on their reasoning and readability of the solution, and were asked to improve their solutions when necessary. The students were awarded bonus points for completing the tasks. Guidance in solving the tasks and reading the course material was offered in daily

drop-in sessions by instructors and the teacher responsible for the course. The students could come to the drop-in sessions when it suited them, and spend there as much time as they wished. There were three lectures per week, and their role was to model the problem solving processes of mathematicians, and link the concepts of the course together. Neither attending the lectures nor completing the course assignments was mandatory. The course was assessed using a midterm exam and a final exam, both having an equal contribution to the final grade of the students.

Several arrangements were made to make it as easy as possible for the students to seek help. The drop-in sessions were organised in a collaborative learning space where guidance was offered approximately 20 hours per week. The learning space is an open space in the main corridor of the department, close to classrooms, school office and the student common room. This means that the students walk through it several times during the day, and it should be very feasible for them to sit down and start working. The instructors wear bright coloured vests, so that it is clear to the students who the teachers are. The tables in the learning space are arranged into groups and act as whiteboards, and the walls are covered with blackboards for the students to share their thoughts with each other and with the instructors.

The instructors were undergraduate and graduate students who were chosen via an interview. Students were used as instructors as there are indications that they are easier to approach than, for example, professors (Fingerson & Culley, 2001). During the course, the instructors went through training by taking part in weekly meetings. In the meetings, pedagogical aspects were discussed, such as how to interact with the students and how to guide them without giving away the answers. The instructors were advised that many of the students do not ask for help even when they need it, and therefore they should walk around and approach the students on their own initiative. Also, the instructors were informed of the importance of encouraging the students.

Data

Participants for this study were students of the course Introduction to university mathematics, which was organised at the University of Helsinki in autumn 2014. To measure attendance, students were asked to answer a short questionnaire regarding their activity during each week as they returned their coursework. The students were asked whether they (1) took part in the drop-in sessions, (2) talked to the instructors, (3) worked together with their peers or (4) attended the lectures at least once during the week. All in all, each student could return up to 12 weeks worth of coursework including weekly activity details.

In addition to answering the questionnaires regarding attendance and activity, the students were given the Motivated Strategies for Learning Questionnaire, MSLQ (Pintrich, Smith, Garcia, & McKeachie, 1993), which is a validated instrument for assessing students' learning strategies and motivations. The students completed the

questionnaire in three phases during the three weeks following the midterm exam. Answering was not compulsory, but students gained bonus points for completing the questionnaire.

RESEARCH QUESTIONS AND METHODOLOGY

Our research questions for this study are the following:

RQ1: How is the students' course performance related to activity?

RQ2: How are the MSLQ factors linked to the students' activity and performance?

For the purposes of this study, we define course performance as total points in the two course exams. Activity is measured through multiple factors: (1) attendance in drop-in sessions, (2) discussing with course instructors, (3) discussing and working with peers, (4) lecture attendance, and (5) returning coursework. We restrict the analysis to the population of those 405 students who gained at least one point in total from the course exams.

RESULTS

Activity and Course Performance

To assess the relationship between course performance and activity, the participants were divided into four quartiles based on their performance in the midterm and final exams. As seen in Table 1, an increasing trend is observable in all of the activity-related categories, and the students in the highest performing quartile are the most active ones across the board.

		Q1	Q2	Q3	Q4	All
	Points	< 19.25	< 28.75	< 37.5	≥ 37.5	1-48
	Students	100	102	98	105	405
Attendance in drop-in sessions	Mean	2.0	3.6	5.4	6.9	4.5
	Std. dev.	2.5	3.6	4.3	4.3	4.1
Discussing with course instructors	Mean	1.0	2.0	3.8	3.8	2.7
	Std. dev.	1.6	3.0	4.0	3.9	3.5
Working with peers	Mean	1.7	3.4	4.6	5.8	3.9
	Std. dev.	2.4	3.6	4.3	4.5	4.1
Lecture attendance	Mean	2.3	4.9	6.7	7.6	5.4
	Std. dev.	2.5	4.0	4.2	3.9	4.2
Returning coursework	Mean	3.8	7.1	9.0	10.4	7.9
	Std. dev.	3.7	3.9	3.5	2.6	4.2

Table 1: Course performance and activity. The participants were divided into four quartiles based on their performance in the midterm and final exams, and the reported activity for each quartile is shown separately.

On average, students in the highest performing quartile reported having come to the drop-in sessions during 6.9 weeks out of the possible 12, whereas in the lowest

performing quartile the corresponding number was only 2.0. In what comes to other types of activity, the trend was similar: Students in the highest performing quartile reported having discussed with instructors during 3.8 weeks compared to 1.0 week in the lowest performing quartile. The highest performing group reported having worked with peers on average during 5.8 weeks, and the lowest performing group during 1.7 weeks. Students in the highest performing quartile reported having attended lectures on average during 7.6 weeks, and in the lowest performing quartile during 2.3 weeks. Coursework was submitted on average during 10.4 weeks in the highest performing quartile, whereas in the weakest performing quartile this happened only during 3.8 weeks.

The standard deviations within the quartiles are high, and Mann Whitney U Test was conducted to determine whether the quartiles differ from each other. It was found that all the quartiles differ significantly from each other ($p < 0.05$) except for the following cases: The quartiles Q3 and Q4 were similar in how often students reported having discussed with an instructor ($p = 0.7742$), worked with peers ($p = 0.0971$), and attended the lectures ($p = 0.1555$). The quartiles Q1 and Q2 were similar in how often students reported having discussed with an instructor ($p = 0.0502$). This means that the highest and lowest performing students differ from each other in what comes to activity and attendance.

Motivated Strategies for Learning and Activity

To assess the relationship between learning strategies and motivations, and the students' activity, the students were asked to complete the Motivated Strategies for Learning Questionnaire. From the 405 students, 164 students completed the full questionnaire. When comparing the performance of the population who answered the questionnaire and the population who did not answer the questionnaire, there is a statistically significant difference between the groups ($p < 0.05$). When combining the quartiles 2, 3 and 4 from the previous section, and comparing their performance to the population that answered the questionnaire, no significant difference exists ($p > 0.05$). This indicates that the following results are related to a population that performed better than the overall population.

Table 2 shows Pearson's correlation coefficient between different MSLQ factors and different types of activity. The following moderate (>0.3) or strong (>0.5) correlations were found: The factors help-seeking and peer learning correlate with collaborative activity of the students (attendance in drop-in sessions, discussing with course instructors, and working with peers). The factors effort regulation, help-seeking, and time and study environment management correlate with lecture attendance and returning coursework. The factors effort regulation, time and study environment management, control of learning beliefs, intrinsic goal orientation, self-efficacy, task value, and test anxiety correlate with exam points.

	Attendance in drop-in sessions	Discussing with course instructors	Working with peers	Lecture attendance	Returning coursework	Exam points total
Critical thinking	0.161 *	0.082	0.060	0.062	0.051	0.121
Effort regulation	0.258 **	0.219 **	0.114	0.309 **	0.359 **	0.391 **
Elaboration	0.227 **	0.201 **	0.119	0.225 **	0.071	0.029
Help-seeking	0.520 **	0.473 **	0.642 **	0.316 **	0.303 **	0.082
Metacognitive self-regulation	0.061	0.028	-0.026	0.108	0.088	0.188 **
Organization	0.150 *	0.128 *	0.026	0.139	0.056	0.018
Peer learning	0.450 **	0.355 **	0.627 **	0.289 **	0.210 **	0.045
Rehearsal	0.030	0.030	-0.039	0.145 *	0.101	-0.050
Time and study environment management	0.281 **	0.269 **	0.125	0.484 **	0.481 **	0.309 **
Control of learning beliefs	-0.043	-0.094	-0.132 *	-0.122	0.012	0.303 **
Extrinsic goal orientation	0.201 *	0.134 *	0.149 *	0.210 **	0.223 **	0.237 **
Intrinsic goal orientation	0.041	0.025	-0.062	-0.007	0.063	0.307 **
Self-efficacy	0.241 **	0.171 **	0.112	0.127	0.297 **	0.687 **
Task value	0.194 **	0.214 **	-0.050	0.123	0.181 **	0.389 **
Test anxiety	-0.127	-0.126	0.004	0.000	-0.152 *	-0.317 **

Table 2: Pearson's correlation coefficients between MSLQ factors and students' activity (* p < 0.05; ** p < 0.01). The coefficients larger than 0.3 are emboldened.

DISCUSSION

This study addressed the connections between students' activity and their course performance, and identified MSLQ factors that are related to activity and performance. A connection between performance and activity is visible: the highest performing students reported that they attended drop-in sessions, worked with their peers, discussed with instructors, and attended lectures more than the weakest performing students. The results are in line with previous findings linking lecture

and classroom attendance with performance (e.g., Moore, 2006; Newman-Ford, Fitzgibbon, Lloyd, & Thomas, 2008). In addition, the results show that also other kind of activity than lecture or classroom attendance is linked to the performance of students. This indicates that it is important to have the students participate and interact with each other and the teaching staff.

The standard deviations of activity within the four quartiles were high. This can be explained by the heterogeneous student population: for example, many of the high performing students never attend instruction as they do not need it, and at the same time some of the poorly performing students show grit by working hard even though they may not pass the course at the first attempt. Even though the standard deviations were high, the highest and lowest performing students differed from each other in what comes to their activity.

The average activity of the students was relatively low. In the case of the highest performing students, low averages can be explained by the students being able to complete the coursework on their own. Also, many of the mathematics students find social encounters difficult, and may therefore avoid, for example, coming to drop-in sessions. In addition, there are relatively many students who work alongside their studies and cannot therefore attend even if they wish to do so. One should also note that the majority of the participants were not mathematics, but computer science or statistics students whose motivation to study mathematics may be low and background knowledge insufficient. Had we investigated mathematics students only the numbers would probably look different. All in all, the results show that more work needs to be done in encouraging the students to participate.

The MSLQ factors give us insight into the reasons behind students' activity and attendance. Students' collaborative activity (i.e. attending drop-in sessions, discussing with instructors and working with peers) was related to the MSLQ factors help-seeking and peer learning. This indicates that the XA method enables interaction with instructors and other students for those students who use help-seeking and peer learning as learning strategies. On the other hand, the MSLQ factors effort regulation and time and study management correlated only weakly with students' self-reported attendance in the drop-in sessions and interaction with instructors, but they were linked to lecture attendance. It seems that in order to attend the drop-in sessions, the students do not need to have high skills in managing their time or controlling their effort and attention. This indicates that the XA drop-in sessions have, to some extent, managed to overcome the time management and motivational issues mentioned in previous studies as a reason for the classroom absenteeism of students (Van Blerkom, 1992; Moore, Armstrong, & Pearson, 2008; Gump, 2006) and for them not seeking for help (Symonds, Lawson, & Robinson, 2008). We suggest that the teaching arrangements and the learning environment created for XA make this possible: the students can come to the drop-in sessions

when they want and the learning space is in the middle of the department and easy to access.

One type of collaborative activity, namely working with peers seems to be special. Many of the MSLQ factors that correlate with attending drop-in sessions and discussing with course instructors do not correlate with working with peers. Such factors are effort regulation, elaboration, time and study environment management, self-efficacy, and task value. This indicates that students work with their peers even though they do not, for example, perceive themselves very good at time management, have low expectations of their ability to accomplish tasks or do not find the course useful. These findings are in line with the results of Karabenick and Knapp (1991) showing that students are more likely to ask for help from their peers than their teachers. This implies that it is important to offer students opportunities to work with their peers. Also, from this view, using undergraduate students as instructors could have benefits, as the students who otherwise tend to avoid help-seeking may find it easier to communicate with an instructor who is almost like a peer to them. This is in agreement with the findings of Fingerson and Culley (2001).

It is known from previous research that low self-efficacy can prevent students from asking for help (Karabenick & Knapp, 1991; Newman, 1990; Ryan & Pintrich, 1997). However, in our study, correlations between self-efficacy and attending drop-in sessions, discussing with instructors or working with peers were relatively small. We suggest that the XA method encouraged the students to seek for help despite of their low self-efficacy. There are several features in XA that contribute towards this aim. According to Newman and Schwager (1993), positive relations with teachers can increase the likelihood of students asking for help, and in the XA method the instructors are advised to approach the students, chat with them and be supportive. Also, there are indications that if the classroom's focus is on understanding and mastery, students are less likely to avoid help-seeking (Karabenick, 2004; Ryan, Gheen, & Midgley, 1998). Indeed, in the XA method emphasis is on mastery rather than proving one's ability. For example, students can resubmit their coursework if their solution is incorrect, and this does not affect the bonus points awarded.

There were some limitations in our study in what comes to measuring the activity of the students. Firstly, students may have different views on what the questions mean. For example, does asking one simple question count as discussing with an instructor? Secondly, our data only shows whether students reported having been active during the week, not the amount of time they devoted to different activities each week. Despite of this limitation, the data give an indication of the perseverance with which the students worked during the course.

There was a noticeable answer bias in the MSLQ, as only a handful of the students who were in the lowest performing quartile completed it. This can, to some extent, be explained by the fact that the MSLQ was given to the students after the first midterm exam, and at that point many of the lowest performing students had probably decided

to quit the course and therefore did not answer the questionnaire. The bias means that we cannot draw conclusions on the whole population.

As the students completed the MSLQ after the midterm exam, one cannot say for sure whether the students' answers have been influenced by the teaching methods of the course. A direction for future research would be to give the MSLQ to the students both at the beginning and at the end of the course to see how their perceptions change when they are exposed to the Extreme Apprenticeship method.

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Ways in which engaging in someone else's reasoning is productive

Chris Rasmussen¹, Naneh Apkarian¹, Tommy Dreyfus², and Matthew Voigt¹

¹San Diego State University, United States, chris.rasmussen@sdsu.edu, ²Tel Aviv University, Israel

Typical goals for inquiry-oriented mathematics classrooms are for students to explain their reasoning and to make sense of others' reasoning. In this paper we offer a framework for interpreting ways in which engaging in the reasoning of someone else is productive for the person who is listening. The framework, which captures the relationship between engaging with another's reasoning, decentering, elaborating justifications, and refining/enriching conceptions, is the result of analysis of 10 individual problem-solving interviews with 10 mathematics education graduate students enrolled in a mathematics content course on chaos and fractals. The theoretical grounding for this work is that of the emergent perspective (Cobb & Yackel, 1996).

Keywords: Decentering, Argumentation, Social Norms, Fractals, Paradox.

INTRODUCTION

Typical goals for inquiry-oriented mathematics classrooms are to foster particular social norms, such as students explaining their reasoning, listening to others' reasoning, and making sense of that reasoning (Yackel & Cobb, 1996). Indeed, such goals for student participation have been central to a long line of recommendations in the United States (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The purpose of this paper is to offer a framework for understanding the various ways in which engaging in the reasoning of someone else is productive for the person who is listening to and attempting to make sense of this reasoning. Prior research has documented ways in which teachers can initiate and sustain such norms for participation (e.g., Lampert, 1990; Stephan & Whitenack, 2003), but most research into the benefits of such engagement focuses on the *students'* thinking, not that of the one engaging in the other's reasoning (e.g., Teuscher, Moore, & Carlson, 2015). While there has been some research into mutual intellectual benefit stemming from peer-to-peer engagements (e.g., Kieran & Dreyfus, 1998), it has not been at the collegiate level. Our work contributes to this surprisingly sparse literature, extends notions identified in disparate settings, and adds nuance to existing notions of engaging and decentering.

The theoretical grounding for this work is that of the emergent perspective (Cobb & Yackel, 1996), which coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical progress is a process of active individual construction and a process of mathematical enculturation. The interpretive framework, shown in Figure

1, lays out the central constructs in the emergent perspective. The within row relationships between respective collective and individual constructs is said to be reflexive, meaning that they are mutually constitutive, evolving together in a dynamic system. For example, Yackel and Rasmussen (2002) analyze individual students' evolving beliefs about their and others' role in relation to evolving classroom social norms. This work speaks to one way in which engaging in the reasoning of others (a social norm) is productive for the individual; namely doing so positively shapes beliefs.

Collective Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 1: The interpretive framework

In furthering the relationships between the constructs in Figure 1, we argue for *across* row relationships. In particular, we take the stance that classroom social norms are also inextricably intertwined with individual mathematical conceptions and activity. In so doing we make an empirically grounded argument for a theoretical connection between the upper left hand cell of the interpretive framework and the bottom right hand cell.

In our broader research program (Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Tabach, Rasmussen, Hershkowitz, & Dreyfus, 2015), we are investigating the coordination between individual and collective processes. In this report, however, we focus on analyzing individual mathematical conceptions and activity in an individual interview setting, with the subsequent goal of coordinating this analysis with an analysis of classroom video-recordings. This report lays a foundation for this subsequent analysis, but the framework for ways of engaging in someone else's reasoning is potentially significant on its own.

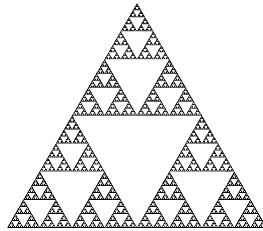
METHODOLOGY

The methodological approach for the larger study falls under the genre of "design-based research" (Cobb, 2000; Design-Based Research Collective, 2003). The study took place in an intact graduate level mathematics course about chaos and fractals with 11 students (10 of whom agreed to participate in individual interviews). Students were (or intended to be) secondary school teachers or community college instructors and all had a bachelor's degree in mathematics. Their masters degree program required a substantial component of mathematics, and the chaos and fractals course qualified as one of their mathematics courses. The course was taught by one of the research team members. Data collected as part of the larger study included video-

recordings of each class session, individual problem solving interviews conducted at the middle and end of the semester, and copies of all student work. In this paper we report on an analysis of the 10 individual, mid-semester problem-solving interviews.

The following question from the mid semester interview is the focus of this analysis:

In class, we discussed the Sierpinski Triangle. How do you think about what happens to the perimeter and the area of the Sierpinski Triangle as the number of iterations tends to infinity?



Follow-up questions:

- One a scale from 1 to 10 with 10 being the most confident, how confident are you about what happens to the area? Can you say more about why you said [confidence number]?
- On a scale from 1 to 10, with 10 being the most confident, how confident are you about what happens to the perimeter? Why do you say [confidence number]?
- A student named Fred claimed the following. Please read it out loud. What do you think about his argument? (Please explain)

Fred's Argument: The computation shows that the perimeter goes to infinity because the perimeter is given by $3 \times (\frac{3}{2})^n$ which increases to infinity as n tends to infinity. But, the perimeter can't *really* be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero.

The Sierpinski Triangle is a fractal, and is the result of an infinite iterative process that begins with an equilateral triangle. Connecting the midpoints of its sides results in another equilateral triangle with sides half the length of the original's and area that is one-fourth of the original's, which is then removed. Repeating this process, *ad infinitum*, results in the Sierpinski Triangle. At each step of the process, the area of shrinks by a factor of $\frac{3}{4}$ and the perimeter grows by a factor of $\frac{3}{2}$. The perimeter of the Sierpinski Triangle can be described by the limit, as $n \rightarrow \infty$, of $P_0 \times (\frac{3}{2})^n$, and the area by the limit, as $n \rightarrow \infty$, of $A_0 \times (\frac{3}{4})^n$, where P_0 and A_0 refer to the perimeter and area of the starting triangle. Thus the Sierpinski Triangle has a perimeter of infinite length and an area of zero. This apparent contradiction comes from the fact that it is a fractal with Fractal dimension $\log_2(3)$, putting it between one- and two-dimensions.

The question was structured so that we would first gain insight into students' own reasoning about the area and perimeter of the Sierpinski triangle, followed by an opportunity for them to engage in the hypothetical reasoning of "Fred." The basis for Fred's reasoning was taken from a whole class discussion that took place several weeks before the interview. Thus, Fred's reasoning is authentic to the students and provides an ideal opportunity for us to subsequently coordinate individual and collective analyses.

While most studies of decentering and engagement have involved interactions between two or more people, ours involves one person interacting with the work of another, who cannot respond. While this setup potentially restricts the ability of interviewees to engage with Fred and his argument, it has certain affordances as well. One affordance is that all interviewees reacted to the *same* statement, allowing us to make direct comparisons. This setup also controls for a variety of other features, such as personal histories, that may influence how people react to each other in face-to-face settings.

The transcripts and student work produced during the interview were open coded using methods from grounded theory (Strauss & Corbin, 1998). This open coding, which was initially conducted by the first two authors then vetted by the other authors to minimize bias and ensure interpretations were grounded in the data, was informed by literature on student thinking about infinity, and in particular infinite iterative processes (Núñez, 1994; Mamolo & Zazkis, 2008), but did not rely on an a priori coding scheme.

The open coding of these interviews revealed differences between students' initial responses and those that followed reading Fred's argument. It also revealed a variety of ways of engaging and responding to Fred. We then supplemented our initial coding, using Toulmin's argumentation scheme (Toulmin, 1969) to analyze the pre- and post-Fred arguments presented by the students. Finally, each transcript was distilled into an argumentation log (Rasmussen & Stephan, 2008), coupled with the primary ways of reasoning being used in each argument and instances of engagement, and supplemented by statements about the mathematics that were not necessarily part of a coherent argument. Logs were analyzed for shifts and relationships, with coding conducted by the first two authors and vetted by the other authors.

RESULTS

Our analysis of students' responses revealed that responding to Fred's argument was a productive experience for most students. There was variation across students with regards to both the extent and nature of their engagement and growth, but we note two major categories of productivity that stemmed from an ability to engage in Fred's reasoning and decenter from their own: elaborating justifications and refining/enriching conceptions of particular mathematical ideas. Figure 2 is intended to capture the relationship between engaging with another's reasoning, decentering, elaborating justifications, and refining/enriching conceptions. Specifically, engaging

with another's thinking can be foundational for (re)engaging with one's own thinking. That is, the act of decentering provided the means for elaborating justifications and refining one's thinking. The intersecting ovals in Figure 2 for these two acts signify the reciprocal relationship between justifying and refining conceptions.

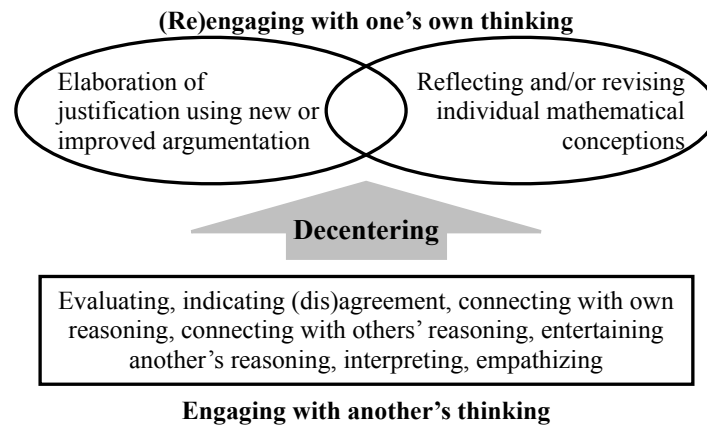


Figure 2: Productivity of engaging in another's reasoning

Since all of the interviewed students were or intended to be teachers at the secondary or postsecondary level, it is particularly interesting to look at their ability to engage with another's thinking. Doing so is foundational to teacher noticing (Jacobs, Lamb, & Philipp, 2010) in which teachers can instructionally build on student thinking. We found that all of the interviewees exhibited the ability to engage with Fred's thinking. We identified the following ways that interviewees engaged in Fred's reasoning: a) evaluating (with or without justification); b) indicating (dis)agreement (with or without justification); c) making connections to their own reasoning; d) making connections to classmates' reasoning; e) entertaining Fred's reasoning; f) interpreting Fred's reasoning; g) diagnosing Fred's reasoning; and h) empathizing with Fred. These ways of engaging provide an opportunity for the individual to decenter. By decenter, we mean putting aside one's own reasoning in an attempt to understand another's reasoning (Steffe & Thompson, 2000; Teuscher, Moore, & Carlson, 2015). Many interviewees, through decentering, engaged or re-engaged with their own thinking in a way that furthered their own thinking. This analysis lays the groundwork for coordinating individual and collective ways of participating in discourse since evaluating (with justification) and indicating (dis)agreement connect strongly to foundational classroom social norms.

In this paper we give a few brief examples of engaging and decentering. Most students gave some indication of agreement or disagreement with Fred's argument, e.g. "I agree with him that the perimeter increases to infinity [...] but I disagree with his second line." This example shows a fairly superficial engagement in which the interviewee attended to Fred's reasoning but viewed it from her own point of view. Other students went further, e.g. "I disagree because we thought about it in terms of

fencing [...] so eventually it's all fence.” The second student's explanation makes it clear that while she has not necessarily built a model of Fred's line of reasoning, she is aware of her own model and believes Fred's is different. This second student then elaborated and improved upon her original argument.

Interviewees also demonstrated a range of depth when engaging with Fred by interpreting his reasoning. Some interpreted Fred's thinking from their own point of view, but others made clear attempts to deduce Fred's reasoning from *his* point of view – in one case an interviewee requested more information about Fred's argument before settling on an interpretation. We saw evidence, across all interviews, that each act of engaging functioned as a potential stepping-stone to decentering, an opportunity that some students took up while others did not. We saw that students who engaged deeply with Fred's thinking and decentered from their own point of view appeared to (re)engage with their own thinking.

As a consequence of decentering, many of the students clarified and even advanced their own lines of mathematical reasoning as expressed by Figure 2. As Fred's argument was in response to a question they had already answered, many reacted by re-explaining or expanding their initial justification. Within mathematical thinking we observed two main subcategories: the elaboration of justification for their claims and the expansion of their thinking regarding the mathematical concepts involved in the task. By elaboration of justification, we mean that students were observed adding new or improved warrants and backings to strengthen their argument or even providing entirely alternative explanations. As an example, one student, Sandor, reacted to Fred's argument by noting that it is *because* the area of the Sierpinski Triangle goes to zero that the perimeter goes to infinity, and explicitly connected the removal of triangles at each recursive step to adding the perimeter of these triangles to the total perimeter. Prior to engaging with Fred's statement, he had treated the two results as essentially separate features of the process – the connection between the two had gone unnoticed or at least unexplained.

With regards to the underlying mathematical concepts, we observed students exploring the nature of infinity, perimeter, and the Sierpinski Triangle itself in greater depth than they had in their initial arguments. Some students appeared to become aware of a distinction between potential infinity (the unending process) and actual infinity (the final resultant state) in their attempts to clarify their reasoning. Many students took the opportunity to define, or re-define, the perimeter of an object. Students also reflected on the fractal nature of the Sierpinski Triangle, noting that it exists “between” dimensions and therefore does not act in the way that a “normal” one- or two-dimensional object might, and that therefore traditional thinking about a perimeter enclosing area is not necessarily valid in this context.

While we identify decentering and mathematical thinking as distinct, we note that they are not disjoint. All of these examples of expanded thinking and reasoning occurred to some extent as a reaction to the thinking of someone else. We posit that decentering functioned as a catalyst for this process. Seeing Fred's argument,

interviewees demonstrated a variety of strategies for engaging with student reasoning, which were taken up with varying depth. Deeper engagement took the form of decentering, which predicated (re)engagement with and growth of their own reasoning. That is to say, the greater the extent to which students engaged with Fred and decentered, the more productive the experience was with regards to their own thinking.

The Case of Curtis

To clarify the constructs and interpretations outlined above, we present the case of a single student, with pseudonym Curtis. We choose this student as an example because of the brevity and clarity of this portion of his interview, as well as the range of constructs identified in his experience with Fred. Figure 3 shows Toulmin analyses of Curtis's pre- and post-Fred arguments, as well as his comment about infinite processes.

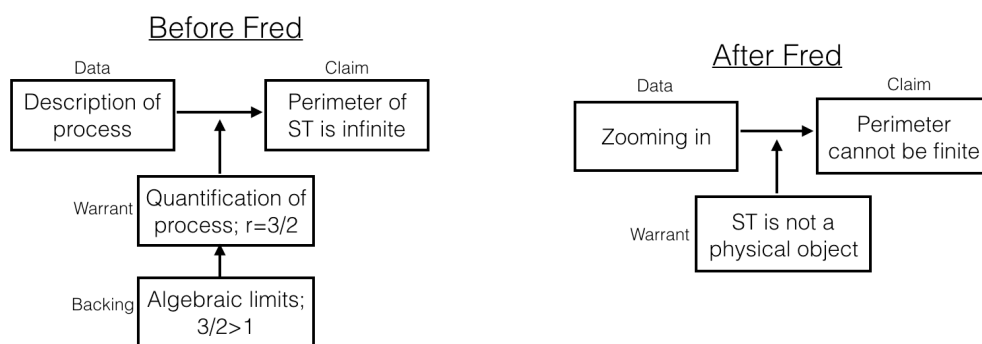


Figure 3: Toulmin analysis of Curtis' arguments.

Toulmin analysis of Curtis' pre- and post-Fred argumentation revealed shifts and changes. A small shift occurred in Curtis' claim: initially he showed that the perimeter is infinite, afterward he showed it could not be finite. This new claim is drawn from different data and is supported by a different warrant. Where initially Curtis used formal/symbolic reasoning, his second argument draws on heuristics and a sense that the Sierpinski Triangle is not a real object. He also brings up the fact that infinite processes do not have a 'final step' after which they reach their final state, something that was not mentioned prior to Fred.

Retracing the emergence of new topics for Curtis, we found that they were directly linked to his engagement with Fred's reasoning, and in particular resulted from his ability to decenter and look at Fred's reasoning in ways not related to his own. Curtis comments that Fred's "logic doesn't work," addressing more than just his faulty claim. The new warrant that Curtis provides, that the Sierpinski Triangle is not a physical object but rather "kind of just a concept," directly addresses an unspoken assumption on Fred's part. It seems that Curtis has identified and reacted to an implicit backing in Fred's argument – that the Sierpinski Triangle is a geometric object that obeys two-dimensional rules. Curtis' diagnosis of a misconception underlying Fred's reasoning implies that he has considered Fred's argument from a

different viewpoint, effectively trying to put himself in Fred’s shoes and understand fully his reasoning.

In addition to presenting a new argument, Curtis presents it in a new style. While his original argument was based in formal limits and notation, his new argument adopts some of Fred’s informal, heuristic, and geometric language. Again, this supports the idea that Curtis is working from Fred’s point of view, rather than his own.

Finally, Curtis’ added commentary about infinite processes comes from his interpretation of Fred’s argument. He says that Fred’s argument is equivalent to there being a final step, a point where something is taken away and the area becomes zero, and notes that this is not how infinite processes work. This seems to address Fred’s data, that the object becomes something with no area.

Altogether, we see that Curtis addresses all the pieces of Fred’s argument (not just the claim) by thinking through Fred’s reasoning (not just comparing it to his own). This includes an implicit backing that Fred does not explicitly state. He does so using Fred’s style of reasoning, and (re)engages with his own reasoning to present a second argument and an observation about infinite processes. Throughout his response to Fred, Curtis addresses Fred’s reasoning and explains why it does not work, rather than simply asserting that his own original ideas are correct.

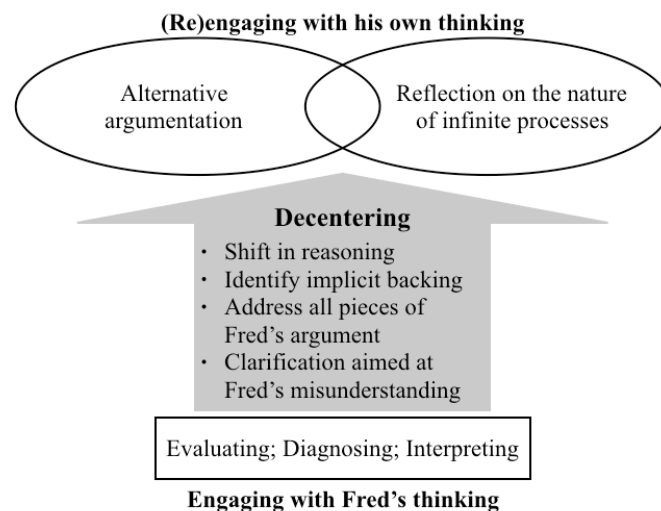


Figure 4: Curtis’ productivity from engaging with Fred’s reasoning.

CONCLUSION

In conclusion, we return to classroom social norms and the ultimate role we envision for our framework. We argue that the ways of engaging we observed in these interviews are closely related to particular classroom social norms. The relevant social norms related to engaging in others’ reasoning include listening to others’ reasoning, attempting to make sense of this reasoning, and indicating agreement or disagreement, with reasons. Moreover, acting in accordance with these norms led, through decentering, to enriched and refined mathematical conceptions and activity.

The case of Curtis illustrates that decentering is an individual cognitive mechanism triggered by engaging with another's reasoning.

Prior work posits a reflexive relationship between engaging in others' reasoning (i.e., social norms) and individual beliefs. In Figure 1, this relationship coordinates the cells in the top row of the interpretive framework. As far as we are aware, the analysis in this paper is the first to coordinate social norms and individual mathematical conceptions and activity. That is, we provide evidence for a relationship between social norms (upper left hand cell of the interpretive framework in Figure 1) and individual conceptions (bottom right hand cell). This importance of this work lies in coordinating different analytic tools that separately address collective and individual phenomenon. Thus, our framework not only contributes to a nuanced understanding of engaging and decentering with another's reasoning, but also leads to links between individual mathematical conceptions and social activity.

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Comparative analysis of learning gains and students attitudes in a flipped precalculus classroom

Matthew Voigt

San Diego State University, United States, mkvoigt@gmail.com

Flipped classrooms are becoming increasingly prevalent at the undergraduate level as institutions seek cost-saving measures while also desiring to implement technological innovations to attract 21st century learners. This study examined undergraduate pre-calculus students' (N=427) experiences, attitudes and mathematical knowledge in a flipped classroom format compared to students in a traditional lecture format. Our initial results indicate students in the flipped format were more positive about their overall classroom experiences, were more confident in their mathematical abilities, were more willing to collaborate to solve mathematical problems, and achieved slightly higher gains in mathematical knowledge.

Key Words: *Flipped Classrooms, Technology Enhanced Learning, Precalculus, Student Attitudes*

INTRODUCTION

The development of online math education has made huge strides in recent years with the creation and wider availability of open source math tutorials such as Khan Academy, Udacity, and Coursera. This has lead traditional institutions to seek time and money saving measures by developing pre-recorded lectures and utilizing problem-based education inside the classroom (Bacow & Bowen, 2012; Mehaffy, 2012); however, little consideration is given to the effects that these changes will have on students' attitudes and academic performance toward the subject of mathematics. One of the key-concepts behind the "flipped classroom" or the "inverted classroom" approach is using technology to offload traditional style lectures to allot more classroom time for problem based exploration and applied learning (Lage, Platt, & Treglia, 2000; Sams & Bergmann, 2012).

REVIEW OF THE LITERATURE

There is a limited amount of international peer-reviewed research available on flipped classroom approaches; however, studies have been increasing in recent years. Preliminary reports seem to suggest that students in flipped classrooms show improved academic success and achieve greater learning outcomes as compared to students in traditional classroom models, (Baepler, Walker, & Driessen, 2014; Love, Hodge, Grandgenett, & Swift, 2014; Mason, Shuman, & Cook, 2013; Wilson, 2013) or at worst does no harm (Mason et al., 2013; McCray, 2000, Bagley, 2014). In addition, student attitudes are fairly consistent and show students view the flipped classroom as promoting their learning (Arnold-Garza, 2014; Scida & Saury, 2006),

increasing confidence in their abilities (Baepler et al., 2014; Kim, Kim, Khera, & Getman, 2014) encouraging social engagement with students and teachers (Baepler et al., 2014; Jaster, 2013; Love et al., 2014), as more relevant to their future career goals (Love et al., 2014) and appreciate the flexibility allowed by online didactic videos (Jaster, 2013); however there is evidence that given a choice, students prefer a traditional model of learning (Arnold-Garza, 2014; Jaster, 2013).

Although recent studies support the use of flipped classrooms, most studies thus far have used small samples sizes, and with the exception of a few conference proceedings (Overmyer, 2013; Wasserman, Norris, & Carr, 2013; Bagley, 2014) most are not specific to the subject of undergraduate mathematics. Since the research on the effectiveness of this pedagogical approach is limited, there are clear gaps in the literature that this study hopes to address. Accordingly, this study is a first step in determining how do students in a flipped learning undergraduate math course compare to students in a traditional lecture course in their:

- Attitudes (motivation, enjoyment and confidence) and beliefs about learning mathematics?
- Experiences and opinions of the course activities and interactions?
- Perceived learning gains and mathematical knowledge?

RESEARCH DESIGN AND METHODOLOGY

Participants were students from four undergraduate pre-calculus II course sections offered at a large research university in the midwestern region of the United States. Two of the courses used the flipped learning model (FL) for instruction and two used the traditional lecture model (TL) for instruction. Each of the course sections met for three hours a week of classroom time and one hour for a Q&A section lead by a graduate assistant. The TL courses used the traditional classroom time to lecture on the classroom material with limited interaction between teacher and students. In comparison, The FL classes used online didactic video tutorials that features a voiceover PowerPoint to present the lecture material outside of classroom and classroom time was then used primarily to complete group (3-4 students) based worksheets with low level practice problems combined with mathematical proofs to derive trigonometric formulas in an active learning classroom.

The research instruments and design methodology parallel the research conducted by Laursen et al. (2014) regarding inquiry-based learning. This large scale study highlighted the beneficial impact of active learning strategies on student outcomes especially for women, low-achieving and first-year students. The first survey instrument referred to as the attitudinal assessment, consisting of 54 questions using a seven point Likert-scale, and was, “constructed on the basis of theory and previous research on mathematical beliefs, affect, learning goals and strategies of learning and problem solving” (Laursen et al., 2014). The second survey instrument is based on a subset of the mathematically focused Student Assessment of their Learning Gains,

referred to as the SALG-M and measures student’s experiences and learning gains using a 5-point Likert scale from (1 –No gains) to (5-Great gains) for each item. The SALG-M instrument was designed to provide faculty with summative and formative information on teaching practices, and has been shown to be a reliable measure of classroom practices and student experiences. The attitudinal assessment pre-survey was administered at the start of the second week of the course and the attitudinal post-survey and SALG-M were administered in the last week of the course. Scores from the multiple choice section of the mathematics department common final examination were used to assess student's mathematical performance. In addition, demographic information including gender, race, class year, college major, previous math courses taken, and GPA were requested.

RESULTS

We received 427 responses (87.5% of enrolled students) from the pre-survey and 300 responses (61.5% of enrolled students) from the post survey. Using the unique identifier we were able to match 214 (43.8% of enrolled students) pre- and post-surveys. Based on prior research from Laursen et al. (2014), a factor analysis was performed on each of the survey items to create composite variables to measure changes in students affect (motivation, enjoyment, confidence), beliefs about learning, and strategies for problem solving problems (See Table 1). In addition composite variables were determined to assess students’ perceptions of the classroom experiences, and self-reported learning gains as a result of the course (See Table 2). A summary of the composite variables and reliability ratings are reported in Table 1 and Table 2.

Table 1: Composite variables of attitudinal and learning behaviors in mathematics

Variable	Description	Reliability <i>Cronbach alpha</i>	
		Pre	Post
Motivation	Motivation to learn mathematics	.761	.771
Interest	Interest in learning and discussing math outside of the classroom	.749	.774
Math degree	Desire to pursue a math major/minor	.838	.822
Math future	Desire to pursue and study for additional math courses.	.536	.672
Teaching	Desire to teach mathematics	-	-
Enjoyment	Pleasure in doing and discovering mathematics	.893	.908
Confidence	Confidence in math and math teaching ability	.828	.859
Math confidence	Confidence in own mathematical ability	.805	.852
Teaching confidence	Confidence in teaching mathematics	.682	.745
Beliefs about learning			
Instructor-driven	Exams, lectures, instructor activities	.687	.689

Group work	Small group presentation and critique of math	.639	.629
Exchange of ideas	Active exchange with other students	.765	.728
Strategies			
Independent	Find one's own way to think and solve problems	.450	.640
Collaborative	Work with other students to brainstorm and solve problems	.717	.683
Self-regulatory	Review and organize one's own work; check one's understanding	.562	.647

Table 2: Composite variables for student experiences and learning gains

Variable	Description	Reliability	
		Post	
Experiences of course practices			
Overall	Overall experience, workload, and pace of the course	.797	
Active participation	Participating in discussion, group work, and explanation of work.	.800	
Individual work	Studying on your own	-	
Lectures	Listen to lectures	-	
Assignments	Tests, homework, feedback on written work	.603	
Personal interactions	Interacting with peers, TAs and instructors	.667	
Cognitive Gains			
Math concepts	Understanding concepts	.906	
Math thinking	Understanding mathematical thinking	.819	
Application	Applying ideas outside math, making math understandable for others.	.828	
Affective Gains			
Positive attitude	Appreciation of math	.812	
Confidence	Confidence to do math	.889	
Persistence	Persistence, ability to stretch mathematical capacity	.781	
Social Gains			
Collaboration	Working with others	.773	
Teaching	Comfort in teaching	-	
Independent Gains	Ability to work on your own	.828	

Linear regression analysis was performed on each of the composite variables in order to determine the magnitude and main effect of classroom format in addition to models controlling for demographic and interaction effects. The results of the main effects model, which are displayed in Figure 1, indicated significant differences for

students' experiences in the classroom, math confidence, and collaborative strategies for problem solving. In addition there were significant differences in self-reported affective, cognitive, and social learning gains, but no difference in independent learning gains (See figure 2). We subsequently discuss the themes that emerged from this initial analysis.

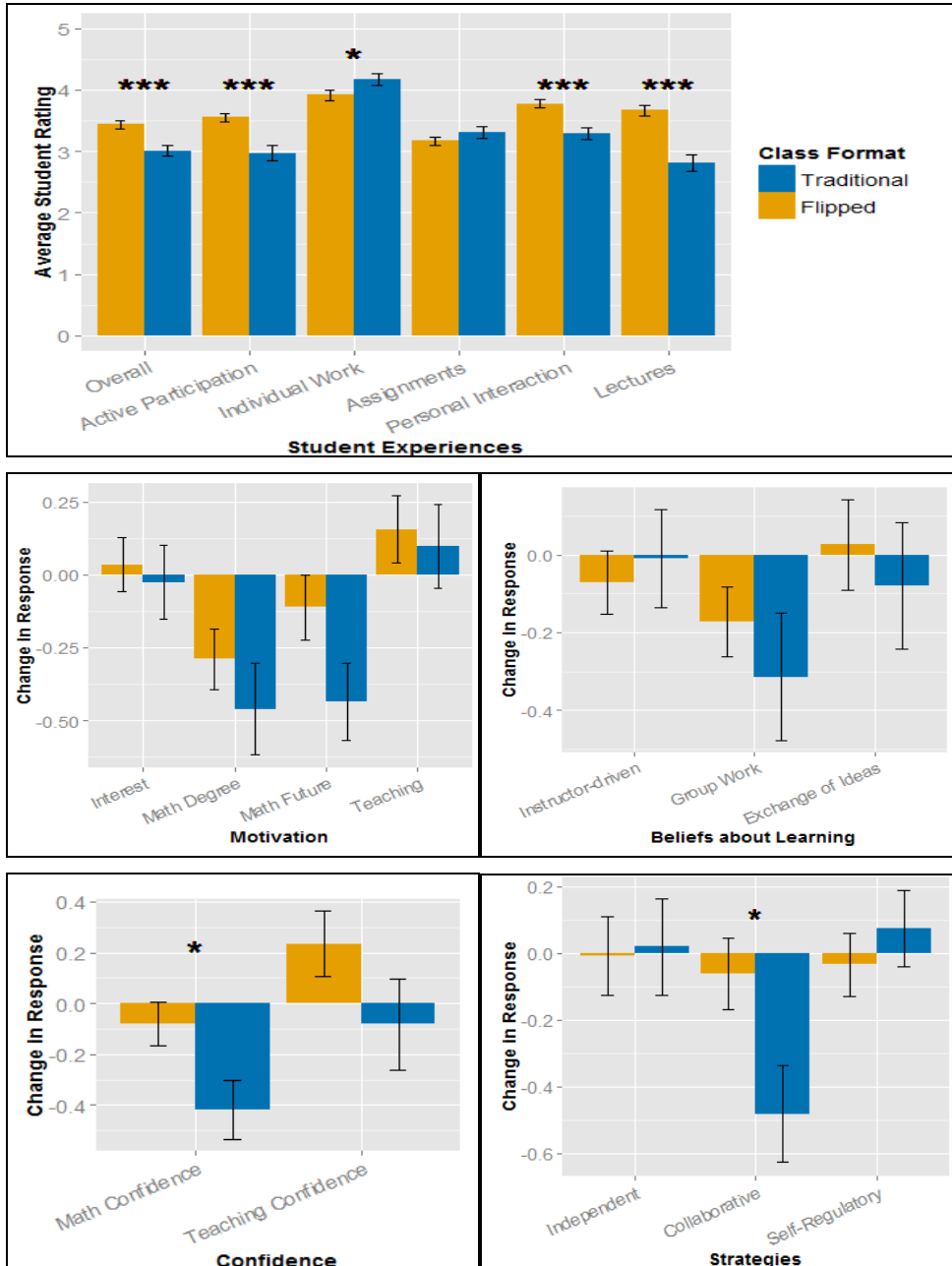


Figure 1: Average classroom experiences and changes in pre and post survey attitudinal variables based on classroom format with standard error bars.

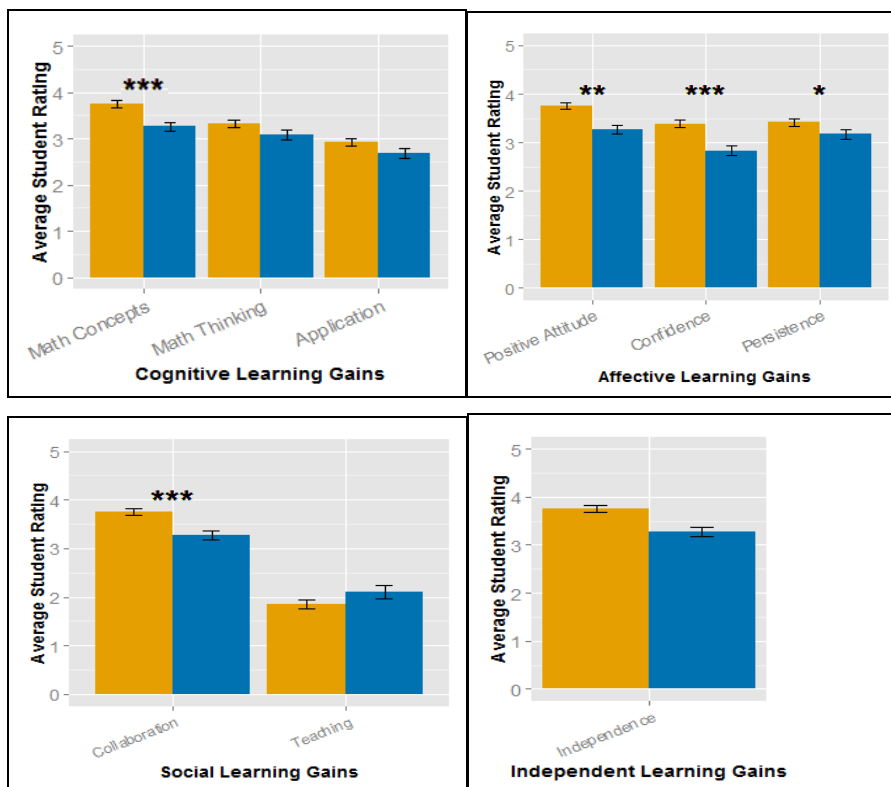


Figure 2: Average rating with error bars for learning gains based on classroom format.

Classroom Experiences

As suggested by prior research, students in a flipped format viewed the overall experiences in the course (workload, pace, and overall approach to the course) as a significantly greater help to their learning than students in a traditional format; however, the research goal was to further investigate the specific components of the course that may have contributed to the overall differential experiences of students in the FL versus the TL format. Active participation (class discussions, group work, explaining work to other students, and listening to other students explain their work), personal interactions (with the instructor, teaching assistant, and peers in the course) and lectures were seen as a greater help to students in the FL format, while individual work such as studying on your own was seen as a greater help to students in the TL format. Assignments were viewed as equally supportive for students in either the FL or the TL format.

In addition to questions about classroom experiences, students were asked, “Would you recommend taking another course offered in the SAME FORMAT as this one?” Contrary to the findings of Arnold-Garza (2014) and Jaster (2013), a large majority of the students (67%) in the FL courses would take the course again in the same format given the choice, compared to a similar but smaller percentage of TL courses students (54%) who said they would take the course again in a traditional lecture format. Further investigation into the make-up of students who would not recommend taking a flipped classroom format, showed a significant difference

$(\chi^2(1, N = 182) = 8.12, p = .004)$ in the gender composition with a larger proportion of women ($N=40$) saying they would not recommend the format as compared to men ($N=15$). The same difference was not present in the traditional class ($\chi^2(1, N = 118) = .145, p = .70$). Although gender and gender interactions with flipped learning were not significant for any of the composite variables, the fact that women were almost three times as likely to indicate a preference for not take the course again in flipped learning format warrants further investigation.

Affective and Learning Strategies Changes

Our results from the attitudinal assessment mirror the results of the MAA national study (Bressoud, Carlson, Mesa, & Rasmussen, 2013) indicating overall students are less confident in their mathematical ability after the completion of the course, but notably students in the FL course had significantly smaller declines in mathematical confidence ($F(1, 210) = 5.44, p = .02$). In addition FL students as a result of the course reported higher affective learning gains including positive attitude ($\beta = -.39, t(282) = -2.92, p = .004$), confidence ($\beta = -.56, t(284) = -4.65, p < .001$), and persistence in mathematics ($\beta = -.25, t(283) = -1.98, p = .048$). We conjecture that there are two contributing elements that resulted in the smaller declines in confidence for the FL students. One notable difference between the FL and TL courses, was the implementation of ten proficiency based quizzes that students had to master in order to pass the course. This mastery based learning approach gives students the opportunity to assert that they fully understand the core topics in the course. In addition to the mastery quizzes the availability of having the online lectures, which our log data shows a majority of students watched multiple times, also provides students with increased scaffolding to support understanding and learning of the course topics.

Students in the FL course also show attitudinal changes in the benefit they see in using collaborative strategies toward learning indicating that they are more likely to seek help from others and share information with other peers ($F(1, 211) = 5.39, p = .02$). This change in collaborative learning strategies we attributed to the reported social gains in collaboration ($\beta = -0.53, t(259) = -2.48, p = 0.01$) due to the course, where FL students reported higher gains in their ability to work well with others, willingness to seek help from others and appreciation of difference perspectives as a result of the course.\

Mathematical Knowledge

Results from student performance on the common math final indicate modest gains in academic performance for students in the FL course ($M=67.2$) compared to students in the TL course ($M=64.7$) format ($F(407,1) = 3.38, p = .067, d = .18$). Although it was not possible to obtain prior mathematical ability, the two course formats had no significant differences between the GPA's, number of college math courses taken, and highest high school math taken for the students, indicating that

the prior mathematical ability among the two course formats were roughly equal. This information coupled with the reported higher cognitive learning gains for math concepts ($\beta = -.48, t(285) = -4.25, p < .001$) for the FL students, indicates the FL format was beneficial for student learning. Future studies should examine if the increases in collaboration and confidence for FL students will translate to better knowledge of higher level mathematical concepts, since we were only able to assess lower-order mathematical thinking on final exam multiple choice items.

CONCLUSIONS AND FUTURE STUDIES

Results from this study are promising for the future implementation of flipped style learning in undergraduate mathematics education. Students generally respond positively to flipped classroom learning experiences, and as a result show increased gains in confidence and willingness to collaborate with others in solving mathematical problems. In addition students show modest gains in mathematical knowledge. These positive trends indicate that flipped learning not only does no harm, but actually benefits students academically and attitudinally.

The next phase in this study will assess the qualitative data obtained through the survey instruments as well as course artifacts in order to understand with greater richness the experiences students had throughout the course, and answer some of the questions raised through our initial quantitative analysis. We seek to understand what factors contributed to the gender disparity in preference for taking a flipped course and whether there exist gains in higher-order mathematical knowledge as a result of using the flipped format. Additionally, we will be collecting longitudinal data to assess the impact this course had on persistence in STEM fields and student performance in subsequent math courses.

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Supporting students gifted in mathematics through an innovative STEM talent programme

Martin Bracke¹, Patrick Capraro¹, Anna Hoffmann², Sören Häuser³, Christian H. Neßler¹, Katherine H. L. Neßler¹ and Andreas Roth¹

¹Department of Mathematics, University of Kaiserslautern, Gottlieb-Daimler-Straße 48, 67663 Kaiserslautern, Germany, knessler@mathematik.uni-kl.de; ²Fraunhofer Institute for Industrial Mathematics (ITWM), Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany; ³ITK Engineering AG, Hahnstraße 70, 60528 Frankfurt am Main, Germany.

Europe is facing an insufficient number of suitably qualified university graduates in the STEM subjects, due to an insufficient number of students choosing to study these subjects and high dropout rates [Eurydice, 2011]. Furthermore, school-age students show a lack of understanding of the value of mathematics, and its potential use for solving real-life problems [Boaler, 2011]. We have been running an innovative talent programme, which gives teams of school students the opportunity to study open-ended and unstructured truly interdisciplinary problems. We present results, which show a remarkable increase in understanding of the students' appreciation of the value of mathematics. We believe such a programme not only teaches students the value of mathematics but will also aid their transition from school to university.

Keywords: STEM education, mathematical modelling, novel approaches to teaching

OUR HYPOTHESIS

Although the number of long-term studies is limited, previous research has indicated that students fail to relate the mathematics they learn in the classroom to the outside world, and feel that the mathematics they learn in school will not be useful for them in their life or career (Boaler, 2011). Research indicates that the understanding of the value of mathematics and enjoyment for the subject can be increased by involving school-age children in mathematical-modelling and problem-solving exercises (Boaler, 2001, Curtis, 2006).

Our hypothesis is that introducing truly interdisciplinary open-ended and unstructured problems, based on real-life STEM problems, at the school-age level increase the understanding of and inclination for the mathematical sciences, as students will see their use in real-life problems. Furthermore, we believe that such a project will support teenagers in their transition to university, by giving them experiences of learning and working independently, rather than learning by rote. As such, we

strongly believe that application of our philosophy within schools could significantly increase the number of qualified STEM graduates within Europe.

THE PROGRAMME

Two years ago we started a programme for teenagers talented in mathematics, which will finish in June 2016. School children from throughout Germany, mainly from schools supported by the MINT-EC network (MINT being German for STEM) were invited to submit an application. From 58 applicants, a rigorous selection process and assessment centre, 24 students, of which 8 are female, were chosen to take part in the programme.

The total programme consists of six separate week-long residential courses, taking place two times a year. The total group of students was split up into four project teams and each team chose a project for the duration of the programme on which they wished to work. The students themselves conceived all of the project goals, initially only being given a vague idea of a subject area in which they could work on. During each residential course, students are free to work on their projects in a relaxed manner as they wish and experts are available, whom the students can ask technical questions, with the aim of this support to be the facilitation of independent student learning. One or two seminars on a mathematical or technological theme are also given at each workshop, for example an introduction to MATLAB, genetic algorithms, Simulink, Raspberry Pi, image processing and metal soldering to name a few.

POSTER CONTENT

The poster will contain further details on the programme, including the four separate projects on which the students are working and pictures of the products they have produced to date. Empirical results from a questionnaire completed by the students will be presented in graphical form, which illustrate the positive effect of the programme in terms of the students' enjoyment of mathematics as well as an understanding of the importance of the subject and its use.

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Difficulties to teach mathematics and beliefs on mathematical modelling by prospective teachers

Richard Cabassut

ESPE-University of Strasbourg , LISEC-EA2310, France,
richard.cabassut@unistra.fr

From a research program on the teaching of modelling we present results related to French prospective teachers. These students have answered an online questionnaire about the difficulties to teach mathematics taking in account biographical variables and beliefs on mathematics, mathematics teaching and mathematical modelling. Heterogeneous beliefs are revealed and the analysis constructs three clusters from difficult to easy relation to different topics on mathematics teaching. The next steps of this ongoing research are mentioned, particularly looking for partners.

Keywords: modelling, beliefs, questionnaire, mathematical teaching, prospective teachers.

RESEARCH TOPIC AND QUESTIONS

From 2016 modelling becomes in the new curriculum one of the six main components of mathematical activity and modelling is already knowledge to be taught in secondary school. For these reasons it is important to train prospective teachers on modelling. Research shows that teachers' conceptions play a key role in the teaching and the learning of mathematics (Philipp 2007). We didn't find research related to France. Our research question is: what are conceptions about modelling in France? In this paper we will focus on the prospective teachers.

THEORETICAL FRAMEWORK AND METHODOLOGY

We will adopt the theoretical framework on conceptions proposed by Philipp (2007, p.259) and on modelling by Maass (2006, p.115). Cabassut and Ferrando (2015) describe how the questionnaire is structured, using different levels of didactic codetermination from anthropologic theory of didactic. The questionnaire is composed of 48 multiple choice questions with sometimes four-point Lickert scale on biography, mathematics conception and modelling conceptions. This online questionnaire was answered between February and May 2015 by 152 French prospective teachers. The statistical analysis is made with the software SPAD that provides frequency table, cross-tabulation and cluster analysis. We adopt an exploratory approach (Tukey 1977) what means we do not need a representative sample.

RESEARCH RESULTS

First we observe a high heterogeneity of answers making difficult the interpretation of clusters. Previous results with in-service teachers (Cabassut, Ferrando 2015) have shown that difficulties to teach modelling are often related to difficulties to teach

mathematics. A majority of prospective teachers consider as difficult the following topics: heterogeneity, assessment of group work, time management, teaching conditions, inquiry based approach. On the contrary a majority consider as easy the following topics: small groups work, assessment, open problem solving. The cluster analysis produce three clusters. The first cluster represents people feeling difficulties in mathematics teaching: trainee, people studying mathematics or sciences, older people, women are overrepresented. The second cluster represents people neutral on difficulties: students who are not trainees, studying no mathematics and no sciences, who don't understand what is modelling are overrepresented. The last cluster represents people finding easy different topics of mathematics teaching: people understanding modelling, studying mathematics, younger people, men, agreeing with our definition of modelling are over represented. We find significant dependences for example between difficulties on heterogeneity and open problem solving, on inquiry based approach and open problems solving or small groups work.

RESEARCH PERSPECTIVE

This research will go on by comparing prospective teachers answers and teachers answers. The cluster analysis will produce paragon who will have semi-structured interviews to clarify the analysis in order to produce a course on modelling for prospective teachers (Cabassut 2015). Partners from other countries are invited to take part to this research in order to compare the role of institutions.

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MetaMath and MathGeAr projects: students' perceptions of mathematics in engineering courses

Pedro Lealdino Filho¹, Christian Mercat¹ and Mohamed El-Demerdash^{1,2}

¹ Université Claude Bernard – Lyon I, France, ² Menoufia University, Egypt
pedrolealdino@gmail.com; christian.mercat@math.univ-lyon1.fr; m_eldemerdash70@yahoo.com.

This poster aims at studying engineering students' perceptions of their mathematics courses. We present the methodology of data collection, the main themes that the questionnaire investigates and the results. The population on which we base this study are partners in two Tempus projects, MetaMath in Russia and MathGeAr in Georgia and Armenia.

Keywords: engineering education, STEM, tempus, MetaMath, MathGeAr.

INTRODUCTION

Mathematics is considered as the foundation discipline for the entire spectrum of Sciences, Technology, Engineering and Mathematics (STEM) curricula. Its weight in the curriculum is therefore high (Alpers, et al., 2013). Several special studies in Europe suggest that competencies gap in mathematics is a most typical reason for STEM students to drop out of study. The overall objective of the Tempus projects, MetaMath and MathGeAr, is to improve the quality of STEM education in the South Caucasian region and Russia, by modernizing and improving the curricula and teaching methods in the field of Mathematics.

After Gaston Bachelard, in Cardoso (1985), an epistemological features evident in the sciences include the aspiration to be objective. From the intuitive perception of a phenomenon, a pre-scientific spirit needs to overcome a set of epistemological obstacles to reach a scientific stage.

METHODOLOGY

To explore students' perceptions of mathematics we produced an online survey to be distributed in all participant countries. The questionnaire has three main dimensions:

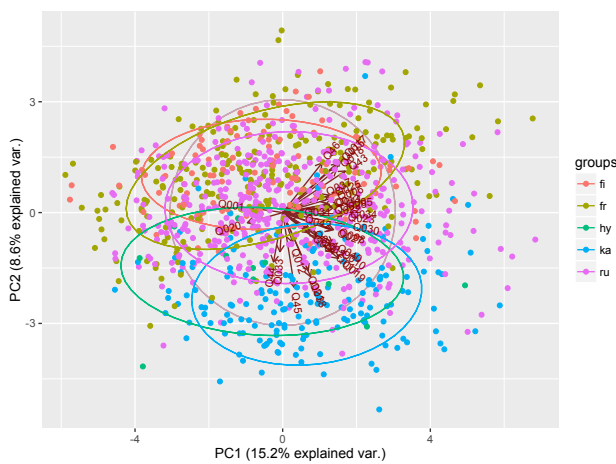
Questionnaire dimensions	Number of Questions
Usefulness of mathematics	8
Teaching mathematics in engineering schools, contents and methods	15
Perception of mathematics	12
Total	35

A total of 35 questions were answered by 1548 students from all participant countries. After collecting the data from the online survey we used the statistical package R to analyze the data and draw conclusions. We performed a *Principal Component Analysis* (PCA) to investigate patterns in the students' responses. In general terms, PCA uses a vector space transform to reduce the dimensionality of large data sets. Using mathematical projection, the original data set, which may have involved many variables, can often be interpreted in just a few variables (the principal components).

Country	N
Armenia	24
Finland	189
France	430
Georgia	285
Russia	612

RESULTS

Here is the plot of the two first Principal Components.



The analysis shows that all students feel that math teaching is too theoretical, not practical enough and has not enough connection with other sciences and the engineer's job reality. Therefore, modernized curricula for engineers should address these issues. On the other hand, Caucasian students tend to perceive that mathematics consists of knowledge rather than competencies, mainly of theoretical interest, with a discrepancy between early practical mathematics and theoretical engineer mathematics. The European students feel that

advanced mathematics is useful, that the role of a teacher is more to help students to apply mathematics than to only transmit knowledge. The Russian students fall in between the two groups and are more diverse in their opinions.

CONCLUSION

Our findings suggest that: teaching should put forward the applications of advanced mathematics and focus on competencies rather than transmission of knowledge; the European countries on the one hand and Caucasian countries on the other are quite aligned but Russian students' perceptions are more spread out and in between those of the European and Caucasian students.

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INDRUM 2016 was organised by the IMAG (Institut Montpellierain Alexander Grothendieck) UMR CNRS UM 5149, with the financial support of the University of Montpellier, the CNRS (Centre National de la Recherche Scientifique) and the Languedoc Roussillon region.



INDRUM 2016 was an ERME Topic Conference.

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INDRUM 2016 was organised under the patronage of the ARDM (Association pour la Recherche en Didactique des Mathématiques), the SMAI (Société de Mathématiques Appliquées et Industrielles) and the SMF (Société Mathématique de France).



List of participants: ALFARO Matthieu (France), APKARIAN Naneh (United States), ARTIGUE Michèle (France), BARALLOBRES Gustavo (Canada), BARDINI Caroline (Australia), BARQUERO Berta (Spain), BERGÉ Analia (Canada), BIZA Irene (United Kingdom), BLOCH Isabelle (France), BOSCH Marianna (Spain), BOURGADE Jean-Pierre (France), BRACKE Martin (Germany), BRIDOUX Stéphanie (Belgium), BROLEY Laura (Canada), BRONNER Alain (France), BRUGUIERES Alain (France), CABASSUT Richard (France), CARRIEGOS Miguel (Spain), CHESNAIS Aurélie (France), CHELLOUGUI Faiza (Tunisia), CHORLAY Renaud (France), CORTELLA Anne (France), DANA-PICARD Thierry (Israel), DESHLER Jessica (United States), DONEVSKA-TODOROVA Ana (Germany), DURAND-GUERRIER Viviane (France), EL-DEMERDASH Mohamed (France), FERRER Josep (Spain), FEUDEL Frank (Germany), FLORENSA Ignasi (Spain), FLEISCHMANN Yael (Germany), FREDRIKSEN Helge (Norway), FUSARO PINTO Marcia (Brasil), GASPAS MARTINS Sandra (Portugal), GHEDAMSI Imène (Tunisia), GIBEL Patrick (France), GONZALEZ-MARTIN Alejandro (Canada), GOODCHILD Simon (Norway), GRENIER Denise (France), GRENIER-BOLEY Nicolas (France), GUEUDET Ghislaine (France), HADJERROUIT Said (Norway), HALBOUT Gilles (France), HASA Jokke (Finland), HAUSBERGER Thomas (France), HOGSTAD Ninni Marie (Norway), HOPPENBROCK Axel (Germany), IOANNOU Marios (Cyprus), JOVIGNOT Julie (France), KIDRON Ivy (Israel), KONDRATIEVA Margo (Canada), KOUKI Rahim (Tunisia), KOUROUNIOTIS Christos (Greece), LAGRANGE Jean Baptiste (France), LALAUDE-LABAYLE Marc (France), LALLOUCHE Mickael (France), LECORRE Thomas (France), MARIN Jean-Michel (France), MARTIN-MOLINA Veronica (Spain), MARTINEZ-PLANELL Rafael (Porto Rico), MATHIEU-SOUCY Sarah (Canada), MILLMAN Richard (United States),

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