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ERME Topic Conference on Mathematics Teaching, Resources and Teacher Professional Development

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Content

Posters

Introduction

Researching Mathematics Teaching, Resources and Teacher Professional Development

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Mathematics teaching and mathematics teacher professional development are areas where research has increased substantially in the last years. For the last ERME conferences, a large number of proposals was related to this research area (e.g., three topic groups were formed at CERME9 in 2015: TWG18 on mathematics teacher education; TWG19 on mathematics teaching; and TWG20 on resources for teaching).

In this ongoing field of research, many issues need further investigation. We need to better understand the underlying characteristics of mathematics teacher education and the professional development contexts that have a positive impact on teachers' professional learning, even with respect to sustainability. Also, further discussion and research are needed on how to *link* research findings and how to *bridge* theoretical and methodological approaches to mathematics teacher pre-service and in-service education.

Studying mathematics teaching goes beyond teachers' classroom behavior. It encompasses teachers' actions and meaning-making as these relate to instruction. This includes, amongst others, task selection and design, classroom communication and assessment as well as the interplay between goals and actions as classroom interactions unfold in the context of broader institutional, educational, and social settings. A central question for investigation is what kind of methodological and theoretical tools are necessary to address this complexity.

In terms of resources, the focus of research for the last decades has been on teachers' beliefs and knowledge. More recently, teachers' identity, tasks, and teaching resources have received attention. Moreover, mathematics teacher educators' knowledge and development has been an emerging field. Aiming at achieving a better understanding, characterizing and/or evaluating the content of teachers' knowledge, several theoretical and methodological frameworks have been developed and discussed. Yet, further discussion seems to be needed in order to better describe the content of such knowledge, its relationships with (and influence on) teachers' beliefs, goals and identity as well as with mathematics teaching.

These three strands (mathematics teacher education, teaching and resources) are far from being disconnected. The ERME Topic conference "Mathematics Teaching, Resources and Teacher Professional Development" (5-7 October 2016, Humboldt-Universität zu Berlin, Germany) served as a platform for investigating in what ways these strands are linked - as regards research questions, methodologies and theoretical perspectives. The International Programme Committee was chaired by Stefan Zehetmeier (Austria), Miguel Ribeiro (Brazil), Bettina Rösken-Winter (Germany), and Despina Potari (Greece).

The conference focused on exchanging participants' knowledge and experiences, and on networking between scholars from different countries and cultures. In sum 69 scholars (60 from Europe) from 16 countries (12 from Europe) participated in this conference and submitted 37 papers and 14 posters. All submissions were peer-reviewed and a selection was made according to the quality of the work and the potential to contribute to the conference themes. Finally, 27 papers and 12 posters were accepted and presented at the conference.

Note: This text is based on: Zehetmeier, Rösken-Winter, Potari & Ribeiro (2016), Mathematics Teaching, Resources and Teacher Professional Development, Newsletter of the European Mathematical Society, pp. 46-47, EMS Publishing House.

Plenary Lectures

From collaborative research to Mathematics Teacher's Specialised Knowledge: a story of knowledge

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In 1999, our research group was approached by a group of primary teachers looking for advice on using problem solving tasks in their teaching. We decided to use this opportunity to follow these teachers' professional development over time. After ten years, we focused our attention on teacher knowledge with the aim of developing an analytical model capable of describing and interpreting the teacher's subject-related knowledge, and exploring ideas for future application in pre- and in-service teacher training. The resulting model is the Mathematics Teacher's Specialised Knowledge (MTSK) model, developed and applied collaboratively by the research group in conjunction with a team of researchers and teachers, and composed of several domains and subdomains, including mathematical knowledge, pedagogical content knowledge and beliefs.

Keywords: collaborative research, professional development, teachers' knowledge, specialised knowledge, beliefs.

INTRODUCTION

After many years of doing research into Mathematics Education, and having competed for grants for numerous research projects at local, regional, national and international levels, and having reflected on the significance of lines of research and research questions, I find myself at times wondering whether it is truly us researchers who define these lines and these questions, or whether they are in fact determined by external influences. In other words, do researchers make inroads into new areas or is it the other way around? In all probability it is an ingenuous question; after all, there are surely many other human activities which are apparently subject to outside forces. We should not be surprised, then, that research (especially into Mathematics Education), like so much other human activity, is channeled according to social and political interests, which at times establish which lines should take priority.

Often such interests are motivated by the multiple areas of knowledge within the university itself as they jostle for predominance. But other times the reason for a particular line of research comes from outside the university context. In this instance it came from the beneficiaries of this research, the teachers themselves. And thus begins the story I would like to narrate about how a group of teachers triggered a major shift in the way the researchers into Mathematics Education at the University of Huelva carried out their research with teachers.

GENESIS OF THE PIC

Our (researchers') interest in teachers and their work dates back many years. In the 80s and 90s this took the form of contributing to in-service training programmes. The first doctoral theses (Carrillo, 1996, published 1998; Contreras, 1998, published 1999) explored aspects of teachers' approach to their work (the way they solved problems, their conceptions regarding mathematics, teaching and learning mathematics and the use of problem solving activities as a teaching approach). Nevertheless, the key moment for us was the founding of the Collaborative Research Project, or PIC (from the Spanish 'Proyecto de Investigación Colaborativa').

This event took place in 1999 when a group of primary teachers approached the Mathematics Education Department at the University of Huelva seeking support for their intention to carry out a problem-solving based methodological innovation at their school. It is not customary for teachers outside the university context to make this kind of approach, the traffic is usually in the other direction with researchers typically attempting to persuade teachers at schools and colleges to participate in their theses and research projects, often finding themselves coming up against a reluctance to be videoed or make a long-term commitment. We can say, then, that we had been presented with a golden opportunity. The teachers had noted learning obstacles in their classrooms and had already identified a problem-solving approach as an ideal vehicle for learning mathematics. Further, they freely recognised that their own backgrounds had not equipped them with the appropriate training to be able to bring the desired innovation into effect.

An obstacle immediately presented itself when they outlined the kind of collaboration they were looking for: essentially they wanted us to provide them with a bank of problem-solving activities for use in class at our earliest possible convenience. The department discussed their proposal and quickly agreed on a counter-proposal. We did not feel professionally comfortable with simply passing on some prêt-a-porter activities, however well-designed and ultimately rewarding they might be, and we were far from convinced that to do so was sufficient in itself to bring about the change in attitude towards mathematics that they were seeking. So instead we suggested creating a work group under the acronym PIC (see above), which would act as a forum for reflection on our conceptions about mathematics and how it might be channelled in class, and in particular about what we could understand by the term 'problem' and an approach structured around the concept. We suggested planning problem-solving based lessons together and then conducting feedback sessions to reflect on how things had gone. This suggestion was accepted, though not unanimously, and several teachers decided not to form part of the PIC. Notwithstanding, the PIC came into being and is still functioning today. In this respect, one has to recognise that the professional motivation of the teachers was and keeps on being stronger than some hindering factors (as the high involvement required by the group, or the level of demand of the designed tasks). No direct implication of the institutions (schools) has been occurred, but the PIC has often received the support by the Andalusian Government.

From their early desire for a bank of off-the-peg problem-solving activities, the members of the PIC came to focus their interest on beliefs, conceptions and the use of problem-solving in Primary Education. In reality, the greatest shift was that, following a short period of working together, the centre of interest became professional development. For the teachers, this meant reflecting on the methodological implications of adopting a problem-solving approach. For the researchers, this meant seizing the opportunity to study the process of professional development among the teachers. The PIC became a collaborative forum for the promotion and study of professional development. And we went from doing research *into* teachers to doing research *with* teachers (Carrillo and Climent, 2002).

THE WORK OF PIC

As mentioned above, the PIC was, and still is, a space within the domain of Mathematics Education devoted to professional development. Nevertheless, we came up against an unexpected obstacle from the start: the teachers made it clear that "we don't want to do mathematics," and were adamantly opposed to taking on any type of mathematical problem or indeed any task that required significant mathematical effort. They had little confidence in their own mathematical abilities and were ill disposed to "go through that kind of suffering."

What is the point of professional development, we wondered, if you are not prepared to pit yourself against some sort of mathematical task? And what role could the PIC play in the promotion (and study) of professional development?

By way of answer, we would say that the PIC is characterised as follows:

> Teachers and researchers share objectives: in the first instance, this refers to the professional development of the teachers themselves, but it includes, too, the development of the university lecturers' research work, as well as their capacities in the field of teacher training.
>
> The work is collaborative: each participant brings with them his or her knowledge and experience of teaching, and shares their needs and interests.
>
> The focus is on teaching and learning mathematics: situations drawn from participants' experience of teaching, and the challenges they face as they go about their job represent the hub around which all discussions in the group meetings revolve.

Within the confines of the PIC, professional development can be conceived of as a gradual improvement in the quality of one's self-reflection (Climent and Carrillo, 2003), and can be measured by the number of elements and the degree of complexity the teacher is able to bring into play in this reflection. Certainly, our objective has never been to study professional development in terms of the teacher's approximation to some pre-established teaching ideal laid down by the researchers.

It was along such lines, then, that the PIC began to take shape. Despite the teachers' reluctance to "do mathematics", they were in fact more than happy to talk about their experiences of mathematics in the classroom. This proved sufficient to initiate a collaborative enterprise that has continued to this day. In truth, the teachers' initial resistance was not unusual as many primary teachers readily admit to being weak when it comes to mathematics and have scant inclination to remedy the situation, tending instead to lean heavily on whatever textbook they use to plug the gaps in their knowledge. What marked this group of teachers out was their decision to do away with the coursebook, or at least to take a highly selective approach to it so as to shift their mathematics teaching towards a problem-solving based approach. It was a significant difference.

Over the time, to the perennial questions of reflection on beliefs and problem-solving were added other areas of interest, which included planning activities using ICT. The composition of the PIC also widened its scope to include lower and upper secondary teachers, students on Master's programmes and in the final year of the degree in primary teaching, and even an education inspector. In this way the PIC met the formative needs of different groups; in particular, it bridged the gap between initial and in-service training. For the researchers, we found we were able to nurture a growing interest in aspects of teacher knowledge, which emerged from the reflections about mathematics triggered by the activities discussed in the PIC sessions.

TEACHER KNOWLEDGE

We (researchers) can identify 2009 as the year in which we became seriously involved in doing research into group members' knowledge, within the context of the PIC, and beyond (studying individual teachers). Our guiding text in this respect was the MKT model (Mathematical Knowledge for Teaching) (Ball, Thames and Phelps, 2008).

Over the course of this period of reflection about teacher knowledge, the teachers in the PIC became aware that their lacunas were holding back their professional development. As said above, the PIC is a collaborative group (Feldman, 1993). The participants share their knowledge and experiences. The work is characterised by a permanent look at the students' needs and learning features. At that time, in the context of the design of tasks, the teachers realised that their knowledge was not enough to approach students' needs accurately. Their commitment in the group, their claims in group sessions and interviews, and their proposals about issues to be dealt with in future projects made it clear. As a consequence, there followed a significant shift in attitude as the teachers recognised they needed to "know more mathematics". This represented a major step forward from the original refusal, although it did not go as far as embracing the opportunity to deepen their knowledge on their own account or facing the challenge of more demanding mathematics. However, it did mean that they were now prepared to leave their comfort zone and garner the necessary mathematics for carry out certain activities in class. Moreover, when they used the term “in class”, they did not understand this as a specific group in a specific year, rather they began to think in terms of the treatment a mathematical item might have over the course of the full educational cycle. This, too, represented a significant shift in attitude, as teachers often tend to “specialise” in certain year groups and forget about what might be happening in others.

Nevertheless, there remains the question of how to interpret “more” (with regard to teachers and researchers) when the teachers said they wanted to “know more mathematics”? Are we talking about quantity or quality? And for that matter, which mathematics? It is our belief that the response to these questions is *specialised knowledge*; that is, the kind of knowledge the teachers already had and which they needed to extend. This naturally led us to draw not only on the work of Shulman (1986) and his original division of teacher knowledge into domains, most particularly Subject matter knowledge and Pedagogical content knowledge, but also the work of Ball *et al* (2008) in applying the model to the case of mathematics teachers, introducing a subdomain of Specialised knowledge (Subject content knowledge) within the domain of mathematical knowledge. We were also influenced by the importance given to knowledge of connections in Rowland *et al’s* (2009) model (Knowledge quartet) and in the work of Fernández *et al* (2010), and Ma’s (1999) notion of packages of knowledge, which gives an integrated vision of knowledge. Taking inspiration from critical aspects of these and other studies, we developed the MTSK model (Mathematics Teacher’s Specialised Knowledge).

THE MATHEMATICS TEACHER’S SPECIALISED KNOWLEDGE MODEL

In the MTSK model (figure 1), in addition to the subdomains of Mathematical knowledge (MK) and Pedagogical content knowledge (PCK), we consider the domain of beliefs (about mathematics and about mathematics teaching and learning). We feel that an extrinsic perspective or organization of MK (in which specialized knowledge is defined as a distinct component) justifies (mathematics) teaching as a profession, but can be difficult to share across different educational systems. By contrast, we chose to adopt an intrinsic organization, in which it is easier and clearer to characterize the subdomains. Moreover, the specialization concerns not only the MK, but the whole model. That is to say, from this perspective, specialization represents mathematics-related knowledge used/needed in/for teaching, regardless of whether it is shared with other professions.

We (researchers) are aware that our own beliefs about mathematics and about mathematics teaching and learning have exerted an influence on the model and likewise our research paradigm, but the same can be said of any research group. We are convinced that lesson observation provides relevant information about the teacher’s knowledge, but at the same time, we recognise that in itself it is not sufficient and needs to be complemented by additional sources, which can include questionnaires, (individual and group) interviews, teacher participation in forums, and so on. We are mindful that in giving priority to lesson observation there is a danger that one's interpretation of events can be overly influenced by what one wants to see in the teacher. The case comes to mind of a teacher carrying out an engaging mathematical activity with his pupils. The researchers took the design of this activity as evidence of his Knowledge of mathematics teaching; however, when asked afterwards about his reasons for using it, his reply could not have been clearer: "it was the next one in the book."

An advantage of the MTSK model is that it can be applied to any educational level. To date, we have carried out studies (all following qualitative methodology) at Pre-school (in progress), Primary (ages 6-11), Secondary (12-15), Baccalaureate (16-17) and University levels. Evidently, in the case of Primary Education, it should be noted that, although the figure of mathematics specialist does not strictly apply, the role of mathematics teacher can be understood as any teacher doing mathematics with their class. At Pre-school, the subject of mathematics does not even exist, but there are elements of mathematics embedded within the syllabus for this level, and the mathematics teacher can likewise be understood as any teacher covering these elements. Various content areas have been studied using the MTSK model: algebra, fractions, functions, geometry (polygons and polyhedra), and infinity. The demarcation of the subdomains in figure 1, and likewise the development of categories pertaining to each of the subdomains, derive from both the literature (as mentioned in the previous section) and specific studies, taking a *top-down bottom-up* approach (Grbich, 2013).

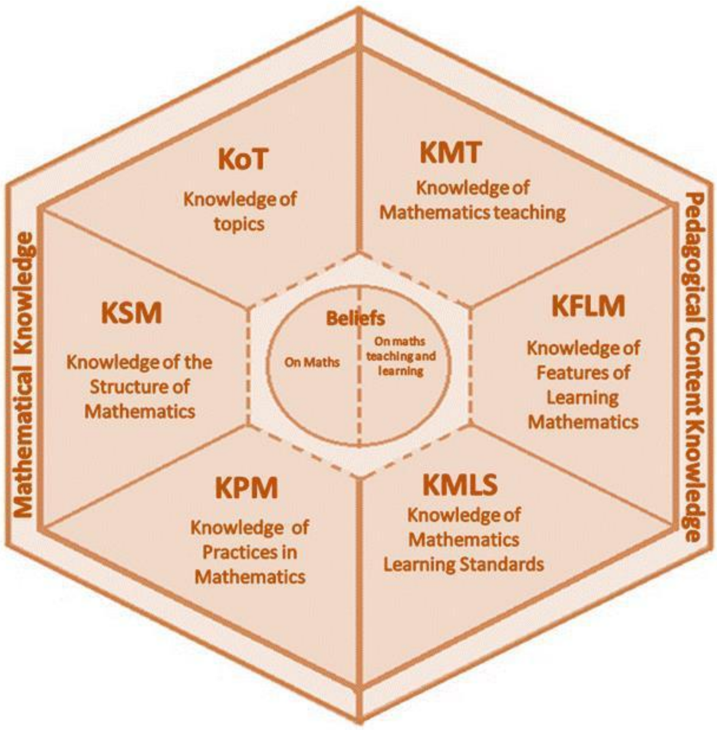

Figure 1: The MTSK model

In order to demonstrate some of the characteristics and descriptors of each subdomain, we will take a concrete example of the model being applied, in this instance to a teacher we shall refer to as Enrique (pseudonym).

Enrique's MTSK

Enrique teaches the 5th year (age 10) at a state Primary School in a small town on the coast in the province of Huelva. There are 25 pupils in his class, of varying levels and from diverse socio-economic backgrounds.

In the lesson extract we will analyse, he aims to guide his pupils towards reaching a definition of polygon. In point of fact, the pupils have already studied polygons in their 4th year, but Enrique is aware that they need to do more work on mathematical definitions and believes they should be capable of arriving at a meaningful definition for themselves. In the follow-up interview to this lesson, he says:

> E (Enrique): Even when they give you a correct definition, they don't have all the possible polygons in mind. What's more, they rarely get to grips with a proper mathematical definition, instead they have a set of properties, which might be enough to define a polygon or might not be.

Various elements of MTSK are brought into play here. In the first part of his utterance, Enrique demonstrates his awareness of his pupils' knowledge gap between the definition of polygon and the set of polygons to which that definition corresponds (*Knowledge of features of learning mathematics* in relation to students' learning difficulties), as sometimes happens when a pupil fails to recognise a concave quadrilateral as a polygon despite the fact that it fulfils all the requirements of the definition. Further, Enrique gives some indication of his knowledge of what constitutes a definition in mathematics (*Knowledge of practices in mathematics* in relation to the necessary and sufficient conditions of a definition). And finally, Enrique's insistence on the pupils developing a meaningful definition is a demonstration of his beliefs about mathematics teaching and learning.

In the lesson itself, Enrique brings a bag to class filled with different flat shapes cut out of card. He invites various pupils to each take a shape from the bag and stick it on the board. He instructs them to divide the shapes into two groups, one on the left and one on the right, but gives no further instructions and no indication as to rationale they should use to form the groups. Nevertheless, those on the left correspond to polygons (including concave polygons) and those on the right to non-polygons (shapes with partial or total curved outlines, including a circle). In this respect, Enrique displays *Knowledge of mathematics teaching* regarding the underlying design of this activity. He has clearly considered a wide variety of flat shapes for his pupils to sort into two groups with the awareness that in doing so the pupils would necessarily have to consider what features are common to each group, something fundamental for developing definitions (and again connected to his *Knowledge of practices in mathematics*).

Following this initial phase, Enrique then tells the pupils to look carefully at the group on the left, and asks if they can remember a name to describe them. After a few moments of pondering, one of the pupils replies that they are called polygons, upon which Enrique tells the class that their task is now to define what a polygon is, in order to do which, they need to focus on the features common to the set of shapes in each group. When a pupil suggests that one of the features common to polygons is that they have corners, Enrique observes that although this is true, there are also shapes in the other group, which have corners. He adds:

> E: When we define something, we try to find the common features, but in such a way that we also exclude the shapes that don't have all the features.

Here we can see evidence for Enrique's *Knowledge of practices in mathematics* regarding the features of mathematical definitions.

The lesson continues with Enrique writing on the board the features suggested by the pupils. In order to ensure that the figures on the right are excluded, he writes a negative feature: polygons do not have curves. He goes on to draw a shape (an open polygonal chain) which fulfils all the features the pupils have provided up to this moment, intending that the pupils should reject it from the set of polygons (as they subsequently do) on the grounds that polygons are closed shapes. He then adds this feature to the list on the board. In this instance, in addition to *Knowledge of practices in mathematics*, Enrique also demonstrates *Knowledge of topics,* with respect to the topic of polygons (definition and properties).

In the next stage of the lesson, Enrique then draws a convex polygon on the board and asks whether the polygon is constituted by the line or what is inside the line. In the follow-up interview we asked what were his reasons for asking this question:

> E: Well, they need to know the difference between a polygon and its outline. Up to now we have just talked about the outline; in the definition we've only mentioned features relating to the sides. I know that a lot of the pupils are not sure about this.

In this instance, two aspects of MTSK come together: *Knowledge of topics* (definition of a polygon and appropriate examples) forms a connection with *Knowledge of features of learning mathematics* (knowledge of typical areas of student difficulty).

At this point, it would be useful to give brief descriptions of the subdomains mentioned so far. *Knowledge of topics* comprises mathematical procedures, properties, rationale, representations and models, as well as contexts, problems and meanings. *Knowledge of practices in mathematics* is about knowing how to proceed in solving problems, how to validate and provide proof in mathematics, the role of symbols and the use of formal language, specific practices in mathematical work (eg, modelling), and how to generate definitions. *Knowledge of mathematics teaching* comprises teaching theories, specific mathematical characteristics of educational materials for teaching a specific content, and strategies, techniques and tasks for teaching mathematical content. *Knowledge of features of learning mathematics* comprises learning theories, strengths and difficulties, modes of students' interaction with the content, and the interests and expectations of learners concerning particular content. We found evidence of Enrique's knowledge with respect to these four subdomains, and additionally evidence of his beliefs about teaching and learning mathematics. Evidence of the other two subdomains was not found in this particular excerpt. However, we can provide an example of *Knowledge of the structure of mathematics* from another extract, in this case drawn from a PIC session in which participants discuss the connections between the treatment of a topic from Pre-school to Secondary level. Various members contribute:

PIC1: When you use a scale, for instance S: 1:500, in Secondary Education, you are bringing in the notion of geometrical proportion between two objects (a rectangle and a room), that is to say, the similarity of shapes.

PIC2: This is based on the conservation of the shape and the existence of a numerical proportion between the lengths of the corresponding sides.

PIC3: It implies an extension of the concept of equivalent fractions, where the numerator and denominator are whole numbers, through the notion of ratio, which is an expression of a multiplicative relation between quantities, like double or triple.

PIC2: These relations offer greater precision in estimating whether one object is bigger than another, something we do in Pre-school.

Finally, the last of the six subdomains, *Knowledge of mathematics learning standards*, is composed of knowledge about learning expectations at different levels, expected levels of conceptual or procedural development at different stages of education, and the different treatment of topics as they are revisited over the full span of the educational cycle. By learning standard, we mean any instrument designed to measure students' level of ability in understanding, constructing and using mathematics, and which can be applied at any specific stage of schooling.

CLOSING COMMENTS

The professional lives of the members of the PIC and of us researchers became interlinked in 1999 (especially who belong to the PIC from the very beginning), constituting a productive forum for sharing experiences and knowledge. Although our research work goes beyond the confines of the PIC, it is fair to say that the PIC has played a decisive role in our professional development.

This paper has illustrated how the MTSK model can be applied, taking the analysis of the knowledge deployed by a primary teacher as an example, and attempting to show

how this knowledge is interconnected and complex. Like any analytical model, MTSK deconstructs the object of study into its constituent parts. As a result, it is able to provide a fine-grained analysis (down to the level of categories and their corresponding descriptors within each subdomain, which space prevents us from presenting here), but at the same time, those of us who carry out research with the model, are aware of the holistic nature of knowledge and are always at pains to underline the interconnectedness of the subdomains.

At another level, MTSK also provides support in planning lessons (including designing activities), whether in the context of in-service training or professional development (such as the PIC), or indeed pre-service training. Whatever the context, it is essential to be able to reflect on the knowledge that is brought into play the teacher concerned.

Among the challenges that we have set ourselves (in addition to continuing to make progress in describing the subdomains and categories, and considering in more detail the meaning of MTSK in Pre-school Education) it to apply MTSK to different topics, to develop a parallel model in other subject areas (one is already under development for the area of Biology, which might well take the name BTSK), improve the picture of interconnections between subdomains, further explore the affective domain (including beliefs), and map out a model of Mathematics Teacher Educators' (Specialised) Knowledge (for which there are already research projects underway).

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Design principles and domains of knowledge for the professionalization of teachers and facilitators - Two examples from the DZLM for upper secondary teachers -

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The DZLM (German Centre for Mathematics Teacher education) is a joint research and development institution of seven German universities to set new standards and prototypes for the professionalization of mathematics teachers. The programmes are developed through research-based design principles, and models for domains of teachers' and facilitators' knowledge and competences. In the following, we will elaborate on DZLM's models and principles by presenting two exemplary courses to implement the new national standards for upper secondary level teachers: one focussing the use of digital tools for teaching and learning mathematics, and one focussing on probability and statistics as a new subject for most teachers.

Keywords: professional development, teachers' competences, facilitators' competences.

PROFESSIONAL DEVELOPMENT - STATE OF THE ART IN GERMANY

Teacher professional development is essential to further develop mathematics teaching (Borko 2004). In recent years, a shift can be stated in better conceptualizing and grounding professional development by means of research. It is no more aimed at eliminating shortcomings, but the development goes more into the direction of a continuous process of professionalization (Rösken-Winter & Szczesny, 2017). That is why the duration and formats of professional development should change from single short courses to courses consisting a mixture of several face-to-face-meetings, as well as blended learning phases for supporting teachers (Fishman et al. 2013). But these conclusions from research findings did not yet lead to be realized in the practice of teacher education in Germany.

To get an idea why these change processes are difficult, it is necessary to briefly describe the educational structure in Germany. Teacher education in Germany is mainly structured in three phases. The first phase at university takes 3,5 to 5 years, depending on students' aims to become a primary or secondary teacher. Although there are standards for teacher education published by the Society of Didactics of Mathematics together with the main teacher association for mathematics, these are not compulsory but just recommendations. There is still a big variety how to conceptualize the education at university as the official guidelines are very vague and allow still many ways of how to decide and realize the content of the education. This is the same for the second part of teacher education, which happens over 18 months at special centers for pre-service-education, run by the regional school administration. During this phase, the future teachers have to teach at a school, partly supervised and assessed, and partly in their own responsibility. In-service education is not compulsory and is offered under the authority of the school administration and by free providers (such as teacher networks, universities, teachers' association). There are currently no standards or guidelines for professional development, and facilitators do not receive specific education to be prepared for this job. They are mainly qualified teachers who are denominated by their governments to act as trainers and facilitators.

The DZLM (German Center for Mathematics Teacher Education) was launched in 2011 with the aim to support und pursue the existing programs and structures for continuous professional development (CPD) nationwide, networking all reforms in this field, and develop new research-based exemplary courses. DZLM is a joined endeavor of different researchers from different universities, collaborating with representatives from school administration of all sixteen Federal States and teachers from school practice. DZLM follows a design-based research paradigm (van den Akker et al. 2006) when designing and researching programs for different target groups of teachers and topics. DZLM offers qualification of facilitators, in-service-teacher-education, out-of-field-teaching and acts as a network platform for information and exchange. For all programs of professionalization - so also for DZLM – the main challenge is the issue of scaling (Coburn 2003). Therefore, our research aims at understanding the change processes and how to overcome problems and obstacles to optimize the programs. Currently DZLM is in the second funding period (2016-2019) with the aim of establishing it as a permanent nation-wide operating institute for research and development in the field of mathematics teacher professionalization.

One important issue for the DZLM was to establish design principles as guidelines for designing and analyzing CPD courses. This has been done in a cooperative process of all DZLM-researchers reviewing the current state of research in the field. Based on this comprehensive literature review six design-principles have been generated to provide criteria of efficient teachers' professionalization:

- **Competence-orientation**: Crucial for effects and efficacy of professionalization is the clear focus on content to improve and deepen teachers' knowledge, and performance in teaching (Garet et al. 2001; Timperley et al. 2007). As an important guideline to address the different areas of relevant content, DZLM has established a framework for teachers and facilitators (see fig. 1) (cf. Lipowsky & Rzejak 2015; Garet et al 2001).

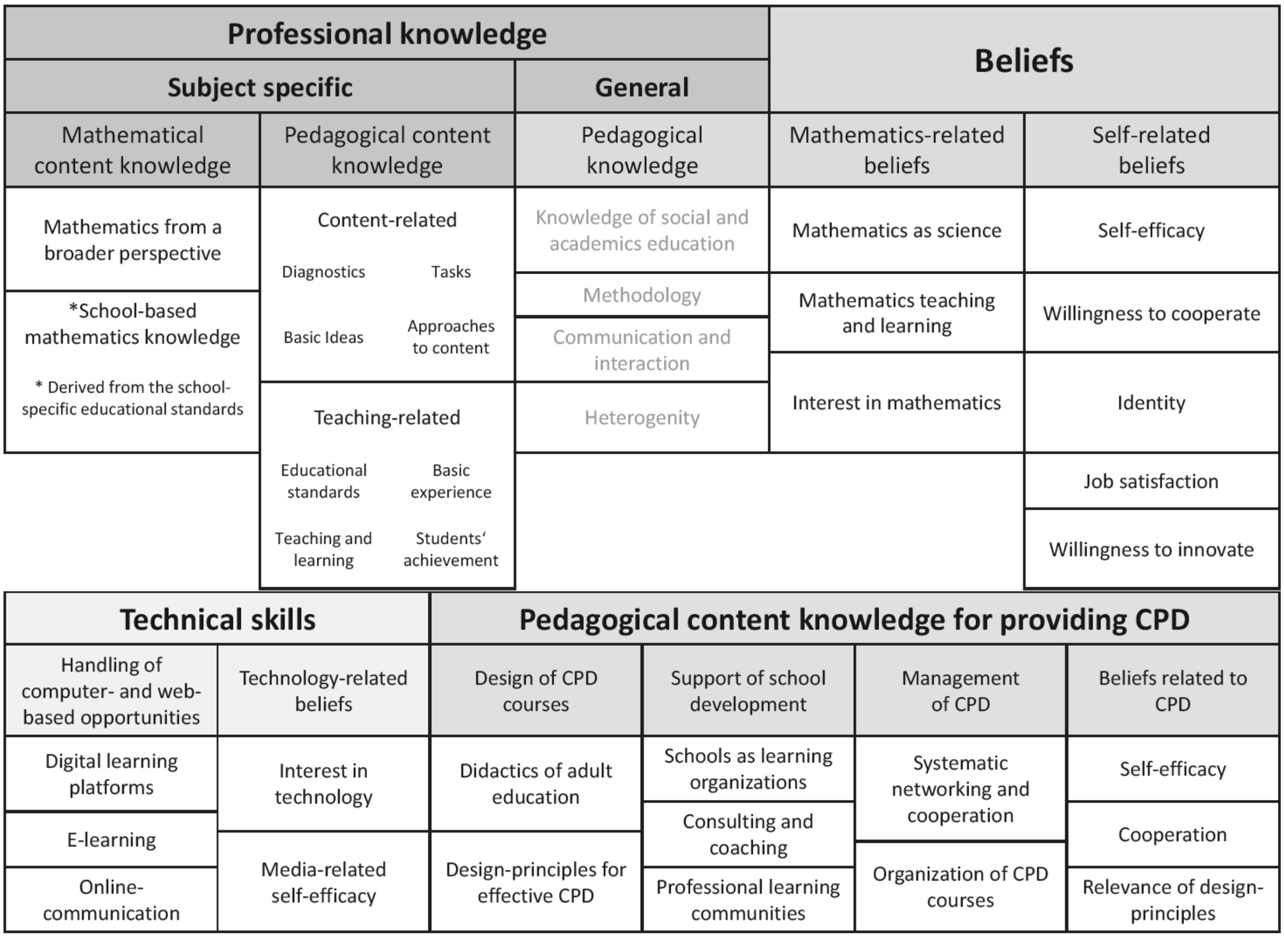

Figure 1: DZLM-Competence framework for PD courses

- **Participant-orientation**: Centering on the heterogeneous and individual prerequisites of participants. Moreover, participants get actively involved into the PD unit instead of pursuing a simple input-orientation (Clarke 1994; Krainer 2003).
- **Stimulation of cooperation:** Motivating participants to work cooperatively, especially between and after the face-to-face phases, ideally sustainable professional learning communities are initiated (Krainer 2003, Bonsen & Hübner 2012)
- **Case-relatedness:** Using cases such as videos of teaching or students' documentations, which are relevant for the school practice, to enable new perspectives, and to realize further dimensions of teaching effects (Borko 2004; Timperley et al. 2007; Lipowsky & Rzejak 2015).
- **Diverse instruction formats:** During PD courses, it is important to realize a mixture of different formats (like lectures, individual and collaborative work).

Also phases of attendance, self-study and e-learning should alternate (Deci & Ryan 2000; Lipowsky & Rzejak 2015.

- **Fostering reflection:** Continuously encouraging participants to reflect on their conceptions, attitudes, and practices (Deci & Ryan 2000; Putnam & Borko 2000).

Taking these principles seriously naturally yields to the necessity to realize CPD initiatives in long-term formats as well (Rösken-Winter et al. 2015).

The following two examples of PD-courses illustrate the work of the DZLM. Both examples are from North Rhine-Westphalia (NRW). It is the biggest Federal State in Germany in terms of numbers of inhabitants (18 millions of 82 million in the whole of Germany). The Federal States are responsible for any educational issues. The nationwide standards in mathematics (KMK, 2012) serve as recommendation, but most of the curricula in the Federal States follow these standards. In the previous years two main innovations for upper secondary level and the final centralized examination (Abitur) have been brought up in NRW. It is on the one hand the introduction of graphic calculators (GC) as compulsory tools (by decree in 2012) in classrooms and examinations. On the other hand the new state curricula in NRW (2014) fixed stochastics (probability and statistics) as an obligatory topic for all students (six months teaching of stochastics in all mathematics classrooms). The main argument for the introduction of the GC was to support a deeper understanding of mathematics by interactive visualization, relieve from routine calculations and routine analyses of data, and by supporting modeling with more realistic examples. Regarding stochastics, in particular the use of the GC for simulations is suggested.

For both topics – an introduction on using and teaching with GCs and on teaching stochastics - the DZLM has collaborated strongly with the educational administration in NRW and realized two PD-courses: “GC compact” and “Stochastics compact”. In the following we present both courses to illustrate the work of the DZLM.

DESIGN PRINCIPLES - REALIZED IN THE PD-COURSE “GRAPHIC CALCULATORS COMPACT”

The DZLM together with the Ministry of Education in NRW were in charge of developing, delivering, and evaluating the long-term professional development (PD) course to integrate graphic calculators (GC) in mathematics classrooms. The project is situated in the context that applying graphic calculators is compulsory in upper secondary level teaching, and in final centralized exam (called “Abitur”) since the beginning of 2014. The design of the course was realized in different design cycles, the first cycle can be characterized as a strong collaboration within a group of teachers, researchers, and one person from the school administration. The course was realized in 2014 - 2015. It consists of four one-day modules (eight hours each) over a half year with phases of own experiences and elements of blended learning in between (mainly networking to exchange materials).

The DZLM design principles served as main guideline for the design from the beginning.

Competence-orientation:

The PD course covers different dimensions of teachers' competencies, which can be summarized in four main goals. The teachers should be able to use a tool in a flexible way, to design tasks integrating the technology, to organize the classroom in a technology-based environment, and to develop appropriate formats and tasks for assessment with the graphic calculator tool. The four modules were dedicated to these four goals: Introduction into working with GCs – Designing tasks by integrating the use of GCs – Classroom organisation in a technology based environment - Assessment. The concrete design of the single modules was based on research results. From the beginning of the course we highlighted relevant subject matter aspects when teaching functions and derivatives integrating technology. For example, we pointed out the importance of developing concept images (Tall & Vinner 1981, Bingolbali & Monaghan 2008) and "Grundvorstellungen" (vom Hofe & Blum 2016) of functions and derivativesm and offered tasks to initiate a fluent use and change between mathematical representations (Duval 2002). Besides these basic aspects systematic evidence is presented on typical student errors, pre- and misconceptions in the field of functions (Swan 1985; Hadjidemetriou and Williams 2002; Barzel and Ganter 2010). Additionally, we always explicated the role of technology as well as possible advantages and burdens when using technology (Barzel 2012). All these goals are made transparent for all participants, thus enabling teachers to clearly see the relation to their own teaching practice, and to increase their motivation while attending the course. The task to introduce the technical facilities during the first module was the task "power flower" shown in Figure 2 (Barzel & Möller 2001). This task served as an investigative open task as well as an example for meaningful tasks when integrating technology, and offered opportunities to reflect on the value of technology concerning the above-mentioned aspects of pedagogical content knowledge. Module 2 offered a sample of modelling tasks in the field of mathematical topics for upper secondary level, also including opportunities of data logging. Module 3 picked up the power flower task (Figure 2) to discuss classroom organisations and the point that technology can either be used to introduce a new topic (e.g. here power functions) or to deepen knowledge during a final phase of exercise. Module 4 focussed on exam situations. Current examination tasks were analysed as to whether the use of graphics is necessary, supportive, neutral or forbidden. Another perspective of reflecting the tasks was the role of the technology, for what the graphics are used for: for discovery learning, for conceptualizing, for enabling individual approaches, for taking over procedures or for controlling. This categorization is also suggested by the current German standards for mathematics in upper secondary level (KMK 2012)

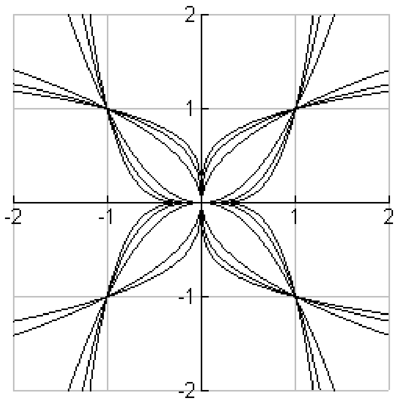

Figure 2: Task to get familiar with the GC : “Create this picture on your screen!”

Participant-orientation:

The participant-orientation combines two challenges: Taking up heterogeneous competences and conditions, and fostering participants’ self-responsibility.

Initially, a preliminary questionnaire regarding the teachers’ conditions, expectations and needs with respect to content and didactical issues can help to adapt the course to the specific target group. All tasks during the course are created to use them in the classroom with students as well. Accompanying material and information about the tasks show possible solutions, typical errors and misconceptions, an idea how and where to integrate the tasks in the learning process, and the relevant role of technology. To foster self-efficacy and self-responsibilities it is important to include a lot of opportunities which activate the participants – for example such as working on tasks in pairs and small groups, and initiating discussions and reflections about the material. Furthermore, at the end of each course, participants were actively involved in providing recommendations for content and methodology that should be included in the following meetings. Between the different face-to-face-meetings of the modules we offered a support-hotline to keep in touch - especially when problems arose.

Stimulation of cooperation

Aiming at sustainable cooperation processes we already stimulated to build professional learning communities (PLC) with teachers from one school or neighbouring schools during the first face-to-face-meeting. This stimulation was accompanied by a short input about the importance and power of intense collaboration in PLCs. The single PLC´s already worked together during the course. For the time after the course we highly recommend working collaboratively: To cooperate when designing tasks for use in the classroom, to share individual values and beliefs, to analyse students` solutions and other cases from the classroom.

Case-relatedness

All modules relate to practical experiences by discussing ideas based on specific cases from classrooms. On the one side, we brought cases into the courses such as specific student results and examples. And on the other side, we asked the participants to bring own cases from their classrooms to provide both a starting point for discussion, and an impulse for reflection. Figure 3 gives an impression of how such cases are used – here to discuss the challenge how students' documentation and language should look like when computer algebra is used. Here, we used the recommendations of Schacht (2017) to distinguish that the use of technical expressions in the documentations can be allowed when learning to get familiar with the technology but that the use of consolidated mathematical language must be used at the end of the learning process.

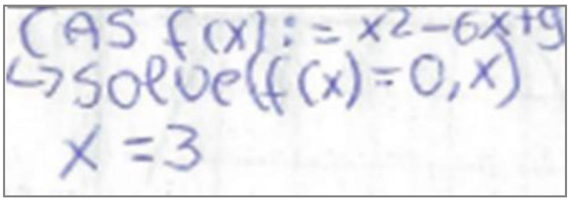

Figure 3: Is this documentation acceptable or not?

Various instruction formats

To ensure active participation and the experience of self-efficacy, various instruction formats are used throughout all face-to-face-meetings. The whole PD-course includes phases of attendance, self-study and e-learning to initiate cycles of input, learning, practical try-outs and reflections.

Fostering reflection

Participants are inspired to become "reflective practitioners" (Schön 1983) by stimulating cooperative reflection as well as self-reflection continuously with respect to tasks, students' solutions and thinking, scenarios of classrooms and on own conceptions, attitudes, beliefs, teaching routines and practices. Participants were encouraged to think deeply about the possible transfer of the teaching material into their own classrooms, and the impact on the own teaching style.

The whole PD-Course was realized in 2014/15 for three groups of teachers at different locations in NRW with about 100 participants. The accompanying research focuses on teachers' beliefs on the use of technology and their self-perception on how and how often they use the technology (Thurm et al. 2017). On the other side, Klinger (2017) investigated students' competencies in the field of function and derivatives to include this knowledge into the PD-courses (Klinger 2017). The current version of the course-material is enlarged now on digital tools instead of graphic calculators and it is published under Creative Commons license on the national DZLM server:

https://www.dzlm.de/fort-und-weiterbildung/fokusthemen/digitalisierung.

DOMAINS OF KNOWLEDGE FOR TEACHERS AND FACILITATORS – THE PD-EXAMPLE "PROBABILITY AND STATISTICS AT UPPER SECONDARY LEVEL"

In this section, we will focus on the design of a PD-course from the perspective of the facets of teachers' knowledge that we addressed. The course also considered the design principles of the previous section, but we will not make this explicit.

Context and overall design of the course

In this second part of our paper, we will illustrate how the content of a PD course was selected and the design was developed based on several circles of implementation and further elaboration. The course we will focus on is the PD course for upper secondary Gymnasium teachers, which we named "Stochastics Compact". Stochastics is used in Germany for the combination of probability and statistics. The course lasted four month with four and later five one day meetings. We started with version 1.0 in 2013 in the state of North Rhine-Westphalia, the current version is version 4.0. A total number of 400 teachers have participated in the various versions of the course.

The versions 1.0 to 2.0 of the course were designed by a DZLM – Team that consisted of teachers, young and senior researchers including the second author of this paper. From version 3.0 onwards we entered into a collaborative project with three facilitators from the federal state of Thuringia and five facilitators from the region of Arnsberg (3.6 million inhabitants) in North Rhine-Westphalia, with whom we developed new versions of the material and jointly used the material in our courses. The collaborative development, implementation and reflection aimed at improving the materials and qualifying the three plus five facilitators at the same time, we call them "project facilitators" in contrast to the other facilitators that will use the material but who were not part of the developmental team. All eight facilitators were experienced teachers that have been active as facilitators since many years, however, long-term PD courses such "Stochastics compact" were new for them. The fact that we brought version 2.0 of the course into the collaboration was a good starting point.

In Arnsberg, the regional administration supported a collaboration that lasted more than three years and three development cycles. The materials have reached a final stage (version 4.0) in October 2017 and are ready for use by all mathematics facilitators of the Arnsberg region. We have published a further elaborated version of a part of the material under Creative Commons license on the national DZLM server (https://www.dzlm.de/fort-und-weiterbildung/fokusthemen/leitideen).

The factors that finally influenced the design of the materials are multifaceted as is shown in Figure 4.

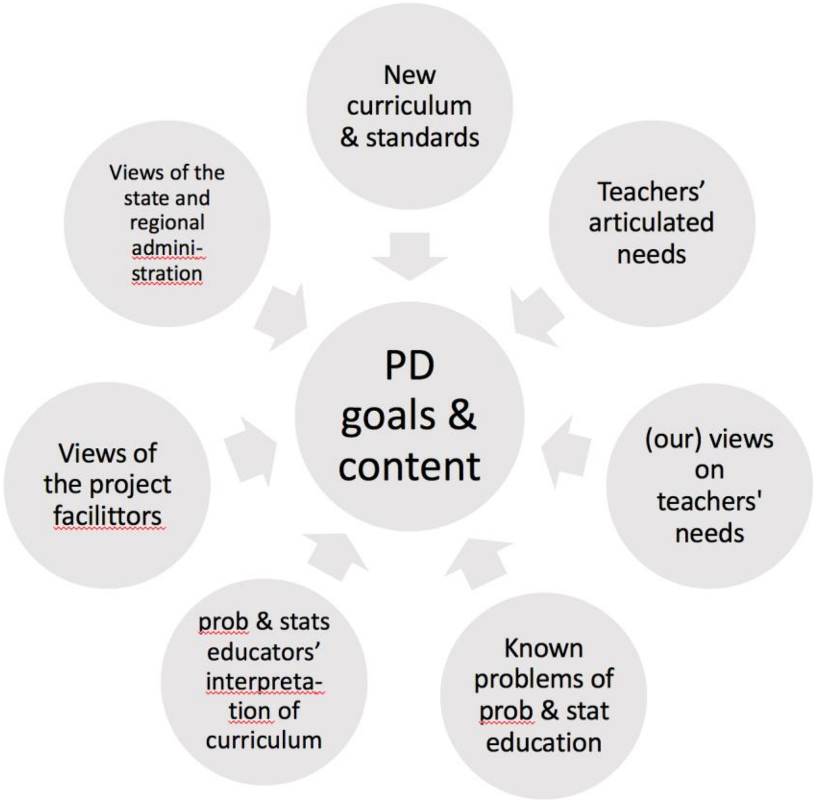

Figure 4: Influencing factors for the PD material

The picture (Figure 4) depicts some tensions between different views of the needs of mathematics teachers. The DZLM team is rooted in the knowledge base and research and development tradition of stochastics education. The new curricula do not take into account all the suggestions and ideas from this tradition, and did not share all the emphases and decisions that were taken when setting up the new curricula in stochastics. Our course is compatible with the new curricula, but tries to influence how these new curricula are interpreted and realized in the classrooms from the perspective of stochastics education. The syllabus of the curriculum allows options for school-based developments and variation and we intend to use this scope for development. We address teachers as independent personalities that we support in developing their own view of stochastics and stochastics education, we do not treat them just as curriculum implementers. We base the selection of PD content on analyses of difficulties of students and teachers and on a view that we consider as "fundamental ideas" for teaching stochastics at upper secondary level (Burrill and Biehler, 2011; Biehler and Eichler, 2015). We build on insights on how technology can be used to support students' learning in stochastics (Biehler, R., Ben-Zvi, D., Bakker, A., & Makar, K., 2013). Moreover, we suggest teaching approaches and material that we had used in university courses for future teachers or that we had tested in experimental classrooms, for instance Meyfarth (2006) on hypothesis testing, Prömmel (2013) on the use of simulations and Wassner et al. (2004) for Bayesian reasoning.

Subject matter didactics: "Fundamental ideas"	Problems of understanding; research on students	Own Research on classroom experiments; practical experiences
Connecting data and chance	Probability concept and relative frequency; laws of large numbers;	Grade 9 - 11
Stochastic independence	Independence as an implicit assumption, and sometimes an inadequate one	Pre-service teacher education
Bayesian reasoning, "natural frequencies approach"	Confounding conditional probabilities, e.g. AIDS-Tests; percentages in media	Grade 9
Binomial distribution as a model	Naïve modelling without assumption checking	Pre-service teacher education
Hypothesis testing	misinterpretations of P- Values, and of results of hypothesis testing in general	Grade 12

Figure 5: Subject matter components of the course and related research

An example of the modul: Connecting data and chance

For making our approach more concrete, we describe an example of the topic "connecting data and chance" (from the first module of the course).

We start with the following "landmark" activity that we are suggesting as a classroom activity when introducing stochastics at upper secondary level: as a challenging problem for students which will also show the power of computer based simulations for solving problems in probability.

Students can choose between two multiple choice tests with two choices in each question (one choice is correct)

Test 1: 10 questions

Test 2: 20 questions

A test is passed if at least 60% of the questions are correctly answered.

If a student just guesses: Which test is easier to pass?

O Test 1 O Test 2 O Equal chances

Figure 6: The 10-20-Test problem.

Activities in the PD course include: teachers guess intuitively, some initial discussion about reasons for the choices, use simulation to decide the question (estimate the probabilities to pass the test just by guessing). In all our courses, all three answers were initially chosen by at least some of the teachers, always stimulating interesting and lively discussions.

We start with simulation by hand (with a coin) where the small sample size usually does not provide a clear answer, and then we move to computer based simulation (with a GC) to get more precise and certain results. Estimating the passing probability will be supplemented by visualizing the whole distribution of "proportion of correctly answered questions" (see Figure 7). This is the basis for integrating the results into an elaborated intuitive view of how the distribution of relative frequency changes with increasing sample size.

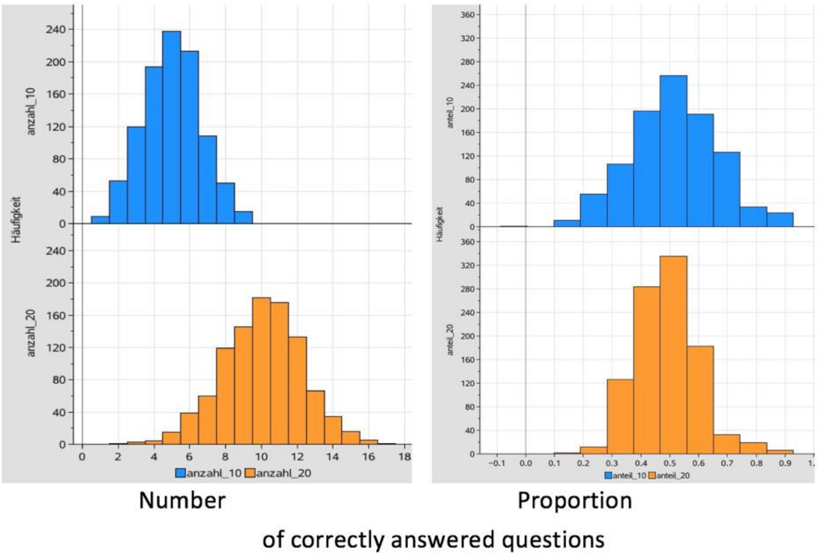

Figure 7: Simulation and visualization of the distributions with the TI Nspire

Some teachers can relate the picture on the right side of Figure 7 to their intuition that the relative frequency tends to be closer to the expected value of 0.5 when the sample size is larger. This stems from intuitions about the law of large numbers, although most of our teachers have never seen such a display as the law of large number is often only visualized as a trajectory, where the relative frequency "approaches" the theoretical probability.

The left side shows the simulated distribution of the number of successes, where the spread is increasing. We support our teachers in relating this to their previous knowledge. The number of successes of guesses during the testing can theoretically be modeled as random variables $X_n$with a binomial distribution, expected values at 5 and 10, and a standard deviation of $\sigma = \sqrt{n \cdot 0.5 \cdot 0.5}$, which increases with n. The right hand side is a simulation of the random variable $Y_n = {}^{X_n}/_{n}$, whose standard deviation is $\frac{\sigma}{n} = \frac{\sqrt{0.5 \cdot 0.5}}{\sqrt{n}}$.

A next step is to widen the question to what will happen, when we further increase the sample size n. Some teachers know that the middle 95% prediction interval around 0.5 can be theoretically calculated as $\left[0{,}5 - 1{,}96 \cdot \frac{\sigma}{n}; 0{,}5 + 1{,}96 \cdot \frac{\sigma}{n}\right]$, which is roughly $\left[0{,}5 - \frac{1}{\sqrt{n}}; 0{,}5 + \frac{1}{\sqrt{n}}\right]$; its width is $\frac{2}{\sqrt{n}}$. The so-called normal approximation of the binomial distribution is used for deriving this interval. This is also called the "one-over-squareroot-of-n-law”. This knowledge is considered as knowledge “at the mathematical horizon” in the sense of Ball and Bath (2009). This cannot and should not become the topic of instruction at the beginnig of the stochastics course, but is important for teachers’ orientation.

We then introduce to our teachers a way for introducing the “one-over-squareroot-of-n-law” just based on simulations and visualizations by means of “the prediction activity”. Based on simulated data, the percentile commands are used to find the middle 95%-interval (Figure 8, left side) and the GC is then further used to explore how the width of this interval depends on the sample size n (see Figure 8, right side).

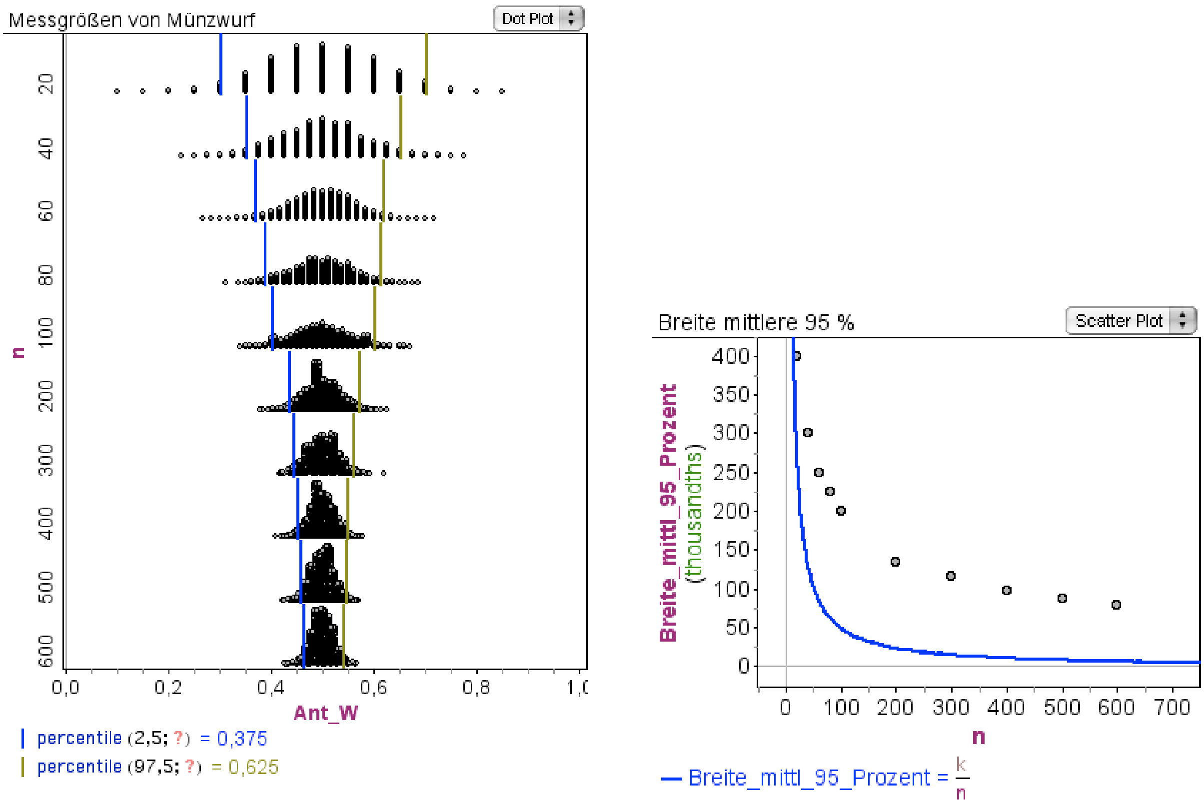

Figure 8: Left side: empirical 95%-prediction intervals
Right side: Trying to fit a curve to the width of the middle 95%: Functions such as $\frac{k}{n}$ do not work for any k; $\frac{k}{\sqrt{n}}$ fits well for k = 2.

It is claimed (without proof) that this law can be generalized to any p and n. For n repitions of a random experiment with success probability p the following inequality holds with 95 % probability for the relative frequencies f_n : $|p - f_n| \leq \frac{1}{\sqrt{n}}$ (95%-prediction interval). We argue that this knowledge is important for students,when they have to relate data and chance: instead of a vague idea that the relative frequency tends to approach the probability p with increasing n, an interval can be provided, in which we can expect the relative frequency with 95% certainty.

We also argue for introducing the inverse statement (with some horizon knowledge on confidence intervals that we cannot elaborate on here). If p is unknown we observe a relative frequency f_n, this value cannot be "far" from the true probability p: $|p - f_n| \leq \frac{1}{\sqrt{n}}$. The practical value for students is that if they simulate n – times and observe n, they can provide a so-called 95% - intuitive confidence interval for p, namlely $\left[f_n - \frac{1}{\sqrt{n}}, f_n + \frac{1}{\sqrt{n}}\right]$.

This knowledge is not obligatory in the syllabus but we argue that a sound dealing with simulations in the classroom requires knowledge about how precisely the unknown probability can be estimated from the relative frequency and how certain this estimation is.

Table 1: Prediction and confidence intervals for standard sample sizes

Sample size	Radius of 95% - prediction interval / intuitive confidence interval
50	± 0,14
100	± 0,10
1.000	± 0,03
10.000	± 0,01

We suggest that teachers at least communicate a rule of thumb table to their students containing interval widths for "standard" sample sizes (Table 1).

Facets of teachers' knowledge and beliefs

We base our course on models of teachers' knowledge, on Hill et al. (2008, p. 377), among others. The authors distinguish Common Content Knowledge (CCK), Knowledge at the Mathematical Horizon (HK), Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Curriculum (KC). This classification however leaves open, what counts as "knowledge" and what the warrants are for the respective knowledge base. We try to overcome the situation that this knowledge is solely based on opinion and experience, and introduce results from research in psychology and mathematics education as evidence for our suggestions and claims.

An extension that takes into account the various facets when we include technology was developed in Wassong and Biehler (2010, p. 2)

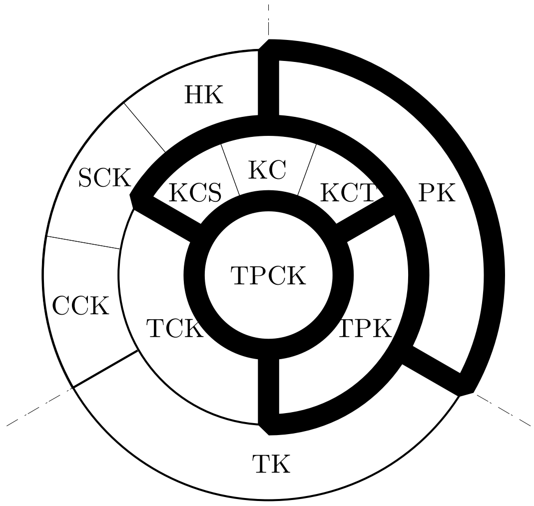

Figure 9: Extended domain map including facets of knowledge on technology

We will illustrate only some of the facets, refering to the above example. TK (Technological Knowledge) includes basic aspects of using the graphic calculator, TCK (Technological Content Knowledge) includes how to use the GC for simulations in stochastics and TPCK (Technological Pedagogical Content Knowledge) includes how to use the GC so that students can develop a better understanding of the law of large numbers through interactive experiments and simulations. KCT includes the suggested activities (10-20-test, prediction activity) and which representations to use for the simulated distributions. We already mentioned the knowledge at the mathematical horizon (HK), that is backgroud knowledge by which teachers can judge whether our suggested simplications are still an adequate elementarization of genuine mathematical content, and why the topics are important to teach. KCS, knowledge of content and students, includes misconceptions concerning the role of sample size. On a practical level, we include a variety of students' answers and reasoning to the 10-20-test problem to prepare teachers what can be expected in the classroom. Moreover the discussion in the PD-course itself - where some teachers have the same misconception at the beginning - is also a source for this knowledge. A mixture of KCS and HK is provided by drawing the teachers' attention to psychological studies, which show the insensitivity to sample size of many students and adults, and the need to better teach this for improving individuals' capacity to adequately reason under uncertainty. We quote the "maternity ward problem", which has the same structure as the 10-20-test problem, form original sources:

> A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies each day. As you know, about 50% of all babies are boys. The exact percentage of baby boys, however, varies from day to day. Sometimes it may be higher than 50%, sometimes lower. […] Which hospital do you think is more likely to find on one day that more than 60% of babies born were boys. (Sedlmeier and Gigerenzer, 1997, p. 36, based on reearch by Kahneman and Tversky, 1972).

We learned however, that making reference to the psychologial literature alone is not always convincing enough for our teachers. So we asked teachers in our course to become researchers themselves in that they should give the 10-20-test problem to a selection of their students. Teachers of the 2014 course asked their students (n = 1163). The results can be seen in Table 2, which convincingly show how widespread these wrong preconceptions are.

Table 2: Students' response to the 10-20-test (n = 1163, convenience sample)

%	**Grade 5 - 9**	**Grade 10 - 12**
Test 1	16	18
Test 2	41	27
Equal chance	43	55

In order to give an impression what can be acchieved by teaching, in Table 3 we refer to the experimental course of Prömmel (2013, p. 493)

Table 3: Students' response to the maternity ward problem before and after teaching

%	**pre**	**post**	**Pre: adequate reasoning**	**Post: adequat reasoning**
Test 1 is correct	26	77	18	59

This result resonates well with teachers experience that even the best teaching will not change all students minds, but that teaching can be successful for the majority of students.

Research on teachers' knowledge before and after the course

What have teachers learned during the course? Our boundary conditions do not make it possible to administer a knowledge test before and after the course. Therefore we use a questionnaire after the course and ask the teachers to subjectively assess their knowledge gain throughout the course (Nieszporek and Biehler, 2017; Lem and Bengo, 2003). This questionnaire covers various facets of teachers' knowledge, for instance CK "I can construct and perform a hypothesis test with fixed significance level?", KCS and KCT "I know typical misinterpretations of hypothesis tests and can elucidate/clarify them?". For assessing self-efficay we use items such as "By participating in the course, I have developed sufficient competencies and have received enough inputs, encouragements and stimuli for the (further) development of materials for my concrete classroom practice".

The result sof these questionnaires are vey encouraging but show a high variability in the answer of the teachers that has to be explained by deeper analyses of our data.

FUTURE PERSPECTIVES

Both courses are being further developed, published and used in other Federal States. The research on the stochastics course will be part of the Ph.D. project of Ralf Nieszporek, who will also focus on how facilitators shape and implement the jointly developed material. Oliver Wagener and Joyce Peters-Dasdemir investigate in their Ph.D. projects how multipliers use the published DZLM-material for the PD-courses regarding the use of digital tools, and how teachers use the materials of the PD-course in their classrooms.

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WG 1 - Working Group:

Teacher Education and Design Principles

Assessing pre-service teachers' competence of analysing learning support situations through a multi-format test instrument comprising of video, comic, and text vignettes

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When assessing teachers' competence of analysing specific classroom situations, the representation format of the classroom situation might impact on the quality of the teachers' answers. However, relatively little is known about the role of, for example, video, comic or transcript-like text formats in assessment instruments. This paper consequently focuses on this research need and presents results from a study which uses a format-aware design. For facilitating connections with prior research, more than 160 pre-service teachers were asked to analyse the use of representations in six learning support situations. The findings support the empirical unidimensionality of the competence construct under Rasch analysis while taking into account the three different representation formats of the classroom situations.

Keywords: vignette-based assessment, competence of analysing, noticing.

INTRODUCTION

Teachers have to draw on their resources (Schoenfeld, 2011) when they need to analyse observations in the classroom – at the same time, the competence of analysing is in itself an important resource of mathematics teachers (cf. Kuntze, Dreher & Friesen, in press): In learning support situations, for instance, teachers have to link criterion knowledge with observations, in order to analyse whether a reaction to a students' question is helpful. A key area of such criterion knowledge is knowledge related to the use of representations in the classroom (Dreher & Kuntze, 2015a), as it is a prerequisite for effectively analysing whether students might encounter problems with e.g. unnecessary changes of representations or with the potential disconnectedness of representation registers. However, the format in which a classroom situation is presented to teachers might make a difference: Whereas Dreher & Kuntze have used text vignettes only, Herbst, Aaron & Erickson (2013) have compared video and animation formats, i.e. formats marked by temporality. To our knowledge, there is hardly any empirical study which is "format-aware" in the sense that the design includes multiple formats (text, comic, video) and the possibility of taking into account their role in the corresponding measurement instrument.

Correspondingly, in order to establish validity with this respect, this study makes an attempt of assessing teachers' competence of analysing the use of representations with a multi-format test instrument coupled with a multi-matrix design in the assignment of formats to teachers, so that the role of the format can be taken into account.

We will in the following give an overview of the theoretical background, present the research questions, inform about design and sample, report about key results and discuss them in a concluding section.

THEORETICAL BACKGROUND: THE COMPETENCE OF ANALYSING THE USE OF REPRESENTATIONS IN THE CLASSROOM

According to Duval (2006), mathematical objects are only accessible through representations that can stand for them in many different ways (Goldin & Shteingold, 2001). In this sense, multiple representations often complement each other and emphasise different facets of the same mathematical object (Duval, 2006). For building up a rich concept image of a mathematical object (Ainsworth, 2006) which facilitates flexible ways of problem solving (Lesh, Post & Behr, 1987), learners have to integrate multiple representations. This points to the core role of changes between different representation registers (Duval, 2006): Changing representations is at the same time a valuable learning opportunity and a potential learning obstacle, as it is cognitively complex and often leads to difficulties in understanding (Ainsworth, 2006; Duval, 2006). Students thus should be supported when dealing with multiple representations: reflecting on and creating connections between different representation registers plays a key role (Duval, 2006; Bodemer & Faust, 2006) and unreflected changes of teachers between disconnected representations may cause understanding problems of students (Sjuts, 2002).

Consequently, teachers have to master the professional requirement of identifying and interpreting relevant observations regarding the use of representations in the classroom (Friesen, Dreher & Kuntze, in press). In particular, for reacting adaptively and optimally to the learners' needs, teachers should be able to analyse how changes between representations take place. Such analysing of classroom situations means that observations are connected with relevant professional knowledge – for instance, specific criterion knowledge may be used for the observations' interpretation (e.g. Dreher & Kuntze, 2015a; Friesen, Dreher & Kuntze, in press).

Analysing classroom situations regarding the use of representations is hence an important competence for mathematics teachers: professional competencies in Weinert's (1999) definition are specific and context-dependent abilities to cope with professional requirements, which is clearly the case here. Studies showing that such analysing is an aspect of teacher expertise (Dreher & Kuntze, 2015a) and that it can be fostered through focused professional development activities (e.g. Friesen, Dreher & Kuntze, in press) further support the relevance of this competence construct.

We see the competence as a hierarchical and one -dimensional variable, as the criterion knowledge (e.g. Duval, 2006) can be seen as a consistent unit which can be applied for analysis in various contexts. The requirements of the situation contexts however may differ in complexity, as teachers might have different preferences for specific representation registers which can interfere with analysis steps (cf. qualitative findings in Dreher & Kuntze, 2015a), i.e. support or impede a critical analysis of the use of representations in the classroom interaction. For instance, we found examples of teachers who were very in favour of a specific representation register and who were not aware of the problem of disconnectedness between representations when the (unnecessary) change into this favoured representation register occurred – their preference of representation registers was so dominant that they did not enter in a criteria-based analysis process (Dreher & Kuntze, 2015a). We conclude from these findings that it might be more difficult for teachers to analyse the use of representations when the registers used by the teacher observed in the classroom situation are among the commonly favoured representation registers.

The role of the representation format of the classroom situation to analyse

Another issue which might influence the complexity of analysing the use of representations in the classroom is the way the classroom situation is represented. As it is almost impossible to reproduce *real* classroom situations identically, classroom situations have to be somehow represented in order to make them available for assessment. Figure 1 shows three representation formats, namely a transcript-like text format, a comic format and two screenshots from a video format representation of the same situation. It is obvious that these formats provide different information, and that they make available identical information elements in different ways. Even if the three representations of this learning support situation have been produced in a structured procedure which aimed at eliminating any contradictions such as different wording or different drawings of representations (cf. Friesen & Kuntze, in press), there are important systematic variations among the different representation formats: Temporality, for instance, makes a difference between video format on the one side and text and comic formats on the other side: For example, the speed issue in interaction is much less visible in a comic and in a transcript. Moreover, the formats

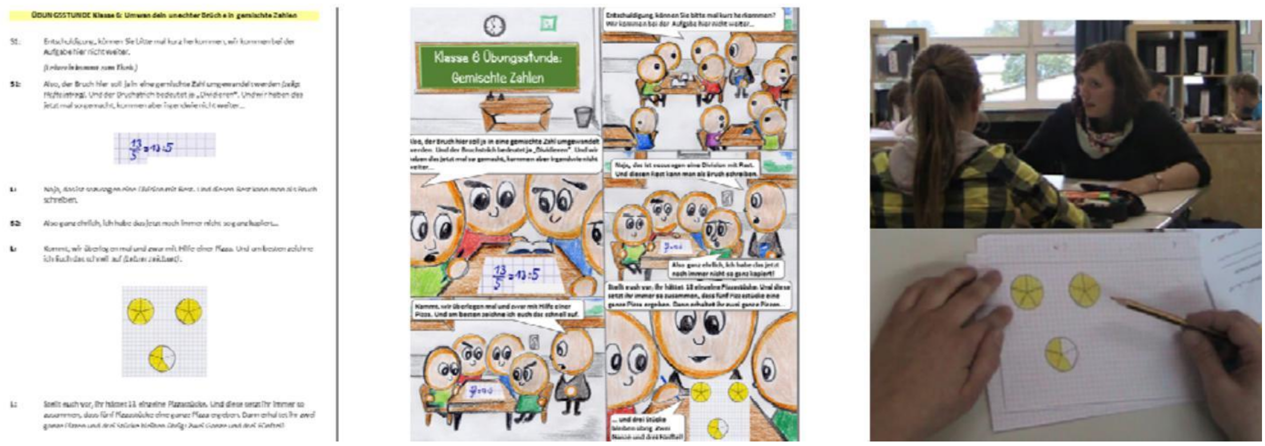

Figure 1: Representation of a learning support situation in transcript-like text, comic and video format (comic drawn by Juliana Egete)

differ in the amount of potentially relevant and irrelevant context information. It might make a difference for the analysis if, for example, the characters in the comic format were all smiling, whereas the colour of the furniture in the background is rather irrelevant for the analysis – and invisible in the text format, for instance. The individual persons in the classrooms are almost absent in the text format, whereas they are more visible in the comic format and appear as real human beings in the video. According to Weinert (1999), aspects of teacher competence are of contextualised nature, so that the amount and methods of contextualisation in vignettes used in assessment instruments may play a role. When assessing competence by referring to professional requirements of teachers as reflected in vignettes (Oser, Salzmann & Heinzer, 2009), format might matter. Even if many researchers discuss the potential of video-based forms of assessment (e.g. Blömeke, Gustafsson & Shavelson, 2015; Seidel et al., 2011; Sherin et al., 2011) there are only very few studies like e.g. the study by Herbst & Kosko (2013) and Herbst, Aaron & Erickson (2013), in which different formats are compared empirically and systematically.

We see analysing as *"an awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation"* (Kuntze, Dreher & Friesen, in press). Classroom situations as represented in different formats can be such subjects of analysis. We assert that the process of analysing is not linear (Friesen, Dreher & Kuntze, in press). In any case, in the teachers' answers, only the results of the analysis will be visible – perhaps only in parts. However, as our focus is on the *competence* of analysing regarding a specific area of criterion knowledge, namely the use of representations, and as the articulation of analysis results is part of this competence, the teachers' answers are very informative. When assessing this competence, we have to be aware that both different classroom situations and format might influence the complexity of vignettes. For this reason, research designs should take this potential interaction into account, so that the potential impact of the representation format is not confounded with the competence construct.

Moreover, among the teachers' perceptions of the vignettes, there might also be other extraneous disturbing factors for analysis: If, for instance, teachers do not perceive a vignette as authentic, this might be detrimental to getting engaged with the corresponding classroom situation. Under this perspective, motivation might play a role as well. The extent to which teachers feel part of the classroom situation or to which they feel to be able to connect with their experience can be considered as further indicators for the facility of teachers' engagement with a classroom situation. For this reason, an assessment instrument should also be aware of perceptions such as authenticity, motivation, immersion, and resonance (e.g. Seidel et al., 2011; Kleinknecht & Schneider 2013).

As the instrument was intended to describe teacher growth during initial teacher professional development, this study focuses on pre-service teachers. From an earlier study with a pilot-like vignette-based subtest (e.g. Dreher & Kuntze, 2015a, b), we were able to extract also expectations related to other possible samples.

RESEARCH INTEREST

In contrast with the key role of representations in the mathematics classroom, empirical evidence about the competence of analysing the use of representations is scarce. Against the background of the considerations above, assessment instruments are needed which take into account the potential role of representation formats. Moreover for such an instrument, it should be examined whether teachers' perceptions are positive as far as authenticity, motivation, immersion, and resonance are concerned.

Consequently, this study aims to find out whether an assessment instrument comprising of vignettes in text, comic, and video format can be used to empirically describe the competence of analysing classroom situations regarding the use of representations empirically with one competence dimension.

In particular, the following research questions are in the foreground:

(1) Do the pre-service teachers rate authenticity, motivation, immersion, and resonance related to the vignettes as positive?

(2) Can the competence of analysing the use of representations in classroom situations be empirically described by a one-dimensional Rasch model?

(3) Are there any systematic differences in the empirical difficulty of vignettes for different vignette formats (text, comic, video)?

DESIGN AND SAMPLE

For assessment, six classroom situations were conceived which all consisted of learning support situations in year 6. According to the situations' design, they start with the teacher being asked for help by a group of students who have a difficulty with solving a given problem and who are using a certain representation (algebraic or pictorial). The teacher reacts with a change of representation without connecting with the representation of the students or encouraging reflection about the connections of the two representations. So in all six cases the reaction is non-optimal according to the theory about the use of representations as outlined at the beginning of this article. The change of representations could for this reason potentially lead to further problems in the students' understanding rather than support it.

These learning support situations were each represented in the formats of transcript-like texts, comics and videos (cf. Fig. 1) . After producing the video vignettes, the text and comic formats were adjusted so that the wording of the dialogues and the fraction representations were congruent (Friesen & Kuntze, in press). For each vignette, the pre-service teachers were asked to what extent the teachers' reaction helped the students regarding the use of representations. The participants' open format answers were coded according to a theory- based top-down categorisation: code "0" was assigned to the answer if it referred at most to representations used by the teacher without making any connections to the students' question/representation, code "1" was used when the answer referred to representations used by both students and teacher and did not mention that the unexplained change of representations might be problematic, and code "2" stands for answers which referred to representations used by both students and teacher and mentioned that the unexplained change of representations might be problematic. The pre-service teachers' answers were coded independently by two raters with good inter-rater reliability (Cohen's κ=0.85).

Moreover for each vignette, the pre- service teachers were asked to answer rating scale indicator items for authenticity, motivation, immersion, and resonance which were adapted from the study of Seidel et al. (2011).

The pre-service teachers were asked to comment on each of the six situations. In a randomised way, each participant received one out of six test booklets as shown in Figure 2, so that the vignette formats were rotated according to a multi-matrix design. The videos were about 1.5 minutes long and could be paused and repeated.

Test booklet number	I	II	III	IV	V	VI
Situation 1	Text	Text	Comic	Comic	Video	Video
Situation 2	Comic	Comic	Video	Video	Text	Text
Situation 3	Video	Video	Text	Text	Comic	Comic
Situation 4	Text	Video	Video	Comic	Comic	Text
Situation 5	Comic	Text	Text	Video	Video	Comic
Situation 6	Video	Comic	Comic	Text	Text	Video

Figure 2: Overview of the six test booklets with rotated vignette formats

The sample consisted of 162 pre-service mathematics teachers (66.9% female; aged on average 21.55 years (SD=2.38)). The pre-service teachers were at the beginning of their professional education (average number of semesters: 1.80 (SD=1.40)).

RESULTS

An important prerequisite for the pre-service teachers' engagement with the learning support situations is that they see the vignettes' authenticity positively, that they are motivated when reflecting about the vignettes, and that they can personally imagine to be part of the situations (immersion) as well as find it possible to connect to their prior classroom experience (resonance). The average ratings concerning these variables were all positive, regardless of the particular classroom situation and representation format, respectively (mean values ranging from 4.1 to 4.7 on a scale from 1 to 6, SD ranging from 0.8 to 1.2). For the first research question, we may hence state in particular that none of the classroom situations was seen as non-authentic and that also from the point of view of the other variables no impeding factors for analysis could be identified.

For answering the second research question, the pre-service teachers' answers were coded and a score per item was assigned according to the codes. Taking all items together, only 25.1% of the answers mentioned the unexplained change of representations and evaluated this change of representation as potentially problematic (corresponding to code "2"). A chi -square test revealed that there was no significant correspondence of vignette format (text, comic, video) and the pre-service teachers' analysis scores related to the classroom situations ($\chi^2(4) = 7.09$).

Based on the codes assigned to the answers, we applied a partial credit Rasch model as partial marks were awarded in an ordered way according to the top-down coding (cf. Bond & Fox, 2015). The six vignettes in the three formats were considered as one item each, resulting in 18 items altogether. The Rasch analysis revealed good fit values for all 18 items ($0.91 \leq wMNSQ \leq 1.16$; $-0.6 \leq T \leq 1.0$), so that all of them fitted sufficiently to the Rasch model (Bond & Fox, 2015). Consequently, the results indicate that the Rasch requirement for unidimensionality is fulfilled empirically and that each item contributes meaningfully to the competence of analysing as imple-mented in the test instrument. The EAP/PV reliability amounts to 0.45, which is rather low, but this has to be seen against the background of missing data by design (see multi-matrix design in Fig. 2) and of the comparatively small number of items, so that the reliability value can be considered as satisfactory (Bond & Fox, 2015).

The Wright map (Figure 3) displays both items and persons located on the same competence dimension (highest values located on the right of the logit scale). As a consequence of the polytomous scoring (codes 0, 1, 2), the Wright map contains two difficulty thresholds per vignette: above threshold estimate 1, scoring code 1 is more likely than scoring code 0 and above threshold estimate 2, scoring code 2 is more likely than scoring code 1 (Bond & Fox, 2015).

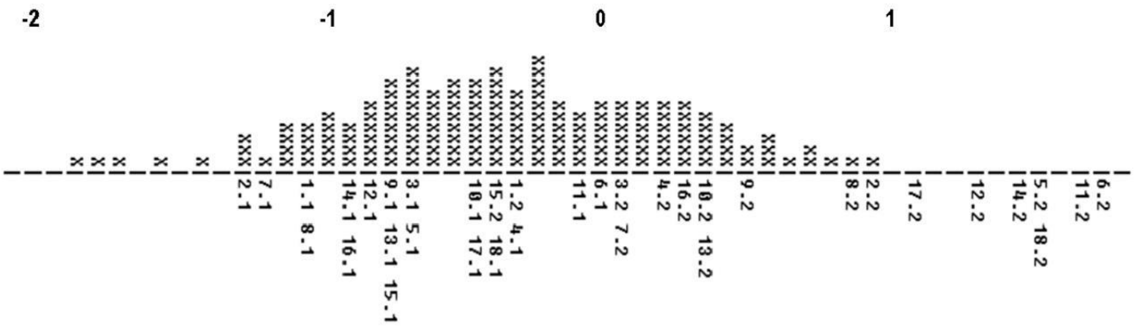

Figure 3: Wright map of the partial-credit Rasch model

Looking at the item difficulty estimates for all 18 items, it can be remarked that the step between code 0 and 1 is empirically easier (mostly negative logit scores) than the step between code 1 and 2 (mostly positive logit scores) – this is consistent with the expectation. The distribution of the persons does not exceed the range of the most difficult thresholds – also this finding conforms our expectations about samples of pre-service teachers at the beginning of their professional education (cf. Dreher & Kuntze, 2015b, as mentioned above).

Turning to the third research question, the Wright map suggests that the presentation format of the six learning support situations does not make a systematic difference for empirical item difficulty. If, for instance, the video vignettes had been more difficult than other vignettes, then the video vignette difficulty thresholds would have systematically appeared more to the right than their comic- and text-format counterparts, which is not the case in the Wright map in Figure 3.

The difficulty estimates can be interpreted as interval data (Bond & Fox, 2015), so that analyses of variance can be calculated for checking whether there is an association of item difficulty and vignette format. In line with the chi-square test reported above, the comparison of text vignettes (items 1- 6), comic vignettes (items 7-12) and video vignettes (items 13- 18) did not yield any significant format-related differences ($F=0.047$, $df=4$; $p= .996$).

DISCUSSION AND CONCLUSIONS

This study's aim was to explore whether it is possible to implement a vignette-based assessment instrument for the competence of analysing the use of representations in the mathematics classroom, taking into account the role of different vignette formats. We were able to build on the work by Herbst et al. (2013): In comparison, the vignette formats included in this assessment instrument (text, comic, video) were very different, as, for instance, Herbst and colleagues had conserved the aspect of temporality across their vignette formats (video and animation formats). We thus consider our choice of vignette formats as relatively wide-spread within the spectrum of possible formats. The results indicate that despite this wide-spread choice of vignette formats, it is still possible to empirically reproduce a single competence dimension, and that in the case of the competence of analysing the use of representations, the analysis difficulty is not systematically determined by the vignette format. In particular, the pre-service teachers' competence of analysing was not connected with item design factors such as temporality, individuality of the persons shown or the context information that were implemented to different degrees in the three vignette formats. We have hence identified an empirically one-dimensional competence construct which is not dependent from the format of the vignettes used in the assessment instrument.

This finding can also be interpreted as supporting the validity of our instrument: The competence construct had been deduced from theory, the vignettes conceived according to theoretical considerations related to analysis requirements, and the representations and the wording of the dialogues had been kept constant throughout the vignette formats even if the information provided in the vignettes varied with respect of other aspects as described above. These representations and the way they were dealt with might thus have been the core subject of analysis, as intended in the instrument's design.

Moreover, the test items' fit to a one -dimensional Rasch model without exception is a very positive finding also for further research: the competence of analysing the use of representations can not only be measured independently from the different vignette formats, but the results facilitate the investigation of factors that make the analysis of the use of representations difficult for teachers. As this study suggests that the vignette formats text, comic and video are comparably effective to assess pre-service teachers' competence of analysing the use of representations, further research focusing on this competence could explore design variations within only one of these formats.

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High inference rating system for an evaluation of metacognitiv-discursive teaching and learning quality

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This paper discusses the design of a rating system for the assessment of the meta-cognitive-discursive quality (MDQ) of a class discourse. It focusses on this system understood as a research tool aiming at a reliable and valid evaluation of MDQ in a class, but it also refers to the usefulness of this tool when applying it as an analytical or diagnostic tool in teacher education. The preliminary results of a research project aiming at an evaluation of this rating system indicate that a two-step procedure designed for its application enhance the accuracy of the discourse analysis and evaluation, and it allows a detailed analysis of the relation between teacher's and students' metacognitive behaviour.

Keywords: metacognition, discursivity, rating system, evaluation.

INTRODUCTION

Despite the recognition of the important role of metacognition in student's learning process (cf. Hattie, 2009; Wang, Haertel, & Walberg, 1990), little is known about the implementation of metacognition in classroom instruction and on relations between teachers' and students' metacognition in a class. Assuming that enhancing learners' metacognition is essential for promoting learning, research on the implementation of metacognition into the school practice, and on supporting teachers in establishing a metacognitive-discursive culture in their classes definitely merits future research (cf. Mevarech & Kramarski, 2014; Depaepe et al., 2010). One challenge for this kind of research had been described by Veenman et al. (2006, p. 10): "Teachers are absolutely willing to invest effort in the instruction of metacognition within their lessons, but they need the tools for implementing metacognition as an integral part of their lessons". This statement raises the issue of the kind of tools that would be appropriate as diagnostic tools for analysing and assessing metacognitive practices of individual teachers in their classes (in teacher-student and student-student interactions), and for identifying strategies for improving them to help the students to become metacognitive learners.

This paper reports on a research project aimed at developing and evaluating a rating system for analysing and assessing the metacognitive-discursive quality (MDQ) of a class discourse (RSMDQ) (Nowińska, in print). RSMDQ can be used in different settings and for various research and practical aims:

- as a *research tool* to evaluate the metacognitive-discursive quality in a given teaching-learning group (e.g. when a relation between this quality and the students' learning success should be investigated),

- as an *analytical tool* in in-service and pre-service teacher education (e.g. to learn how to analyse a classroom discourse, identify variables influencing the effectivity of the discourse with regard to various aspects of the learning process, and to learn how to design effective classroom discourses),
- as a *diagnostic tool* in in-service teacher professional development programmes (e.g. to analyse and diagnose the metacognitive-discursive quality in a particular class of an individual teacher, and to find strategies for improving its potential for promoting learners' metacognitive behaviour and their learning process).

This paper explains the design of RSMDQ and exemplifies its use as a research tool but it also discusses the usefulness of RSMDQ in the other two settings.

METACOGNITION IN A CLASSROOM DISCOURSE

The origin of research in metacognition in mathematics education lies in learners' difficulties in *solving problems*, and is closely related to the question of how to learn and teach solving non-routine problems. When applying to *learning* mathematics in a class, metacognition refers to a broader spectrum of activities then during problem solving. The groundwork for the operationalisation of the concepts of metacognition with the objective of making it understandable und evaluable in terms of empirical observations of classroom situations has been done by Cohors-Fresenborg and Kaune (2007), as they constructed a category system for an interpretative, transcript-based analysis of metacognitive and discursive activities (CMDA)[1] in a class discourse. Their paper presents this category system, explains and exemplifies in detail the use of it. CMDA does not differentiate between metacognitive processes understood as cognition about (one's own or of the others) cognition – in particular when problem solving – and cognition about the results from cognition (calculation, verbal or written information, argumentations, questions). Also in the second case the purposeful application of such cognitive behaviour at the appropriate moment results from metacognitive thoughts, and reflects the intention to control and understand the given calculation, information, argumentation or question. According to this conceptualisation, the objectives of metacognition[2] in learning mathematics are, for example, to *plan* the use of mathematical tools, methods, and representations to justify an argumentation or to explain an idea; to *control and evaluate* the accurateness of argumentations, the adequateness of external (e.g. formal) or internal representations of mathematical concepts, the correctness of the use of tools and procedures; to *reflect* on the ways of reasoning, defining or proving, and on similarities and differences in conceptions and arguments. Since a learning process in a class can only lead to a deep understanding of concepts, representations and

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[1] The complete German version of CMDA is presented in Cohors-Fresenborg, Kaune, & Zülsdorf-Kersting (2014).

[2] Metacognition in this context is decomposed into *planning*, *monitoring* and *reflexion.* The colors used here refer to the colors used in CMDA. In the same way two colors are used for *discursivitiy* and *negative discursivity*.

tools, if the planning, monitoring and reflection related to them are connected accurately to the matter discussed and take students' ways of thinking into consideration, *discursivity* is needed to facilitate the productive use of metacognition in classroom discussions (Cohors-Fresenborg et al., 2014, p. 7). Discursivity means activities carried out to improve the precision and accuracy in a discourse: orchestrating single utterances to a comprehensible discourse unit and orientation on students' ways of thinking. Examples of discursive activities are precise (re-)for-mulating and comparing of learners' ideas, strategies, conceptions and misconceptions, connecting them with precisely represented mathematical concepts or argumentations, and also confronting the learners with problems regarding the intended precision. On the contrary, *negative discursivity* means activities with a negative influence on the precision and accuracy in the discourse. Examples are: the use of inadequate vocabulary or superficially clear sentences with an unclear sense, incorrect logic structure of an argumentation, and bringing into the discourse an alternative idea without references to what has been said and discussed before.

This broad perspective on metacognition in a classroom discourse was used to design RSMDQ. It makes it possible to investigate relations between teachers' and students' metacognitive activities and the mechanisms facilitating and hindering the effectiveness of metacognition in everyday teaching and learning situations.

RATING SYSTEM FOR AN EVALUATION OF MDQ IN A DISCOURSE

In the ongoing research project[3], the Group Cognitive Mathematics at Osnabrueck University (in cooperation with the University of Kassel and The German Institute for International Educational Research - DIPF) works on the development and evaluation of a rating system for a *video-based* analysis (by a *category system*) of metacognitive and discursive students' and teacher's activities in classroom discourse, and for the evaluation (by *rating scales*) of the MDQ of the analysed discourse. By means of a generalizability study (Cronbach et al., 1972; Praetorius et al., 2012) the reliability of the designed rating system (RSMDQ) will be evaluated, and a decision study will be conducted to determine how many lessons from a given teaching-learning group, and how many raters would be needed to get reliable statements about the MDQ of the discourse in this group.

Ideas for the design of the rating system

The design principles for the development of RSMDQ result from the fact that metacognition and discursivity are intertwined and can be carried out with a different local quality (e.g. elaboration, precision, relevance for the discussed question). Both constructs have to be analysed in teacher-student and student-student interactions, and their potential to promote learners' metacognitive behaviour and an understand-ding use of mathematical tools have to be analysed. Consequently, it is necessary to

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[3] The project is supported by Deutsche Forschungsgemeinschaft under reference Co96/8-1.

look at several complementary aspects of the interactions and to integrate them in a complex assessment. This can only be achieved by a *high inference* rating system, but the use of such a system demands complex qualitative decisions and a high degree of the necessary conclusions. This leads, in general, to a reduced reliability of the assessment (Praetorius et al., 2012). An improvement of the reliability through reducing the focus of the evaluation to *observable* metacognitive behaviour aspects does not make sense. Right from the design stage for such a rating system, it has to be prevented that, as a consequence of the pressure to get a satisfactory reliability, only the surface structure of the lesson is evaluated and the deep structure neglected. This could provide incorrect predictions as to what extent a 'good' surface structure of a lesson can promote an understanding learning and lead to sustainable results of this process (cf. Nowinska, 2011).

In the ongoing research project a new idea to cope with this research problem was developed. To obtain reliable assessments, despite the needed complexity of the interpretative discourse analysis, the rating process is designed as a *two-steps* procedure. The decision was made, not to dramatically reduce the complexity of the category system CMDA but to adapt it for a *video-based* analysis, and to use it as an analytical tool for a detailed interpretation of the discourse in the *first step*. The same rater uses his interpretation as a basis for the global evaluation of MDQ of the whole discourse in the *second step*. The obligation for an elaborated interpretation causes that the rater deals with the videos very intensively, and therefore it can be expected that the evaluation of MDQ will be reliable and accurate.

The idea of using these two steps seems to play a crucial role contributing to the usefulness of RSMDQ as a research tool, and also as an analytical and diagnostic tool. The result of the first step provides a detailed 'map' of metacognition and discursivity in class interactions and makes these constructs 'visible' for researchers, raters and teachers. This enhances the accurateness of the subsequent evaluation, and helps an individual rater or teacher to better understand the weak and strong aspects of an individual teacher's efforts in promoting metacognition and learning process with understanding. To make this usefulness comprehensible to the reader of this paper, a *reference example* will be used in the following sections. The subsequent analysis of this example shows how the 'map' of the metacognitive and discursive activities identified in the given discourse can be analysed with RSMDQ to find an accurate evaluation of MDQ.

The reference example

The example presents transcript-excerpts from a discussion in a grade 7 classroom in one German secondary school. In the previous lesson the teacher (T.) introduced two types of equivalent transformations (ET) to solve linear equations by writing them on the board. Up to the end of the lesson the ETs have not been explained and justified. The first (second) ET regards 'the addition and subtraction of the same number (term) on both sides of an equation'. Two equations have been solved by a sole application of these transformations. In this lesson the students have to solve the equation 4-x=6.

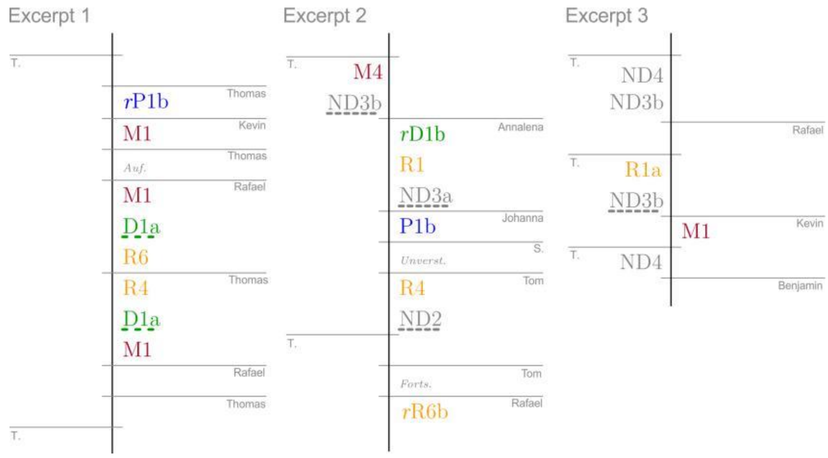
Thomas	The task was 4-x=6. I thought, we could bring x to the other side of the equal sign, and this would make the task easier. Namely, quite simply, plus x, then one has 4=6+x. And now one can see, if one wants to have only x, one has there also the six, hence minus six. And then it is 2=x.
Kevin:	I would say this is correct.
Thomas:	Rafael.
Rafael	I would say, the result, er, I would say it is correct, but I do not know how you got the idea. I did not understand how you, er, how you got 4=6+x. *[a few minutes later:]*
T.	How about the others? This could call in your minds the second type of the equivalence transformations. (8sec) Have a look into your notes. *[one minute later:]*
Johanna	Actually, I only calculated 4-6. I got 2. *[...]* I do not understand why one has to do all these complicated steps with x, if one can just calculate 4-6.
T.	Johanna, we are looking for the number x. We have already said it several times during last lesson. We are trying to change the equation, to transform it so that at the end we get the answer to the question: What is x? On the left side, there is minus x. This does not satisfy Thomas.

Thomas suggests (and justifies) a plan to solve the equation and to make it easier first. Kevin and Rafael control the result obtained by Thomas. Rafael gives a critical reflective question concerning Thomas' plan. Johanna reflects on her difficulties in understanding the sense of Thomas' idea, and justifies her critical remarks by pointing to an easier way of solving the equation. The transcript shows that the learners are autonomous in planning the way for solving the equation, in critical controlling of the use of mathematical tools, and in reflecting on the sense, usefulness and complexity of these tools. They also try to understand what their classmates think. Their single metacognitive activities indicate a great potential for understanding the idea of solving equations, but – since the different learners' conceptions of solving an equation have not been elaborated and compared with each other – they do not produce a comprehensible discourse unit. The teacher does not initiate deep reflection on the mathematical activity. Instead of that, he only points to the type of ET (written on the blackboard) that has to be used there.

The first step: local categorizing of metacognitive and discursive activities

In the first (video-based) coding step the rater interprets each students' and teacher's contribution. He decides whether a given contribution indicates metacognitive and (negative) discursive activities, he interprets the kind of these activities, and describes them with codes from the category system. This interpretation is based on the category system adapted[4] for this purpose from Cohors-Fresenborg and Kaune (2007). The choice of one category from the category system demands from the rater a careful consideration with regard to alternative categories, and a justification which one of them is the most adequate in the given case. Thus, the rater sets for himself a local interpretation of the discourse, and gets an overview of the kind and quality (precision and elaboration) of each individual activity on the one side, and of the coherence of the whole discourse on the other side.

One result of the categorizing is presented to the rater in a form called *category line*. It can be considered as a map or an abstract representation of the discourse process and its metacognitive and discursive content. The category line is to be read from top to bottom. For each student's and teacher's contribution there is a short horizontal segment with the name of the speaker. The segments for student's contribution are represented on the right side of the vertical line, and these for the teacher (T.) on the left side. Under each segment there are codes for metacognitive and discursive activities identified by the rater in the respective contribution. The following figure shows three *excerpts* from the category line generated to for the entire 10-minutes long discourse including the discussion shown in the reference transcript.



The *first* excerpt shows many metacognitive and discursive activities on the students' side; due to the absence of codes on the left side one knows that they are carried out without a demand given by the teacher. This could be interpreted as a great potential for understanding the mathematical activities discussed in this lesson. The *second* piece shows metacognitive activities combined with negative discursive actions. The teacher's monitoring activity M4 is not coherent, and it does not explain the understanding difficulties signalized by the students (ND3b). Two students' activities do not precisely refer to the structure of the given equation (ND3a), and include incorrect vocabulary (ND2) hindering the understanding of what is meant. The learners try to explain there the sense and the validity of the ET suggested by Thomas (R1, R4, *r*R6b). Due to the negative discursivity their efforts do not contribute to clear the problematic issue. The *third* piece shows many codes for negative discursivity indicating that the teacher does not take students' difficulties into consideration when explaining the equation (ND4, ND3b). The *missing* reflective and discursive intervention on his side hinders the

[4] The further developed version of this category system can be found here: http://www.mathematik.uni-osnabrueck.de/fileadmin/didaktik/Projekte_KM/Kategoriensystem_EN.pdf

understanding of the mathematical activity carried out in the discourse. The local *absence* of metacognition and discursivity has to be considered in the evaluation of MDQ to make this evaluation accurate, and to draw valid conclusions from it.

This requires a high degree of the necessary analytical reasoning. Taking into consideration only the observable fact that the learners are quite autonomous in practicing planning, monitoring and reflection would hinder the validity of the intended evaluation and its explanatory value with regard to the expected students' mathematical understanding (like in the study in Depaepe et al., 2010).

The second step: global assessment of MDQ of a classroom discourse

MDQ of a classroom discourse is evaluated by means of seven high inference rating scales. Each of them consists of a guiding question (*GQ*) focusing rater's attention on aspects to be analysed and evaluated complementary, and of several answers to the *GQ* describing in detail how these aspects are reflected in the discourse. Different answers describe qualitatively different situations. *Their order reflects the increasing quality of the discourse with regard to the relevant aspects*. The rater has to choose the one that best describes the given situation, and to justify the choice. In the following, the guiding questions will be explained. To each of them the answer for the entire 10-minutes long discourse including the discussion in the reference transcript will be given in brackets, written in *italics*, e.g.: (*Answer no. 1 out of 4*).

The *first GQ* focuses on teacher-students interactions and their potential to facilitate learners' autonomy in practicing metacognition. There are four answers to it. The answer no. 1 describes the case that metacognitive activities are carried out almost exclusively be the teacher, and it cannot be indicated that the teacher is aimed at fostering metacognitive skills by the learners or – alternatively – only a few metacognitive activities are practiced by the teacher and by the students, and no effort is made to use these activities accurately to better explain and understand the subject of the discussion. In the case explained in answer no. 4 the learners are autonomous in practicing and regulating metacognition, and they make effort to precisely elaborate the subject of the discussion. (*Answer no. 4 out of 4*)

The *second GQ* focuses on justifications combined with metacognitive activities, on efforts made with them to better understand the subject discussed, and on promoting learners' autonomy in justifying. It should be assessed whether justifications are practiced and valued as being important in the culture established in the class. There are four answers, analog to these to the first *GQ*. To choose the right answer one has to take into consideration the extent to which the teacher and the learners really make efforts to precisely explain and elaborate the content discussed. Hereby it is important to distinguish between the syntactic form of a justification and the content of it. An utterance with the formal form of a justification does not necessarily have any relevant explanatory content in the given context. Such utterances are called 'pseudo justifications'. Situations with a high number of such 'justifications' left without critical comments and corrections hinder the development of the reasoning skills of the learners. (*Answer no. 2 out of 4*)

An accurate evaluation of MDQ has to differentiate between lots of disconnected teacher's and students' metacognitive and discursive activities, and an orchestrated discourse producing accurate explanations and justifications for issues discussed in the class. The *third GQ* focuses on the interplay of the metacognitive and discursive activities carried out, on their potential for understanding the subject-specific issues discussed in the class (questions, tools, methods, argumentations) and for organising and systemising mathematical knowledge in students' minds. The first answer refers to a discourse without any productive use of metacognitive and discursive activities. The second answer describes the case that the understanding can only be indicated by an individual learner. The third refers to the case that the interplay of these activities contributes to a deep understanding in the class. (*Answer no. 2 out of 3*)

Discursivity is in the focus of the *fourth GQ*. It evaluates to what extend the discourse integrates learners' ways of thinking, and is aimed at making students' and teacher's utterances comprehensible for others and accessible for further analysis regarding individual ways of thinking and reasoning, or differences between what was said or written and what was meant by that. (*Answer no. 1 out of 5*)

The *fifth GQ* deals with negative discursivity and with efforts made to prevent it. The answers to this GQ describe the extent to which negative discursivity hinders the reciprocal understanding in a class and the understanding of the subject-specific issues (tasks, tools, methods or ways of reasoning). (*Answer no. 1 out of 5*)

The *sixth GQ* focuses on stringently guided discourse units called 'debates'. The answers to this GQ vary between situations without any (even short) debate, and between situations with at least one long debate guided by the learners and characterized by the use of discursive and metacognitive activities with justifications. The other two middle answers refer to situations with only short and not elaborated debates guided by the learners or to situations with a longer debate guided by the teacher. (*Answer no. 1 out of 4*)

The quality of the classroom discussion can change dramatically if a challenging and complex issue is being discussed. The cognitive challenge of such an issue must be stated more precisely and clear in order to find appropriate tools, methods, and ways of reasoning to elaborate the issue. This requires the use of elaborate metacognitive and discursive activities, and the inclusion of a meta-knowledge with regard to the subject matter. The *seventh GQ* focuses on situations with challenging and complex issues, and on the efforts made by the teacher and by the student to orchestrate the individual utterances, arguments, ideas and conceptions into a coherent discourse unit. This GQ plays an important role in comparing MDQ of two classes discussing issues with different complexity, and also in a long-term evaluation of MDQ in an individual class. The answers vary from the case without complex issues or with an ‘intellectual chaos’ when discussing such issues, to situations with noticeable efforts made by the teacher or by the learners to find appropriate methods and ways of reasoning to elaborate the given issue. (*Answer no. 2 out of 4*)

DISCUSSION

The evaluation of the 10-minutes long discourse including the reference transcript leads to the following assessment. For the first *GQ* the highest answer (no. 4 out of 4) has to be chosen because the learners are autonomous in practicing metacognition, and make efforts to understand the mathematical activity being the core issue of the lesson. This assessment is supported by the category line showing lots of codes for students’ metacognitive activities. Nonetheless, this remarkable, observable characteristic of the discourse does not automatically lead to a high quality of other aspects of MDQ being substantially relevant for understanding the discussed mathematical tools and formal representations. For the second *GQ* answer no. 2 (out of 4) has to be chosen: there are only a few mathematical justifications and they seem to play no relevant role in the classroom culture. For the fourth and fifth *GQ* answers no. 1 have to be chosen: the discourse does not respect students’ difficulties, questions and ways of thinking. Consequently, answer no. 2 (out of 3) for the third *GQ* states that no relevant understanding processes have been initiated in the class. Such processes can only be indicated in the case of one single student. And, furthermore, due to the intellectual chaos and the absence of any attempts to clarify the reasons of the fundamental understanding difficulties externalised in students’ critical remarks and questions, the rater has to choose the answer no. 2 (out of 4) for the seventh *GQ*.

This evaluation leads to the following conclusion. A complex analysis was needed to describe and evaluate the metacognitive and discursive activities in the discourse including the reference example. The question whether the learners are autonomous in practicing metacognition had to be split from the question whether their activities promote an understanding learning process. Also the question whether the learners take the responsibility for managing the discourse and the use of mathematical tools had to be split from the question whether the discourse produces a coherent mathematical argumentation. Furthermore, it was substantially crucial to split the question whether the teacher allows the students to be responsible for solving the equation from the question whether he acts co-responsible for the quality and results of students’ activities. None of the separated questions can give an accurate and valid evaluation of the MDQ of the discourse. Quite to contrary, these aspects have to be analysed and evaluated complementary.

The implementation of the two-step rating procedure – in which the rater first locally and precisely analyses metacognitive and discursive activities, and in which he gives the global evaluation of MDQ immediately after a sophisticated interpretation of the results from the first step – is a promising design to fulfill the requirements of reliable and valid assessments. It also proved as an effective method in educating raters and pre-service teachers in conducting a detailed discourse analysis, and in differentiating between strong and weak aspects of observable sight structures and of deep structures of teaching-learning situations. This opens new possibilities for research aiming at implementing metacognition in classes, and in enhancing its effectiveness in promoting students' understanding in learning mathematics.

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Using deliberative dialogue to analyse critically the role of families within a master program for future teachers of mathematics

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This paper aims to discuss the curriculum of teacher-training programmes for pre-service mathematics teachers drawing on a three-year long research within a master's programme in mathematics teaching for middle and high school pre-service teachers. We found the advantages on using deliberative dialogue to discuss how a group of pre-service teachers analyse the role of families during the mathematics learning school process. We conclude that teacher-training programmes developing competences may help future teachers to deal with potential conflicts when performing their work as teachers of mathematics. Although the 'practicum' is a short period, we have evidence of discussions regarding a developmental influence on social citizenship transversal competencies for teacher development.

Keywords: professional development, citizenship competence, deliberative dialogue

INTRODUCTION: SCOPE AND PURPOSE

We aim to encourage future teachers of mathematics to improve their professional practice by equipping them with didactical tools based on a rigorous theoretical foundation (Zavlasvsky & Sullivan, 2011). According to previous studies, for future teachers it is important to be aware of the increasing democratic participation of family members through out-of-school mathematical practices (Diez-Palomar, 2015). The Spanish education system express that mathematics teachers should be competent not only for understanding mathematical practices, nut also includes other general professional competencies such as having tutorial skills to make a bridge between the school system and family involvement. Our aim in this paper is to describe some changes observed among pre-service teachers (for middle and high school), during the three-year lifespan of an inter-university training program in Spain by introducing social/ecological aspects during didactical analysis in training courses. We expect to provide solid criteria to redesign mathematical tasks to allow family (and other members of the school community in a broader sense) to participate within the process of learning in mathematics.

THEORETICAL BACKGROUND

We start from a context in which traditional models of instruction still dominate mathematics education especially at the high school and University level (Barquero et al 2011). Most high school mathematics teachers see mathematics as a rigid and fixed body of knowledge, and they think that their responsibility is just to transmit this knowledge to their students. In such a framework, the intentionality of the deliberative dialogue (Skovsmose &Valero, 2001; Serradó, Vanegas & Gimenez, 2015) is to facilitate the establishment of explicit democratic mathematics practices for future teachers involving families and learn from it. We assume deliberative dialogue as a communicative episode considering: (a) the reasons or lack of reasons for people's preliminary opinions and judgements before actually making a final statement, (b) the pros and cons of possible decisions before actually making them, and (c) the benefits and losses of possible courses of action, before engaging in them." (Serradó et al., 2015, p. 291)

In fact, we know that two main variables influence mathematics participation and deliberation in rich tasks: the activity proposed, and the management. Regarding the management, interactive groups developed within the learning communities (Valls & Kyriakides, 2013) provide evidence that family involvement opens the possibility to increase learning interactions in [and out] of the classroom; such kind of practices improve students' mathematics achievements (Díez-Palomar & Cabré, 2015). Learning communities draws on the participation of members of the educational community (in a broad sense, including families, neighbours, or anyone who want to collaborate with the school), to implement *successful educational actions* –SEAs– (Flecha, 2012). Family involvement is one of those SEAs. Epstein (2011) has pointed out the difference between parental engagement, parents' participation, and family and community partnership, when talking about family and mathematics involvement. When parents (or other family or community members) want to support their children in learning mathematics, most of the times they have to go back to their schooling to remember their "mathematics". In doing so, some conflicts emerge due to the changes in terms of curriculum that we have experienced in the last decades. Those "changes" (*mathematics reform*) make difficult for parents to help children with their school assignments (Hoover-Dempsey & Sandler, 1997). Some families start to recognize that they can no longer help their children. In such cases, family and community involvement depends on the large task to build individuals' identities to become what Civil (2001) calls "resources", as leaders or facilitators of their children learning. Drawing on our experience doing research in this domain, we can recognize that almost all future teachers are unaware of these ideas, and have pre-judgements about the role of families in relation to the school system.

In order to address that situation of [potential] conflict we want to introduce a theoretical framework, the deliberative dialogue that enables us to address the difficulties in clarifying the fuzzy border between participation, deliberative dialogue and their children's mathematical learning. We understand that deliberate dialogue on mathematics classroom should enable vertical and horizontal forms of communication that support face-to-face and on-line dialogue and negotiation through deliberative interaction involving teachers, students and family members. The role of the teacher's change is associated to deliberate dialogue and crucial when it comes to management, power, responsibility and judgement (Serradó, Vanegas & Giménez 2015).

In our study we call 'professional tasks' those that we propose to future teachers in order to encourage them to perform didactic analysis and to develop the associated competencies. We understand 'competence' as the ability to design, apply and evaluate sequences of learning by means of didactic analysis techniques and involving the use of quality criteria. It is generally hypothesized that someone may reflect and improve their competence in terms of the analysis of mathematical classrooms, in order to make best use of the opportunities for being a teacher as 'teacher enquirer' (Mason & Johnston-Wilder, 2004).

In this paper we discuss data emerging from a teaching research project based on an inquiry and reflective practice framework in which we design and implement diverse teacher training cycles as teaching experiments that include the competence "family involvement" within the training program. From the outset, the development of these cycles, was explained by means of six types of professional tasks (Giménez, Font & Vanegas, 2013):

> Analysis of mathematical practices, objects and mathematical processes; Analysis of didactic interactions, conflicts and norms;
>
> Evaluation of tasks and classroom episodes using quality criteria, including ecological issues;
>
> Design and implementation of a lesson in their [pre-service teachers] period of internship;
>
> Analysis and evaluation of the suitability of the didactic implemented unit; Improvement of their lessons designs (for future implementation).

According to this perspective, the mathematical activity plays a central role in terms of systems of operative and discursive practices. From these practices, the different types of related mathematical objects emerge; building cognitive or epistemic configurations within them. Problem-situations support and contextualize the mathematical activity; languages (symbols, notations, and graphics) may serve as tools for action; arguments justify the procedures and propositions embedded within the concepts. Such a perspective is compatible with methodologies based upon critical mathematics education perspectives (Skovsmose &Valero, 2001).

METHODOLOGY

This proposal was conducted in the frame of the Master's Final Project (MFP) in a program of teacher training conducted at the University of Barcelona. In this paper, we draw on data collected concerning the tasks of type c

We collected video-recorded observations of two sessions devoted to the interaction analysis and two sessions for ecological suitability analysis including two deliberative dialogues about family involvement and future teachers' reflections at the end of the workshops. The sample includes two groups of 35 pre-service teachers.

At the beginning, future teachers made many naive comments related to the role of interactions and norms when analysing classroom episodes and about the role of families when we talked about teacher interactions and norms. Before that, future teachers designed and implemented tasks to see mathematical objects and processes, showing constructs as cognitive and semiotic conflicts and epistemic obstacles. In doing so, a new criteria for valuing mathematical quality was introduced (mathematical richness of processes, coherence, analysing the errors, connectedness, etc.) (Giménez, Font & Vanegas & 2013). We use some case studies emphasizing the role of contextualisation against formalism, referring episodes from practicum experiences of previous future teachers.

To observe the role of interactions, several episodes were analysed to describe types of norms (socio-mathematical, epistemic, or simply interactive norms. It was introduced the tools of ontosemiotic perspective for suitability criteria (Godino, 2014). Two classes of measurement were analysed by using the model of Scott and Mortimer, not only looking for patterns of models of management, but describing the role of the teacher.

For the purposes of this paper, we focus on analysing a school mathematical practice with the introduction of equations to have a first didactical analysis. We see the student conflicts, norms and interactions (Font, Planas & Godino, 2010). After that, we selected a professional task in which we start with the description of a school session where the teacher explains to a group of parents two ways to solve an equation with a single unknown (ax+b=c). The teacher uses the “balance” metaphor to solve the equation by searching the arithmetic solution by compensation processes.

We frame the discussion with the pre-service teachers by presenting an episode of a group of parents participating in a mathematics workshop addressed to families. This workshop was oriented to teach them strategies in order to help their children to solve homework at home, as well as providing them resources (manipulative, handouts, etc.) to use them with their children. This is a regular way that parents use to become engaged in their children’s education (Hoover-Dempsey & Sandler, 1997).

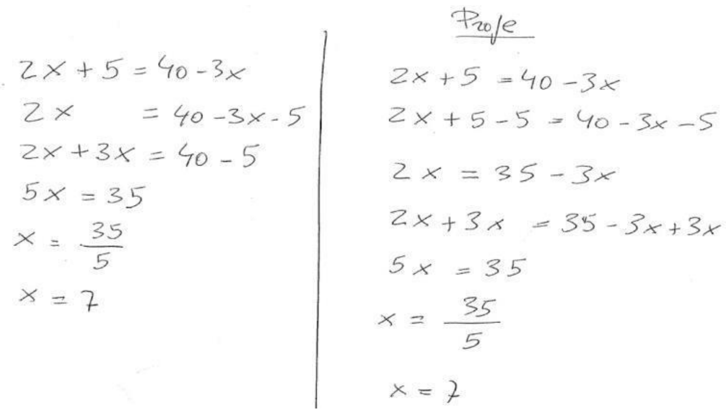

Figure 1. Two ways of solving equations

In that episode the teacher was using the blackboard to explain two different common ways to solve equations (Figure 1). He was working [teaching] with 5 mothers and 1 father. First, he asked the participants in the group how they would solve the equation 2x+5=40-3x. The teacher wrote some participants' procedures in the blackboard and a used a short debate for epistemic analysis purposes. We talk about mathematical objects and processes involved, relate this episode with a previous episode about equations analysed in a previous session, and identifying some potential cognitive conflicts.

Then, he narrated how some teachers explain the same topic in the school and how to focus on meanings. He wrote this procedure on the blackboard (right side in the figure 1), alongside the participants' solution. Two participants complained that they did not understand what the teacher was doing; they claimed that they feel confused One of them protested: "where are all these 'fives' coming from?" All participants claimed that was the first time for them to see the "balance" metaphor to solve equations.

The teacher explained that, in order to eliminate the "five" in the expression 2x+5 (written on the left side of the equation), and isolate the unknown, he needed to take away ("sacar" in Spanish version) "five" from both sides of the equal sign. "If not, the balance scales [in this case meaning the instrument] will lose its *balance* [meaning equilibrium]". The teacher was using the metaphor of the balance scales to justify his answer as well as explaining the meaning of an equation and the solution process. After he had finished his explanation, the following dialogue took place:

Teacher: What do you think? It is OK?
Mothers: Yes… great (the mother who asked the question talks loader).
Mother: It is 'cos at home we did not understood it…
Teacher How come?
Mother:… at home I did not understand it in this way. This that you just explained to us, my daughter was telling me "Mum, we have to put this here!", so I said: "where do you put it?" because I knew it [how to solve the equation] in the other way… the old way (moan in the background, like approving her words), and I was not able to understand it because there? was no explanation in the book [the textbook].
Teacher: But, now do you understand it better?
Mother:(voices of other mothers saying "yes") more or less. What happens is that here it? is very easy [in the workshop]… but I… (she starts laughing and making with her hands showing that sometimes tasks are very difficult for her). mimics
Teacher:…well… the other way is the same… but you must…
Mother:(at the same time) Now I understand, because, because…
Teacher:(at the same time)… for everyone…

Mother: … she explains that she does it that way, and I did not know how to explain it…

Parents use the procedure of moving all the "numbers" to one side of the equal sign, and the unknown to the opposite side. This was a regular teaching strategy used in textbooks of the seventies, to introduce the properties of equalities.

The participants then discussed the features emerging from this episode. In the analysis of the didactic interactions, as well as the conflicts and norms, some claims about the role that families should play within the learning process at home emerged during the deliberative process; in fact, the pre-service teachers projected their own social representation about "family role in supporting their children' learning process" within their own discourse. After that, it was discussed how parents interact with their children when solving mathematics items. "It was completely new for me" told a future teacher. "We are learning what it is an important mathematics consideration, but nobody explained us what happens at home. The difficulties are really the difference understanding, not the procedure" (FT 17). Using other episodes, the trainer discussed that the differences among parents and school, could interfere in children's learning, because the argument "My teacher taught me in a different way, so yours is wrong" could be used by individuals to justify an answer that is wrong, independently from the way used to get the result.

We describe the future teachers change and behaviour by means of content analysis and social analysis by observing illocutionary assumptions and their role during the debate according Epstein (2011) ideas. The belief of the importance of the responsibility in family (and other community members) participation, as a democratic value, and the role of teacher in such an environment can be analysed through a theory of argumentation for Mathematics Education purposes (Garuti & Boero, 2002) and of decomposition of the speaker's role (Krummheuer, 2007). Here the speakers are future teachers when discussing about the role of parents, but also the parents in the episode. We see the global arguments, to observe commonalities, and local arguments about individual approaches.

DISCUSSION

As the practice of analysing episodes is not strong, many expected mathematical comments disappear with future teachers. Nevertheless, future teachers identify the consequences of the balance metaphor and take away procedure. For instance if you have negative numbers, you cannot remove. Thus cognitive analysis during training process seems not to be enough, but necessary.

After presenting the episode, an interesting discussion among the pre-service teachers happened. Some of them explained that when children are 13-14 years old, many families are no longer able to help their children to solve their homework tasks. "Many parents don't remember how to solve equations". One of the pre-service teachers (FT) claims:

FT3: I would like to say that almost half of us have work with children, and all of us have met parents who say the same that this woman [in the video]; that they do not understand what we (as teachers) say, or that there is not communication at all. Thus, in the triangle [the triangle teacher-student-content] something is missing. In fact, you always must look for external help.

After that, another of the pre-service teachers claimed:

FT 5.: about this question… regarding the issue that a father or a mother who wants to help their child, and they see that it is like the Babel tower, right? That mathematics is like that… like I know how to solve the problem, but it is not like… this is the symbolic language… but why? If parents must adapt themselves to the new methodologies… Understand them is OK… I mean, to understand the method that your father used to explain the lesson to you?, and use it to solve the equations; but from the other side, it is also important to see that not always can we expect that teachers can see how parents solve [tasks] at home. We cannot be always that way in the classroom. What I see more complicated in the other issue: and it is something that happened to me once, because my father is a professor in the university, and some friends of us use to say that he is very good. So when I was child, and I went home, I used to tell my mom, I am going to work with him… and he used to say: hummm… hummmm… this way to explain things is very weird, the way we used years ago is better… and was forced to solve the tasks again and again (…) But a person is not a machine… Do we have to ask them to start studying again? Do they have to come back to the school to be able to understand the language we use in the school?

As the discussion advanced, the deliberative process produces a change in the opinion of the participants in the session. They start to question the “norms” already accepted about the role that families must play in terms of their children’ mathematics learning. “It is important that parents can help the students, knowing what they are doing” (FT7) “parents must know about maths”(FT 14). When analysing the data, we identify a social network, because many people talk in the group, and respond to a previous statement, as a chain of arguments. Mathematics appear after some requests, never immediately.

As global arguments, we found that some future teachers affirm that parents should not interfere within teachers' work; they just have to do "their work", which is "to make sure that their children attend school and do the homework every day." Other participants disagreed with these types of claims.

The conflict presented when discussing the episode, provokes a reaction against the role of a "parent" involved in school topics. Participants argued to include such kind of considerations in the definition of competences that teachers must develop / acquire in a teacher-training programme (to learn how to teach mathematics in middle and high school). One of the participants claims:

FT 12: (with experience working in high schools): Now that I saw that video, I
would do many things differently. At least I would be more aware
about what is the problem that parents have.

In fact, deliberative democrats specifically accentuate the need of recognising mathematical properties as generalised processes. There is a need of arguing for convincing in a way that different views have to be adjusted or confronted by means of argumentation in order to decide our common destiny on mutually acceptable terms (Englund, 2006). This is what happens in the pre-service teacher debate, giving opportunities for qualitatively better deep understanding about how to best involve students and parents, and move beyond superficial comments about the issue. In fact, "the most important is not to help the parents to know how to solve the equations, but to understand that some of them do not understand the meaning of the metaphors that we use to explain" (FT 23).

FINAL COMMENTS

After three years of training experience, we found much evidence to suggest that future teachers really transform their attitude towards using a "didactical approach" to inform their [future] professional work as teachers.

On the other hand, we recognized the final master degree programme as the starting point for developing research competency for future teachers including other type of aspects, in addition to the instrumental, epistemological and cognitive components of the "competence to teach mathematics". In fact, it gives opportunities for students to learn and recognize problems related to their professional context. The results show that deliberative debates can improve didactical analysis, when analysing the quality of interactions of future teachers. It also improve knowledge about sociocultural and critical aspects of professional development (Skovsmose & Valero, 2001).

Case study served not only for epistemic issues, but also for ecological suitability. The deliberate dialogue promoted during the professional task, contributes to problematize some of these social representations, opening possibilities to re-design the school mathematical tasks to include families' points of view.

In addition, future teachers need to spend adequate time to discuss the analysis of norms not only within mathematics classrooms, but also to understand out-of-school parental involvement. In summary, this study contributes to our understanding of some ecological variables that influence professional development. Even in the school practicum, future teachers learn from the debates during the Master in terms of amplifying the indicators of quality criteria (Godino, 2014) before the classroom practice. The future teachers reflect about mathematics classroom interactions in different ways. Following our research findings we intend to promote "didactical analysis" beyond the banality, considering classroom situations as an integral and dynamic system evolving in time, promoting autonomous mathematical thinking and independent validation of results as future teachers (Laborde, Perrin-Glorian, Sierpinska, 2005).

NOTES

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Prospective teachers' interpretative knowledge: giving sense to subtraction algorithms

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The process of interpretation and assessment of students' mathematical productions represents a crucial aspect of teachers' practices. In such processes, teachers rely on the so-called interpretative knowledge, which includes particular aspects of their mathematical and pedagogical knowledge, their view of mathematics, and their values. In this paper, we analyze and discuss prospective primary teachers' interpretative knowledge gained through their assessment of different subtraction algorithms.

Keywords: prospective teachers' interpretative knowledge; prospective teachers' beliefs; subtraction algorithms.

INTRODUCTION

In practice, teachers are required to continuously interpret students' mathematical behaviors, speech, and productions. In this process of interpretation, teachers introduce some *affective* aspects, including their beliefs, their values (for example, in the evaluation of the seriousness of an error), and their expectations (Liljedahl and Oesterle, 2014). Closely linked with the affective aspects, teachers' knowledge is also exhibited through interpretation and evaluation processes. In previous studies, this type of knowledge was referred to as *interpretative knowledge* (e.g., Ribeiro, Mellone, & Jakobsen, 2013; Ribeiro, Mellone & Jakobsen, to appear) and has emerged as a potentially significant construct both for researchers in mathematics education and for teacher educators.

Indeed, through the observation and discussion of teachers' interpretations of students' mathematical productions (and comments), researchers can gain insight into teachers' mathematical and pedagogical knowledge, beliefs, values, and expectations. On other hand, it allows teacher educators to develop significant discussions and mathematical knowledge with prospective teachers by highlighting the potential for mathematical exploration through students' productions, especially those containing errors or proposing non-standard solutions (Borasi, 1996).

Within this framework, we have developed a wider project aiming to access mathematical teachers' knowledge, beliefs, values, and expectations implicit in these processes of interpretation—during the initial as well as continuous education. Imbedded in such project, a particular kind of tasks has been conceptualized and implemented. One of the core aspects of the nature of such a task is rooted in asking (prospective) teachers to give sense to pupils' productions (some of which can be considered incomplete, containing errors, or simply based on non-standard reasoning) in response to a posed problem, as well as provide them with constructive feedback (e.g., Ribeiro et al., 2013). The work we have conducted to date has mainly focused on mathematical teachers' knowledge. Here, on the other hand, also teachers' beliefs are explored, in order to broaden our understanding of the nature and factors that influence prospective teachers' reasoning and argumentation when giving meaning to students' productions. In particular our analysis shows how some prospective teachers' beliefs about mathematics, together with their lack of knowledge about the mathematical proprieties at the roots of algorithm, prevent them to appreciate the correctness of an algorithm different from the "traditional one".

The study of the arithmetical operations and the relative algorithms is one core aspect of most primary school curricula around the world (e.g., NCTM, 2000). Nevertheless, the approach, the focus, and the algorithms related to the whole number arithmetic, in some cases, differ from one country to another. Such diversity of algorithms and of the mathematical rationality sustaining them can be perceived as a source for deepening teachers' beliefs and understanding of not only the algorithms, but also the whole number arithmetic in general. Indeed, if from one side we agree with Bass (2015) when he mention that "*A numerical computation, of a say a sum of two numbers, is not about understanding what the sum means. Instead, give two numbers A and B in notation system S, a calculation is a construction of a representation of A+B in same notation system S*." (p. 11). On other side, we consider that the navigation among (between) different algorithms of one same operation can enhance the opportunity to unpack both the different meanings of the operation as well as the features of the notation system of representation. This was one of the reasons that motivated us to conduct inquiry into the subtraction algorithm(s). Indeed, the knowledge and awareness of the mathematical aspects (such as the properties of the arithmetical operations or the decimal positional representation of numbers) involved in arithmetic operations, as well as the relative algorithms, are perceived as a crucial aspect of (primary) mathematic teachers' knowledge.

THEORETICAL FRAMEWORK

In the last decades, the research in mathematics education has emphasized the need to consider affect in the interpretation of the teaching/learning process of the mathematics. In particular, Thompson (1992) underlines the role of teachers' beliefs in classroom practices: beliefs that Philipp (2007, p. 258) defines as "the lenses through which one looks when interpreting the world". In this context, Grootenboer (2008, p. 479) refers explicitly to "the pervasive influence of beliefs on teaching practice" and scholars debate about how to recognize their central role also in programs devoted to mathematics teachers' development.

In their overview of the literature, Liljedahl and Oesterle (2014) underline as on the one hand beliefs are organized in systems, on the other hand the different types of beliefs systems that may affect teaching: beliefs about mathematics, beliefs about the teaching of mathematics, beliefs about the learning of mathematics, beliefs about students, beliefs about teachers' own ability to do mathematics, to teach mathematics, etc.

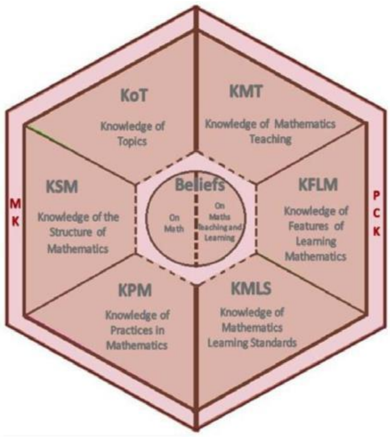

Figure 1. MTSK conceptualization.

Complementary to the role of beliefs and values in teachers' practices and mathematical understanding (as well as knowledge development), interpretative knowledge is perceived as one core element of the content of teachers' knowledge. Such interpretative knowledge is deemed to support teachers in giving sense to students' productions, always perceiving such productions as learning opportunities, even when they are non-standard or contain errors (e.g., Ribeiro et al., 2013). Such knowledge would allow teachers to develop and implement ways to lead students in building knowledge, starting from their own reasoning, even when it differs from that expected by the teacher.

The development of pupils' mathematical knowledge starting from their own reasoning, in our view, is possible only if the teacher activates a real process of interpretation, shifting from an *evaluative listening* and to a more *flexible hermeneutic* listening activity (Davis, 1997). In particular, in our framework the teacher's evaluative listening is conceived as process trough which the teacher sees if there is a fitting between pupils' productions and the mathematical scheme of correct answers he/she has. While a real interpretation process, linked also with Davis's notion of *hermeneutic listening* (1997), is linked to teacher's flexible attempt of redrawing a mathematical learning path that embodies pupils' productions. This vision makes our notion of interpretative knowledge different from other mathematical teachers' knowledge conceptualization, in the sense that errors/non standard reasoning are not conceived as something to avoid. Rather this framework puts errors/non standard reasoning at the core of mathematical teachers' knowledge as source to capitalize that really shapes the dynamics in mathematics educational process (Borasi, 1996).

Aimed at framing the relationships between interpretative knowledge and beliefs, we ground our work in the Mathematics Teachers Specialized Knowledge—MTSK—conceptualization (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013; see Figure 1). Indeed, in accordance with such approach, all of the teachers' knowledge is specialized, and teachers' beliefs are considered a core aspect influencing, and being influenced by, teachers' knowledge. Such beliefs are rooted in all their previous experiences, both as students and as teachers undergoing their initial (and continuous) teachers' education. Moreover, these beliefs not only affect teachers' attitudes and actions, but also have a direct and crucial link with their mathematical knowledge (MK, considered in the left part in Figure 1), thus shaping their perception of the mathematics education process(es) (pedagogical dimension, depicted in the right part of Figure 1).

Although the MTSK considers six sub-domains of teachers' knowledge, in the scope of this work, we only address two of the MK sub-domains. In particular, concerning the context of subtraction in the set of natural numbers N, we focus on the Knowledge of Mathematical Topics (KoT) and the Knowledge of the Structure of Mathematical (KSM).

KoT includes teachers' knowledge pertaining to the definition and justification of the mathematical content (e.g., the difference between a and b, when $a > b$, corresponds to the search of the unique c that satisfies the equation $c + b = a$); properties, issues, and associated procedures such as algorithms; different forms of representation (e.g., decimal positional representation of numbers, columns or linear arrangement of algorithms); phenomenology (e.g., comparing, taking away, and compensation, see, for example, Fuson et al., 1997).

The KSM refers to teachers' knowledge of an integrated system of connections. Such system allows teachers to understand and develop advanced concepts from an elemental standpoint, as well as elemental concepts from approaches considering an advanced mathematical standpoint. Concerning subtraction in N, it is related to, for example, the same operation in other number sets; subtraction involving other mathematical entities (e.g., algebraic variables, vectors, matrices, functions); the potential transition from the elemental aspects of subtraction in N to other advanced aspects such as, for example, the use of finite-difference methods in finding the solution of differential equations.

When considering KoT and KSM pertaining to interpretative knowledge, the content of such sub-domains should allow teachers to look for the potentialities embedded in students' productions and comments (even if students are unaware of them). For example, when giving meaning to different subtraction algorithms, such knowledge should allow teachers to perceive, understand, and appreciate each of the different mathematical aspects required to explain the different steps followed by the student to find the solution.

Obviously, when teachers' beliefs about mathematics (Liljedahl & Oesterle, 2014) are exclusively linked with a procedural, instrumental (Skemp, 1971) approach to/view of mathematics (also due to the set of experiences they have been immersed in), such beliefs implicitly shape the ways they perceive the content of their own KoT and KSM and what they deem necessary to be included in these sub-domains. The aim of this study is to explore the relationship between teachers' beliefs and their revealed KoT and KSM in the light of the interpretative knowledge. We hypothesize that teachers' beliefs related to an instrumental vision of mathematics (Skemp, 1971) can be an obstacle to their interpretation of students' productions if these differ from that anticipated by the teacher.

METHOD

In this study, we explore the nature of beliefs, KoT, and KSM revealed by a group of Italian prospective primary teachers when solving a particular interpretation task in the scope of a Mathematics Education course in which the second and the third author were the educators.

In particular, our sample included 40 prospective primary teachers in the third year of the five-year professional primary teacher training program provided in Italy. The task was administered during one of the first sessions of the course. It commenced by instructing the prospective teachers to find a solution to a given subtraction and afterwards to pose problems involving such operation (Figure 2).

> Consider the following subtraction: 51−17.
>
> a) Find the result and explain verbally how you obtained it
>
> b) Pose two problems that involve this operation

Figure 2. First part of the task.

After completing this first part of the task, prospective teachers were given another sheet containing seven pupils' productions to the same problem. The prospective teachers were asked to reflect and comment on the mathematical correctness (and adequacy) of these productions, and to propose possible feedback that could be given to each of the seven pupils in order to support their mathematical learning. For brevity, we focus our attention on three of the pupils' algorithms only (Figure 3), along with the corresponding prospective teachers' comments and reactions.

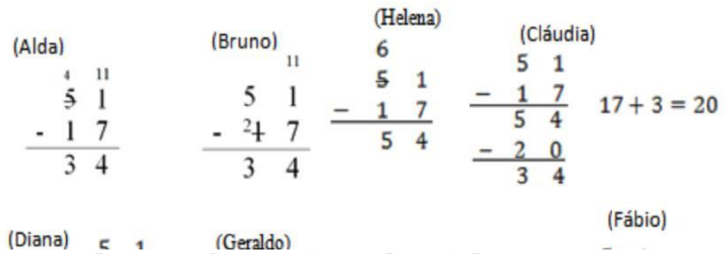

Figure 3. Three subtraction algorithms/representations.

Each of the pupils' algorithms included in the task have been selected with a particular rationale. In particular, concerning the three discussed in this work, Alda's algorithm (the one traditionally used in Italian schools) was included in order to access prospective teachers' beliefs and aspects included in their KoT and KSM when discussing and giving meaning to such "traditional" algorithms. Bruno's and Claudia's algorithms (the first is essentially rooted in the decimal representation of numbers and the properties of subtraction, whereas the second is grounded in the handiness of working with tens) were included to discuss prospective teachers' knowledge and ability to interpret and grasp the correctness of algorithms that differ from their preferred solution (Alda's algorithm) and the emerging beliefs in this process of interpretation and sense given.

We commence the analysis with a qualitative discussion on teachers' beliefs that emerged in the evaluation of the pupils' algorithms shown in Figure 2. Next, we intertwine this discussion with the contents of their KoT and KSM, whereby our analysis is grounded in the argumentation they present when giving meaning to the algorithms provided (e.g., reference to subtraction properties, definitions, representation issues or advanced mathematical aspects). Finally, we present a more quantitative analyses of the links between teachers' KoT and KSM that sustain their ability to interpret students' solutions.

DISCUSSION

All the 40 prospective teachers' answers converge on considering Alda's solution as "mathematically adequate." In ten several cases, the judgment of *adequateness* is related only with the consideration that Alda has solved the subtraction in the same way the prospective teachers would, as noted in the following comment:

> *"Alda's solution is based on correct mathematical reasoning, and is the same as the one provided by me."*
>
> *"For me, the adequate solution is Alda's solution because it is also how I perform the subtraction"*

Moreover, in ten cases, the adequateness is attributed to the fact that Alda's algorithm is the "traditional" one, i.e., the one "learned at school," as evident in the following answer:

> *"Alda solved the subtraction in an adequate way. She firstly subtracted the 11 from the 7 (by borrowing a ten) and then she subtracted 1 from the 4 (the 5 became 4 because it loaned a ten to the units) I think that the reasoning is 'adequate' because the procedure followed to solve the problem is the traditional one /the one taught in the school."*

This last prospective teacher's answer is based on considering Alda's algorithm adequate. It is rooted in recognizing it as the "traditional" approach, expressing it using the same wording used when learning it at primary school. These two facts provide this prospective teacher the guaranty of correctness– no references to subtraction properties or number representation issues are considered important.

None of prospective teachers' interpretations of Alda's algorithm provides a reference to the subtraction definition, properties, or potential different meanings. In this sense, the provided interpretations of Alda's algorithm allow us to recognize a very basic prospective teachers' KoT, as they do not seem to know the actual rationale underpinning the algorithm (as all are using the mnemonic they have learned while primary students). Alternatively, it is possible that they find referring to such rationales unimportant or irrelevant (as seen, in some cases, they just mention that, as Alda's algorithm is the same as their solution, it must be an adequate one).

On the other hand, it is important to highlight that none of the other algorithms is assessed as *adequate*, even if some are deemed *correct.* This discrepancy suggests that prospective teachers do not consider *correct* and *adequate* as synonymous. In particular, 15 teachers considered Bruno's and Claudia's solutions *inadequate* or even *wrong*, for various reasons, mostly because they differ from the approach they know (they named traditional), as noted below:

> *"The answers given by the other children are inadequate because they don't reflect the traditional solving method for the subtraction."*
>
> *"I think [Bruno] doesn't understand the action of taking away."*

In this last comment, the prospective teacher refers only to one of the subtraction meanings (taking away), thus revealing the need for more extensive work on developing prospective teachers' KoT, revealing also her beliefs about mathematics concerning the uniqueness of a process to find the solution. In this sense, the link between beliefs about mathematics stemming from their previous experiences ("there is only one correct answer—the traditional one") and their revealed KoT is evident.

In some other cases, alternative solutions are considered *confusing*:

> *"The calculation is very personal. The result is correct, but the solving method is not very clear [Claudia]."*
>
> *"Alda performed the calculation correctly. Claudia and Bruno confused me, I don't know . . ."*

These comments point to the low interpretative ability rooted in the belief that only "the traditional algorithm is correct" (probably related to a more general belief concerning the uniqueness of a correct mathematical answer). Moreover, in all 15 answers that consider Claudia's and Bruno's solutions inadequate or wrong, no references to issues that could be include into the content of KoT are made. Finally, in these answers, very few attempts were made to recognize the student's purpose or strategy.

In fact, only 17 of the 40 prospective teachers provided an answer accepting, without negative comments, Bruno's or Claudia's productions. Nine of these answers mainly focus on the validity of the algorithms based on the use of the so-called "invariantive"[1] property of subtraction in order to justify the proposed procedure:

[1] The invariantive property refers to the fact that the difference between two numbers does not change if the same number is added or subtracted to both the original numbers.

"Claudia added 8 units to 51 and to 17; she therefore used the invariantive property."

"I think that Claudia's reasoning is correct because it is based on the invariantive property in order to make the calculation easier, owing to the use of 'rounded' numbers."

Another critical point of our inquiry concerns prospective teachers' proposals of possible feedback to these students. It is interesting to note that very few teachers suggested any feedback and those that did (feedback perceived as explaining the "correct" process) tended to provide a set of actions aimed at explaining the traditional algorithm, or using the meaning of subtraction as "taking away," as noted below:

"I could help the children with stimulating questions as: What is the subtraction? What does taking away mean?"

Moreover, regarding the KSM, which we consider an important knowledge to refer in mathematics teacher education, it is important to underline that, in prospective teachers' answers, we recognized none references to KSM. Finally, our analyses showed that in solving this task in the complex system of beliefs (Liljedahl & Oesterle, 2014) the beliefs about mathematics seemed crucial, in particular the pne according with only "the traditional algorithm is correct". As we observed probably this is related to a more general belief concerning the uniqueness of a correct mathematical answer and in our analysis it appeared always intertwined with a presence of a poor KoT.

CONCLUSION

Aimed at deepening our understanding of the type and nature of potential links between prospective teachers' interpretative knowledge and beliefs, we designed a particular mathematical task. This task required the prospective teachers to interpret different students' subtraction algorithms and provide feedback to those they consider incorrect. The choice of the task follows the MTSK theoretical framework, which underscores the importance of ensuring that tasks utilized in teacher education are directly connected to the work of teaching. Moreover, we recognized the potential of arithmetic operations algorithms to bring out insights about the prospective teachers' views and understanding of mathematics and its teaching.

One of the findings stemming from our work pertains to the fact that majority of the prospective teachers' answers are rooted in a very firm belief about mathematics that only the "traditional" algorithm—the one they are familiar with (Alda's one)—should be considered correct. Moreover, even in this case, the revealed KoT is at the level of description of the different steps of the procedure, thus revealing prospective teachers' difficulties in arguing about the mathematical reasons of the correctness of the traditional algorithm.

Almost half of the prospective teachers deemed the traditional algorithm as the only correct one, as we could in statements like "*Alda's algorithm is THE right one*." The use of the definite article "the" highlights the prospective teachers' believes on the uniqueness way for finding the answer to the given subtraction: it should be interesting to investigate if perspective teachers believe that – in general – a mathematical task has an unique answer and an unique way to be solved. Such belief about mathematics is present in prospective teachers who also show a poor KoT, which determines a very narrow space of solutions to the posed problem (Ribeiro *et al.*, 2016). This example evidences the role of mathematical knowledge and the view/belief of what means mathematically epistemologically incorrect answers, are a relevant obstacles in the development of a strong interpretative knowledge.

On the other hand, in many of the prospective teachers answers which consider Bruno's and Claudia's algorithms inadequate, we found that subtraction is described assuming only one of its three possible meanings, like in the statements "*I think he [Bruno] doesn't understand the action of taking away.*". This finding supports, once again, the idea that the failure of the interpretation process is intertwined with the content of teachers' KoT. In that sense, it will also be important (and interesting) to deepen our knowledge of the meanings of subtraction presented in the problems teachers posed in the first part of the task. It was also interesting to observe the differences in use (and the corresponding beliefs) of the terms "correct" and "adequate" by some prospective teachers. Indeed, according to some study participants, Bruno's and Claudia's algorithms are correct but not adequate. This finding clearly requires further investigation.

Moreover, it is important to underline that almost half of the prospective teachers who grasped the correctness of Bruno's and Claudia's algorithms used the elements of KoT (in particular the invariantive property) to justify their correctness. This finding, yet again, reveals the essential role of KoT in the development of the interpretative knowledge.

We showed that the absence of key elements of KoT together with particular beliefs about mathematics prevent (prospective) teachers to trigger these processes. Our proposal to place the idea of interpretative knowledge as the core of the Mathematical Knowledge for Teaching, underlines a need of a different mathematics education culture that induces teachers to activate real interpretation processes and to use/capitalize pupils' answers as sources. A possible further step is to clarify the essence of the interpretative knowledge and to identify possible key experiences that trigger in teachers' practices attitudes oriented to a real listening and interpreting of pupils' answers. In this direction, it is important to highlight that our proposal is also conceived as a potentially effective approach to working on teachers' beliefs and knowledge. Indeed in our classes analyzing and discussing with prospective teachers the kind of task we presented here, recognizing with them the correctness of pupils' answers previously labeled as incorrect, reflecting with them upon their own different reactions and evaluations, we have often been observing interesting (prospective) teachers' changes in beliefs and knowledge developments on mathematical critical issues. But further research is needed for analyze and document this other part of our work as teacher educators.

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Problem creating tasks to develop teacher's competencies

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In this presentation, an experience about creating new problems carried out by teachers in their classrooms is analysed. The teachers use a strategy specifically designed to change a given problem posed in a concrete class episode. Form such experience problem creating is found a way of contributing to the development of didactic and mathematics competencies of teachers

Keywords: problem creating, professional development, teacher competencies, teacher training.

INTRODUCTION

Problem posing takes increasing attention in recent years. It was recommended that the teachers offering opportunities to know about formulating problems from a giving situation. The best way to do this is to adopt an inquiry-based learning approach. Obviously, teachers should have developed their problem creating skill to be able to work in this way with their students. Many authors underline the importance of the relationship between problem solving, problem posing and problem creating by including problem creating in teacher training programs (Singer, Ellerton and Cai, 2015). An open problem is how to articulate theoretical notions of Ontosemiotic perspective of constructing mathematical objects and processes with problem posing in teacher training courses in order to produce good mathematical and didactical problems promoting professional development.

The future teacher needs to be able to modify some proposed problems in order to get a richer mathematical activity, being aware of their mathematical benefits. It should be part of growing the capacity of analysing didactically the mathematics activity (Rubio, 2012). We expect that having in mind such tools for designing and didactical challenging problems motivates teachers' interest in creating problems and in developing their capacity for crafting problems in ways that serve teaching and learning. Thus, the aim of the paper is to explore the use of a strategy for engaging pre-service and early career teachers to enrich mathematics problems after didactical and epistemic reflection by means of problem transformation. In particular, to find which mathematical and didactical benefits emerge when using instructional strategies in teacher training courses as a case study design. Thus, two questions are involved: (a) Which are the characteristics of a strategy that uses reflection as the core of promoting that better proposed problems appear, and (b) which mathematical and didactical benefits emerge when using a didactical analysis reflection.

THEORETICAL FRAMEWORK

Problem posing has been usually interpreted as the generation of a new problem or reformulation of a given problem; as the formulation of a sequence of mathematical problems from a given situation; or as a resultant activity when a problem is inviting the generation of other problems. Authors as Silver referred to problem posing as involving the creation of a new problem from a situation or experience, or the reformulation of given problems (Stoyanova and Ellerton, 1996). Instead of that we consider creating mathematics problems being a process which let us obtain a new problem from a given one (problem's *variation*); or from a situation (problem's *elaboration*). The situation can be like it is presented in the reality, or configured as a part of the problem's elaboration (Singer & Voika, 2013).

Some researchers try to integrate problem posing ideas and didactical analysis for teacher training purposes, analysing the benefits of qualified joint reflection and aspects associated with its development, using problems with fractions by using semantic analysis as a reflective analysis (Ticha and Hošpesová, 2013). We assume creating problems as related to complex processes considering knowledge base, task organisation, heuristics and schemes, group dynamics, and individual considerations (Koichu & Kontorovich, 2013). It is also important valuing aspects of the proposed problems in order to see a mathematical improvement (Sengül & Katranci, 2014) even because future teachers had difficulties for characterising conceptual aspects.

It is clear the power of transforming mathematical tasks according variations by promoting teachers being sensitive and recognising how to use in the classroom (Milinkovic, 2015). Mathematical content knowledge is necessary, but our hypothesis is that transformations in problem posing can improve content knowledge by means of didactical analysis, and also contributes to increase didactical competencies of teachers. We identify more deep approach when using suitability criteria proposed by OSA, considering the analysis of epistemic issues; cognitive; normative; interactional; emotional and ecological issues to influence task design (Gimenez, Font &Vanegas, 2013).

In this paper, problem's variation instruction (Malaspina, Font & Mallart, 2015) is a content analysis based strategy to integrate above proposals for improving competence of didactical analysis for future teachers or in service teachers. It consists of first exposing teachers to a class episode. When analysing the possible mathematical difficulties solving the problems included during the episode, we also notice the didactical requirements to improve the solving process. A pre-problem is a new proposal statement that try to satisfy such didactical needs. In order to develop this perspective of problem creating, we consider four key problem elements (Malaspina, 2013): Information, Requirement, Context and Mathematical Environment. As a second redesign, we introduced a more detailed epistemic analysis, using the tools of suitability criteria, in order to recognise the power of knowing the configuration of objects and processes following OSA. A post-problem is a new proposal to improve the problem by finding easier problems responsive to difficulties students had, and harder problems to challenge students to generalize key ideas beyond simply answering the initial problem. This global instructional strategy is called ERPRP because, it starts by facing a class episode (E), first reflection (R), producing a problem (P), introducing tools from didactical analysis (R), and producing a better problem (P). In such a framework, we would like to show that such strategy helps and stimulate the ability to create mathematics problems, through modifying a given problem, considering mathematical and didactic aspects. Therefore, problem creating by using transformations of previous problems is a contributing way to the professional development for future teachers.

METHODOLOGY

We have chosen a qualitative ethnography study with the proposal of a starting strategy, and exemplification as cases study from 2013 to 2015 with three groups of 25 prospective teachers participating in problem solving courses in Peru, Ecuador and Spain. The first step consisted of choosing a topic and designing some easy and motivating problems as starting points to create new problems by changing some math concepts or ideas assuming ERPRP strategy already described. All the proposed pre and post-problems are analysed by means of content analysis to see which are the mathematical and didactical new ideas learnt behind the proposals. We use next section to present some paradigmatic examples to reveal the power of the phases of the strategy used as a qualitative analysis, and some of the mathematical and didactical benefits drawn.

DISCUSSION

At the beginning, the problem creating experiences had been performed with pre-service teachers as a part of the mathematics courses with a strategy ERPP, in which initial reflection, pay attention to analyse mathematical difficulties, doing two steps of modification problem posing. The positive experiences of the individual work and of the group work were the basis to design the strategy above explained.

We spent two hours only on developing a starting problem creating workshop. We made a short exposition about problem creating, including some examples of problems created in previous workshops. We presented a previously elaborated problem to the students presenting a concrete class episode of a teacher T. In this episode, the trainer describe some of pupil reactions when solving the problem. Each future teacher created its Pre problems individually. Group discussion plays an important role of this first strategy. We redesigned such initial strategy in order to include another reflection moment using suitability tools to improve challenging pos-problems. In a second experience, a theoretical based reflection is a new phase. Pos-problems are the final step to be analysed. Let us see some research results of mathematical and didactical benefits by means of some examples.

The role of the initial Episode.

The main issue of proposing an episode instead a problem, is to see problem statement in a real professional class-context. In fact, the position of a teacher is not only being a problem solver, but a problem inquirer. We see it in the following example of proposal.

> The first week of July a shop called MARKET sold all the products without any discount; the second week, applied discount of 20% on all the items; and the third week, added discount of 15% . It was announced as the GREAT DISCOUNT OF 20%+15% ON ALL THE PRODUCTS. You have to study whether the third week of July the shop called ALFA sold products with 35 its % discount on prices of the first week is true or not. After a few minutes: Most of the students say yes, it is true. Juan and Carla say no because the discount of the third week was less than 35%. Maria says that the discount of the third week was 68%.

The role of individual reflecting. Future teachers usually explain that they have similar difficulties to the students in the episode. Just some of them can solve the problem discussing about the multiplicative structure of a discount.

The role of Pre-Problem creating and group discussion. The main value of pre-problem is to contribute to a better comprehension of the situation presented in the episode. It gives opportunities for starting a didactical analysis. Let us see some *pre problems* posed by teachers to help in the process of creating new problems after discussing the problem above cited. The research group analyse all the productions, to observe the hypothesis of mathematical and didactical purpose in each proposal. It gives opportunities for identifying the background of future teachers. The problems creation starts individually at the beginning and discussed afterwards in groups. All the groups of future teachers solve the problems created, and the explanation's resolution is part of a socialization process with all the participants. Following the examples, we notice that in some cases, the author's idea when the teacher posed the problem was considering a price very easy to calculate its percentage in order to help pupils focus their attention on the total discount

> Rosa pays a sum of 100 "new soles" for a shirt, with discount of 20% because of ending bargain sales and with an additional discount of 10% thanks to having the shop's card. What percentage did Mary take off on buying the shirt? (FT1)

In other cases, the author was interested in showing the students another point of view of the total discount. It is not only a simple sum of percentages. In order to achieve this objective, the author had chosen this problem because it posed a total discount (100%) very little intuitive.

> In a clearance sale, a shop applies discount of 50% on all its textiles during a week, and the following week applies an additional discount of 50%. What is the total percentage discount applied during the second week? (FT2)

In some cases, a group of future teachers develop a common problem trying to help pupils to distinguish between the money paid and the discount. This seemed to be the confusion of the student called Maria in the situation. Apparently, she had done well her calculations but she did not distinguish between the money paid (68% of the initial price) and the total discount (100 - 68 = 32%).

> Rosa pays a sum of 100 “new soles” for a shirt, with discount of 20% because of ending bargain sales and with an additional discount of 10% thanks to having the shop’s card.
> a) How much does the blouse cost to Maria if she buys it during the second week?
> b)What is the percentage representing the second week’s price with the blouse’s price without discount?
> c) What is the blouse's total percentage discount during the second week?

Observing the examples proposed, there is a need of focusing, on solving the problem giving a justified answer, but to understand the mathematical content or property distinguished or “specific” in such a proposed initial task. Only when the groups of future teachers think more than the mathematics topic involved, we can see didactical growing. In general, future teachers are worried about what they do not know, and they could learn from others. The pre-problem also help to recognise management aspects by doing a context analysis. Future teachers know about facing children’s difficulties.

The role of Post-Problem creating.

At the beginning of creating post-problems, many future teachers thought that it is important to conserve the structure by finding similar problems to the given problem, with other prices and in some cases, considering three successive discounts; basically, with quantitative modifications in the data. The future teachers imagined that the main aspect to modify is the computation problem and the particular process of solving the problem. One of the future teachers tell us “it is a problem of discounts”. However, they carried out more enriched problems via transformations when they formulated post- problems, even without a second reflection.

In some groups of future teachers, it appear the need that children should reinforce the comprehension of the fact that the total discount is not a simple sum.

> *Pedro and Juan bought a shirt each one. Pedro bought it with a discount of 20% plus another additional discount of 20%. Juan bought it with a discount of 30% plus another additional discount of 10%. Who did obtain the greater discount? (Gr 1)*

Another group thought it was interesting to pose situations about cumulative percentage, considering charges and not only discounts. Its solution requires a better understanding of the percentage concept.

> There is a shop where if you want to pay after enjoying the product 30 days with a card, the price increases 10% more. And if you want to pay after 31 days but before 35

days, there is a surcharge of 5%. If Juan bought in this shop on August 20th and paid on September 23rd, what did he pay for percentage of surcharge? (Gr 2)

Bearing in mind the importance of the socialization aspect, in all the cases we have kept an extra mathematical context but we need also to create problems in an intra mathematical context. Generalization allows us working in this context. Generalisation appear as being a new statement with higher mathematical value.

If the shop called BETA knocks off end of the season of p%, plus an additional of q%, what is the percentage of the total discount in relation to the price without any discount? (Gr 3)

In this case, or similar ones, the problem lets illustrate in an easy way the discount of r% applied to the sale price of a product (x) through a composition of linear functions: $f(x) = \left(1 - \frac{r}{100}\right)x$

It is clear the role of this phase is to see what is behind a problem in terms of promoting mathematics learning. If there is a discussion about "particularisation/ generalisation" or contextualisation/ de-contextualisation, we can see the teachers growing their didactical analysis competence. And consequently, promoting more rich problems.

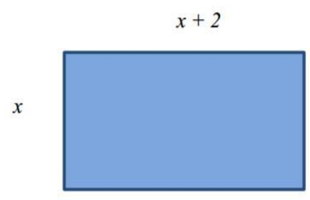
The power of discussion when analysing the content

Let us notice the influence of the mathematical content example and the classroom discussion showing the value of such reflection to improve mathematical content knowledge, by using successive problem transformations. Working with second degree it was presented the following short episode.

If we multiply the age of Charles 3 years ago, times the age that Charles will have after 5 years, we obtain 48. Which is actually the age of Charles? To solve the problem some students wrote $(x + 9)(x - 7) = 0$ to conclude that the actual age of Charles is 7 years old, because the other solution is negative.

One group wrote the factorisation as $f(x) = x^2 + 2x - 15 = (x + 5)(x - 3)$. They feel the main issue is to identify the second-degree equation to solve the problem, and then plan the following post-problem, introducing contextualised situation.

Which can be the dimensions of a rectangular room if the area must be maximum of 15 square meters being the length two meters more than the other size? (post Gr 5). Then, they draw the following design



And then they wrote the following inequality $x(x+2) \leq 15$, and they conclude that the result are the points of the interval]0 ; 3] In such case, the teacher ask the participants to make explicit the relation between the solutions and the given function f. After some minutes without any proposal, we propose to use another register, not an algebraic equation, but a graphic register.

Therefore a variation of the problem was suggested.

> Which can be the dimensions of a rectangular room if the area must be maximum of 15 square meters being the length two meters more than the other size? Ilustrate the solution using the graphic of a quadratic function. We ask the teachers to create a new problem using the graphic of linear functions above cited.

The need of epistemic analysis. The adapted new scheme ERPRP.

During the first experiences, having the post-problem the trainer was almost satisfied. In fact, the mathematical object more difficult to analyse was the mathematical argument in front of expressions and terms. Only six future teachers distinguish the arguments used during the solving process. A half of the future teachers talk about propositions and procedures.

When we introduced an epistemic analysis during the redesign, the future teachers noticed more mathematical aspects than before. For instance, many of the future teachers talk about generalisation, and give explanations about the need for analysing maximum or minimum when the problem needs to use a second-degree equation. It is the case of the problem of second-degree, in which a sequence of new post-problems appear, to introduce the role of connecting representations when introducing mathematical objects. Let us see an example of starting problem (Malaspina, 2013).

> *Present the graphic of a function f given by the function $f(x) = x^2 + 2x - 15$ using the graphs of two affine functions.*

The future teachers used both graphs of $g(x) = x+5$ and $h(x) = x - 3$ as you can see, using geogebra. A first reaction was to obtain points of the product by multiplying the corresponding ordinates of the points of the graphics of g and h. Nevertheless, it was suggested to do a more wide and global analysis, and more qualitative, using key points.

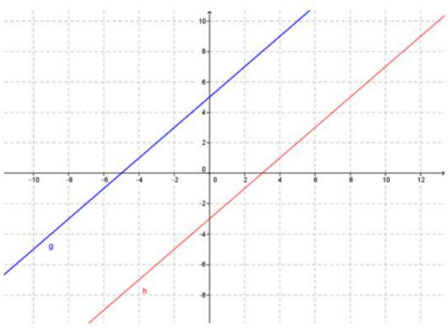


Figure 1. Observations about the function approach

The trainer asked to find some points of g and h in which we have points of the function f. They discover the graph must include the points (-5; 0), (3; 0) . Therefore g(-5) = 0 and they conclude that multiplying by every number the result must be zero. With similar argument, they found that f(-5) = 0 y f(3) = 0. It was also proposed to find the sign of points for f according the signs of g and h. They conclude that for

$\neq -5$ and something similar for $\neq 3$. They tell us "when x < -5, the graphs of g and h are below the x axe. In consequence, the product is positive and the graph of f, when x<5, will be up the x-axe. Similar analysis give to the conclusion that when -5 <x<3 the graph will be below the y=0, and for x > 3, the graph of the function f will be above y=0

Using ideas about the continuity of f, they discover that the graph is a curve passing through (-5; 0) and (3 ; 0); decreasing till certain point of the interval]-5 ; 3[; increasing the points after. When they used geogebra to observe their intuitions, they found that the result is according the intuition.

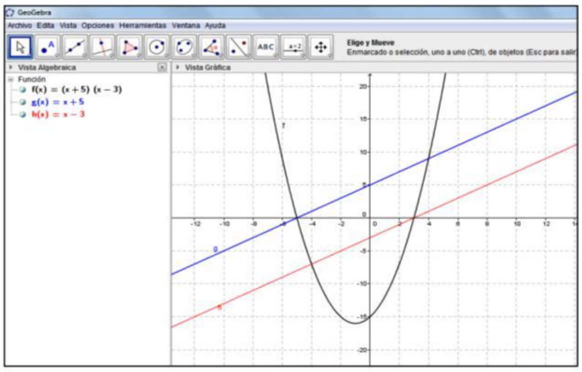


Figure 2. The parabola as a product of two right lines.

After that, the future teachers identify that new statements can be proposed. Didactically speaking, the revised experiences offer also evidences in which the future teachers identify and notice tools for understanding students' practices and difficulties. According final master comments of pre-service teachers in Spain, we found many examples in which they learn from students, using the strategies related to problem creating activities. They told us about "how interesting was to see that now I understand why 14 years-old students have difficulties to understand that a parabola is the product of two right-lines". The percentage's theme and equations theme are very favourable to create problems in an extra mathematical context and it suggests a great diversity of imagined situations in the created problems. This reveals the authors' advances in the mathematical object management, in the reality observation and in elaborating tasks to go deeper into the subject to solve the problem created.

The cases presented are paradigmatic examples showing the emergence of empowerment of future teachers, using different kind of transformations: quantitative (changing numbers), qualitative (the problem deals with discounts and increases), relational (the information is shown to make easier the meditation over possible wrong answers) and in some cases, it is a piece of information added or the requirement is extended. Problem creating as a redesigned process related to one concrete theme contributes to deal it deeply. It provides opportunities to relate mathematical ideas and representations to get an insight to involve students into intra-mathematical connections. In our examples, creating problems within a reflective process gave opportunities for relating algebraic situations to geometrical graphical interpretations unknown for the teachers. Such interventions give opportunities of reflecting about intra and extra mathematical connections. But at the same time, future teachers talk about interactions, the role of contextualisation, to overcome the magisterial class, and the role of mathematical debates.

FINAL REMARKS

Creating problems give opportunities and benefits to challenge future teachers to claim for powerful understanding of connecting representations, assuming the role of problem posing as a positive way for critical math understanding. The use of techniques of relating concepts, by changing the problems, in order to see the identification of a need for helping students to improve their own understanding.

As a part of the challenges posed by this research on creating math problems in mathematics education contexts, we see evidences in which creating math problems on a given topic activates new learning processes that favour intra mathematics connections with other fields of knowledge and reality. The intervention of the researcher contributed more to focus upon the theoretical perspectives of problem creating "There' sees not one path and everybody has their personal path that they can discover and that's what makes it fun. That's the adventurous part of mathematics, the creative part of mathematics and we miss that in the way mathematics is taught." (Manjul Bhargava; Fields medal 2014)

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Education of facilitators: Cooperation to compensate challenges of PD-courses for facilitators?

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While recent studies focus on PD-courses (Professional Development-courses) for teachers, only little is known about PD-courses for facilitators. One main challenge for facilitators is the balance between different types of knowledge in PD-courses for teacher. Some facilitators tend to focus exclusively on practical knowledge and good examples, which cumber the process of scaling up PD. PD-courses are one core element to educate facilitators, but they are limited in temporal resources, so that not all aspects can be in focus on depth. This paper focuses on the strength and limits of the cooperation between facilitators and how this cooperation can complete PD-courses for them.

Keywords: facilitators, scaling up, multipliers, forms of knowledge, cooperation.

THEORETICAL AND EMPIRICAL BACKGROUND

Starting points for PD-courses for facilitators and scaling up PD

In Germany Professional Development (PD) – courses are lead by facilitators. These facilitators are more or less experienced teachers that are interested in PD. In Germany no binding standards for the education of facilitators exist and quite often they have none education for their work as facilitators. For this reason PD-courses for facilitators are very important in Germany. The idea of such courses is to prepare the facilitators for their own PD-courses for teachers.

Recent studies focus on the design and effect of PD-courses for teachers (e.g. Timperley et al., 2007). The formulation of Design-Principles for effective PD is one result of this research (Barzel & Selter, 2015). However, only little is known about PD for facilitators. As a first starting point results from cognate scientific disciplines and research fields are used, also preliminary findings about effective PD for teachers, but empirically grounded findings about the adaption for facilitators are missing. Another starting point are research findings on adult education. Typical for adults is, (a) that they already learned a lot during their life and have that in mind in new learning situations, (b) that they focus on the practical use of new knowledge and (c) that they have status specific behavioural expectation (Geissler, 2001). Empirical findings on the adaption of the principles of adult education for facilitators are also missing. Only some studies focus explicitly on facilitators (e.g. Borko et al., 2015).

Although recent studies do not focus on facilitators' PD, its relevance is undoubted. If innovations shall be implemented into school practice, PD-courses for facilitators are one out of three possibilities to reach a high number of teachers and to scale up PD. This way of scaling up is called Cascade Model. Learning in Professional Learning Communities (PLCs) and E-Learning PLCs are the possibilities to scale up (Maaß &

Artigue, 2013). In the research project the focus is on scaling up in the Cascade Modell, because it offers most likely the chance to implement innovations from research into practice at schools.

Coburn (2003) defines the following four quality criteria to evaluate the process of scaling up, which can help to identify more or less successful adaptions from the PD-course for facilitators to the PD-courses for teachers. For Coburn (2003) this process is successful if the facilitated innovations are understood and realized (a) in *depth* and (b) *sustainable*. Also (c) as much people as possible should be reached (*spread*) and (d) the target group has to make the innovation its own (*shift in reform ownership*). The more the four interdependent criteria are fulfilled, the more successful the process of scaling up is. This relevance does not change the fact that even if "the issue of 'scale' is a key challenge for school reform, yet it remains undertheorized in the literature" (Coburn, 2003, p. 3; Rösken-Winter et al., 2015).

The relevance of different roles for facilitators

The situation in Germany is that facilitators are typically more or less experienced teachers and work part-time as facilitators. Because institutionalised education for facilitators does not exist (yet), facilitators are quite often marginally skilled. They work in two roles at once, which can lead to role conflicts. Especially situations in which the expectations as teachers and as facilitators are mutually exclusive are problematic. But even if the two roles are not problematic for the facilitators, the evaluation of PD-courses for facilitators should always take into account that the participants have these two roles. Table 1 exemplifies this aspect in adaption of Lipowsky and Rzejak's (2012) model for the effects of teachers PD for facilitator (similar to Guskey, 2000).

Facilitator in the role as…

…facilitators:		**…teacher:**
F1: the response of the facilitator		T1: the response of the teacher
F2: the learning process	←	T2: the learning process
F3: the planning of PD-courses	→	T3: the planning of lessons
F4: the facilitation of PD-courses		T4: the facilitation of lessons
F5: the learning of teachers		T5: the learning of students

Table 1: Adaption of Lipowsky and Rzejak's model for teacher PD for facilitators

Types of knowledge for PD

The differentiation of types of knowledge can be a starting point to determine the aims and contents of PD-courses for facilitators. Several authors already defined relevant types of knowledge for PD for teachers. Shulman (1986) claimed that a "conceptual analysis of knowledge for teachers would necessarily be based on a framework for classifying both the domains and categories of teacher knowledge […]

and the forms for representing that knowledge" (ibid., p.10). For him, the often-cited domains and categories are subject matter content knowledge, pedagogical content knowledge and curricular knowledge, and the forms of knowledge are propositional knowledge, case knowledge and strategic knowledge. The focus of this paper is on Shulman's PCK, which will be further differentiated. Fenstermacher (1994) distinguishes knowledge facts in another way than Shulman does. He focuses on the differences between knowledge as a result of teaching experiences and knowledge as a result of research on teaching. While drawing on Fenstermacher's approach, Cochran-Smith and Lytle (1999) differentiate the owner of knowledge further. For Cochran-Smith and Lytle knowledge can be constructed in three ways: knowledge-in-practice (teachers experiences: e.g. practical knowledge), knowledge-for-practice (knowledge from science: e.g. categories for practices) and knowledge-of-practice (knowledge from reflections). All of these types of knowledge are important for the process of scaling up: e.g. without knowledge-in-practice innovations would not reach practice in schools (table 1), because the use-oriented teacher would not accept the whole PD (Geissler, 2001); e.g. without knowledge-for-practice teachers could not implement ideas on their own (shift in reform ownership) and reach sustainability; and e.g. without knowledge-of-practice the innovations are not understood in depth. For facilitators it is important to keep in mind, that all three types of knowledge have specific interdependent contents for the role as a facilitator and the role as a teacher.

But facilitators, especially those with little experiences as facilitators, tend to focus on knowledge-in-practice in their PD-courses for teachers (Wassong i.p.; Zwetzschler et al., 2016). Thereby they fulfil the well-known expectations of teachers in PD-courses to learn practically useful aspects (e.g. Geissler, 2001). This focus of the facilitators even tends to fit their reflection and practice in their role as teachers – the more they reflect their lessons in categories of knowledge-for-practice, the more they facilitate those categories (Zwetzschler et al., i.p.).

Relevance of perceived self- and group-efficiency for facilitators

The connection between job-performance and the degree of experiences as facilitators hints to a further aspect: the self-efficiency, the individual beliefs to successfully overcome challenges by oneself (e.g. Bandura, 1997). Khursid et al. (2012) point out that a lower degree of job performance and perceived self-efficiency is typical for novice teachers in comparison to experienced teachers. This implemented development is accomplished by the development of group-efficiency, the beliefs of a group to successfully overcome challenges as a whole (e.g. Bandura, 1997). Schmitz and Schwarzer (2002) point out, that the longer people are part of a team, their perceived group-efficiency decreases in comparison to the increasing perceived self-efficiency. However, it is not yet known in how far these connections and implemented developments can be adopted for the performance and development of (novice) facilitators.

The aim of this paper is to better understand challenges of novice teachers and which connection to the construct of self- and group-efficiency exists.

RESEARCH QUESTIONS

One possibility to overcome the described challenges lies in immersing facilitators in both roles (as a teacher and as a facilitator) in depth, but this would last a (unrealistic) long time. Typical PD-courses for facilitators endure only some days with distance phases in between and cannot realize both aspects in depth. Therefore, the facilitators have to find ways to compensate this gap in their education. As all facilitators worked together with colleagues and appreciated this cooperation, the construct of perceived group-efficiency seems to be relevant to understand their job performance - so the following research questions are pursued:

Q1 In how far does facilitators' self-efficiency of knowledge-in-practice, knowledge-for-practice and knowledge-of-practice influence their cooperation and their perceived group-efficiency?

Q2 In how far can the cooperation with colleagues foster the intended learning processes of the PD-course towards knowledge-in-practice, knowledge-for-practice and knowledge-of-practice for facilitators?

METHODOLOGY OF THE CASE STUDY

Data gathering

To answer these research questions sicxteen semi-structured interviews (of 45-120 minutes each) were conducted. All interviewees took part in a PD-course for facilitators and facilitated PD-courses for teachers afterwards. All forms of knowledge (in-, for- and of-practice) were part of the PD-course for facilitators. The interview questions dealt with the general design of their PD-course for teachers, their PCK of the content, their experiences in the PD-course for facilitators and their experiences with the topic as teachers. To get further insights, we also simulated parts of a planning process of a PD-course for teachers. Eleven of these interviews were part of a project on facilitators in cooperation with Kim-Alexandra Rösike, Bärbel Barzel, Susanne Prediger and the author, the others were conducted by the author.

Data analysis

All interviews were audio recorded. A qualitative content analysis (Mayring, 2015) was conducted by paraphrasing aspects according to the research questions. Selected parts were transcribed and analysed in detail by Vergnaud's (1996) theory of conceptual fields. This theory offers "a fruitful and comprehensive framework for studying complex cognitive competences and activities and their development" (ibid., p. 219). Theorems-in-action as "proposition[s] that [...] [are] held to be true by the individual subject for a certain range of situation" (ibid., p. 225) and concepts-in-action as "categories [...] that enable the subject to cut the world into distinct […] aspects and pick up the most adequate selection of information" (ibid.) were the focus of the interpretative analysis. So the theory of conceptual fields enabled a deeper understanding of the facilitator's thoughts.

EMPIRICAL INSIGHTS FROM TWO CASES

To exemplify the results, prototypical empirical insights of two representative facilitators are presented. The first facilitator is Greg. He is an experienced teacher with only little experiences as a facilitator. The second one is Julia, who is experienced as teacher and facilitator.

Greg: Cooperating to compensate gaps in knowledge-for-practice

Greg focuses in his PD-courses for teachers primarily on knowledge-in-practice. Thereby he fulfils teachers' expectations. So sustainable scaling up in depth and with a shift in reform ownership probably will not happen due to his exclusive focus (Zwetzschler et al., 2016).

In the following sequence Greg explains his needs as a facilitator:

Greg:	That is certainly uh as I said also the part of pedagogical content knowledge based on uh let me say of the theory, wished not only examples, whether even more, what is the extent, what's the PCK idea, the advantage of theory uh beginning, let me say in the first main part, that you know there – even more in depth […] That you just get more experienced in that aspect.

Greg wants to learn more about PCK, playing for him a role as a background theory. PCK as a theory explains for him the extent and the idea behind examples. It is a type of knowledge that categorises his action. So his need can be condensed as a need for knowledge-for-practice: he wants to be more experienced in knowledge-for-practice. Other paragraphs show, that he has no needs for knowledge-in-practice. His perceived low self-efficiency with respect to knowledge-for-practice and high self-efficiency with respect to knowledge-in-practice matches the analysed problem of his PD-courses: the main focus is on knowledge-in-practice. But knowledge-for-practice is also part of his PD-courses for teachers, although his prior knowledge and the PD-course for facilitators are insufficient for him.

In the following sequence Greg explains how he prepares his PD-course for teachers:

Greg:	And that is the way how we did it […]. We chose a topic, that matched our work [at school], that was our topic after the next, so that you could prepare
Interviewer:	hmhm
Greg:	and uh that's indeed the way how it is here either. I said: So now you can individualise and differentiate
Interviewer:	yes
Greg:	So you take aspect uh out of your textbook, and than I also talked to Ben, I said: Ben, you already did lots of PD-courses for teacher
Interviewer:	hmhm

Greg: Yeah, I said: Do you have something against, uh, if you send me one of your versions, that I can design my own by it, what the content is about. And yeah, then Ben sent me that and then I did with the staff, no matter if it is about individualising and differentiating or productive practicing, took that and worked with it.

Interviewer: hmhm

Greg: as basis, as scaffold

In the first three paragraphs Greg describes his preparation for PD in his lessons at school. He tries to get practical experiences in his lessons, which he can use afterwards in his PD-courses for teachers. His experiences prepare himself for knowledge-in-practice in PD-courses for teachers. Thereby he perceived high self-efficiency with respect to knowledge-in-practice. In the last three paragraphs Greg adds a second well-established aspect of his preparation for PD-courses for teacher. He asks his experienced colleague Ben for his materials. These materials are the basis for Greg and help him to identify and organise the content. So the ideas of Ben help Greg to compensate his gaps in knowledge-for-practice. The combination of his practical experiences and the theoretical ideas and structure of Ben guide Greg in creating his PD-course for teachers. This helps him to overcome individual problems with knowledge-for-practice. So his perceived group-efficiency is in general higher than his perceived self-efficiency.

Julia: Cooperating not needed – her own ideas are good enough (or better)

Julia balances in her PD-courses for teachers the different forms of knowledge and mediates competently between teachers' experiences and aims of the courses (Zwetzschler et al., 2016).

In the following sequence Julia speaks about her needs as a facilitator, also in relation to cooperating with colleagues:

Julia: Well, if there would be anything that I would like to learn, than I would claim it.

Interviewer: hmhm

Julia: But I realised in a PD-course for teachers, that I, that we did in a team – a new facilitator of the team came along

Interviewer: hmhm

Julia: She is also experienced in PD-courses for teachers [

[…]

Julia: How great the extent was, what she prepared, right?

Interviewer: hmhm

Julia: It began with little notes on each table, where they, well – things like that, that aren't important for me in PD-courses for teacher

Interviewer: hmhm

Julia: Things I don't need, but they are good for colleagues. I think, that in the cooperation with others I often realise aspects missing aspects in my work […] Hmhm. I believe, that if, if I could consciously name something, that I need

Interviewer: hmhm

Julia: Then, I would consciously get that and uhm – break me in

In the paragraphs Julia reports on her needs as a facilitator. Directly at the beginning and again at the end she claims that she would try to overcome needs if she had some. This shows her beliefs of having no (important) needs at the moment. Her perceived self-efficiency as a facilitator seems to be very high. In the middle she talks about experiences of cooperation. In those situations, she realizes differences between her work and the work of others. She claims, that those moments can be starting points for her to identify further needs as a facilitator. But in the end she dissociates herself from the other facilitator, by judging the differences as irrelevant for her. In contrast to the benefit of the cooperation for Greg, it stays suspect if Julia judges the cooperation as beneficial. For her the perceived collective-efficiency seems to be lower than her perceived self-efficiency.

RESULTS AND DISCUSSION

The knowledge of facilitators and the perceived self-efficiency seems to match the benefit of collegial cooperation and perceived collective-efficiency for the different forms of knowledge: Greg perceived high self-efficiency with respect to knowledge-in-practice, whether his perceived self-efficiency with respect to knowledge-for-practice was relatively low. Although his perceived self-efficiency with respect to one type of knowledge is high and one is low, he focuses on cooperation and his perceived general group-efficiency is high. Instead, Julia's perceived self-efficiency is high with respect to knowledge-in- and knowledge-for-practice and her perceived group-efficiency is low. She doesn't focus on cooperation. Table 2 shows these matches.

	knowledge-in-practice	**knowledge-for-practice**	**focus on cooperation**
Greg	Self-efficiency(high)	Self-efficiency (low)	✓
	∨	∧	
	General group-efficiency (high)		
Julia	Self-efficiency(high)	Self-efficiency(high)	✗
	∨	∨	
	group-efficiency (low)	group-efficiency (low)	

Table 2: Matches: self- and collective-efficiency for forms knowledge and cooperation

These results lead to the following theses: As long as the perceived group-efficiency with respect to *one* form of knowledge is higher than the perceived self-efficiency, the facilitators prefer cooperation.

It seems as if the facilitators compensate gaps in forms of knowledge by cooperation, which supports the adaption of the already explained findings about self- and group-efficiency for the group of facilitators: Novice facilitators seem to have a lower degree of perceived self-efficiency in comparison to experienced facilitators. And the more experienced facilitators are, the lower the perceived group-efficiency is.

This qualitative study shows first matches between forms of knowledge, specific competences and self-efficiency with collective-efficiency and cooperation. Further research is needed to broaden these qualitative results and confirm them quantitatively.

Another aspect for further research is the quality of cooperation. In the case of Greg the collaboration enabled him to prepare and facilitate PD-courses for teachers, but his courses focus excessive on knowledge-in-practice, impeding scaling up in depth (Coburn, 2003). This illustrates that the cooperation could not enhance the PD-course. Greg needs further learning situations. Aspects of a PLC like the mutual visitation and shared visions could support his learning process (Hord, 1997, for visitations: Khursid et al., 2012). As a consequence of these results focussing on the Cascade Modell for scaling up should be reconsidered: if especially facilitators with gaps in forms of knowledge tend to cooperate with colleagues – how can we take this systematically aspect into account and improve the quality of the cooperation?

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Design Research with a focus on content-specific professionalization processes: The case of noticing students' potentials

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Abstract. *This mainly methodological paper presents a specific approach of design research, called 'Design Research for teachers with a content-specific focus on professionalization processes'. Its three main characteristics are: (1) 'for many' rather than 'with some' teachers, (2) content-specificity, and (3) focus on teachers' processes. The approach and some typical outcomes are exemplified by the case of a project which fosters secondary teachers to notice students' mathematical potentials. The case is discussed with respect to general issues.*

Keywords. *Design Research for Teachers' Professional Development, Qualitative Research on Professionalization Processes, Noticing, Mathematical Potentials.*

1. ADOPTING DESIGN RESEARCH FOR TEACHERS

1.1 Design research as established research methodology with big variety

Design Research is a widely established research methodology for enhancing and investigating students' learning. It is especially strong when two aims are to be combined: (1) designing learning arrangements for classrooms and (2) investigating the initiated learning processes and contributing to local instruction theories (Bakker & van Eerde, 2015). Although design research approaches share common characteristics (e.g., interventionist, theory generative, iterative, ecologically valid, and practice-oriented, cf. Cobb et al., 2003), a big variety of approaches exists (cf. the 52 case studies documented in Plomp & Nieveen, 2013). These approaches differ in their reasons for doing design research, their types of results, their intended roles of the results for educational innovation, their scales, and their background theories (cf. Prediger, Gravemeijer, & Confrey, 2015a). Our Dortmund research group follows a topic-specific approach which allows to account for different mathematical topics in detail (Prediger & Zwetzschler, 2013) with a focus on learning processes (ibid.; Prediger et al., 2015a). This approach is now adapted to designing and researching environments for teachers' professional development.

1.2 Adopting design research *for* many teachers, not only *with* some

Zawojewski et al. (2008) suggested extending the research methodology of Design Research from students to teachers' professional development (PD) "in order to under-stand both, how teachers develop in their practice and how to design environments and situations to encourage the development of that practice" (Zawojewski et al., 2008, p. 220). Meanwhile, many teacher educators have described impressing individual professionalization effects of *design research with teachers*, for the exclusive minority of teachers privileged to be part of design research teams (Smit & van Eerde, 2011; Bannan-Ritland, 2008) . Although this is without doubt the most intensive PD setting, it is not realizable for scaling up, since many teachers have no access to intensive collaboration with researchers. However, scaling up for reaching many teachers throughout whole Germany is the critical long-term goal for the first and third authors' work in the DZLM, the German National Center for Mathematics Teacher Education (Rösken-Winter, Hoyles & Blömeke, 2015).

Thus, this article suggests complementing the approach of design research *with some* teachers by *design research for many teachers,* taking into account that professional development for scaling up requires well-founded, robust designs for classrooms and PD courses (Burkhardt, 2006; Swan, 2007). Whereas the individual PD work of *researchers* with a selected group of teachers can be based on spontaneous, intuitive decisions in deep discussions, a robust design for PD conducted by *other facilitators* also needs to be grounded on a solid theoretical base, to anticipate possible challenges of the content to be learnt and typical professionalization processes. This calls for the next two characteristics, content-specificity and process-focus.

1.3 Content-specificity and focus on teachers' professionalization processes

So far, the growing body of research on conditions and effects of PD is mainly focused on *pedagogical principles* for PD programs (e.g., Timberley et al., 2007). But for ro-bust designs for scaling up, also a good theoretical base for the *content* of the PD course itself is relevant, which cannot be taken for granted (Prediger, Quasthoff, Vogler, & Heller, 2015b). Specifying what teachers should learn in which perspective about a certain content (e.g. a mathematical topic or noticing students' difficulties) usually re-fers to the current state of research on classroom practices or teachers' professional knowledge for this content. This reference can be substantiated by also taking into ac-count typical teachers' perspectives, which can be reconstructed when qualitatively investigating content-specific professionalization processes.

In their research survey on PD research, Goldsmith et al. (2014) emphasize the need to focus on teachers' professionalization *processes* rather than only on quantitatively measurable effects. Even if they have not found much research on processes yet, they collect indications that teacher learning "is often incremental, nonlinear, and iterative, proceeding through repeated cycles of inquiry" (ibid, p. 20). As the research gap is even bigger for *content-specific* research results, it is a major aim of the approach presented here to provide fine-grained insights into teachers' processes of professionalization on different specific PD contents. For this aim, the most appropriate approach is the adaptation of topic-specific design research with a focus on learning processes (elaborated for classrooms in Prediger & Zwetzschler, 2013; Prediger et al. 2015a). Adapted to the level of teachers, we call it *Design Research for teachers with a focus on content-specific professionalization processes*.

1.4 Four intertwined working areas for PD Design Research

Figure 1 shows the four working areas that are *iteratively* connected in the design and research process, adapted from Prediger and Zwetzschler (2013) for PD design research. The four working areas comprise (a) specifying and structuring PD goals and contents in hypothetical intended professionalization trajectories, (b) developing the specific PD design, (c) conducting and analyzing design experiments in PD settings, and (4) developing contributions to local theories on professionalization processes.

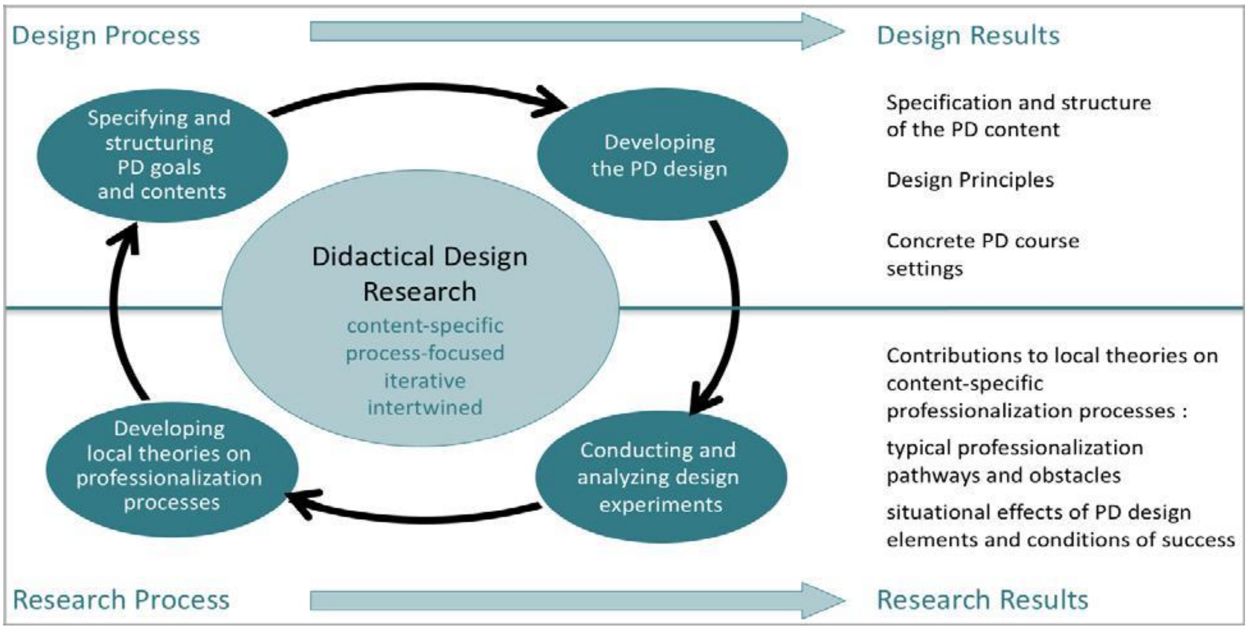

Fig. 1: Working areas and results of Design Research for teachers with a content-specific focus on professionalization processes

The areas are *intertwined* in the sense that each cycle builds upon results of previous cycles across the areas. Corresponding to the two general aims of Design Research, design results and research results have equal importance: The design results comprise the PD course settings as well as their backgrounds, a specified and structured PD con-tent and refined design principles. The local theories are developed to underpin the concrete products and to be generalized by accumulation over several projects. Contributions to local theories on content-specific professionalization processes can be expected with respect to typical individual pathways and obstacles, means for support in the PD setting as well as their effects and contextual conditions of success.

2. THE CASE OF DOMATH, A PD DESIGN RESEARCH PROJECT ON NOTICING STUDENTS' POTENTIALS

For illustrating the approach, we briefly give some insights into the dual design research project DoMath (working on student and teacher level, here focused to the teacher level). The project addresses secondary school mathematics teachers who intend to develop their competences for noticing and fostering students' mathematical potential. Due to space limitations, we focus mainly on noticing rather than fostering.

2.1 Goals, structure, and background of the DoMath PD program

Goals and structure. The DoMath program for classrooms adopts a wide, dynamic and participatory conceptualization of mathematical potentials (Schnell & Prediger, 2016, following Leikin, 2009), addressing specifically those (often underprivileged) students not yet identified as talented. The classroom instructional design therefore builds upon whole class enrichment settings with rich, self-differentiating open-ended problems (ibid.). Teachers become sensitized to notice students' potentials in the rich situations and to adaptively foster the noticed potentials by facilitating supportive interaction.

PD programs in DoMath span over several months in action and reflection settings of material-based video clubs (Sherin & van Es, 2009). In the PD sessions, teachers reflect on classroom video-clips and student products stemming from their teaching experi-ments with the jointly prepared whole class enrichment settings (Rösike & Schnell, in press). The preparation includes their own mathematical inquiries as well as anticipat - ing students' ideas. The typical structure of the PD program is visualized in Figure 2.

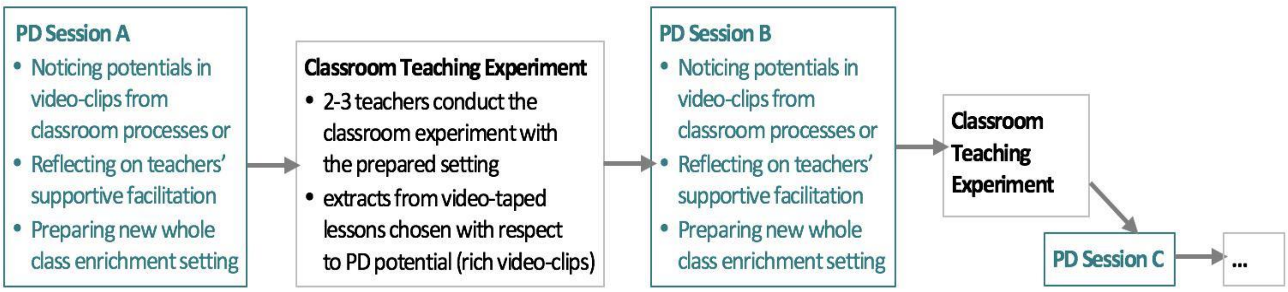

Fig. 2: Structure of the PD program with sessions and intermediate classroom experiments

Background. The general PD content *noticing* has been characterized in several re-search studies: They emphasize the need for teachers to overcome deficit-oriented modes on students and the necessary shift from product- to process-oriented perspec-tives (Empson & Jacobs, 2008). By the construct of professional vision, Sherin and van Es (2009) conceptualize noticing by three subconstructs, (I) selective attention, (II) knowledge-based reasoning underlying the actions and (III) interpreting specific events in terms of broader pedagogical principles.

In the specific case of noticing students' mathematical potentials with a dynamic and participatory conceptua-lization of potential, all three subconstructs are impor-tant. For uncovering hidden potentials, the process per-spective in a non-deficit-oriented mode is hypothesized to be an important precursor for extending the selective attention and widening the repertoire of possible actions (cf. Fig. 3 for the intended professionalization trajectory which corresponds to a hypothetical learning trajectory in other design research approaches, cf. Prediger et al., 2015a).

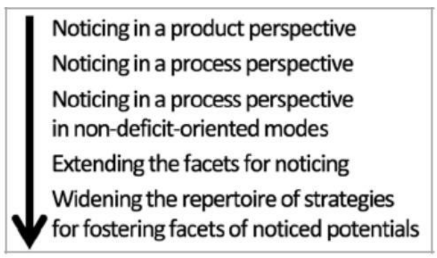

Fig. 3: Intended profession-nalization trajectory

2.2 Project design in three iterative cycles with mini cycles

Overall project design. The DoMath PD program is developed and investigated in an ongoing PD design research project in the described approach (cf. Section 1) from 2014 to 2018. Three iterative cycles of design experiment series are conducted in 2014/15 (with 5 teachers in 6 PD sessions over 12 months), 2015/16 (with 20 teachers in 6 PD sessions over 10 months) and 2016/17 (planned with 20 teachers in 2-3 longer PD ses-sions over 6 months). Between the PD sessions of one design experiment cycle, mini cycles of investigating processes allow immediate refinement of the program. During the mini-cycles of the first two cycles, the relatively vague intended trajectory matured into a more detailed specification of a model for noticing students' potentials (Schnell & Prediger, 2016). Later, this refinement of the underlying content-specific theory will allow pursuing the long-term aim to develop a PD course for scaling up with facilitators within the DZLM.

Methods for data gathering. Most classroom teaching experiments and all PD sessions are videotaped, as well as some individual video sessions between the third author and 1-3 teachers each. The individual video sessions complement the data from group discussions during PD sessions as they allow deeper insights into the individual professional visions.

Methods for data analysis. Based on the sensitizing subconstructs of professional vision (I – III) and theoretically derived facets for identifying potentials (in Fig. 3), the interpretative methods for analyzing transcripts from the video data aimed at developing a category system (1) for specifying demands and challenges in teachers' noticing and (2) for reconstructing individual pathways in professionalizing the noticing. The excerpts presented here stem from the ongoing analysis of professionalization contents and pathways of design experiment cycle 2 and are based on 33 hours of video material (13 h PD sessions, 16 h of their classroom interactions, 4 h individual discussions of video).

2.3 Exemplary insights into teacher's diagnostic perspectives

Effect of the design element Video. As described by others (e.g. Sherin & van Es; 2009; Empson & Jacobs, 2008), analyzing videos in the PD sessions turned out to be a design element which successfully initiates the shift from product perspectives (focusing only on the outcome of student work) to process perspectives (focusing on the richness of processes even if the outcomes stay incomplete, cf. Fig. 3). The shift also seems to be stable in the teachers' classrooms.

Effect of the design element Focus Question. As the teachers of the first mini cycles kept deficit-oriented modes for a long time, we revised the program starting by analyzing videos with a focus question "What kind of potentials can you discover in the processes of the students?". The effect of the focus question was substantial in the second cycle: from the first PD session on, the second teacher group adopted a process perspective in mainly non-deficit modes.

Accounting for obstacles and teachers' perspectives. However, the process perspective did not automatically lead to focusing hidden potentials and searching for strategies to foster them. Instead of thinking about strategies to foster uncovered seeds of situational potential, the teachers showed and discussed mainly strategies to help students to solve the open-ended problems. In consequence, the noticing mainly focused on students' processes of coping with the task (or why they could not cope well). This can be illustrated by the following excerpts of data:

After watching a video clip of two female students working on an open task about several derivatives (grade 12), Sonja, one of the video-watching teachers in the third PD session, says

Sonja Where they have problems is with verbalizing what they found out – especially mathematically correct verbalizing. So, I think they did understand the principle, but [… not the relevant pattern behind it].
And well, you have to justify or formulate it in a more differentiated way.

(Cycle 2, PD Session 3, Clip 'Derivatives', transcript line 78, min. 16:48)

Within her analysis of the video-clip, Sonja points out what the girls would have needed to accomplish the problem. She emphasizes what they reached and the discursive obstacles they need to overcome. Sonja' perspectives is an instance of what we researchers later decided to call the *process-coping perspective* (see below): Although already overcoming purely deficit-oriented modes and focusing on pro-cesses, Sonja does not yet focus on potentials. As our teachers often adopt this perspective, we needed to include it into the model and consider it as rational choice, since it is teachers' responsibility to support the students in coping with the task (or their acquisition of competences or knowledge). Hence, it is also a direct successor of the product perspective.

The process-coping-perspective often coexists with the *potential indicator perspective* which we have reconstructed when the teacher implicitly poses her- or himself questions like "Which situational indicators for students' potentials can we identify?". For example, the teacher Stephanie analyzed a video clip of four students (grade 8) working on a problem-solving task

Steph That is really a good way of abstraction. They generalize very well at this point. Also, how they stay at it. They know now, they have the odd numbers and now they think about how to adjust the stairs [of numbers]. […] Thus, they communicate well with one another and then generalize really well. There is a lot of potential.

(Cycle 2, Individual discussion of video clip 'Stair problem', transcript line 45, min. 18:42)

Stephanie also reconstructs steps in the coping perspective, but beyond that, she identifies students' way of abstraction as an indicator for their mathematical potential. At the same time, the way she and some colleagues talk about the students signals that she conceives potential here as students' stable disposition rather than a dynamically emerging and disappearing moment in the situation which requires teacher's efforts to be stabilized.

It was a longer discussion in the research team to reconstruct the backgrounds for these observed obstacles. After having re-analyzed also other transcripts, we realized the need to differentiate the process perspective which is still too vague in the hypothesized learning trajectory (Fig. 3). The result of several reconstructions and discussions was a refined perspective model (Fig. 4) which allows to take into account the teachers' perspective and to structuring of the PD content which was not adequately grasped by the earlier learning trajectory in Fig. 3 (cf. Schnell & Prediger, 2016).

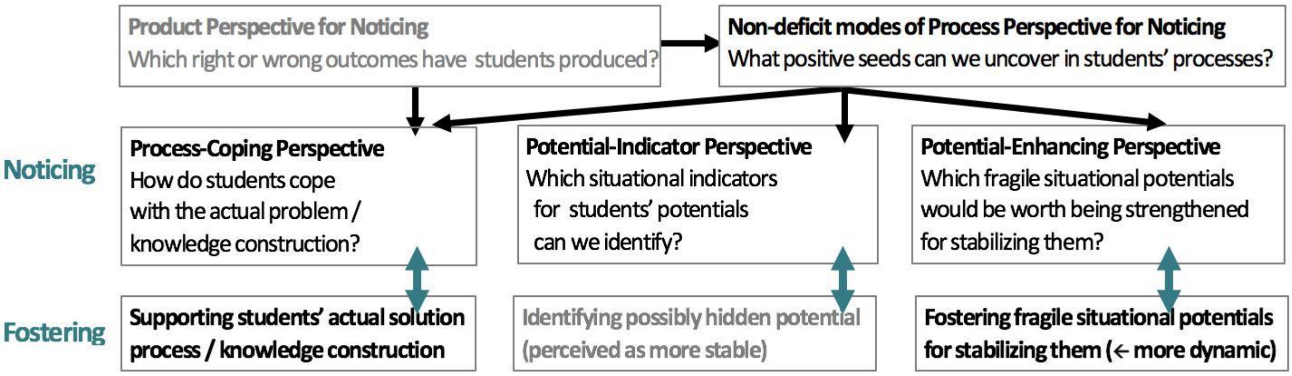

Fig. 4: Refined structure of PD content: Perspective model for noticing and fostering potentials

The last perspective, at which the PD programs aims, is now called the *potential-enhancing perspective*, asking for fragile situational potentials which would be worth to be strengthened in order to stabilize them. This perspective would allow fostering potentials, but in the beginning, teachers rarely adopt this perspective. A condensed fictional prototype of this focus of selective attention would be:

Teach That is really a good way of abstraction, they generalize very well at this point. I tell them how this as a brilliant approach. Hoping, they get used to doing it more often.

Some teachers, especially Henry, can adopt the *potential-enhancing perspective*, and even explain what he should NOT do in order to foster the situational potential:

PD leader [...] Would you have liked to give them an impulse, if you would have been there?

Henry Yes, I do find it great. So I noticed for myself that it works quite well even if I don't give any prompt. I notice that I, as teacher, would have quickly felt the need to say 'oh, look here, what happens here? The three here.' And now I think you sometimes give them too little time so that they can unfold their ideas in peace. That it needs a lot of time […] Because I find they gave the right impulses themselves.

(Cycle 2, Individual discussion 2 of video clip 'Stair problem', transcript line 79-80, min. 14:55)

In total, the refined model specifically includes the following observation: what teachers selectively notice is highly connected to what they intend to foster: As long as the main goal is supporting students' actual processes of working on a given task, it is rational to stay in a process-coping perspective (cf. Fig. 4). The potential-indicator perspective looks at indicators for students' existing potentials displayed in a certain situation. While it is important in our teaching approach, it cannot help fostering students when potentials are perceived as pre-existing and more stable dispositions. In contrast, sensitive strategies for fostering (still fragile) situational potentials in order to stabilize them in the long run require a potential-enhancing perspective of noticing. It is this perspective which teachers adopt the least often in the beginning of the course and successively learn to adopt during the discussion of fostering strategies. Rather than linear, teachers' navigation during the professionalization process is forward and backward, since they need to coordinate different perspectives at the same time.

2.4 Exemplary design results and research results

By the case of the DoMath project, we can exemplify typical design and research results of typical PD design research projects as listed in Fig. 1.

Research results. Although the existing literature provided consolidated knowledge of the general structure of teachers' noticing and general pedagogical principles for enhancing them (Sherin & van Es, 2009; Blomberg, Renkl, Sherin, Borko & Seidel, 2013), little was known about the specific content, noticing students' hidden mathematical potentials based on our dynamic and participatory conceptualization of potential. Thus, the empirical research on teachers' processes was necessary to itera-tively refine a local theory on this PD content and individual pathways to approaching it. First research results are condensed in the *perspective model for noticing potentials* (cf. Fig. 4). It provides a content-dependent language for describing typical professio-nalization pathways and obstacles. Of course, the reconstructed insights into effects of specific design elements like focus questions are not yet generalizable, their transferability to other contents should be investigated in further research.

Design results. The research results on effects of specific design elements have itera-tively influenced the design of PD sessions within the mini cycles and between the big cycles. However, we have only achieved first steps for the long term goal of designing a *PD program with robust materials* that can be used for scaling up, i.e. for facilitators who have not joined our programs themselves. For this purpose, the theoretical founda-tion is crucial, and in this sense, the *specification and structure of the PD content* based on the perspective model is also an important design result which will guide a manual for facilitators. With respect to pedagogical *design principles*, the project has mainly confirmed existing work (e.g. Blomberg et al., 2013) and found content-specific ways for their realization, a design result which is far from trivial.

3. DISCUSSION

Although design research *with* teachers on the student level is an excellent setting for professionalizing *some* teachers, this paper pleads for extending the approach for reaching *many* teachers. In the presented approach, design experiments take place in PD sessions, not in classrooms alone. PD design research adds to usual PD program development a much more intense, video-based analysis of teachers' professionalization pathways during and between the PD sessions, by own teaching experiments and their video-based reflection in small groups. The reconstruction of teachers' individual professionalization pathways allows gaining profound insights into the structure of the PD content: in our case, the process perspective had to be split for understanding teachers' pathways (cf. perspective model in Fig. 4).

Like every analysis of individual learning pathways, such an analysis has always the risk to be deficit-focused, devaluing the perspective of the learning teachers. Thus, systematically taking into account the teachers' perspectives and its inner logic invites to search for a synthesis between teachers' and intended perspectives and helps to overcome the risk of deficit-orientation (Prediger et al. 2015b). In our case, we had to accept the process-coping perspective as a natural and important perspective which should coexist with the potential-enhancing perspective.

The methodological control of the interpretative data analysis procedures is paramount for achieving profound empirical results. This means respecting the quality criteria of transparency, intersubjectivity and openness for phenomena outside the theoretical input. However, quality criteria in design research go beyond these classical methodological criteria, as they also comprise relevance and practicability of the design, generalizability of the results by accumulating over several projects and ecological validity of the complete setting (Cobb et al. 2003). For the concrete project, the generalizability of the research results is not yet achieved since the process is only at the beginning. However, its preliminary results are encouraging to pursue this aim.

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Pre-service teachers' learning to notice students' fractional thinking: The design of a learning environment through a Learning Trajectory

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The skill of noticing has been identified as an essential component that expert teachers must acquire. Therefore, research about how teacher educators can design learning environments to support the development of this skill has emerged. In our research, we focus on how pre-service teachers learn to notice students' fractional thinking. In this paper, we describe a learning environment designed to promote the development of preservice teachers' noticing, taking into account a student' Learning Trajectory of fractional schemes.

Keywords: noticing, fractional schemes, learning trajectories, pre-service primary teachers' learning.

THEORETICAL BACKGROUND

The NCTM (2000) has claimed the need to base teaching on students' thinking. In order to achieve this goal, teachers need a greater flexibility in recognising students' mathematical thinking and students' learning progressions. In this sense, the skill of noticing has been identified as a critical component of mathematics teacher expertise (Sherin, Jacobs, & Philipp, 2011) emerging research issues about how teacher educators can design learning environments to support the development of this skill (Wilson, Mojica, & Confrey, 2013).

The skill of noticing

Mason (1998; 2002) stated that "noticing is a movement or shift of attention" (Mason, 2011, p. 45). Mason distinguished between *accounting of* a phenomena and *accounting for* it. *Accounting of* a phenomena implies a neutral description "as objectively as possible minimizing emotive terms, evaluation, judgements, and explanation" (p.40) while *accounting for* implies to "offer interpretation, explanation, value-judgement, justification, or criticism" (p. 41) of this phenomena.

Mason (2011) identified different ways in which people can attend (p.47):

Holding wholes implies attending to something without discerning details.

Discerning details is picking out bits, discriminating this from that, decomposing or subdividing and so distinguish and, hence, creating things.

Recognizing relationships is becoming aware of sameness and difference or other relationships among the discerned details in the situation.

Perceiving properties is becoming aware of particular relationships as instances of properties that could hold in other situations.

> *Reasoning in the basis of agreed properties* is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms.

"In mathematics, the shift from recognizing relationships to perceiving properties is often subtle but immediate for experts and yet an obstacle for students" (Mason, 2011, p.47). This perspective emphasises the importance of identifying the relevant aspects of the teaching-learning situations (discerning details) and interpreting them (recognising relationships) to support instructional decisions (reasoning in the basis of agreed properties).

Previous research has shown some characteristics of pre-service teachers' development of this skill using video clips (Coles, 2013; van Es, & Sherin, 2002), or using classroom artifacts (Fernández, Llinares, & Valls, 2012; Sánchez-Matamoros, Fernández, & Llinares, 2015). For example, the use of video clips enables teachers to rebuild classroom interactions in chronological order (accounts of) for later interpreting them providing evidence (accounts for) without judgement (Coles, 2013), and supports changes in pre-service teachers' level of reflection. Using classroom artifacts also helps pre-service teachers to recognise and use the mathematical elements for identifying students' mathematical thinking (Fernández et al., 2012; Sánchez-Matamoros et al., 2015).

Our study is embedded in this line of research and focuses on the development of pre-service teachers' noticing of students' mathematical thinking. Research has shown that when pre-service teachers attend to students learning progressions in a mathematical domain, they are better able to make decisions about next instructional steps (Son, 2013; Wilson et al., 2013). In this context, students' learning trajectories (Battista, 2012) can assist pre-service teachers in identifying learning goals for their students, in anticipating and interpreting students' mathematical thinking and in responding with appropriate instruction (Sztajn, Confrey, Wilson, & Edgington, 2012).

Particularly, the focus of our research is how pre-service teachers learn to notice students' fractional thinking. With this objective, we have designed a learning environment to promote the development of pre-service teachers' noticing, taking into account a students' Learning Trajectory of fractional schemes. In this paper, we describe the learning environment designed.

A Learning Trajectory of fractional schemes

Corcoran, Mosher, and Rogat (2009, in Wilson et al., 2013) postulated that learning trajectories and the instructional processes are linked and deserve attention since learning trajectories:

> … provide teachers with a conceptual structure that will inform and support their ability to respond appropriately to evidence of their students' differing stages of progress by adapting

> their instruction to what each student needs in order to stay on track and make progress toward the ultimate learning goals (p.19).

Previous research has shown that pre-service teachers' knowledge (Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Mojica, 2010) of students' learning trajectories allows them to have into account students' mathematical thinking when taking instructional decisions. A Learning Trajectory consists of three components: a learning goal, learning activities, and a hypothetical learning process (Battista, 2011). A Learning Trajectory includes descriptions of learning activities that are designed to support students in the transition through intermediate stages to a more sophisticated level of reasoning.

The Learning Trajectory of fractional schemes, in our study, has been characterised taking into account empirical studies of how student's reasoning about fractions develops over time (Battista, 2012; Steffe, 2004; Steffe, & Olive, 2009). While schemes is a construct that is used with the aim to model students' cognitive structures, operations are seen as "mental actions that have been abstracted from experience to become available for use in various situations" (p. 46) and are considered as the key components of schemes (McCloskey, & Norton, 2009). The operations considered in our Learning Trajectory of fractional schemes are: unitizing, partitioning, disembedding, and iterating. We characterise the Learning Trajectory considering these operations, the configuration of the schemes that children develop in the field of rational number reasoning proposed by Steffe (2004) and the development of students' reasoning about fraction through levels of sophistication proposed by Battista (2012).

The learning goal of the fractional schemes Learning Trajectory is derived from the Spanish Primary Education's curriculum: giving meaning to the idea of fraction and its different representations and, understanding the meaning of fractions operations. This learning goal highlights two key aspects to achieve: *a)* the transition from the intuitive meaning of splitting into congruent parts to the idea of fraction as part-whole taking into account different representations, and *b)* the construction of the meaning of operations with fractions.

In relation to **students' learning process**, we consider six different levels of students' fractional reasoning based on the following mathematical elements. In relation to the transition from the intuitive meaning of splitting into congruent parts to the idea of fraction as part-whole: (i) the parts are congruent. The parts could be different in form but congruent in relation to the whole, (ii) a part could be divided into other parts, (iii) consider a part as an iterative unit, (iv) the inverse relationship between the number of parts and the size of each part: a greater number of divisions of the whole makes each part smaller (keeping the same whole).

Related to the construction of the meaning of operations with fractions: (i) the parts must be congruent to join / separate, (ii) repeat a fraction to construct a fraction, "n times a/b", (iii) fraction as an operator "a/b of c/d", (iii) division as a measure "how many a/b are in B” or “how many a/b are in c/d”, (iv) the remainder of a fraction division.

The six levels of students’ fractional reasoning are:

> At level 1, students have difficulties in recognising that the parts in a fraction must be congruent.
>
> At level 2, students recognise that the parts could be different in form but congruent in relation to the whole. This allows them to identify and represent fractions in a continuous context but they have difficulties with discrete contexts. They also begin to use some unit fractions as an iterative unit (i) to represent proper fractions (although they have difficulties with improper fractions) and (ii) to solve some fraction addition problems with the same denominator (although students can have difficulties in considering the relation between the part and the whole to justify the meaning of the fraction addition).
>
> At level 3, students identify and represent fractions in discrete contexts recognising that the groups must be equal. They also recognise that a part could be divided into other parts. When comparing fractions they recognise the inverse relationship between the number of parts and the size of the part. They can use a part (not necessarily the unit fraction) as an iterative unit to represent proper ($f<1$) and improper ($f>1$) fractions. They can also reconstruct the whole using any fraction as iterative units (continuous and discrete contexts). In addition, they use intuitive graphical representations to add/ subtract fractions with different denominators.
>
> At level 4, students can solve simple arithmetic problems with the help of a guide or support. They can obtain equivalent fractions and represent operations graphically. When adding or subtracting fractions with different denominators they understand that the parts must be congruent to join/separate although they need a guide that allows them to choose correctly the unit. They identify the fraction as an operator “*a/b* ***of*** *c/d*” in the multiplication, and they develop two types of reasoning; (i) division as a measure and (ii) division as a partition, in the division.
>
> At level 5, students can operate and solve arithmetic problems symbolically, identifying patterns. They can graphically justify what they do but in simple situations. At this level, they are capable of interpreting the remainder on a division of fractions.
>
> At level 6, students understand how algorithms for fraction operations work and can use pictures to explain the operations. They do not need a guide to represent fraction operations.

THE DESIGN OF THE LEARNING ENVIRONMENT

The learning environment consists of five sessions of 2 hours each. In the first two sessions, we introduce the mathematical elements, and pre-service teachers solve and analyse primary school activities such as representing and identifying proper and improper fractions in a continuous or discrete context. The aim is that pre-service teachers identify the mathematical elements in the fraction tasks.

The aim of the other three sessions is that pre-service teachers learn to interpret students' mathematical thinking based on the Learning Trajectory and to propose instructional actions in relation to students' mathematical thinking. We provide pre-service teachers with primary school students' answers with different levels of fractional reasoning to three different fraction tasks (identifying fractions, comparing fractions, and adding fractions). Pre-service teachers have to interpret students' fractional reasoning using the information of the Learning Trajectory of fractional schemes.

We have designed the three tasks following the same structure: a primary school activity, three primary school students' answers with different level of fractional reasoning and the next four questions (Figure 1).

C1- Describe **the task** taking into account the learning objective: what are the **mathematical elements** that a student needs to solve it?

C2- Describe **how each pair of students has solved the task identifying** how they have used the *mathematical elements* involved and the difficulties they had.

C3- What are the characteristics of students' mathematical thinking (Learning Trajectory) that can be inferred from their answers? Explain your answer.

C4- How could you respond to these students? Propose a **learning objective and a new activity** to help students progress in their understanding of fractions.

Figure 1. Questions that pre-service teachers have to answer related to each task

These questions focus pre-service teachers' attention on relevant aspects of primary school students' answers (discerning details) identifying the relevant mathematical elements; on interpreting these answers (recognising relationships between the mathematical elements and the students' mathematical thinking) and on supporting instructional decisions (taking into account the students' mathematical thinking). Following, we describe one of the three tasks of the learning environment. The task that corresponds with identifying fractions (task 1).

Task 1- Identifying fractions

The task consists of an identifying fractions activity (Figure 2), the answers of three pairs of primary school students to the activity and the four questions of Figure 1:

1. Choose the figures below that show ¾. Explain your answers.

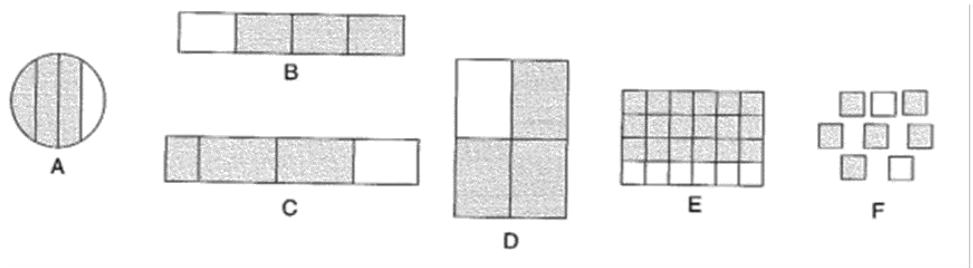


Figure 2: Activity of identifying fractions (adapted from Battista, 2012)

While students are solving the activity Júlia (the teacher) observes how the different groups are solving the activity. Júlia realises that students are using mathematical elements of fractional schemes allowing her to identify students who have difficulties.

Xavi and Victor's answers

Júlia: *Please, Xavi and Víctor, what is your answer?*

Víctor: *Mmmm, well we think Figures A, B, C and D represent three-quarters.*

Júlia: *Xavi, do you agree with Víctor?*

Xavi: *Yes, A, B, C and D are divided in 4 parts, and 3 are shaded.*

Júlia: *Is everyone okay?*

Joan and Tere's answers

Joan: *No*

Júlia: *What do you think?*

Tere: *We believe that Figures B and D are three quarters because they are divided into four equal parts and three are shaded. Figures A and C have 3 parts of 4 shaded, but the parts are not congruent...*

Júlia: *And Figure E? What do you think about Figure E?*

Joan: *Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded.*

Tere: *Sure, it is not three-quarters.*

Júlia: *And the F?*

Both *It is not a fraction. In figure F, there are only 6 shaded squares.*

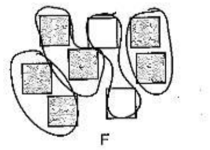
Felix and Alvaro's answers

Júlia: *Do you agree with the answer of Joan and Tere? Is there anyone who has a different answer? Félix and Álvaro, what do you think?*

Félix: *Well ... yes. We agree with Joan and Tere answers related to figures A, B, C, and D but we think differently about figure E...*

Júlia: *What do you think? Could you explain your answer?*

Álvaro: *Well ... mmm sure. If you look each line of Figure E, each line has 6 squares, and as there are 3 lines shaded of the 4 total lines then it is three quarters. In addition, ... Figure F also represents three quarters because if you group the squares in groups of 2, you get 4 groups of 2, and there are three groups shaded.*



Álvaro and Félix answer to Figure F

In the next section, we discuss the answers we expect from pre-service teachers to each question.

Answers that we expect from pre-service teachers to task 1- Identifying fractions

C1- *Describe **the task** taking into account the learning objective: what are the **mathematical elements** that a student needs to solve it?* In this task, pre-service teachers should identify the following mathematical elements:

This is a task of identifying proper fractions (f <1) that includes various representations of the whole (continuous contexts: a circle and a rectangle, and a discrete context: little squares). The parts, in figures A and C, are not congruent but are congruent in figures B and D. These figures are included to determine the students' understanding related to the mathematical element: the parts have to be congruent. On the other hand, the inclusion of figures E and F, provides the possibility that students mobilise the idea that a part could be divided into other parts (the idea of equivalence of fractions).

C2- *Describe **how each pair of students has solved the task identifying** how they have used the mathematical elements involved and the difficulties they had.* Pre-service teachers should identify that:

Xavi and Víctor identified as ¾ the first 4 figures (A, B, C, D). Therefore, they considered B and D as representations of 3/4. This suggests that they counted the parts in which the whole is divided and then counted the shaded parts. They did not take into account that the parts in which a whole is divided must be congruent (when they considered figures A and C). Furthermore, they did not identify E and F as 3/4, indicating that they had difficulties in considering that a part could be divided into other parts (equivalence). The answers of Xavi and Victor show that they considered fractions as ¾ as counting the number of parts in which the whole is divided and then counting the shaded parts, regardless of the congruence of the parts. Furthermore, they did not understand that a part could be divided into other parts. These characteristics indicate that Xavi and Víctor are at level 1 of the Learning Trajectory.

Joan and Tere used properly the idea that the parts should be congruent (considering that figures A and C are not ¾). Their answer in relation to figure E (Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded) and figure F indicates that they did not consider the mathematical element that a part could be divided into other parts. The answers of Joan and Tere show that they recognised that the parts in which the whole is divided must be congruent in continuous contexts, but still did not recognise that a part can be divided into other parts (idea of equivalence) indicating that they are at level 2 of the Learning Trajectory.

Finally, **Álvaro and Félix** used the idea that the parts should be congruent (they did not consider A and C as ¾) and their answer in relation to figures E and F indicates that they considered that a part could be divided into other parts.

C3- *What are the characteristics of students' mathematical thinking (Learning Trajectory) that can be inferred from their answers? Explain your answer*. Taking into account the mathematical elements identified in each pair of students' answers

Students / Mathematical elements	Víctor & Xavi	Joan & Tere	Félix & Álvaro
The parts should be congruent	NO	YES	YES
A part could be divided into other parts	NO	NO	YES

Pre-service teacher could identify the next characteristics of the Learning Trajectory for Álvaro and Félix:

The answers of Álvaro and Félix show that not only they were able to recognise that the whole should be divided into congruent parts but also they acknowledged that a part could be divided into other parts. This last characteristic allows them to recognise equivalent fractions in both continuous and discrete contexts, indicating that these students are at level 3 of the Learning Trajectory.

C4- *How could you respond to these students? Propose a* ***learning objective and a new activity*** *to help students progress in their understanding of fractions*. Pre-service teachers have to identify the characteristics of the transition in the Learning Trajectory to propose the new activity and the learning objective. For instance, they could propose the next learning objective for Xavi and Victor:

Xavi and Victor are at level 1 of the Learning Trajectory so they should start to recognise that the parts could be different in form but congruent, and begin to use unit fraction as an iterative unit.

FINAL REMARKS

We designed this learning environment to develop pre-service teachers' noticing of students' mathematical thinking in the domain of fractional schemes. We used a Learning Trajectory of fractional schemes as a theoretical reference. Our purpose is to provide pre-service teachers with different students' answers to let them to frame practical situations through the cognitive processes of attending and interpreting the students' mathematical thinking. We hypothesise that this kind of knowledge will allow pre-service teachers to move from evaluative comments, based on the correctness or incorrectness of the students' answer, to more interpretive comments based on evidence taking into account the characteristics of the important mathematical elements evidenced in the students answers. Furthermore, it allows them to provide instructional activities coherent with how students are thinking.

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Designing a web-based professional development toolkit for supporting the use of dynamic technology in lower secondary mathematics

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Cornerstone Maths was designed to support wide-scale student learning of key mathematical concepts using dynamic digital technologies. Moving the project to scale (over 150 schools) has necessitated a rethinking of the design of the professional development component to provide more appropriate support for 'within school' implementation and for scaling among all the teachers of mathematics in a school. We report the outcomes of the first phases of design research through which we have used our empirical research to inform the design of a web-based 'Cornerstone Maths Professional Development Toolkit' created to achieve the afore-mentioned goals, describe some preliminary findings in terms of its use by different teachers and set out our plans for the future.

Keywords: lower secondary mathematics, dynamic mathematical technology, professional development, landmark activities, mathematical pedagogic practices

INTRODUCTION

The context of a longitudinal project in England, *Cornerstone Maths*, which aims to support wide-scale student access to dynamic mathematical technologies to enhance mathematical understanding of 'hard-to-teach' topics in lower secondary mathematics, has necessitated a highly connected approach to the three important themes of the conference: mathematics teaching; resources; and professional development. The evolution of these three elements has been central to the design research methodology that led to the definition of the *curriculum activity system* that comprised: dynamic web-based software; student workbook and teacher guide; and teacher professional development (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Vahey, Knudsen, Rafanan, & Lara-Meloy, 2013). The outcomes of this earlier work have been widely reported (Clark-Wilson, Hoyles, Noss, Vahey, & Roschelle, 2015; Hoyles, Noss, Vahey, & Roschelle, 2013). Although there is evidence of successful scaling of the Cornerstone Maths teaching approaches in particular school settings, as our focus has shifted to try to understand and theorise on the 'products and processes of scaling' (Clark-Wilson, Hoyles, & Noss, 2015), our research lens is now trained firmly on the nature of the specific mathematical knowledge for teaching that underpins classroom implementations of Cornerstone Maths in ways that retain fidelity to the original design principles. Furthermore, our extensive classroom observations are enabling an articulation of the nature of teachers' mathematical knowledge for teaching (MKT) and associated mathematical pedagogic practices (MPP) of teachers as they develop both confidence and competences in their classroom uses of the technology with their students.

THEORETICAL FOUNDATIONS

In keeping with the three themes of the conference, this section summarises the theoretical foundations of the current research and its focus on a need to better understand the important components of teachers' MKT and associated MPP when designing professional development to support teachers to work with dynamic mathematical technologies in classrooms.

Mathematics teaching with technology

The design principles of the Cornerstone Maths curricular activity system are deeply rooted in a number of seminal research projects, through which the efficacy of the teaching approaches with technology were explored and established. The three curriculum units address the following topics:

- **Unit 1 Linear functions**. Drawing on the seminal research of Jim Kaput (Kaput, 1987; Tatar et al., 2008), the unit addresses the following key mathematical ideas: coordinating algebraic, graphical, and tabular representations of linear functions; $y = mx + c$ as a model of constant velocity motion; the meaning of m and c in the motion context; and velocity as speed with direction.
- **Unit 2 Geometric similarity**. The use of sliders to explore multiple instantiations of geometric figures within dynamic environments (Hollebrands, Laborde, & Sträßer, 2008) is central to the design of the unit, which addresses: identifying variants and invariants in shapes that are mathematically similar, including identification of the scale factor of enlargements and the particular conditions for congruency; and recognising the important one-to-one geometric correspondence of sides and vertices in mathematically similar polygons.
- **Unit 3 Algebraic patterns and expressions**. The ESRC/EPRSC-funded MiGen project[1] developed the microworld, 'eXpresser' and researched its impact on students' understanding of algebraic variable and generalisation within the context of geometric patterns (Mavrikis, Noss, Hoyles, & Geraniou, 2013). This software and tasks informed the design of the Cornerstone Maths unit, which addresses: recognising the geometric structure of algebraic patterns (seeing the general in the particular); naming and linking variables; and modelling algebraic equivalence through the different ways of seeing a pattern.

Each unit of work includes between 2-4 weeks of curriculum work, which schools implement as 'replacement units' within their localised 'scheme of learning'.

Conceptualising mathematics teachers' mathematical knowledge and practices with technology – the landmark activity

In our work, we use Thomas and Palmer's 'Pedagogical Technical Knowledge' (PTK), as a frame that incorporates 'the principles, conventions and techniques required to teach mathematics through the technology' (Thomas & Palmer, 2014, p. 75). PTK combines teacher factors such as instrumental genesis (Artigue, 2002; Guin & Trouche, 1999; Verillon & Rabardel, 1995), mathematical knowledge for teaching (MKT, Ball, Hill, & Bass, 2005) and teachers' orientations and goals (Schoenfeld, 2008) in a model as shown in Figure 1.

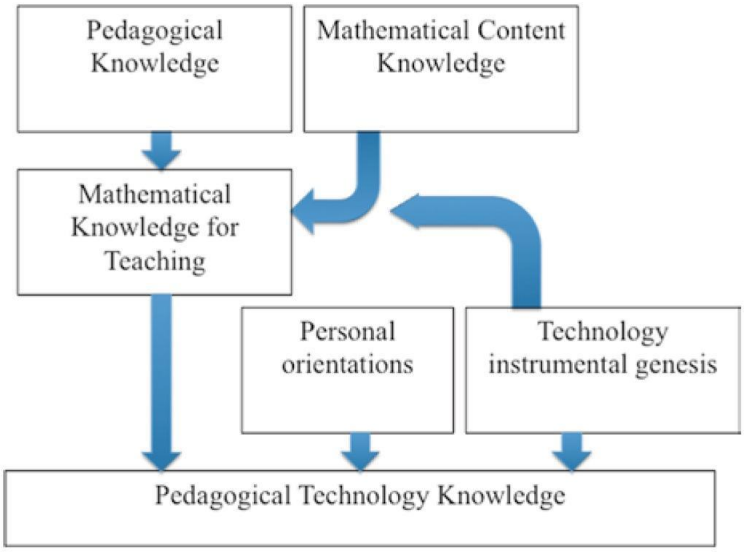

Figure 1: A model for the framework of PTK (Thomas & Palmer, 2014, p. 76)

Important to us was that PTK acknowledges teachers' personal orientations and the epistemic value of the tool, two elements that are absent in alternative frameworks. For example, in TPACK (Koehler & Mishra, 2009), teachers' technological knowledge is conceptualised as separate from the learning of the subject, drawing from the 'Fluency of Information Technology' as its theoretical base. Clearly teachers do need skills other than those around mathematical learning (for example classroom management with technology and some basic appreciation of the laptop, tablet or accessing the web) but this is not our main concern.

However, as we began to apply Thomas and Palmer's PTK in our work, we became interested in how it might be used to make sense of, and characterise teachers' mathematical pedagogic practices (MPP) with technology in classrooms. For this we looked to the work of Selling, Garcia and Ball (2016) who, in research to develop a framework for the design of items to assess teachers' MKT, have defined the 'mathematical work of teaching framework' (MWT) as a set of 'actions with and on objects' that relate to: 'mathematical representations; structure and explanations (including justifications and reasoning); and explanations (includes justifications & reasoning)' (ibid, p. 87).

We adapted this framework to take account of the use of digital tools and to devise the following set of pedagogic practices that we could use to both analyse classroom observation data and to inform the design principles for the Cornerstone Maths Professional Development (PD) Toolkit (See Table 1).

	Mathematical Pedagogic Practices with Technology
Explanations (includes justifications & reasoning)	**Comparing explanations that involve hypothesised or real actions as expressed with the digital tools** to determine which is more/most valid, generalisable, or complete explanation. **Critiquing explanations that involve hypothesised or real actions as expressed with the digital tools to improve them** with respect to completeness, validity, or generalisability.
Mathematical structure	**Determining, analysing, or posing problems as expressed with the digital tools** with the same (or different) mathematical structure. **Analysing structure in students' technological work** by determining which strategies or ideas are most closely connected with respect to mathematical structure. **Matching investigations with structure** as expressed by the digital tools. **Connecting or matching representations** as expressed with the digital tools.
Representations	**Analyzing representations** by identifying correct or misleading representations in a text, talk, written and technological work. **Selecting, creating, or evaluating different representations** as expressed by the digital tools. **Verbalising the meaning of representations** as expressed by digital tools and how they are connected to key ideas.

Table 1: Mathematical Pedagogic Practices for teaching mathematics with dynamic technology.

Scaling teachers' access to professional development

Our earlier work used research findings from the scaling of Cornerstone Maths in hundreds of English mathematics classrooms to articulate the 'processes and products of scaling' (See Table 2). By products, we mean the quantifiable measures that indicate the 'spread' of the Cornerstone Maths innovation across and within schools. The 'processes', or the means through which this spread is achieved, are both contextually and culturally located, with each process interpreted differently depending on the prevailing mathematical culture in classrooms and associated institutional factors.

Theme	Products	Processes
Geographical reach	a) Number of schools involved	a) Development of web-based curriculum activity system. b) Development of teacher community.
	b) Number of local hubs involved	c) Development and maintenance of regional hub-based offer of professional support. d) Development of school clusters, supported by project team leading to development of local hubs with local Cornerstone Maths project lead.
School buy-in	c) Improved student attainment	e) School-devised methods to evaluate students' outcomes
	d) Number of whole departments involved	f) Development of school-based PD. g) Support to embed Cornerstone Maths within local of schemes of work.
	e) Wider use of the materials	h) Teacher use of the materials beyond their original project commitment. (e.g. with older classes or revision classes).
Penetration in mathematics department	f) Number of participating teachers in each school	i) Development of a lead practitioner (who may be the subject leader). j) Development of peer-support for participating teachers.

Table 2: The products and processes of scaling Cornerstone Maths in hundreds of classrooms in England

Previous phases of Cornerstone Maths research involved PD that was face-to-face and online (asynchronous/synchronous) – focusing on Processes (a), (b) and (c) to achieve impacts related to Products (a) and (c) (Clark-Wilson & Hoyles, 2015; Clark-Wilson, Hoyles, & Noss, 2015).

However, we had research data from one school that evidenced that it had accomplished Processes (f), (g) and (h) to achieve impacts in relation to Products (d) and (e). Consequently, our attention turned to the design of a PD Toolkit that could directly support schools with some experience of Cornerstone Maths to develop their own collaborative, school-based PD to enrol other mathematics colleagues for within-school scaling. Furthermore, the design of the PD Toolkit was informed by prior research into teacher professional development in England that highlighted more effective practices thus:

> One successful approach involves **collaborative communities of practice** of teachers working to enquire into their professional practice. Such communities are often kick-started and sustained **by outside expertise**, provided by maybe a 'trainer' or a university educator. The most successful professional development **pays attention to the development of the subject (mathematics or science), itself** and particularly student learning. (de Geest, Back, Hirst, & Joubert, 2009, p. 38)

In our case the 'outside' expertise was not to come from an outsider to the department, but would be provided by a member of the department, a 'Cornerstone Maths champion', who had already participated in Cornerstone Maths PD and, most importantly, taught and evaluated Cornerstone Maths 'landmark activities' with positive outcomes[2]. Thus the toolkit has 'authenticity' in that it supports their co-planning and provides links to their everyday practice, with opportunities to reflect on students' work and classroom activities (See the ZDM Special Issue on 'Evidence-based CPD: Scaling up sustainable interventions', Roesken-Winter, Hoyles, & Blömeke, 2015).

Critical to the design of the Cornerstone Maths Toolkit is the assumption that Cornerstone Maths teachers will be self-motivated to select from its resources to use with colleagues in their own departments. Thus the development of the toolkit is the object of iterative and collaborative design research to address the research question, what (digital) professional development content, activities and structures can best support school-based PD concerning Cornerstone Maths?

METHODOLOGY

The Cornerstone Maths PD Toolkit is a set of diverse web-based resources for secondary mathematics departments to support school-based PD leading to embedding Cornerstone Maths units within the school's localised schemes of work. Our focus for this paper is the design research undertaken to produce the toolkit: the design principles and first description of the toolkit. We focus on the Cornerstone Maths Unit 3 on algebraic patterns and expressions.

Our design research methodology involves the following phases:

- Systematic analysis of questionnaire data from all Cornerstone Maths teachers that asked them to outline their current (and anticipated future) PD needs (n= 127).
- A review of other PD toolkits and their design (e.g. mascil[3], FaSMEd[4], EdUmatics[5]).
- Interviews with self-nominating Cornerstone Maths champions (n=9) during which they critiqued and enhanced early PD toolkit designs.
- A case study of one school that had successfully implemented all three Cornerstone Maths units in its localised schemes of work.

- Observations of teachers involved in collaborative PD using Cornerstone Maths Toolkit resources.

FINDINGS

The analysis of teacher questionnaires indicated some common ‘PD needs’, which included:

- Mathematical tasks that supported teachers to reflect on appropriate mathematical content and progression for each of the curriculum topics (developing both mathematical and pedagogical aspects of MKT).
- Short video clips and guidance materials that introduce and support teachers’ instrumental geneses, which includes consideration of how teachers can, in turn support and develop *students*’ instrumental geneses.
- Exemplar students’ digital and paper/pencil productions embedded within professional tasks for teachers.

To date we have completed the first three phases and have designed a draft toolkit that includes a set of resources from classroom practice and student responses derived from landmark activities that provoked ‘transformational’ discussion among teachers and students, alongside more general background to Cornerstone Maths and evidence of its effectiveness. A design challenge is to provide opportunities for participating teachers to develop the composite elements of PTK in engaging and meaningful ways. Whilst we chose not to make these elements explicit within the PD toolkit design, we have mapped the elements that are specific to each of the Cornerstone Maths topics, software and teaching materials. For example, the definition and linking of algebraic variables is fundamental knowledge to support teachers’ technology instrumental geneses within Cornerstone Maths Unit 3.

From the work to date, we conjecture that that opportunities for (and perceptions of) collaborative, departmental-based PD vary from school to school due to a range of factors derived from different sources; the overall structure of the school, the experience of the subject leaders to name but two. We intend to probe these factors further in case studies in a schools selected according to their different profiles in order to tease out which resources teachers select from the Cornerstone Maths PD Toolkit to use in their departmental PD and why, and, ultimately the success or not of any in school scaling. We anticipate reporting some tentative findings at the ERME conference in October 2016.

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NOTES

1. The MiGen project was funded by the ESRC/EPSRC Teaching and Learning Research Programme (Technology Enhanced Learning; Award No: RES-139-25-0381).

2. We use our own construct of 'landmark activities', which as those which indicate a rethinking of the mathematics or an extension of previously held ideas – the 'aha' moments that show surprise – and provide evidence of students' developing appreciation of the underlying concept (This construct is described in more detail in Clark-Wilson, Hoyles, & Noss, 2015).

3. The mascil toolkit "designed to support the delivery or facilitation of professional development for teachers of mathematics and science" http://mascil.mathshell.org.uk/

4. The FASMEd toolkit to support teachers teachers in the use of formative assessment with low achieving students. https://toolkitfasmed.wordpress.com/

5. EdUmatics online PD resource for secondary mathematics to learn to use and integrate technology within their classrooms. http://www.edumatics.mathematik.uni-wuerzburg.de/en/

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Teachers' informal professional development on social media and social network sites: when and what do they discuss?

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Abstract: In recent years, a digitally extended context for teachers' professional learning has arisen. Digital gadgets (smart phones, etc.) alongside the development of social media and social network sites change how people interact and work together, and, hence, teachers initiate and orchestrate their own professional development on the Internet. In this paper we report on an on-going three-year study and show some of the prospects of conducting research on mathematics teachers' informal professional development on social media and social network sites, and, furthermore, discuss the need for theoretical and methodological development.

Keywords: Teachers' professional development, Social media, Facebook, digital.

INTRODUCTION

In communities where digital gadgets (e.g., smart phones, applets, laptops, etc.) are in common use, people change how they work, interact and communicate. The arena for professional development has accordingly transformed and branched out into the Internet. In the literature we now find studies examining social media and social network sites as a means for teachers' professional learning and knowledge-sharing (e.g., Al-Oqily, Alkhatib, Al-Khasawneh, & Alian, 2013; Bissessar, 2014; Borba & Llinares, 2012; Hew & Hara, 2007; Liljekvist, 2014; Manca & Ranieri, 2014; Pepin, Gueudet & Trouche, 2013; Rutherford, 2010; van Bommel & Liljekvist, 2015). The findings show that teachers use different forums on the Internet, such as, Twitter, Web sites, personal blogs, and Facebook, as resources to share and develop pedagogical subject-matter knowledge, to ask for and give pedagogical advice, etcetera. Thus, the arena for professional development of teachers has changed. Teachers not only engage in traditional forms of professional development activities, such as, taking courses, reading books, and participating in the local school colloquium. They also engage in new forms of professional development made possible by the evolution of the Internet. Online courses, web-seminars and other formal professional development are widely spread nationally as well as internationally.

Another trend is also evident: Teachers initiate and orchestrate their own professional development on the Internet. This phenomenon promotes reflection upon social media and social network sites as a means for teachers to regain ownership of their professional development (cf., Issa & Kommers, 2013; Ranieri, Manca & Fini, 2012; van Bommel & Liljekvist, 2015). Further, Issa and Kommers (2013) discuss the shift in educational practice of professional development: from a transfer role into a developmental role. They raise the question of how teachers will reposition themselves into learning communities for mutual learning – a question closely linked to the Call for this ERME Topic conference: What are the characteristics of professional development contexts that have a positive impact on teachers' professional learning? With this paper we want to encourage the ERME Topic Conference ETC3 to discuss the digitally extended context of teachers' professional learning. We argue for the need to know more of what kind of impact social media and social network sites have on, for instance, mathematics teachers' knowledge-sharing, and their meaning-making in relation to improvement of instruction and assessment. Drawing on an on-going three-year study, we show some of the prospects of conducting research on mathematics teachers' informal professional development on social media and social network sites, and, furthermore, discuss the need for theoretical and methodological development.

BACKGROUND

Teachers share their professional life with other teachers, that is, their colleagues. But, the forums on social media and social network sites suddenly give another meaning to who a colleague is and when a conversation with such a colleague can take place: "the current evolution of social media and social network sites transforms the day-to-day practice, the lived experience, and with whom we share similar experience" (Liljekvist, van Bommel & Olin-Scheller, accepted)

Discussing school-related issues together with colleagues is, of course, not a new phenomenon, neither is the reading of subject-related magazines or books, nor taking courses for professional development. However, social media and social network sites give new opportunities for (mathematics) teachers, that is: new sources to draw upon (Ruthven, in press). Courses previously given on a certain day, in a certain place and time, can now be taken online, at your own pace and place. Further, reading to acquire new knowledge can now imply reading books, but also, for instance, reading other teachers' blogs.

Professional development can take place in different ways, forums and arenas. In order to understand these different forms of professional development, including both digital and non-digital alternatives, we argue that it is of importance to take the issue of ownership into consideration. In the case of so-called monologic professional development (book, lecture, etc.) the author, or lecturer has the ownership of the content and form of the professional development whereas in more dialogical forms of professional development (courses, collegial dialogues) the participating teachers gain ownership of content and form (Issa & Kommers, 2013).

Besides the issue of ownership, we need to consider the source itself. Teachers use different sources to customize their professional development, that is, books, (online) courses, blogs. Ruthven talks about re-sourcing and explains that re-sourcing teaching not only should be thought of in terms of 'using conventional resources in new ways' (Ruthven, in press), but also could include drawing on new sources. New resources can incorporate ownership by the author (for instance webinars), but can also mean that teachers gain ownership by creating their own homepage or blog, by giving input in sites for teaching resources (e.g. lektion.se) or by using social network sites like Twitter or Facebook.

As mentioned previously, professional development on social media and social network sites has been studied. However, our point is that we know hardly anything about professional development initiated by teachers themselves, about the informal discussions between teachers. Drawing attention to this lack, the call for the ERME Topic Conference ETC3 states: "We need to better understand the underlying characteristics of mathematics teacher education and the professional development contexts that have a positive impact on teachers' professional learning, even with respect to sustainability". As the previous invisible collegial dialogues now become visible (van Bommel & Liljekvist, 2015; van Bommel, Liljekvist & Olin-Scheller, 2015), the study presented here aims to focus on the non-researched online informal professional development.

THE ON-GOING STUDY

At CERME9, three suggestions for possible foci for our study were discussed. Mapping the arena of professional development of teachers on Facebook, inquiry into the collective knowledge, and issues regarding extended workplace learning (van Bommel & Liljekvist, 2015). All foci where considered to be of interest. As a start of our study we have argued for that these type of Facebook groups can be looked upon as an arena for professional development (Liljekvist, van Bommel, Olin-Scheller, accepted). Teachers use new technology, changing the arena for professional development. However, in this paper we will consider 'mapping the arena', as it is the focus of our three-year research project at this initial point. Mapping the arena was decided upon, as it would give clear insights into the informal professional development within the groups on social media and social network sites: 1) When do teachers discuss with others? and, 2) What do teachers discuss and reflect upon within such groups? We want to emphasize that all groups are initiated by the teachers, not by us as researchers. The overall question suggested at CERME was: In what way can our study inform our research community regarding the (informal) professional development of mathematics teachers?

Setting

Our study is conducted within the social network site Facebook. Studying teachers' communication in [Swedish] Facebook groups is of interest for different reasons. Firstly, teachers create and join groups initiated by themselves. They thereby create an informal form of professional development where they themselves have initiated and formed the content of the professional development (Bissessar, 2014; Liljekvist, 2014; van Bommel & Liljekvist, 2015) A second reason to focus on the informal professional development on Facebook is the number of participants. Groups can exist with 45 participants up to 30 000 participants. The size of the groups is growing, as well as the number of groups. In Sweden, with around 130 000 teachers in total, such Facebook groups constitute a substantial part of the teacher population and studying this phenomenon is therefore of interest, as the large number of active teachers indicates an enduring professional development on the social network site.

Conducting a study on social network sites requires special attention to ethical issues. It is a matter of maintaining public trust in researchers as well as the possibility to conduct research in online environments in the future. The study has been approved in the local ethics committee of Karlstad University.

When the studied groups on social network sites are large, it is more likely that the members look upon the communication as public, and, hence the topic discussed may not be delicate, as, for example, physical and psychological health and socio-economical personal issues (Ellison & boyd, 2013; Knobel, 2009; Little, 2002; Roberts, 2015). The members of the Facebook groups in the study have chosen a specific domain in which to engage and the theme in the group is not on delicate issues (cf., Roberts, 2015). In this study, therefore, ethical considerations must also be given to how the intervention (i.e., exploring the group activity) per se disturbs the communication pattern, the trust and the evolving norms, the participation pattern, and so on, in the group studied (Ellison & boyd, 2013; Knobel, 2009; van Bommel & Liljekvist, 2015).

Taking the ethical issues into consideration, we have adopted both qualitative and quantitative methods in this study and this is explained and set out in the next section.

When do teachers discuss?

In order to detect when teachers discuss with others on the social network site, a quantitative method was applied. All status-updates were registered during one year (2015). The time of day when teachers posted their status was noted, with a distinction made between working and non-working hours. Holidays, weekends etc. were included as far as possible.[1] Furthermore, boundaries for working hours and non-working hours had to be set. We decided that working hours were between 08.00 and 17.00.

Arguments for which groups to choose were guided by two principles: The Facebook groups had to be in mathematics or Swedish, representing the two largest subjects at school. Furthermore, the groups should be large, with more then 2 000 members. All status-updates during one year (2015) were registered for all selected groups. In this paper the focus will be on the Facebook groups in mathematics.

In order to answer the question when teachers discuss, three different timeframes are of importance: (i) global frame – on a yearly basis, (ii) local frame – on a weekly basis, and, (iii) micro frame – on a daily basis, related to working hours in working days. For the micro frame we have to keep in mind that it is not merely the time of day that is of interest. That is, a status posted at 10:17 am on a Saturday is treated differently in this analysis then a status posted on a Thursday at 10:17 am. Below we present some facts about the groups that are under analysis: in total nine groups, with each between 2000 and 11000 members. Accordingly, the number of statuses vary between the groups: between 82 and 3048 statuses.

(i) Globally the posts are spread over the year with some peaks, clearly corresponding to the school calendar. There is a peak in the weeks before the summer holidays. A similar peak is visible in the two weeks before the Christmas holidays. Likewise, the activity is low during the Easter, summer and Christmas holidays. Before school starts, at the end of the holidays, the activity in the groups goes up again and remains relatively stable.

(ii) Locally, we see that Mondays–Thursdays score the highest number of posts (around 70%, just over 17% per day). Fridays and Sundays score a bit lower (12% per day) and the lowest activity appears on Saturdays (around 7% of the posts).

(iii) On a micro level the data show what time of day teachers post their statuses. Around 35% of the posts are made during working hours, and just over 65% of the posts are made outside working hours. The groups have an activity of around 20% during weekends and holidays.

The above gives us an insight in the activity patterns of the teachers in these Facebook groups. These activity patterns raise new questions, given the large amount of activity outside working hours – to what degree do teachers feel free to use Facebook as part of their work? How legitimate is Facebook considered to be as a tool for professional development? It also raised questions concerning the content: are there specific topics that are discussed at certain instances during a year? In the next section, we cite some of the content discussed in the studied Facebook groups.

What do teachers discuss and reflect upon?

As the ERME call states, numerous frameworks have been developed "aiming at achieving a better understanding characterizing and/or evaluating the content of teachers' knowledge". In this study we have chosen Shulman's PCK-framework (Shulman, 1987) in order to categorize the content of the posted statuses regarding the second research question – what do teachers discuss and reflect upon? Hence, it is a way of characterizing the content in the posted statuses and comments, not a way to measure teachers' actual knowledge. PCK was seen as a framework that made it possible to categorize posts in all groups – irrespective of the school subject in focus. So far, we have looked at mathematics and the subject Swedish language, but in case the study will be scaled up, it will be possible to use PCK also in other subjects.

Shulman (1987) described different categories of teacher knowledge, of which three categories have been most influential: Pedagogical Knowledge, Content Knowledge and Pedagogical Content Knowledge (cf. Ball, Thames & Phelps, 2008). For each of the categories of Shulman's framework, some examples[2] taken from our data are given to show in what way the framework is used to categorize the content of the statuses. The statuses are analysed by the researchers and a comparison of the analysis is made to ensure intersubjectivity. During the coming months, a stratified sample will be taken from the groups and a categorization using all (under)categories of Shulman's framework will be made.[3]

Each specific Facebook group aims at a specific domain to engage in, and each of its members has actively chosen to become a member of that specific group. The groups of interest in this study are the groups with subject specific interests, and issues related to the subject (pedagogical content knowledge, and content knowledge) are therefore to be expected. The category pedagogical knowledge, however, addresses issues that are not related to any subject, but merely to teaching in general. Even though the groups in our study are subject-specific groups, pedagogical issues do come up. Status 1 below concerns teachers' preparation for upcoming national tests. What do teachers do to prepare for this non-routine teaching practice?

Status (1): To all of you who will conduct the national tests. How do you prepare? Do you read the teacher's guide? What else?

The following status concerns the arrangement of the teaching of pupils with special needs. Just as in status 1, the experiences of others are asked for. A detailed description of the situation is given: 'we would like to get away from…', 'we would like to accomplish…' and a request concerning alternative teaching arrangements is posed.

Status (2): We have had a discussion at school on how to organise the special needs support. We would like to avoid the phenomenon that some pupils get stuck once they start there. We would like to accomplish a more dynamic way of working where pupils get help for a limited period and then can go back to the regular classroom teaching. Does anyone know good models for organising such support?

Both status 1 and 2 are related to the *how* of teaching – and are classified as *pedagogical knowledge*. Some of the posted statuses do not address the how, but only address the what. Status 3 shows a post where a member wants to have help with a specific mathematical term and it purely concerns *content knowledge*.

Status (3): I am familiar with a concept but don't know the Swedish name for it: 'interior angle'. How can I translate it into Swedish? I have looked on google, searched on the internet, tried google translate but cannot find anything suitable. Wikipedia is not useful either for a translation here.

Regarding the third category, pedagogical content knowledge, members post statuses asking for advice, for instance. The member in status 4 is asking for advice concerning technical tools appropriate for his/her pupils. A description of specific demands, as well as a description of the needs of the pupils are given. The teacher's qualified description indicates awareness of *pedagogical content knowledge*.

Status (4): I am looking for a calculator where all input appears and stays on the screen. Will use it with pupils with difficulties in mathematics. Can someone recommend a good model?

In the following status, the member also asks for advice, this time concerning the teaching of a mathematical topic (division). This status clearly belongs to the category *pedagogical content knowledge* as it concerns the teaching of a specific content.

Status (5): How do you introduce division in year 2?

It is interesting to note that the question in status 5 might seem to be a general question, but a closer look at the status reveals that the teacher is not asking for help regarding division in general, but only wants to know about the introduction of the topic. The teacher also specifies the age group (year 2). The comments on this post show a thread addressing different aspects. Comments 1 and 2 both relate to the mathematical content (multiplication and division), comment 1 through an example, and comment 2 through a more general description. Further, comment 2 gives advice on the *how*: laboratory work. Finally, comment 3 relates to pupils' difficulties when learning the topic: pupils can, but are not able to write it. All three responses are classified within the category *pedagogical content knowledge*.

Comment (1): Think first double and half; multiplied by 2 and divided by 2.

Comment (2): Show the relationship between multiplication and division. Laboratory work with blocks or other objects

Comment (3): Word problems, connect to math language. Pupils know, but are not able to write it, then demonstrate how multiplication and division are related. Most often challenging and a fun way of learning. Good luck!

Besides asking for help, members also share experiences and status 6 shows such a shared experience. The member starts with a clarification of the mathematical topic in focus and at what level it is treated (*pedagogical content knowledge*). Further, this member shows what was done in class and illustrates the results with a picture.

Status (6): Today I worked with definition, axiom, theorem and prove with my pupils who take course 1b at upper secondary school. To prove Pythagoras theorem with a 'puzzle' was an immense success and created understanding. (Followed by four illustrative pictures)

Statuses 1-6 above have been used to give an insight into the data and exemplify our categorization into the three categories described by Shulman. As stated before, the results regarding the question on what teachers discuss and reflect upon are preliminary and will be adjusted in later versions of this paper. Furthermore, this part of the study will continue during 2016 and parts of 2017. In the next section, we raise some points for discussion, some processed, and some preliminary, and further study will show if these points are of relevance or not.

CONCLUSION

Social media and social network sites change the arena of professional development of teachers. This implies a changing role of teachers in learning communities (Issa & Kommers, 2013). Our study gives insights into the underlying characteristics of digital and informal professional development . The when and what of teachers' input in social media and social network sites have been exemplified in this paper in relation to Shulmans' framework (1987). We deceided to distiunguish between working hours and non-working hours and the activity patterns do raise a question: when do teachers have time to plan and reflect? The activity patterns within the groups indicate that teachers' professional development partly takes place outside working hours. Why? To what extend are teachers free to use this new arena? To what extend are social media included as part of a working day? We are aware of the fact that not all posts are to be looked upon as professional development, moreover posts differ in form, content and depth. As we have indicated, our study is on-going and we aim at deeper further insights later on, in terms of insights into the *quality* of the teachers' status and comments in the Facebook groups. Such quality for instance can be measured by looking at the coherence within a post (statuses and comments together). But maybe of greater importance are insights into the *impact* of Facebook groups and other professional development on social media and social network sites. Furthermore, it will be possible to look at the impact of formal professional development initiated by the state, through looking at the threads and topics addressed in the Facebook groups.

The quality of teachers' professional development has been the subject of other studies. The digitally extended context for teachers' formal and informal professional development, however, has not been taken into consideration and we want to invite the ERME Topic Conference ETC3 to discuss this extended context for teachers' professional learning through this paper. When digital gadgets are in common use, there is no longer a distinct border between teachers' interaction with their colleagues in their local school and their interaction with colleagues in social media and social network sites.

NOTES

1. School holidays sometimes vary between different regions and are therefore not always possible to account for.

2. In line with the conference theme, the examples given in this paper are taken from the Facebook groups related to the subject mathematics.

3. New results will follow during spring, meaning that an updated version of the results will be given in newer versions of the paper and during the conference

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WG 2 - Working Group:

Teacher Resources and Research Methods

ASSESSMENT OF CANDIDATES' MATHEMATICAL KNOWLEDGE ON ADMISSION TO PRIMARY EDUCATION DEGREES: A REQUIREMENT?

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Our attention is focused on the mathematical knowledge of students enrolling for Primary Teaching Degrees in Catalan universities. We present the preliminary steps of a study[1] which aims at developing a mathematics test that should be included in the official entrance examination for applicants to the teaching degree starting September 2017. After briefly introducing the concept of Basic Mathematical Knowledge (BMK) and determining the content to be assessed in the entrance examination, we present a pilot test conducted on 291 students in order to evaluate their BMK. Our results not only evidence the candidates' BMK inadequacy, but confirm the need to consider mastering BMK as a requisite for admission to the Primary Teaching Degree.

Keywords: entrance examination, initial teacher education, basic mathematical knowledge.

INTRODUCTION

To this date, University entrance exams in Spain are identical for all degrees, without specific tests for each type of study. However, the new legislative change which regulates education in Spain (LOMCE – Ley Orgánica para la Mejora de la Calidad de la Educación, i.e. Organic Law for the Improvement of Quality in Education), passed in November of 2013, established a validation of 'baccalaureate' and allowed university campuses to design their own tests for University entrance. Therefore, it is essential to find a more precise way of establishing what these tests aim to measure for the entrance to each degree offered.

In the case of the degrees in Education, these tests have not yet been defined. However, in Catalonia, both social media and academics are recently paying an increased attention to the need to improve pre -service teacher training. However, those in charge of political decisions need to be convinced that a test on mathematical content knowledge is a necessary part of an entrance examination for accessing a primary teaching degree. Therefore, the first results of our research allows us to stress out that it is necessary to develop a test.

Therefore, in this study, we suggest considering the evaluation of candidates' basic mathematical knowledge (BMK) on entrance to University to start their training in Primary School Teaching. Thus it is not only essential to determine the form and content of the BMK, but also to prove that candidates' BMK sufficiency cannot be taken for granted.

The TIMMS study (Third International Mathematics and Science Study) evidences differences and deficiencies in the mathematic knowledge of students of several countries. Spain ranked below the average of participating countries of the European Union and the OECD. This fact brings to light the need to revise the teaching of mathematics in the Spanish educational system and suggests that it is paramount to provide a good initial education to future primary school teachers in order to improve this situation.

With results obtained by TIMMS as starting point, TEDS-M (Teacher Education Study in Mathematics) was created, an international comparative study about the knowledge acquired by future mathematics teachers in primary education and compulsory secondary education after their initial training. The aim of TEDS-M was to analyse the differences between initial training programmes and their impact on the education of future teachers. Despite the low number of participating countries in this study and the differences between the training programs of each of them, it brings evidence that better results were obtained in those countries where education in mathematics is more specialised. In this respect, Lacasta and Rodríguez (2013) have documented nominal relations between the level of mathematical knowledge of educators and their level of knowledge for mathematics teaching, mathematical content being the main requisite for a good understanding of how to teach mathematics.

The interest in discovering knowledge for the teaching of mathematics has promoted the evaluation of its content and, particularly, the evaluation of future teachers' knowledge (Norton, 2012; Senk et al., 2012, Walshaw, 2012). However, there is little research describing the mathematical knowledge of students at the start of their training to become teachers, therefore the evaluation of such knowledge is a challenge for their educators (Linsell & Anakin, 2012).

Our objective in this communication is to introduce the concept of BMK and present the results of a research that motivate the need to study it. We present a first theoretical approach to this concept and shortly present how the content required for assessment in entrance examinations was fixed. Our proposal is supported by the expert knowledge of researchers who are also experienced educators of primary school teaching students. Subsequently, we present some partial results from a pilot test designed to evaluate such knowledge that was administered to 291 first year students of the Primary Education Degree at the Universitat Autònoma of Barcelona. Our empirical results allow us to justify the importance of establishing a BMK to be mastered as a requisite to enter teacher training.

BASIC MATHEMATICAL KNOWLEDGE

Shulman (1986; 1987) stressed the importance of content knowledge, defining the latter as the amount and organisation of knowledge of the subject, pointing out that content knowledge requires going further than being familiar with facts and concepts of the subject, it also requires the understanding of its structures. According to Fennema and Franke (1992), Knowledge of Mathematics includes teacher knowledge of the concepts, procedures, and problem-solving processes, the concepts underlying the procedures, the interrelatedness of these concepts, and how these concepts and procedures are used in various types of problem-solving. These authors coincide with Shulman when stating that teachers shouldn't only know mathematical procedures but should also understand the concepts underlying these procedures.

Later on, Ball, Thames & Phelps (2008) used the model suggested by Shulman and elaborated the MKT model (Mathematical Knowledge for Teaching), that was created to describe the knowledge of in-service teachers. One of the main axes of this model is Shulman's Content Knowledge, which they called Subject Matter Knowledge. The MKT model proposes the division of Subject Matter Knowledge in three subdomains: Common Content Knowledge, Specialized Content Knowledge and Horizon Content Knowledge. Common Content Knowledge is the knowledge that every adult that has received mathematical training should have and is used in a wide range of contexts (Ball, Hill & Bass, 2005). Specialized Content Knowledge includes an understanding and mathematical reasoning inherent to the teacher. Horizon Content Knowledge is mathematical knowledge that students will be learning in the future. Again focussing on in-service teachers, Rowland (2008), based on observation in the classroom, proposes the Knowledge Quartet to describe mathematics teachers' knowledge as having four dimensions: foundation, transformation, connection and contingency. In particular, the foundation dimension includes, among others, the propositional knowledge on which teachers support their practice.

In the last decade, several researchers have contributed with nuances or new proposals to the established ideas and have helped to consolidate and expand existing concepts. However, all the concepts introduced in the previous paragraphs refer to teachers' knowledge needed for the actual practice of teaching while we are not even dealing with novice teachers, but with teacher-students. It cannot be expected of students who start their degree to have received a previous education that provided them with a deep understanding of the mathematics concepts studied or an outlook oriented towards conferring their learning to others.

Therefore, in Castro, Mengual, Prat, Albarracín and Gorgorió (2014) we introduced BMK as the disciplinary mathematical knowledge that students need in order to benefit from their courses in mathematics and mathematics teaching during their education to become teachers. It is important to note that we are referring to students that have not even started their training as teachers, and we suggest that BMK should be a requirement for their pre-service training. BMK would be the initial disciplinary knowledge on which to build throughout teacher students' training, to attain the mathematical and pedagogical knowledge required to start their professional practice. As educators of teachers, we take BMK as the mathematical knowledge starting point for our courses, which should be based on a thorough knowledge of elementary mathematics, being the foundation that would support the building of a structurally robust training.

BMK should be the basis on which to build Shulman's Content Knowledge (1986, 1987) and of Fennema and Franke's Knowledge of Mathematics, but we cannot expect Shulman's idea to be equal to BMK in its entirety. In Ball et al.'s MKT model, BMK is part of common knowledge and the starting point for development of the knowledge of the horizon, since the education the students have received before reaching University level should have allowed them to deal with more advanced knowledge than what they are going to teach in Primary School. Similarly, we believe that we may require our students to know the basis and terminology of the mathematics they have been taught during their previous schooling. Therefore, we consider BMK to be part of the foundation component of Rowland's Knowledge Quartet.

Similarly to the outlook presented in this study, Linsell & Anakin (2013) claim that the models developed to describe the professional knowledge of the teacher have limitations when it comes to the knowledge analysis of beginning undergraduate students. Linsell & Anakin (2012) propose the concept of Foundation Content Knowledge to refer to the knowledge of mathematical content that future educators possess when starting their training programme. This type of knowledge includes as inseparable conditions, both conceptual knowledge and methodological knowledge. The characteristics of Foundation Content Knowledge are related to the ability to model, modify, reason and confirm, the implementation of multiple representations, making generalizations, working with real numbers and understanding basic facts, amongst other aspects.

Our notion of fundamental mathematical knowledge differs little from the Foundation Content Knowledge of Linsell & Anakin (2012), since they refer to the knowledge of future educators at the beginning of their training. However, our research distinguishes itself from that of these authors already at a preliminary stage, given that we wish to determine the knowledge required at the start of undergraduate teacher training, which we have termed BMK. We aim to ascertain the content of the latter by consensus between experts in order to evaluate the BMK. Linsell & Anakin evaluate the knowledge students actually have in order to end up describing it as insufficient, possibly as a comparison to the desired amount of non-explicitly stated knowledge.

METHODOLOGY

After developing a preliminary theoretical approach to Basic Mathematical Knowledge, we set out to establish the mathematical content domains to which it refers. We focus on the mathematical content prescribed by the curriculum of the Spanish compulsory education –Numbers and Arithmetic, Space and Shape, Relations and Change, Measure, and Statistics and Randomness– since we do not expect an encyclopedic knowledge from our students, but wish to verify whether they possess a solid basic knowledge.

In parallel, while developing the criteria to fix the exam content, we set out to elaborate a pilot diagnostic test that should be the first step towards a tool to assess students' BMK. For this purpose we revised different pre-existing tests aimed at the evaluation of mathematical knowledge of teachers in different moments of their training or professional development. Some of the aforementioned tests include TIMMS, TEDS-M, items from the Texas Mathematics Educator tests, as well as the activities employed by Linsell & Anakin (2012) in their study.

These test items are designed with an open-question format to avoid suggesting possible answers to the students, as may be the case when using a multiple-choice question format. The questions aim to evaluate mathematical knowledge at three different levels: reproductive, applicative and relational. Finally, we selected twenty five activities that comprised a balanced test with respect to content blocks and levels of mathematical knowledge. Some of these exercises will be shown later on, together with the results obtained.

The aforementioned test was handed to 291 students of the first year of the Primary Education degree at the Autonomous University of Barcelona (UAB) who had not yet taken any lessons in mathematics or mathematics teaching. The minimum University-entrance examination grade to enter these studies at the UAB is the highest required amongst the 8 degrees in Primary Education of the different Universities in Catalonia. On the other hand, the minimum entrance grade is located at the 81st percentile (77 of 421) in relation to all degrees offered at Catalan Universities. Therefore, we can state that not only have our students successfully passed their educational stages previous to University admission, but have also obtained higher University-entrance qualifications than students entering many other graduate courses.

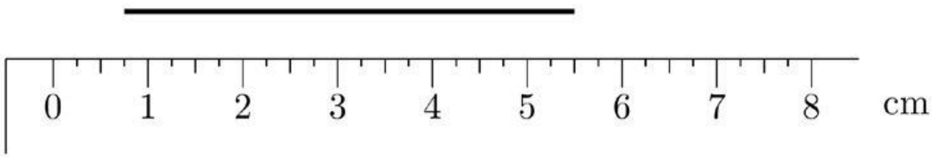
SOME RESULTS

As follows, we present the analysis of some of the data from the answers of the 291 students to three of the questions included in the aforementioned test.

The main interest of our study is focussed on determining the type of background content desired for students recently admitted to the Degree in Primary Education and diagnosing mistakes made in their learning process. For this reason, we wish to make a quantitative analysis of the type of mathematical content these tests reveal.

Measuring a segment with a ruler

In one of the questions of the test, the students were given the following image and were asked to establish the length of the segment.



The following table summarizes the students' answers to this question:

Answer	Frequency	Percentage	Answer	Frequency	Percentage
NC	5	1.7%	4.8 cm	7	2.4%
4.3	11	3.8%	5 cm	11	3.8%
4.3 cm	11	3.8%	5.3 cm	11	3.8%
4.5 cm	6	2.1%	5.5 cm	4	1.4%
4.6 cm	9	3.1%	5.75 cm	14	4.8%
4.7	4	1.4%	6 cm	4	1.4%
4.75	35	12.0%	Other	53	18.2%
4.75 cm	106	36.4%	Total	291	100.0%

Table 1. Answers to question "Measuring a segment with a ruler"

The data on the table show that the correct answer, 4.75 cm, is also the most frequent one, 35.7% of the students. We also see that 11.2 % of the students give the number resulting from the measurement, 4.75, but they do so without units. For what refers only to the use of units, 75.1% of them use the appropriate ones, 2.8% of the students do not use any, and 0.4% use the wrong units such as cm^3. It is important to note that we have identified 44 different answers for this question, suggesting that the use of open questions does not condition the students' response. However, the most discouraging answers are those where the result given –twice 9.5cm, 19cm, 25.5 and 47– is bigger than the length of the ruler on itself –8cm.

Perimeters and surface areas

One of the test questions asks the students to calculate the surface area and perimeter of a square with 7 cm sides and of a circle with a radius of 6 cm. Table 2 shows the different categories into which we have organized the answers, the number of answers that fall into each of them and the percentage for each category considered.

	Square		Circle	
Answer	Area	Perimeter	Area	Perimeter
No answer	39 (13.4%)	63 (21.6%)	147 (50.5%)	182 (62.5%)
Correct	115 (39.5%)	174 (59.8%)	38 (13.1%)	42 (14.4%)
Correct calculations - wrong units	102 (35.1%)	27 (9.3%)	32 (11.0%)	16 (5.5%)
Interchange area and perimeter	15 (5.2%)	5 (1.7%)	7 (2.4%)	2 (0.7%)
Wrong for other reasons	20 (6.9%)	22 (7.6%)	67 (23.0%)	49 (16.8%)

Table 2. Answers to question “Perimeters and surface areas”

The results in table 2 show high indices of “unanswered” questions, and a clear ignorance of the calculation process for the surface and perimeter of a circle. In particular, it is worth noting that 30 of the students, 10.3% of the total, do not employ the number ‘pi’ for neither the calculation of the circle’s surface area nor for that of its perimeter.

Added to the misunderstandings between the concepts of surface area and perimeter, there are errors in terms of the units used when giving the answers, especially for surface areas. The results obtained when considering only the units in the answers (without considering the calculated figure provided) are shown in table 3.

	Square		Circle	
Answer	Area	Perimeter	Area	Perimeter
N/A	39 (13.4%)	63 (21.6%)	147 (50.5%)	182 (62.5%)
Correct Units	130 (44.7%)	192 (66.0%)	64 (22.0%)	74 (25.4%)
Incorrect Units	70 (24.1%)	1 (0.3%)	25 (8.6%)	2 (0.7%)
Without units	52 (17.9%)	35 (12.0%)	55 (18.9%)	33 (11.3%)

Table 3. Use of units in the answers to question “Perimeter and surface areas”

Contextualised problem with verbal formulation

The formulation of another question of the test is the following: “When going on a school’s outing it is required for children to be accompanied by adults. Each adult can be responsible, at the most, for a group of 16 children. In an outing with 54 children, how many adults are needed to accompany them?”

Table 4 summarizes the answers of the students to this question and shows their relative and absolute frequencies.

Answer	Frequency	Percentage	Answer	Frequency	Percentage
No answer	17	5.8%	4	154	52.9%
3	41	14.1%	4 with errors	20	6.9%
3.375	24	8.2%	5	6	2.1%
3.4	12	4.1%	Other	10	3.4%
3.5	7	2.4%	Total	291	100.0%

Table 4. Answers to question "School's outing"

We can see that 52.9% of the students answer this question correctly, calculating the ratio in excess, in order to take into account the context of the formulation. However, some of the students give an answer of 4, based on erroneous calculations or invalid arguments (6.9% of the total students). On the other hand, we also find a trend that groups questions 3.375, 3.4 and 3.5, in which the students consider the result of the division to directly be the answer to the question, thus overlooking the interpretation of the situation exposed in the formulation of the problem. These answers represent 14.8% of the total. There is another group of students (14.1%) who give the answer of 3 adults who should accompany the group of children. These students do not consider the possibility of finding the ratio of the division in excess and act without understanding the context of the problem, using a ratio which does not cover the number of children.

CONCLUSIONS

Our students at the start of their Degree in Primary Education at the UAB have successfully passed their studies previous to University but with an incomplete BMK, according to the results obtained in our empirical study. Specifically, we have documented a lack of competence related to BMK, in aspects that have to do with units of measurement, or the contextualisation of mathematical knowledge.

There may be an implicit agreement among teachers of mathematics and mathematics education in Catalonia about what constitutes the BMK required of our students, but this has never been explicitly stated. Therefore, when it comes to the knowledge used as starting point for training in Degrees in Primary Education, it is paramount to clearly set out what is expected of our students. From a teacher-training point of view, results such as those exposed herein evidence the need to commit to the improvement of our students' understanding of elementary mathematics, in order to successfully face subjects related to its teaching.

The results obtained show that many of our students have not developed an adequate construction of mathematical knowledge during their previous education, and are therefore not able to reproduce those processes they learned by heart and without searching for their meaning within a practical context. Many of the students who enter University may have possibly forgotten the elementary mathematics they once studied. Therefore, we would agree with Fennema and Franke (1992) and Linsell and Anakin (2013) on the fact that the knowledge students carry with them at the start of their training may possibly be characterised by memorising and standardised problem-solving and is far from the one we, as teachers' educators, would expect them to have.

The proof of the lack of elementary mathematical knowledge of students at the start of their teacher training justifies the notion of Basic Mathematical Knowledge and suggests the need to keep working not only towards its characterisation, establishing its form and content in order to evaluate it, but also towards developing and validating a tool to assess the BMK of candidates entering a Primary Teaching Degree. As a conclusion, if we had to give a short answer to the question posed in the title of the paper – Should the assessment of candidates' mathematical knowledge a requirement for on admission to primary education degrees? –our answer would be yes, despite the practical and political implications of taking a decision of such importance.

NOTES

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Reliability of the adapted mathematical knowledge for teaching number concepts and operations measures

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Measuring and understanding teachers' mathematical knowledge for teaching is accepted as an important element for effective mathematics teaching and learning, but mathematics teacher education do not naturally attend to and make sense of these. We describe, in part, a validation study in which 212 pre-service teachers participated in a previously adapted mathematical knowledge for teaching test with an aim of establishing validating and reliability of the adapted measures and items. We discuss the test and items' performance using classical test theory. The results show low overall internal consistence and identify eight very problematic items out of forty-six. We also give possible reasons for the low internal consistency. The study offers hope for defining standardized tools for assessing mathematical knowledge for teaching in Malawi.

Keywords: Mathematical knowledge for teaching, pre-service teacher.

INTRODUCTION

According to the Organization for Economic Cooperation and Development (OECD) education systems should equip all humans in a modern society with knowledge, skills and tools to stay competitive and engaged (OECD, 2016). In terms of Mathematics learning, education systems need to equip all learners with at least knowledge and skills about number concepts and operations. This is one of fundamental topics that is a prerequisite to understanding mathematics in later stages of schooling or appreciating mathematics in the society. To teach mathematics effectively, primary school teachers do not only need to understand the fundamental mathematical concepts, but they are also expected to have a mastery of the art and science of teaching mathematics (Reid, 2011).

The need for mathematics teachers to have a robust mathematical knowledge is undeniable. However, researchers (e.g. Chitera, 2011; Jurdak, 2009) observe that most mathematics teachers are not well prepared to teach mathematics in terms of their subject knowledge. Ball, Thames and Phelps (2008) built on Shulma's work and developed two constructs of subject knowledge namely: content knowledge (CK) and pedagogical content knowledge (PCK). For them, CK comprises teachers' comprehension of the subject matter to be taught and PCK is teachers' comprehension of how to make subject matter learnt by students. In our case, therefore we consider CK as entailing the mathematics content to be taught as prescribed by the curriculum, while PCK as the knowledge that the teacher needs in order to organise and present the mathematics content into forms that can be easily understood by learners, and explain the connections between them.

While there seems to be an obvious distinction between CK and PCK, Hill, Ball and Schilling (2008) contend that the distinction is rather not obvious at the lower level of mathematics teaching and hence they developed the term mathematical knowledge for teaching (MKT). Different schools of thought have emerged about CK, PCK and MKT since the work of Shulma, and Ball and colleagues. However, researchers generally agree that knowledge for teaching is precursor to learning and that this knowledge is affected by, among other factors, teacher development programmes (e.g. Blömeke, Suhl, & Kaiser, 2011). Furthermore, Kleickmann et al. (2013, p. 93) report that the extent to which different teacher education programmes influence knowledge for teaching or the extent to which the knowledge for teaching changes among different populations of teachers remain unclear. In Malawi context, for example, there is an evident need for standardized tools for assessing teachers' understanding of important mathematical concepts, ideas and skills, and the level to which teachers are mathematically prepared for the work of teaching. Research in this direction could assist in informing the educational system to provide relevant professional development for mathematics teachers and teacher-education programs through improved curriculum in terms of content and corresponding pedagogy.

Primary teacher Education in Malawi

The current primary school teacher education programme in Malawi referred to as Initial Primary Teacher Education (IPTE) is a two-year pre-service education programme delivered in two phases of one academic year in length each. Phase one is a full-time residential programme characterised by student teachers attending ten taught modules and about a half module of microteaching and teaching practice experience. The second phase is a one academic year of school-based teaching practice (TP) supervised by school mentors and teacher educators. At the end of the TP, student teachers return to college to sit for national exams set by the Malawi National Examination Board (MANEB). The teacher education curriculum in Malawi addresses both content and pedagogy hence has some aspects of CK and PCK (MIE, 2010). Like in most teacher education programmes, the student teachers' CK mostly develops during the first phase of the IPTE programme. During this phase, student teachers are also formally introduced to mathematics related PCK by their lecturers. Furthermore, the students have more opportunities of developing both CK and PCK through micro-teaching and school experience in demonstration primary schools based within the colleges. However, Kunje, Lewin and Stuart (2003) report that the curriculum is ineffectively delivered with material mainly packaged "as facts to be learned and assessment regimes reinforcing this recall-based orientation to curriculum" (p. 112). They also observe that the school-based phase "is peripheral rather than central to the curriculum". Consequently, the development of knowledge for teaching is greatly compromised among student teachers.

Mathematical literacy is important for developing countries like Malawi (Tsafe, 2013) if she is to alleviate poverty among her people and to strategically position herself for development. Unfortunately, results from regional studies by the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) show that Malawian learners continue to perform significantly below the regional average in mathematics including on items related to number concepts and operations (SACMEQ, 2011). Influenced, in part, by such comparative studies and the need to have a relevant curricula at all levels of schooling, Malawi has revised and released new curricula recently. Specifically, in September 2015 the Ministry of Education introduced a science based secondary school curricula which puts emphasis on science and mathematics. Despite these changes and new curricula direction, it is unclear how well the IPTE programme prepares teachers to teach mathematics given the persistent poor performance in primary school mathematics.

To what extent do primary school teachers understand mathematics in general and number concepts and operations in particular, and how well do the teachers exhibit mathematical literacy seem important questions to advance in the Malawi context. However, there could be a number of barriers to conducting research to answer these critical questions. One obvious barrier has been and remains the unavailability of appropriate tools with which to assess teachers' knowledge for teaching in Malawi.

Although, there are a number of diagnostic tools available to assess knowledge of mathematics, most of these were developed for specific audiences or are content focussed and do not assess ability to teach mathematics. The Learning Mathematics for Teaching (LMT) measures developed in the US focus on the work of the mathematics teacher (Hill, Ball, & Schilling, 2008). Specifically, the LMT-NCOP items measure conceptual understanding of number concepts and operations, and how a teacher can react to specific situations related to NCOP in a classroom. This renders the LMT-NCOP a potential and meaningful tool for assessing knowledge for teaching numbers concepts and operations of both practicing and pre-service teachers in Malawi. Even though the LMT-NCOP measures have been adapted for use in Africa, Ghana, it is important that we understand how individual NCOP items behave in our context. Therefore, the main purpose of this study was to investigate how the adapted items would function in Malawi.

METHOD

The LMT measures for assessing teachers' mathematical knowledge for teaching number concepts and operations were developed as part of the Learning Mathematics for Teaching (LMT) project at the University of Michigan. The measures assess multiple aspects of teacher knowledge including subject matter knowledge and pedagogical content knowledge (Hill, Ball, & Schilling, 2008). Number concepts and operations as part of the school mathematics curriculum have remained a central part of the LMT measures. We selected LMT items from existing forms for adaptation by aligning the items to the IPTE mathematics curriculum and adapting them to the Malawi context (Kasoka, Kazima, & Jakobsen, 2016).

Kane's (2013) framework for validating tests was employed. Kane remarks that when tests are adopted or adapted, they are used on populations that are generally different from the intended in terms of culture, academic experiences, linguistic abilities, social and economic experiences among others. We were therefore mindful of the need of the adapted measures to be fair and meaningful in the Malawi context while at the same time to provide scores with same meaning as intended in the source context (Kane, 2013). Kane postulates a six component validation argument for exporting tests as follows: (i) domain definition, (ii) evaluation, (iii) generalization,
(iv) explanation, (v) extrapolation and (vi) utilization. A rigorous attempt was made to address these components during the adaptation process (see Kasoka, Kazima, & Jakobsen, 2016) as summarised in Table 1. An instrument consisting of forty-six adapted items was administered to pre-service teachers from one teacher education college.

Data Collection

The participants of the study were 212 first year students from one primary teacher education college of whom 61 were females and 151 were males. Primary school teachers in Malawi do not specialize to teach specific subjects. Consequently, all first year students at the college participated in the study. All the participants had completed secondary school and held a school certificate of education. The college was purposively selected because we had initially worked with the lecturers at the college during professional development activities. Approval to conduct the study was obtained from the college management and all the participants were briefed about the study and its objectives. We administered the adapted form in one sitting with the help of mathematics lecturers from the college.

Data Analysis

To evaluate the functionality of the adapted measures, we used classical test theory (CTT) and item response theory (IRT). In this paper we focus on CTT results and descriptive statistics. For all the forty-six items, Cronbach's alpha reliability coefficient and descriptive statistics were computed. Furthermore, for each item on the form, we sought four item characteristic indices namely: item difficulty index, upper-lower (U-L) groups discrimination index, point-biserial discrimination index and change in Cronbach's alpha coefficient with the item removed.

The first index we sought was item difficulty index. The item difficulty index measures the proportion of participants that responded to the item correctly. The greater the number of participants that chose the correct response or key, the less difficult the item was. While Mehrens and Lehmann (1991) suggest different acceptable difficulty levels of items based on the number of response options, Allen and Yen (1979) argue that acceptable item difficulty ranges from 0.26 to 0.75 regardless of the number of response options. Since the items have different numbers of response options, we opted to use Allen and Yen range.

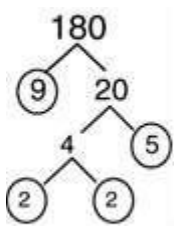
<table>
<tr><th>Activity</th><th>Description</th></tr>
<tr><td>1. Adopt or adapt</td><td>Examined the source instruments and measures in terms of structure and content in line with Malawi context</td></tr>
<tr><td>2. Selecting measures/items</td><td>Aligned the content, tasks and skills in the measures/items with the IPTE mathematics curriculum and syllabus.</td></tr>
<tr><td>3. Linguistic translation</td><td>While Malawi uses English as a language of instruction at all levels of education, we modified some US language terms to terms commonly used in Malawi.</td></tr>
<tr><td>4. Cultural contextualization</td><td>Made changes relating to general cultural context by replacing US names of people, places and objects with familiar names.</td></tr>
<tr><td>5. School contextualization</td><td>Modified the content of the measures to reflect IPTE curriculum, syllabus, and textbooks e.g.
US

Malawi<table><tr><td>9</td><td>180</td></tr><tr><td>5</td><td>20</td></tr><tr><td>2</td><td>4</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
</table>

Table 1: Summary of the adaptation process

The second index we used is the U-L discrimination index. According to Kelley (1939), and Hopkins (1998), the U-L coefficient shows the difference between the percent of test takers whose raw scores are above some set limit (i.e. in the upper group) and responded correctly to an item, and the percent of those whose raw scores are below some set limit (i.e. lower group) and responded correctly to an item. For them, the limits are set at the 73^{rd} and 27^{th} percentile points for upper and lower limits respectively. To determine the U-L discrimination index, we therefore divided the participants into upper 27%, middle 46% and lower 27% groups. Although, some researchers (e.g. Taiwo, 1995) have reported U-L discrimination with 25% cut-off point, Kelley argues that the U-L discrimination index stabilises when the 27% cut-off point is used. To calculate the U-L indices, we divided the difference between the number of participants in the upper and lower groups who responded correctly on an item by 57 (i.e. 27% of 212) and used Ebel (1965) to interpret the indices: > 0.4 –

Satisfactory discrimination, > 0.3 – Fair discrimination, > 0.2 – Need some revision and < 0.2 – Item must be removed or totally revised.

The last two characteristics we considered are correlation based. In this regards, we first assessed the item discrimination using point-biserial correlation. The correlation coefficients were obtained to understand the relationships between the score on an item and the total score from all items. As pointed earlier, the internal consistency reliability of the items collectively was determined by the Cronbach's alpha. We therefore sought changes in the value of the Cronbach's alpha if an item was deleted from the form to decide whether or not the deletion would increase the internal consistency. IBM SPSS Statistics was used for these analyses.

RESULTS AND DISCUSSION

If the adapted measures are to be used for inference about teachers' knowledge, then it is important that they must be both reliable and valid. Our focus in this paper is reliability as a step to determining the validity of the measures. Internal consistency reliability was evaluated by Cronbach's alpha. Analysis of all items produced Cronbach's alpha of 0.581(N=111). The value is below the generally acceptable level of 0.7 (Pallant, 2007). This suggests that the internal consistency of the items collectively is poor and the items appear not to be measuring the knowledge for teaching number concepts and operations. When we examined the changes in alpha (see Table 2) after deleting an item, seventeen items produced positive changes. The maximum change recorded was 0.028 from item 21c. Therefore, the deletion of any items does not significantly improve the internal consistency. While we understand the implication of the low value of Cronbach's alpha, Pedhazur and Schmelkin (1991) argue that reliability values ought to be evaluated by taking into consideration the specific circumstances of a study before condemning the measures due to lack of reliability. For instance, they argue that if test takers are not knowledgeable, they tend to respond rather randomly hence affecting the reliability of the measures. All the participants involved in our study were pre-service teachers in their first year of teacher education. Consequently, most of them did not have any teaching experience to draw on as they responded to the items. This is evident as Cronbach's alpha was calculated based on N = 111 out of 212 participants. This shows that almost half (101) of the participants did not respond to at least one item. While the analysis fails to show good reliability of the items based on the alpha value, we are of the view that this is mostly attributable to the participants' characteristics than the items.

The difficulty level of each item was compared to the range proposed by Allen and Yen (1975). Out of the forty-six items in the form, seventeen were below 0.26 and only one was above 0.75. Item 17 was the easiest with a difficulty index of 0.82. This item was attempted by all the participants and 82% (174) of the participants provided a correct response. The most difficult item was item 24 with a difficulty index of 0.01. Only 1% (2) of the participants responded correctly to this item. With an average difficulty of 0.31, the items were generally hard for the participating pre-service teachers. Given that the adaptation related the items to the IPTE curriculum, student text books, and instructors' guides (see Kasoka, Kazima, & Jakobsen, 2016), the overall low difficulty level would be as a result of the characteristics of the participants rather than a mismatch between the items' content and curriculum.

Item discrimination analysis involved establishing the U-L discrimination index and point-biserial correlation coefficient for each item. U-L indices for twenty-two items were outside Ebel's (1965) recommendation minimum of 0.2. Only one item (item 1) had a negative U-L discrimination index (-0.12) which shows that more low performing pre-service teachers (in lower group) responded to the item correctly than their high performing counterparts (in upper group). The least discriminating items were items 6, 7, 8d, 10, 12a, 24 and 25. The difficulty indices for these items were all below 0.26 (see Table 2). This shows that very difficult items discriminated poorly and hence problematic. Moreover, removal of most of them negatively affected the alpha value. Item 5c was the most discriminating with a U-L index of 0.68.

All the items were analysed to examine the tendency of the participants responding correctly to an item and scoring highly overall. This was done through the point-biserial correlation. The results showed that some items did not correlate well with the total score. Twenty items did not show expected correlation with the total score. These had point-biserials of less than the desirable 0.3 for 'good items' (Pallant, 2007). However, Pallant advises researchers to use a minimum threshold value for point-biserial and recommends 0.15. Considering Hopkins' (1998) observations that discrimination indices above 0.1 indicate "fair discrimination" and that any positive discrimination indices above this value may be accepted because they show that there is a higher probability of a high performing examinee select a correct response than a low performing examinee. Hopkins is, nevertheless, concerned with negative point-biserial indices because they show that low performing examinees are more likely to select a correct response than high performing examinees as in items 1, 6 and 11a (see Table 2). This may suggest that the items have some underlying errors which are preventing high performing participants from responding correctly. The three items are therefore obviously problematic and must be removed.

None of the statistics we have computed so far show the performance of individual response options to understand their contribution to the overall quality of the items. All problematic items were therefore, subjected to further analysis to examine the performance of the response options in terms of their popularity among the participating pre-service teachers. We use item 6 to exemplify this analysis. Item 6 was intended to measure teachers' ability to explain the divisibility rule for 4. This item had a difficulty index below the overall mean of 0.3 and its U-L index was 0.00 which shows that it could not discriminate between the upper and lower groups. In terms of the usefulness of the response options for item 6, response option a) had positive point-biserial which suggests that high performing participants were more likely to select this as a correct response than low performing participants. This response option was an ineffective distracter. Eighty-four participants selected this option, 25 of whom were among the 27% most knowledgeable and 16 were among the 27% less knowledgeable. The behaviour depicted by this distractor is expected of a key (correct) response. The key for this item was b) and had a negative point-biseral which is uncharacteristic of a key response. The other two response options c) and d) appeared to be effective distracters given that they both returned negative point-biserials (see Table 3). This suggests that the participants who picked these response options were mainly among those who exhibited low overall performance.

Item No.	Change in alpha	Difficulty	U-L Discrimination	Point-Biserial
1	0.027	0.72	-0.12	-0.049
5c	-0.048	0.50	0.68	0.521**
6	0.016	0.11	0.00	-0.036
7	0.024	0.24	0.05	0.238
8d	-0.012	0.13	0.07	0.209
10	-0.001	0.25	0.09	0.204
11a	0.023	0.38	0.14	-0.043
12a	0.020	0.16	0.07	0.213
17	0.003	0.82	0.16	0.324
21c	0.028	0.25	0.18	0.189
24	-0.001	0.01	0.04	0.233
25	-0.002	0.11	0.05	0.244

***Significant at the 0.05 level. **Significant at the 0.01 level**

Table 2: CTT results for selected items

Option	Code	N			Proportion	Point-biserial
		Total	Upper	Lower		
a)	0	84	25	16	0.4	0.145
b) ***Key***	***1***	***23***	***7***	***7***	***0.11***	***-0.036***
C)	0	36	6	13	0.17	-0.055
d)	0	60	13	22	0.28	-0.066

Table 3: Further analysis of item 6

In summary, item 6 requires further examination in terms of how it was phrased and its content in relation to Malawi curricula. It might also be important to examine the knowledge background of the participants. The key and response option a) fail to make it a quality item.

CONCLUSION

In conclusion, we note that although the study has some limitations mainly to do with the homogenous nature of the sample used, helpful evidence was shown that the adapted measures are a promising tool for use in Malawi albeit the low internal consistency. Furthermore, the current results and performance of the adapted measures are a good basis to speculate that the participants lacked knowledge for teaching number concepts and operations, and to suggest a methodological change in investigating MKT among pre-service teachers in Malawi. The pre-service teachers seem to have responded to the items rather randomly than influenced by their knowledge. It sounds logical therefore to suggest that there is need for further evidence from a less homogenously population using a longitudinal study. The study has found that there are low levels MKT for NCOP among pre-service teachers and this has possible implications on teacher education programmes. We have also found that not all adapted items (e.g. item 6) function well and may need to be improved or removed. While there is evidence to support the use of the measures, our analysis showed some inconsistencies between item difficulty, discrimination and point-biserials. However, we noted that most items that were problematic based on difficulty index, they also fell out of range on the other two indices.

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Development of motivation referring KCS and KCT in professional development courses

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In this paper, we discuss pre-test's results of a research project aiming to investigate the impact of a professional development course on teachers' motivation and teachers' pedagogical content knowledge (PCK). For this purpose, we propose a brief review of research on teachers' professional development. We further refer to the main characteristic of the professional development course, i.e. to integrate examples of students' solutions of tasks that the teachers were asked to collect in distance phases of the course. Afterwards, we discuss the main theoretical constructs for our research, i.e. motivation, PCK and a combination of motivation referring to different aspects of PCK. Based on an adapted questionnaire motivation in this context is measured in two professional development courses. Results of pre-tests (at first meeting) show that teachers consider situations to acquire PCK as challenge and are interested to do it on a medium level.

Professional development, pedagogical content knowledge, motivation, beliefs

INTRODUCTION

There is a consensus that university studies and internships are not enough to prepare future teachers for all challenges with which they will be confronted in their professional career (Mayr & Neuweg, 2009). For this reason, professional development (PD) is understood as being a key factor for innovating and reforming mathematics teaching in school (Garet, Porter, Desimone, Birman & Yoon, 2001). However, research referring to teachers' professional development implies that PD must have several characteristics for being effective (Desimone, 2009).

For example, Kedzior and Fifield (2004) sum up characteristics of high quality teacher PD: If a PD course is content-focused, extended, enables active learning (Garet et al., 2001), supplies follow-up support or reflecting students' learning results (Franke et al., 1998) teachers' learning will be sustainable (see also Timperley, 2007, 2008).

However, Yoon et al. (2007) criticised the absence of studies that focus on different aspects of efficient professional development courses in an experimental or quasi-experimental setting. For this reason, the actual efficiency of aspects of successful PD like reflecting student learning results is not clear (Lipowsky 2010, 2011). For example, the research program of cognitively guided instruction (CGI) by Franke, Carpenter, Fennema, Ansell & Behrend (1998) is an often cited example of an effective

PD program. However, CGI fits several of the characteristics of effective PD. Since this is the case for several PD programs that are found to be effective, the impact of a specific characteristic of a PD course on teachers' knowledge, beliefs or motivation is not investigated yet.

Therefore, a research project named PROFIL (http://profil.ph-bw.de/wiki/Hauptseite) is started in Germany to prove different aspects of professional development programs. The main purpose of this project as a part of PROFIL is to investigate the impact of a characteristic of a PD course that we call "reflecting on students' learning results" (cf. also Timperley, 2008) that was indicated to be an effective characteristic of PD programs. "Reflecting on students' learning results" could be a characteristic of PD courses that include more than one face-to-face-meeting and a distance phase between the face-to-face-meetings: Teachers were introduced in a specific issue of mathematics teaching in the PD course and develop tasks or lessons referring to this specific issue. In distance phases they integrate the developed tasks or lessons in their teaching and collect examples of students' works, e.g. solution of tasks. The solutions are sent to the facilitators of the PD course. Some of the students' works are integrated in the next PD course as topic of reflecting students' learning results referring to a specific mathematical subject.

In our study, we focus on the impact of this reflecting on students' learning results on the teachers' learning in PD courses, and, especially on the teachers' pedagogical content knowledge, beliefs and motivation. In this paper we report about measures on motivation according to pedagogical content knowledge and results of the pre-test. For this reason, we analyse the construct of motivation itself, i.e. its value in different groups or correlations among different aspects of the construct of motivation.

THEORETICAL BACKGROUND

An impact of a PD course could firstly refer to teachers' professional knowledge. According to Shulman (1986) pedagogical content knowledge (PCK) is a central part of teachers' professional knowledge. Further, PCK can be divided in "knowledge of content and students" (KCS), "knowledge of content and teaching" (KCT) and "knowledge of content and curriculum" (Ball, Thames & Phelps, 2008). We are going to consider primarily KCS and KCT in this study. These aspects of teachers' knowledge include on the one hand knowledge about students' mathematical concepts and typical mistakes or misconceptions (KCS). On the other hand, it consists of knowledge referring to most appropriate representations of mathematical concepts, decisions about teaching style or basic ideas for mathematical concepts (KCT).

Besides knowledge, motivation is a crucial part of teachers' mathematics related affect (Hannula, 2012) and, thus, a crucial part of teachers' professional lives. For this reason, a main aim of our study is to investigate teachers' motivation. Motivation in general is used to explain human behaviour. Accordingly, it determines orientations of goals, persistence and intensity of actions (Schiefele & Schaffner, 2015). For this reason, motivation is considered as a psychological process which initiates, maintains, directs and evaluates actions to achieve a positive valued state (Dresel & Lämmle, 2011; Rheinberg & Vollmeyer, 2012). Following Lewin (1946) motivation emerges out of the combination of components of a person and environmental attributes. Based on this model, Rheinberg, Vollmeyer and, Burns (2001) conclude that every performance, especially learning performance, requires motivation.

For the learning performance of teachers' in professional development courses, we refer to the construct of actual motivation considered by Vollmeyer & Rheinberg (1998) that consists of four dimensions: probability of success, apprehension of failure, challenge and interest. The first three sub-dimensions belong to achievement motivation. A central aspect of this motivation type is the comparison with an achievement scale. On that account, an individual compares itself with an individual, a social, an objective or an external determined norm (Heckhausen, 1974). Atkinson (1957) divided achievement motivation into an approaching and an avoiding component which is considered as "motivation to approach success" and "motivation to avoid failure". Both components belong to an expectation component.

By contrast, challenge and interest are considered as a value component. Therefore, challenge depends on subjective success probability. According to individual performance a person estimates task difficulty. Challenging tasks were considered as those that have medium difficulty. The learners are able to handle these, but they have to make an effort (Rheinberg & Vollmeyer, 2012).

The fourth sub-dimension "interest" is characterized by the person-object-theory of interest (Vollmeyer & Rheinberg, 1998). In this context interest is conceptualized as a special relationship between a person and an object (i. e.. physical things, tasks, special topics of knowledge or activities) that is characterized by positive emotional states and high subjective object values (Dresel & Lämmle, 2011; Krapp, 1999). Based on a survey involving 287 teachers, Schiefele, Streblow and Retelsdorf (2013) confirmed three dimensions of teacher interest. These belong to all three dimensions of teachers' knowledge (content knowledge, pedagogical content knowledge and general pedagogical knowledge). *Subject interest* is considered as interest referring to content knowledge taught in the classroom. Besides, it includes knowledge beyond school related content knowledge. By *pedagogical content interest* the authors understand interest to prepare lessons well, to acquire new methods for teaching and to read specialized literature. *Pedagogical interest* belongs to every aspect of pedagogy in school situations.

Eccles and Wigfield (2002) suggest a positive relationship between expectation (probability of success) and value (interest) component. According to these two aspects, we focus in this paper on the following two questions referring to motivation:

(1) Which effects show reflecting on students' learning results in professional development courses on teachers' achievement motivation towards KCS and KCT?

(2) Which effects show reflecting on students’ learning results in professional development courses on teachers’ interest towards KCS and KCT?

METHODS

In this study we want to investigate, whether reflecting on students’ learning results (see above) is a feature of sustainable and effective professional development. For this reason, we follow a quasi-experimental design whereas “reflection on student learning results” is the independent variable and motivation the dependent variable.

Referring firstly to the design of the whole research approach, we use a pre-post-test design which includes three groups of teachers. The first one takes part in a professional development course with reflecting on students’ learning results. The second also takes part in a professional development course without reflecting students’ learning results. Instead of reflecting on students’ learning results, these teachers examine the issues of a theoretically driven task design. For example, in a current course, the first course got an input about problem solving. They developed problem solving tasks referring to given criteria. In the distance phase of the PD program, the teachers were asked to give the problem solving tasks to their students, to collect students’ solutions, and to send these solutions to the facilitators. The second group follow the same program. However, they were not asked to collect students’ solution, but to improve the tasks on the basis of the teachers’ overall impression of the lessons where the tasks were integrated. The third group, i. e. the control group, do not get any intervention.

In this paper we only refer to the first two groups. The first group consists of 21 teachers and the second group consist of 15 teachers. The teachers of the two groups stem from different towns, but stem from the same region in Germany. Thus, the school system, the curriculum and partly also the teachers’ education is identical or at least similar. The geographical distance made it difficult for teachers to interact. Both groups showed a similar distribution of gender and age. For this reason, although we did - for pragmatic reasons - not used a randomization, there was no hint for systematic differences between both professional development groups in our quasi-experimental study. For the region we chose (the German federal state “Sachsen-Anhalt”) there is further the specific characteristic notable that most of the teachers in this region have none or little experiences with PD.

In the first meeting in the PD course, the first and second group completed a questionnaire involving items concerning motivation referring to KCS and KCT and concerning KCS and KCT and also beliefs. The second test will take place at the last meeting of the professional development course. There is a six-month time span between pre- and post-test which is equal to the overall duration of the PD course. In this paper, we restrict the focus on the pre-test and further to results referring to the teachers’ motivation.

The instrument for measuring motivation is based on a questionnaire from Rheinberg et al. (2001) which measures actual motivation. To measure motivation according to KCS and KCT we adapted the questionnaire. Therefore, we developed seven tasks which represent specific educational situations (see figure 1). The dimensions KCS consists of three situations, KCT consists of two situations and there are two situations for general pedagogy. The first version of the adapted measure was tested to reduce the number of situations as well as the number of items for every situation (first version includes eighteen items for each situation). Based on the results we deleted a situation in each pedagogical content dimension. So the completed questionnaire consists of the following situations: Investigating students' mistakes, creating a diagnose test, solving students' social problems, finding appropriate basic ideas and finding appropriate representations for mathematical concepts.

To shorten the number of items in each situation, we excluded one sub-dimension of actual motivation. As a result, the completed questionnaire consists of items referring to apprehension of failure, challenge and interest. The participants have to rate the items on a seven point Likert-scale with categories from "I do not agree" to "I fully agree". There are five items for interest, five items for apprehension of failure and four items for challenge. The following figure shows an example of a situation of motivation referring to KCS (creating a diagnose test) with items of the sub-dimensions.

Diagnose test

Introducing the topic of derivation, you are forced to make a diagnose test to measure students' performance on functions. Therefore, you create a diagnose test with eight tasks. For example, one task is:

Explain what a function is.

Please, rate the following statements:

	I do not agree						I fully agree
1. I like it if it is a puzzle to create tasks for a diagnose test. (*interest*)	1	2	3	4	5	6	7
2. For me it is embarrassing to create tasks for the diagnose test that are not appropriate. (*apprehension of failure*)	1	2	3	4	5	6	7
3. For me, creating tasks for a diagnose test is an appreciated challenge. (*challenge*)	1	2	3	4	5	6	7

Figure 1: Example of a situation measuring motivation according to KCS

The adapted questionnaire of Rheinberg et al. (2001) was tested by the authors in pilot studies and results showed good reliabilities of the subscales measured by Cronbach's alpha. In particular, the values of Cronbach's alpha differed between 0,593 and 0,929 in pilot studies. These are in common with the values for reliability calculated by Rheinberg et al. (ibid.) Therefore, we used the questionnaire in our present study.

RESULTS

At this moment, we are able to report results of the pre-test from both professional development courses. So we present the results of motivation referring to KCS and KCT and also referring to general pedagogy for these teachers. The following table reports means and standard deviations for teachers who are in professional development course with reflecting on students' learning results:

	KCS			KCT			general pedagogy		
	I	F	C	I	F	C	I	F	C
	4,03	2,39	4,69	4,67	2,47	4,89	4,45	2,82	4,99
	1,01	0,80	0,98	1,09	1,19	1,00	1,25	1,37	0,99

Table 1: Mean and standard derivation for teachers (PD with reflecting on students' learning results)

The table illustrates that the means of apprehension of failure (F) are similar in each category (KCS, KCT and general pedagogy). Further analysis of the differences between the means does not show any significant discrepancy. We gained similar results for the sub-dimension "challenge" for every category. In contrast, the means of interest referring to KCS and KCT differ significantly. Although there is a significant difference, the interests concerning KCT and KCS are highly correlated ($= 0{,}91$; $< 0{,}01$). For this reason, we made a deeper analysis of correlations. The results of these calculations are shown in the following table 2:

		KCS			KCT			general pedagogy		
		I	F	C	I	F	C	I	F	C
KCS	I	-								
	F	-,34	-							
	C	,66**	-,35	-						
KCT	I	,91**	-,22	,66**	-					
	F	-,35	,61**	-,48	-,20	-				
	C	,86**	-,15	,83**	,86**	-,23	-			
	I	,63**	-,19	,47*	,53*	-,35	,59*	-		
	F	-,49*	,62**	-,41	-,33	,79**	-,23	-,45*	-	

Table 2: correlations between all sub-dimensions with are significant * < , or ** < ,

The results show that the correlation between general pedagogical interest and KCS is

0,628 (KCS) and the correlation between general pedagogy and KCT is 0,529. Although the correlations are lower, they are also significant ($< 0{,}01$ for KCS respectively $< 0{,}05$ for KCT). Referring KCS, the mean of interest was significantly lower than the mean of challenge, and significantly higher than the mean of apprehension of failure. Analysis of the means in the other sub-dimensions show similar results. One exception is the relation between the means of interest and challenge in the sub-dimension KCT. In contrast, interest correlates significantly with each dimension of challenge ($< 0{,}01$) according to both dimension of PCK motivation.

With regards to achievement related motivation referring to KCS and KCT there are also significant correlation between them. In special, the correlation in the subscale challenge between KCS and KCT is 0,833 which is significant at $< 0{,}01$. In addition, the correlation in the subscale apprehension of failure between KCS and KCT is 0,611 which is also significant at $< 0{,}01$.

The results of the other teachers taking part in the professional development course without reflecting on students' learning results are shown in the following table 2:

	KCS			KCT			general pedagogy		
	I	F	C	I	F	C	I	F	C
	3,81	2,57	4,88	4,71	2,51	4,95	4,77	3,42	5,08
	1,27	0,99	1,13	0,50	1,05	1,00	0,90	1,18	1,04

Table 2: Mean and standard derivation for teachers (PD without reflecting on students' learning results; R = representation and BI = basic ideas)

Teachers' ratings from the second professional development course show similar means and standard deviations compared to the first group. Nevertheless, the main results of the second group are in common with those of the first group.

DISCUSSION

The main purpose of this paper is to determine teachers' motivation referring to KCS and KCT at the beginning of the professional development courses. This aim needs an analysis of the results referring different aspects of motivation and also correlations of different motivational aspects. Our first results show that values of motivation are similar in both groups, but there are some differences between them. Overall, the means of challenge and apprehension of failure demonstrate a positive perspective of teachers' achievement motivation. Teachers of both groups tend to be motivated for approaching success (Atkinson, 1957). Especially, the participants perceive the situations for KCS and KCT as challenges. According to interest, the means are on medium level for KCS and at a higher level for KCT. This is consistent in both groups. It could be explained by their existing knowledge structure and low PD experience.

Teachers' qualification based on seminars at the university and practical seminars. The main purpose of these were to prepare teachers for teaching mathematics. Therefore, they concentrate on representations, basic ideas, methods to teach, etc. For this reason, teachers interest could be high in situations that are referring to these topics.

The results have shown correlations between aspects of achievement motivation and interest which are significant in most cases. This is in common with the assumption of Eccles and Wigfield (2002), because they suggest a positive relationship between interest and probability of success. Our results support this assumption in both groups for teachers. Although there is a distinction on a theoretical level, our data involving high and significant correlations did not show a distinction. Especially, the correlation between interest of KCS and interest of KCT suggests that these sub-dimension of pedagogical content interest could be only one dimension. The results of apprehension of failure and challenge support this assumption, because there are also high correlations which are also significant. These results of the pre-test have to be investigated to verify if the assumption is the correct.

FUTURE RESEARCH

Based on the data of the pre-test, we want to investigate the differences of teachers' motivation after taking part in the PD program. We assume that motivation of teachers who take part in PD with reflecting students' learning results will increase in the sub-dimension KCS. Especially, we expect a growth in interest referring to the appropriate situations, because in the course teachers investigate students' learning results. Therefore, they are engaged in identifying students' concepts of mathematical ideas and constructs, which is a part of KCS. In contrast, we assume that motivation of teachers who do not reflect on student learning results will increase in the sub-dimension KCT. In their PD course they focus on instructional ideas. So it is possible to expect that their pedagogical content knowledge about these ideas increases. In consequence, the motivation referring to this sort of knowledge should increase, too. Apart from both dimensions of pedagogical content knowledge teachers do not engage in general pedagogical situations. Therefore, they do not acquire new knowledge and experience according to pedagogy. As a result, their motivation referring to general pedagogy should not increase.

We hypothesize, that a change in motivation could in consequence result in a change of teachers' knowledge and beliefs, because the teachers will initiate activities around the knowledge of students, if they have higher interest (Rheinberg & Vollmeyer, 2012; Schiefele & Schaffner, 2015). This assumption is also in common with the results of Rheinberg, Vollmeyer and Burns (2001).

The results indicate that motivation to acquire knowledge of content and students and knowledge of content and teaching coincide in one dimension. Therefore, there may be only a dimension "motivation referring to pedagogical content knowledge". These assumption is in common with the differentiation of Schiefele et al. (2013). Further investigations are necessary to prove the structure of motivation scale.

The study also includes measures of beliefs about teaching and learning mathematics as well as measure of KCS and KCT. So further analysis should provide insights in the network between the three categories of teachers’ professional competence. Especially, the comparison of the results of pre- and post-test should ensure whether reflecting students’ learning results is a feature of effective professional development. The analysis will show if the student-centred perspective enhances teachers learning (cf. Franke et al., 1998). Besides, there will be qualitative interviews with teachers of the course within reflecting students’ learning results is a feature. Based on this data we want to provide insights in teachers’ feelings and cognition referring to teaching out of a student-centred perspective and their motivation to analyse students’ learning results.

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Pedagogical context and its role in articulating mathematical knowledge for teaching

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This paper examines the design of tasks for developing and assessing mathematical knowledge for teaching, in particular the role of pedagogical context. It argues that pedagogical context plays a vital role in shaping the reasoning involved in generating correct responses and in the articulation of mathematical knowledge for teaching more generally. It concludes with suggestions for more fully specifying the design of tasks to developed and assess mathematical knowledge for teaching.

Keywords: pedagogical content knowledge, mathematical knowledge for teaching.

Compelling examples of mathematical knowledge for teaching (MKT) (Ball, Lubienski, & Mewborn, 2001; Ma, 1999) and evidence associating it with improved mathematics teaching and learning (Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Hill, Umland, Litke, & Kapitula, 2012) have sparked interest in making it central in the mathematical education of teachers. Despite this interest, programs still focus mostly on disciplinary knowledge rather than MKT. What is needed for a more solid shift in teacher education and professional development are robust tasks for developing and assessing MKT. This observation leads to a basic challenge: although an initial set of MKT tasks have supported the development of some measures, large numbers of compelling new tasks have not been readily forthcoming.

Several challenges hamper progress. To better understand these challenges, we have found it useful to reflect on the design of effective MKT assessment tasks used in large-scale evaluation projects. According to Hill, Sleep, Lewis, and Ball (2007), a key innovation in assessment studies that yielded demonstrable effects was the inclusion of pedagogical scenarios that frame mathematical problems situated in practice. From an examination of items on the National Teacher Examination (NTE) used in the United States in the 1980s, which described pedagogical context yet failed to measure consequential knowledge, they point out two ways in which the inclusion of pedagogical context can go awry. First, for many items, they found that the pedagogical context was merely “window dressing” (p. 119) — because the item would measure essentially the same knowledge if the pedagogical context were stripped away. Second, at another extreme, they discussed items that lacked a defensible solution because ambiguity in the pedagogical context allowed more than one professionally defensible answer.

To understand the design and functioning of pedagogical context in MKT tasks, we conducted talk-aloud interviews with research mathematicians and mathematically knowledgeable and experienced teachers.[1] Based on analysis and our continued efforts to support people in sriting MKT tasks, this paper extends the observations of Hill et al. (2007) to explain how pedagogical context matters in MKT tasks and argue that it plays a fundamental role in articulating MKT. We begin by describing our interview study and what we learned about how pedagogical context shapes the mathematical work of responding to MKT tasks and the MKT assessed. Finally, we argue that our analysis, together with our experiences in supporting others in writing MKT tasks, suggests that pedagogical context is essential to articulation of MKT, both in MKT tasks and more generally in the identification of MKT.

LEARNING FROM PERFORMANCE AND MIS-PERFORMANCE

In previous work validating MKT assessment items, our group found that research mathematicians, often from their missteps, revealed much that otherwise might be presumed trivial or remain tacit in our understanding of the work involved in responding to MKT items. Additionally, highly experienced and knowledgeable teachers often expressed aspects of the work that otherwise might have remained unrecognized and unnoted. In this study, we interviewed about 60 experts with 26 items from the Learning Mathematics for Teaching (LMT) (Hill et al., 2005), Measuring Effective Teaching (MET) (Phelps Weren, Croft, & Gitomer, 2014), and Diagnostic Teacher Assessment in Mathematics and Science (DTAMS) (Saderholm, Ronau, Brown, & Collins, 2010) instruments.[3] We analyzed both the text of items and interview data. In these analyses, we engaged in a logical analysis and professional vetting of the work of teaching (Hoover, Mosvold, Ball, & Lai, 2016).

For a lesson on comparing fractions, Mr. Howard wants to choose a model that will make it easy for his students to compare a wide range of fractions, in problems such as:

Which is larger, 2/3 or 3/5?

Which is larger, 1/6 or 3/16?

Which is larger, 2/7 or 3/10?

Of the following models, which would <u>best</u> serve his purpose?

a) Drawings of round pizzas
b) Drawings of rectangles
c) Pattern blocks
d) Money
e) These models would work equally well to compare a wide range of fractions

Figure 1: An example choosing representations item.

In an initial stage of the project, 46 interviewees read aloud and talked through the solution of 11 LMT items. (See Figure 1 for an example item.) The content of the items was from upper elementary and middle school topics in areas of whole number operation, rational number, and proportional reasoning. Twenty-seven interviewees were research mathematicians selected from the participant list of the annual Joint

Mathematics Meeting of the Mathematical Association of America and the American Mathematical Society. Many were eminent mathematicians from highly ranked research mathematics departments. Nineteen interviewees were expert teachers identified by nationally recognized leaders of professional development as most likely (within the United States) to know MKT. All had at least ten years of teaching experience. Many were themselves in leadership positions, but all had taught or been actively engaged with students and teachers in schools within the last five years.

After coding the pedagogical context of the items, we recorded how each element of the pedagogical context might be used in relation to decision points produced from analysis of the interviews (Figure 2). (An "X" indicates that the element of pedagogical context plays a role in making the mathematical observation in such a way that deleting it would remove the grounds for making the observation. The "X" is bolded for an element that is primary for the observation.)

Mathematical observations contributing to the solution	**Purpose**	**Problems**	**Models**
1. Patterns blocks and money work well only for restricted sets of fractions (halves, thirds, fourths and halves, fifths, tenths respectively), while circles and rectangles are more flexible.	X	--	**X**
2. Drawings introduce construction issues (imprecision, more room for error, …), circles even more problematic than rectangles (especially with denominators that are odd or multiples of odd numbers).	X	--	**X**
3. Rectangles are readily aligned for easy comparison (or sub-divided vertically for one fraction and horizontally for the other, yielding comparable pieces).	X	--	**X**
4. Other fractions may present difficulty or require special consideration (e.g., large or prime denominators, pairs of fractions that are not evaluated by other means, …).	**X**	X	X

Figure 2: Coding of the use of elements of pedagogical context for decision points identified from the narrative of competent performance for the example item.

We then replicated this process with 4-6 interviewees for each item for 10 items from the MET and 6 items from the DTAMS instruments. Analyses of interviews formed the basis for writing narratives for the role of pedagogical context in shaping the MKT assessed by each item. We found that items across all instruments had a teaching purpose/task and provided some form of instructional materials, records, and examples (such as example problems, student work, manipulatives or instructional representations, student explanations, and classroom dialogue) and that these elements of pedagogical context played a prominent role in supporting the mathematical observations that contributed to answering items correctly. We then analyzed the work across items to produce generalized narratives for items with the same task of teaching. (See Figure 3 for two examples.)

SHAPING THE MATHEMATICAL KNOWLEDGE MEASURED

We found that, in items designed to measure MKT, pedagogical context is needed in reasoning toward correct answers. It is used in making mathematical observations which, taken as a whole and vetted professionally, provide evidence for defensible answers. Together, the pedagogical context and supported observations provide a cogent characterization of the MKT reasoning involved in responding to the item.

More than this, the teaching purpose, a crucial element of the pedagogical context, often provides important orientation and sense of direction for the mathematical work involved. For instance, the task of choosing a model to compare fractions leads one to noticing which fractions are and are not easy to represent with different materials and what is involved in using the model for comparison. Instead of being asked simply to decide which of two fractions is larger, this MKT item asks for comparing the complexity of using different models. To persist with this task and to have a sense of how to judge the complexity of using a model, what it means to compare, and how to know when sufficient distinctions have been made, one needs to know the purpose — in this case, choosing a model that makes it easy for students to compare a range of different kinds of fractions. The example comparison problems provided in the item give a sense of the range of comparisons to consider. The set of models constrains the scope of the work and frames the set of issues to be considered.

Looking across items with similar tasks of teaching provides further generalization. Below is the generalized description for the task of choosing representations and for a second task of choosing examples (Figure 3). Notice that for choosing representations the first step involves sizing up a range of issues that might be pertinent, which then serves as a guide for knowing what to pay attention to when experimenting with the use of different representations.

Choosing representations

1. Recognizing the features and relationships prominent in the design of the objects being considered.
2. Considering how to use each representation for the purposes.
3. Considering and running through sensible test cases.

1. Tracking on the instructional purpose for the exercise (e.g., introduce a procedure, assess student understanding, provoke error, highlight a special case, encourage multiple approaches, etc.).
2. Considering the features of or what happens with particular numbers or examples by working through the given problems, playing with different ways that students might solve them, and determining what is different mathematically about the examples and how these differences might impact students' thinking, their approaches to solving the problems, or the mathematical issues that might arise.
3. Identifying what feature of the example addresses that instructional purpose and whether aspects of the examples obscure or get in the way of the instructional purpose.

Figure 3: Narrative of MKT reasoning involved for two example tasks of teaching

Our analysis across items led to three observations about the role of pedagogical context in shaping knowledge measured by MKT items, which we will use to support our argument for the role of pedagogical context in articulating MKT as a domain. Our first observation is that pedagogical context shifts tasks from being disciplinary mathematics tasks to being pedagogical mathematics tasks. Figure 4 summarizes key characteristics of disciplinary mathematics tasks as compared to pedagogical mathematical tasks. In the example task, the pedagogical context shifted the nature of the task from that of comparing fractions, which is the students' mathematical task in the context, to a pedagogical mathematics task of comparing models, where comparing fractions is a subordinate task carried out in the service of comparing models. Comparing models is not a pedagogical task just because it may have a pedagogical aim; it is a pedagogical task because the chosen model should work on a set of comparisons such as those given and should be easy to use. The implicated pedagogical mathematical work is figuring out how to use the given mathematical representations to carry out the example mathematical comparisons and deciding which numbers might pose thornier mathematical challenges. In the same way that two numbers shape the work of comparing fractions, the pedagogical context of a collection of fractions to be compared shapes the pedagogical mathematics task of choosing representations. A different set of fractions to be compared might have changed the work of which representation to choose.

Disciplinary mathematics tasks	*Pedagogical mathematics tasks (or mathematical tasks of teaching)*
Compare fractions Compute Solve a problem Justify a solution Identify structure	Choose representations Analyze errors Appraise nonstandard work Solve in different ways Follow others' thinking Size up incomplete reasoning

Figure 4: Contrasting disciplinary tasks and pedagogical mathematics tasks

Second, pedagogical context situates pedagogical mathematics tasks in contexts that require doing mathematics while holding onto and coordinating with pedagogical purpose. For instance, a person engaged with the pedagogical mathematics task of *choose representations* might specialize the purpose to *choose a model that will be easy for students to use*. Or, *appraise nonstandard work* might specialize to *which potential interpretation of thinking best fits with nonstandard student work*. The pedagogical purpose provides the basis for doing the work of the item; this basis is not mathematically determined. Exactly which fractions need to be compared? Which approaches to comparison might students find easy or hard?

Instead of changing the problem from a mathematics problem requiring mathematical knowledge to a pedagogical problem requiring pedagogical knowledge, the pedagogical purpose shifts the nature of the cognitive demand associated with the mathematical problem. It introduces potentially competing agendas and a need to track on purpose while engaging in mathematical work. In our analysis, this was particularly evident in the contrast between mathematician and teacher interviews, where mathematicians often lost track of pedagogical purpose in ways that led them astray, while teachers facilely tracked on and used pedagogical purpose to navigate decisions. Mathematicians would worry about not having determinant information when experienced teachers would have a sense of what is sufficient for answering underlying mathematical questions for the purpose at hand, even if it is not complete information. In addition, mathematicians often struggled to hold on to the question being asked, drifting off to other questions, often back into doing the mathematics problem given to students or exploring mathematical ideas seen as related to those problems but not related to the MKT question being asked. Our point here is that frequent missteps, despite displaying sophisticated disciplinary mathematical knowledge, made more apparent the distinctive character of the mathematical work required when carrying out that work with regard for pedagogical context.

Third, the pedagogical context establishes a basis for an orientation and character for mathematical reasoning distinctive to teaching as professional work. We have mentioned the way in which the pedagogical purpose of choosing a model to compare fractions provides an orientation for the mathematical work, giving it purpose and a sense of direction. Consistent with the two examples characterized in Figure 3, many of the items required doing mathematical work while heeding pedagogical purpose. It is as if, more than pedagogical knowledge or skill, pedagogical heed is required in responding to MKT items.

ARTICULATING CONTENT KNOWLEDGE FOR TEACHING

Our observations demonstrate ways in which the pedagogical context provided in well-designed MKT items shapes the MKT being assessed. In this section, we argue that the formulation of the pedagogical context is what *articulates* content knowledge for teaching – it gives expression to MKT.

The word *articulate* comes from the notion of dividing into distinct parts, which taken together convey a more complex sense of the whole. It can mean to pronounce clearly, but also to joint something — to formulate in an article or articles or to express or convey (a thought) by means of language. It is this sense that pedagogical context provides a means of expressing or conveying MKT that we mean here.

The analysis in the previous section demonstrated the role of pedagogical context in shifting the focus from disciplinary tasks to pedagogical mathematics tasks, associating tasks with pedagogical purpose, and establishing a particular orientation and character for mathematical reasoning distinctive to teaching. One way to interpret this is that without the pedagogical context these items would be limited to the domain of other mathematical tasks that typify disciplinary work (e.g., traits in Figure 4) and would fail to assess the distinctive knowledge and skill known to be associated with increased learning. In other words, the doing of mathematical work (such as comparing fractions) while keeping in mind a purpose (of choosing a representational model) and attending to what is involved (in using a model, as one uses it or talks about it) is common in teaching, but uncommon in the discipline of mathematics. For instance, a disciplinary impulse can lead one to focus on the mathematics problem given to students or to explore variations or generalizations of a mathematical problem (a distraction that played out in many of the interviews with mathematicians), losing track of the need to interpret the mathematical validity of a student's confusing approach or generate a mathematical problem with a solution satisfying specific criteria. These latter tasks typify MKT, and it is pedagogical context that allows for their expression and that thus makes visible the articulation of the task of teaching, such as shown in Figure 3.

Our analysis is limited to sampling from items that have been produced to date, with a set of features of pedagogical context that is likely narrow. For instance, student background is not a prominent feature and plays a minor role in the items analysed. This is likely a result of narrowness of existing items and likely to change as scholars continue to expand work in this arena. For instance, Goffney (2010) has pointed out the mathematical demands of equitable teaching and Wilson (2016) has explored the development of assessment items to measure such knowledge in relation to dual language learners. Despite these limitations, we propose that the role of pedagogical context is important in the development of tasks to support the development of equitable teaching and that lessons from the above analysis can provide valuable guidance.

We close by offering a suggestion about how MKT might be articulated in the work of specifying the design of MKT tasks, in line with an approach developed by Illustrative Mathematics. Their approach requires not only writing a problem, but providing a commentary (and sample solutions). Consider the item in Figure 5.

Ms. Seidel is introducing the distributive property. To motivate her students, she wants to give them an example that will focus their attention on how using the distributive property can simplify computation. In which of the following examples will the use of the distributive property most simplify the computation?

a) 12 x 29 + 12 x 38 = ___
b) 17 x 37 + 17 x 63 = ___
c) 13 x 13 + 15 x 15 = ___
d) 16 x 24 + 16 x 24 = ___

Figure 5: Choosing examples item

Based on a narrative for doing the task (Figure 6), a commentary might be written (Figure 7), where the commentary characterizes the MKT that the task is intended to develop or assess, intended use or the task, and the pedagogical context provided in the scenario. The production and review of such a commentary provide powerful tools for collaborative efforts to develop MKT tasks, where explicit statements about rationale for pedagogical context significantly enhance development, review, and professional sanctioning.

1. Tracking on the fact that the instructional purpose for the example is to focus students' attention on how using the distributive property can simplify computation.
2. Considering different ways of evaluating the expressions and of using the distributive property and what these imply about what it means to simplify the computation, including recognizing the following: the most reasonable way of using the distributive property in (a) yields $12(29 + 38) = (12)(67)$, which reduces the computation from two to one application of multiplication; the most reasonable way of using the distributive property in (b) yields $17(37 + 63) = (17)(100)$, which reduces the computation from two non-trivial applications of multiplication to one simple one; it is not clear how to use the distributive property in (c); and although there are numerous quantities that could be factored out of the two terms (to similar effect as in (a)), none significantly simplifies the complexity of the multiplication to be done (use of doubling can be made with or without the use of the distributive property).
3. Recognizing that in problems such as these the distributive property does not avoid multiplication, but does allow for regrouping quantities into powers of 10, which greatly simplifies multiplication in a base ten system, and that (b) is the only one that affords this opportunity.

Figure 6: Narrative for the MKT reasoning involved in choosing examples item.

Examples shape instructional opportunities, however crafting and choosing good examples requires mathematical dexterity and skill in doing mathematical problems while tracking on instructional goals. This task asks for an example in which the distributive property can be used to simplify computation significantly. The purpose of this task is to see whether teachers flexibly consider different ways of evaluating the expressions using the distributive property and, simultaneously, what these imply for efficiency of the computation. It requires recognizing that the distributive property does not avoid multiplication, but does allow for regrouping quantities into powers of 10, which greatly simplifies multiplication in a base ten system. The task is currently written as a multiple-choice item for assessment. But it also can be used for launching a discussion about the nature of examples for which the distributive property is useful.

The mathematical task of teaching is choosing examples, but the teaching scenario needs to create a realistic need for choosing an example that requires the distributive property. In this scenario, the pedagogical purpose is to motivate learning of the distributive property. In particular, the scenario proposes motivating the distributive property by giving an example that will focus students' attention on how using the distributive property can simplify computation. This means that the example needs to provide a sharp contrast in the extent to which the computation is simplified by using the property relative to not using it. The examples given in the options in this task are selected to create such a contrast, where only option (b) significantly reduces the complexity of the multiplication. The instructional setting of introducing the distributive property contributes to a sense that the scenario is realistic.

Figure 7: Commentary for choosing examples item.

Through this process, task developers can encode implicit hypotheses about what matters about the pedagogical context when teachers face particular content problems of practice. Teachers who are able to use the pedagogical context in tasks as a resource for responding to tasks demonstrate knowledge in a way that simulates teacher knowledge use in teaching; their reasoning with the pedagogical context can be used to scrutinize and make visible the implicit hypotheses, iterate item development, and refine articulations of mathematical tasks of teaching and MKT assessed.

Bringing together content and pedagogy has been a persistent theme in conversations about the content-knowledge education of teachers over the last 50 years. However, taking stock of scholarship on content knowledge for teaching, Graeber and Tirosh (2008) remind us that, while the concepts of pedagogical content knowledge and content knowledge for teaching are useful, the union of content and pedagogy remains elusive. Beyond introducing complexity and challenge for writing MKT tasks, pedagogical information plays a non-trivial function in tasks designed to develop and assess professionally situated mathematical knowledge by articulating constrained instances of the relationship between content and teaching that is at the heart of the notion of MKT. Ball (2000) characterizes the "intertwining of content and pedagogy" as a continuation of Dewey's (1964/1904) effort to find the "proper relationship" between theory and practice. Our growing understanding of the role of pedagogical context in the design of and reasoning within MKT tasks is beginning to give us a better understanding of the "*proper relationship*" between content and pedagogy in characterizations of content knowledge for teaching.

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2. We were denied access to COACTIV items and release of TEDS-M items occurred after we completed interviewing.

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Relationships between the knowledge and beliefs about mathematics teaching and learning of two university lecturers in linear algebra

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In this study we explore the potential relationships between the specialised knowledge of two university lecturers in linear algebra and their beliefs about mathematics teaching and learning. We scrutinise the lecturers' knowledge with the aid of the model Mathematics Teacher's Specialised Knowledge (MTSK), from which the subdomains Knowledge of topics (KoT), Knowledge of mathematics teaching (KMT) and Knowledge of features of learning mathematics (KFLM) enable us to establish associations with their beliefs. We found that the beliefs manifested by these two lecturers about methodology and subject significance are related to their KoT in terms of procedures and applications, KMT in terms of examples for teaching, and also to KFLM in terms of student errors.

Keywords: specialised knowledge, beliefs, university lecturer, linear algebra.

INTRODUCTION

Amongst studies into mathematics education, teachers' knowledge has received special attention. There has been found to be a connection between this knowledge and the teacher's beliefs, and both contribute to the quality of teaching. There have been few studies directed towards the relationship between university lecturers' knowledge and their beliefs about teaching and learning mathematics. We presented the findings of one study, focusing on the specialised knowledge deployed by a lecturer when teaching the topic of matrices and determinants, in a paper at the CERME meeting in the Czech Republic (Vasco, Climent, Escudero-Ávila, & Flores-Medrano, 2015). The paper presented here develops this line of research, examining possible relationships between the specialised knowledge displayed by two linear algebra lecturers, and their beliefs about teaching and learning mathematics.

MATHEMATIC TEACHERS' SPECIALISED KNOWLEDGE

In the last few decades, teachers' knowledge has attracted a great deal of interest from researchers, and various frameworks have been developed for the purpose. In our case, we have made use of the model *Mathematics Teachers' Specialised Knowledge* (MTSK) (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013). This model was developed in response to the difficulties experienced in applying the model *Mathematics Knowledge for Teaching* (MKT) (Ball, Thames, & Phelps, 2008), specifically with respect to the characterisation and demarcation of the different subdomains into which it is constituted, and aims instead to represent the specialised nature of teachers' knowledge as an integral and inseparable element of that knowledge.

The MTSK model comprises two knowledge domains: *Mathematical Knowledge* (MK) and *Pedagogical Content Knowledge* (PCK). In addition to these two domains, and in a relation of mutual permeability with them, are teachers' beliefs about mathematics and how the subject is taught and learnt. These operate on teachers' classroom practice. MK is itself constituted by the subdomains *Knowledge of topics* (KoT), *Knowledge of the structure of mathematics* (KSM) and *Knowledge of practices in mathematics* (KPM). For its part, PCK includes the subdomains *Knowledge of mathematics teaching* (KMT), *Knowledge of features of learning mathematics* (KFLM) and *Knowledge of mathematics learning standards* (KMLS). For details of the subdomains and categories making up the framework, readers are directed to previous CERME presentations (Carrillo et al., 2013; Vasco et al., 2015) and related papers (Vasco, Climent, Escudero-Ávila, Montes, & Ribeiro, 2016). In this paper we limit ourselves to a brief overview of the subdomains KoT, KMT and KFLM, as it is these which allowed us to establish the relationships between the teachers' specialised knowledge and their beliefs about teaching and learning mathematics.

KoT is defined as a thorough, grounded knowledge of mathematical content. It comprises the categories of *phenomenology and applications* (knowledge of phenomena associated with the meanings of a mathematical topic and ways a topic can be applied), *properties and fundamentals* (knowledge of properties which fulfil a mathematical objective or are necessary to carry out a procedure), *representations* (knowledge of the different ways a topic can be represented), *definitions* (knowledge of descriptions and characterisations of a concept, including related examples, along the lines of the constructs *concept definition* and *concept image* developed by Tall & Vinner, 1981), and *procedures* (knowledge about how, under what conditions and why something is done, and the key features which result in doing it).

KMT, as the name suggests, concerns knowledge about the teaching of mathematics and includes the following categories: *theories of teaching* (that is, specific to mathematics education), *material and virtual resources* (books, whiteboards, software, and so on as tools for teaching mathematics), and *activities, tasks and examples for teaching* (knowledge of examples for teaching a mathematical idea and their potency for the topic in question). It is important to distinguish between teachers' knowledge of examples relating to definitions, which, as noted above, pertain to KoT, and those relating to examples for teaching, which are the provenance of KMT, and allow teachers to highlight features specific to particular topics. In this respect, Bills et al., (2006) note that the deployment of examples in class is complex, involving careful choice in directing learners' attention adequately so as to reach the right generalisations.

The authors classify the different types of examples into generic examples (examples of concepts and illustrations of procedures), counterexamples (in support of an opposing hypothesis or proposition relating to a concept, procedure or step in a demonstration), and non-examples (of use in demarking the boundaries of a concept or procedure, or in demonstrating the conditions of a theorem). In considering how to maximise the effectiveness of the use of examples in teaching mathematical concepts, Blanco, Figueiredo, Contreras, & Mellado (2011) focus on the attributes *transparency* and *variation*. Transparency refers to degree to which the key aspects of the example are evident to the learner, such that it becomes a point of reference. In the case of variation Watson & Mason (2005) consider *dimensions of possible variation*, along which certain aspects or details might change while remaining within the limits of being a recognizable example of the concept in question (different ways of approaching the same concept).

Finally, KFLM is knowledge about how mathematics is learned. The chief focus is not on the student, but rather on mathematical content as the object of learning. The categories included here are: *learning styles* (knowledge of theories of the cognitive development of the student), *areas of strengths and weaknesses associated with learning* (that is, of the student in regard to the content), *the students' forms of interacting with the content* (knowledge of student strategies), and *the student's motivation with regard to mathematics* (knowledge of students' expectations about the content).

BELIEFS ABOUT TEACHING AND LEARNING MATHEMATICS

Teachers' knowledge and beliefs interact (Charalambous, 2015), and these interactions can lead to a better understanding of both aspects. In this paper we present possible connections between the subdomains of the specialised knowledge of two linear algebra teachers and their beliefs about the mathematics teaching and learning.

We are aware that different positions can be taken in relation to the terms "beliefs" and "conceptions". As the individual's set of incontrovertible personal truths, beliefs are an important means of taking account of the affective aspects of the teacher's personality. Conceptions, on the other hand, can be considered as a conceptual substrate, influencing all aspects of a cognitive nature and playing a key role in determining the teacher's thinking and actions (Ponte, 1994). In this study we will use the term "beliefs" to refer indistinguishably to conceptions and beliefs (as discussed above), although aware that our focus is chiefly cognitive.

In order to study the lecturers' mathematics teaching and learning beliefs (MTLB), we drew on the classifications in Carrillo & Contreras (1994) for tendencies (*traditional*, *technological*, *spontaneous* and *investigative*) and categories (*methodology, subject significance*, *learning conception*, *student's role*, *teacher's role*). Each category has corresponding indicators by which teachers' beliefs can be inferred. The aim is to arrive at an interpretative description of the teacher's beliefs, rather than assign the teacher to any specific tendency. For example, in the category *methodology*, the following indicators describe teachers' orientation in terms of classroom practice: 'repetition of [N type] exercises characterise classroom practice' (*traditional* tendency); 'the exercises aim to reproduce logical thought processes and, in consonance with this, error analysis by the students (*technological* tendency). In the case in hand, both descriptors help us to describe the conceptions of Carlos and Jordy regarding their classroom practice, with a view to identifying associations with their specialised knowledge, without delving into their corresponding teaching tendencies.

METHODOLOGY

This is a qualitative study, taking a case-study design (Yin, 2003). It focuses on two lecturers teaching linear algebra in the first year of a degree course at the University, and who, for the purposes of this study, will be referred to as Jordy and Carlos. We attempt to provide an answer to the research question, 'What interaction is there between the specialized knowledge of two linear algebra lecturers and their beliefs about mathematics teaching and learning?' In this paper, due to limitations of space, we will focus principally on the knowledge which is manifested in the use of examples. The two lecturers were chosen for their willingness to take part in the study and our intrinsic interest in the teaching cohort to which they belonged, with a view to subsequent participation. Jordy is a graduate of the Educational Sciences faculty, specializing in mathematics, with 22 years' experience in teaching mathematics at secondary level and 9 years at the university. Carlos is a geologist with 17 years' experience of teaching mathematics in the university.

Data collection was carried out via class observations (using video recordings) and semi-structured interviews. The topic of *matrices, determinants and systems of linear equation* was chosen to be observed as it was first in the study programme for the course in linear algebra, and essential for tackling subsequent topics. The analysis of these data followed the procedures of content analysis (Bardin, 1996), sifting through the teachers' actions and statements for evidence of knowledge pertaining to the appropriate MTSK subdomains, and beliefs about mathematics teaching and learning, using throughout the categories outlined above – from the MTSK model in the case of knowledge, and the analytical tools in Carrillo & Contreras (1994) in the case of beliefs.

RESULTS

Tables 1, 2 and 3 summarise the relationships between the teachers' knowledge and their beliefs. With regards to *methodology*, both lecturers pursued procedural objectives with their classes, demonstrating an interest in their students gaining mastery of the operations and algorithms associated with the mathematical content in question. Jordy and Carlos ascribed an instrumental end to teaching matrices (*subject significance*), of interest because they provide solutions to systems of linear equations or to find the value of a variable. This is made clear in their interviews:

Jordy: [In answer to what he wanted his students to learn about matrices] To be able to do the required operations, pose problems involving matrices and know how to solve them. Because matrices allow you to solve systems of equations.

Carlos: When we're talking about matrices, we're talking specifically about variables, and there are different procedures for finding variables when we apply matrices.

Evidence of knowledge about scenarios requiring the use of the content and potential applications (KoT – *phenomenology and applications*) is demonstrated in the case of Jordy through mathematical situations (solving systems of equations, fundamentally), and in the case of Carlos through some situations which could arise in real life. The kind of knowledge displayed in these instances would seem to be consistent with their beliefs about the *subject significance* of matrices (that is, serving practical ends, with high value given to the ability to reproduce the content further on in time). Below are the relevant excerpts from their interviews:

Jordy: Now we need to turn our attention to vectors, and after that other areas based on matrices, so you see that this content is important because of what comes later in terms of calculus and differential equations.

Carlos: I give the students practical examples from real life, so that they can see that matrices are not just applicable to a specific science, but to any area, such as nutrition, sport, or a factory.

There was also evidence of associations between the teachers' knowledge about KoT – *procedures* and their beliefs about teaching – *methodology*. Jordy chooses not to present the content as a unitary procedure; instead, he sets up a series of exercises which aim to reproduce the logical processes involved. Carlos, on the other hand, does decide to present the content in this way, and the lesson activity is marked by the repetition of exercises. In setting up the class in this way, Jordy manifests a depth of knowledge pertaining to the category KoT – *procedures*, which leads him to go beyond a plain exposition of the procedures in order to underline the reasons underlying their use. In the case of Carlos, the knowledge in evidence is of the type KoT – procedures, *How is something done*, and hence his knowledge would appear to be external to mathematics, attuned to an eminently utilitarian view of the subject.

Categories and indicators of beliefs	MTSK
Methodology: replicating patterns of thought (Jordy) and mechanical (Carlos). *Subject significance:* Orientation (applicability), Objective (informative, utilitarian) (Jordy and Carlos)	KoT-*procedures*: *How is something done?* (Jordy and Carlos) and the reasoning underlying procedures (Jordy). KoT- *phenomenology and applications*: applications to mathematics itself (Jordy) and applications to real life (Carlos).

Table 1: Associations between beliefs and KoT of Jordy and Carlos

There are also associations between Jordy's knowledge of examples for teaching (KoT – *examples for teaching*) and his beliefs about *methodology*. Jordy seems to conceive of the role of the lecturer as that of replicating the process of building knowledge via a selection of examples which allow different features of the content to be highlighted. The extract below is from his introduction to the topic of multiplying matrices:

Jordy: In order to add matrices we need to meet a condition. What was it?

Student: They need to have the same dimensions.

Jordy: They need to have the same dimensions. In order to multiply matrices we also need to meet a condition. If we take matrix A $\begin{bmatrix} 2 & 3 & 1 \\ 4 & -5 & 0 \end{bmatrix}$ What are the dimensions of this matrix?

Student: Two rows, three columns.

Jordy: The dimension is 2x3. To multiply two matrices we need the number of columns in the first matrix to be the same as the number of rows in the second matrix. If A looks like this, B has to have three rows, it doesn't matter how many columns it has. Let's imagine that matrix B is a column matrix $\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ It can be multiplied with this one, the only condition is that it has three rows. The dimensions of tis matrix are . . .

Student: 3x1

Jordy: If the matrix doesn't meet this condition, then as you say it cannot be multiplied […] [The lecturer writes two matrices, 3x2 y 3x4.] What we do is we take the first row of matrix A and we multiply it by the items in the first column of matrix B […].

Jordy generally presented the content using three of four examples, in this case avoiding entering into the topic of multiplying square matrices of the same order so as to make it clear to the students how important it is to define the dimensions of the matrices to be multiplied in order to determine whether it can be carried out or not. Jordy employed a wide variety of generic examples (Bills et al., 2006), which meant that he could focus students' attention on the more salient features and on aspects providing a wide range of images. We associate this knowledge of examples with *Knowledge of Mathematics Teaching* (KMT). The examples used by Jordy show evidence of transparency (Blanco et al., 2011), given that the matrices A(2x3) and B(3x1) used by the teacher are aimed at directing learners' attention towards salient aspects of the content (the dimensions of the matrices). In addition, the teacher draws attention to aspects that might vary in the examples employed (*dimensions of possible variation*, Watson & Mason, 2005), in this case the possible dimensions of matrix B in relation to A (leading to the examples B(3x2) and B(3x4)).

Furthermore, Jordy's conceptions are aligned with a kind of classroom practice favouring what could be called replicative structured exercises (that is, the student exercises are designed to encourage them to replicate the logical thought processes). He warns students about potential pitfalls along the way. This conception, considered in *methodology*, is associated with his knowledge about errors and difficulties in learning (KFLM – *areas of strengths and weaknesses associated with learning*). Hence, when dealing with the topic of multiplying matrices he advises the students to define the dimensions in order to avoid falling into the common error of applying the algorithm for adding matrices, multiplying (in the case of matrices with the same dimensions) elements in corresponding positions (as he explains in an interview):

Jordy: The error that they can always fall into first is thinking that they have to multiply number by number according to the position it's in. So, at least in this case, multiplying matrices, I tend to bang on about dimensions.

Thus, his knowledge of students' learning difficulties with respect to the content (KFLM) is associated with his knowledge about examples for teaching content (KMT) and his conceptions about *methodology* (replicative structured exercises, with emphasis on warning students about pitfalls). At the same time, his knowledge of examples is related to his conception of *teacher's role* (presenting content by means of replicating the process of its construction).

Categories and indicators of beliefs	MTSK
Methodology: replicating patterns of thought with emphasis on errors Teacher's role: deliver content by replicating process of knowledge construction	KFLM-*weaknesses associated with learning*: transposition of algorithm for adding matrices to multiplication; failure to note dimensions of matrices KMT-*examples for teaching*: variability of examples

Table 2: Associations between Jordy's beliefs and his KMT and KFLM

In Carlos' case, his knowledge about the category of *examples for teaching* (KMT) is intertwined with his beliefs about *subject significance*, that is, that the point of the subject is essentially informative with practical applications in everyday life. To this end, the lecturer develops the work on matrices around a set of exercises intended to reproduce real life situations (with the matrices representing, in this case, models of cars for different routes and with different fuel efficiency).

Carlos: I always give you practical examples so that you get an idea of how they can be applied […] You can see here that we've got two tables with information about different situations. The first gives different routes followed by four makes of car (3x4), while the second table shows the petrol consumption for each of the models over the five week days (4x5). If we put them into matrix form will they be the same or not?

Student: No, they won't.

Carlos: No, but that isn't what matters in this case. Here, the layout they're in is the right one to multiply them because what we do is multiply the rows by the columns; so the number of columns in the first matrix coincides with the number of rows in the other one […] We have to multiply the element which is in the first row, first column of table T by the element which is in the first row, first column of table G [...].

By going through exercises of this kind, the need to define the operation (the multiplication of matrices) emerges, and this approach is consistent with the lecturer's beliefs regarding framing what needs to be learned in practical terms. In this case, in terms of the lecturer's beliefs, methodology would seem to be based on the repetition of such exercises, with less emphasis on the logical processes involved and on errors (unlike the case of Jordy). The lecturer's KoT is evident in procedures (*how it is done*, essentially) and phenomenology (*application of content to other fields*).

On the other hand, regarding his pedagogical content knowledge, Carlos displays KMT with regard to examples for teaching. His knowledge, which would seem to be primarily related to the practical purposes to which the contents can be put, does not directly correspond to any of the categories in Bills et al. (2006), so much as to phenomenological examples. The example is transparent (Blanco et al., 2011) in the sense that directs' students' attention and illustrates a procedure (the product of matrices).

Categories and indicators of beliefs	MTSK
Methodology: mechanical repetition of exercises Subject significance: purpose (informative and practical)	KMT-*examples for teaching*: examples of applications to real life

Table 3: Associations between Carlos' beliefs and his KMT

CONCLUSIONS AND IMPLICATIONS FOR FUTURE RESEARCH

Having established the relationships between them, we can see how the knowledge and beliefs of the two lecturers studied interact. Their conceptions enable us to better understand their knowledge and vice-versa. Their conceptions about *methodology* and *subject significance* are consistent with the KoT (regarding procedures and applications) which each of them manifests in their practice. Likewise, their conceptions about *methodology* (class practice activities with exercises which replicate patterns of thought versus activities focusing on mechanical repetition, with possible emphasis on errors), and *teacher's role* indicate an association with the KMT that each demonstrates (in terms of knowledge of examples for teaching); in the case of emphasising error avoidance, there is an association with their KFLM (in terms of knowledge of errors). Both aspects (beliefs and knowledge) are reflected in the elements they give emphasis to in the course of their teaching. Hence, dealing with the multiplication of matrices, one of the lecturers chooses to emphasise the conditions required to apply the procedure and the avoidance of potential errors, the other the applications to real-life situations.

Teachers' knowledge and ways of thinking are essential to the teaching of mathematics, and need to be understood in order for teachers to be helped to improve their practice, and consequently their students' learning (Chapman, 2015). We have carried out an initial approach towards establishing relationships between teachers' knowledge and their conceptions about the teaching and learning of mathematics, and this contributes to our understanding of how and why teachers do what they do in the classroom. Detailed analysis of teachers' knowledge and conceptions about mathematics teaching and learning can help us gain a broader vision of the relationships between these two constructs. Furthermore, it would be interesting to explore further where the associations that have been uncovered lead, how they might influence, for example, the teacher's beliefs in generating specialized knowledge.

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CONCEPTUAL ISSUES IN DEVELOPING A FRAMEWORK FOR EXAMININGTEACHERS' KNOWLEDGE

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Abstract

When a researcher is required to consider the conceptual framework underlying a new model of professional knowledge, various questions are likely to come to mind: What are the authors proposing? What are the consequences of these proposals? What do they contribute? These questions are directed atexplicit elements of the conceptual framework. Nevertheless, there is another question which emerges: What theoretical position have the authors taken in developing the framework in this particular way? This is the question which we aim to address in this paper, shedding light not only on the MTSK model itself, but also on how it came into being.

Keywords: Mathematics Teachers' Specialized Knowledge; Teacher knowledge frameworks; Mathematical Knowledge for Teaching

INTRODUCTION

In the past three decades, a multiplicityof representations of teachers' knowledge have emerged (e.g. Topology of professional knowledge (Bromme 1994); Knowledge Quartet (Rowland, Turner, Thwaites, & Huckstep, 2005); Knowledge for Teaching (Davis & Simmt, 2006); Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008); TEDS-M framework (Tatto et al., 2008); Mathematical Proficiency for Teaching (Schoenfeld & Kilpatrick, 2008); Didactic-Mathematical Knowledge (Godino, 2009); COACTIV project framework (Baumbert, Kunter, 2013); Mathematics Teachers Specialized Knowledge (Carrillo, Climent, Contreras, Muñoz-Catalán, 2013); Knowledge Atom System (Scheiner, 2015)). These models have largely been grounded in Shulman's (1986) work, and have focused principally on two of the domains of teachers' knowledge his work drew attention to (subject matter knowledge and pedagogical content knowledge).

When one considers the number and diversity of such models, questions about their design and development naturally emerge, in particular with respect to the authors' assumptions. Sometimes these questions are given clear answers, but at others they remain implicit. Lerman (2013) claims that the authors of some models "*do not make any overt commitment on their position, though readers can infer where they stand*" (p.629). Specifically, his reflection concerns whether theapproach underlying aparticular modelis individual or social.

In this paper, our aim is to give an account of the choices and assumptions involved inthe process of developing the Mathematics Teachers' Specialized Knowledge model, MTSK (Carrillo, et al., 2013), so as to allow the readers, in keeping with Lerman’s charge, to know where we stand.

Mathematics Teachers’ Specialized Knowledge

This section provides an overview of the MTSK model, giving particular attentionto the manner in which professional knowledge is organised in the model, so that subsequent observations in the following sections concerning the conceptual structure of the model can be better appreciated.

The aim of the model is to provide an interpretative framework from which to explore our understanding of the knowledge deployed by teachers in the course of their teaching and related activities. It is this very professional dimension of the knowledge in question which, from this perspective, is designated “specialised”, whenever the connection to mathematical content is clear. As a result, the model excludes all those elements which can be considered as pertaining to the domain of general pedagogical knowledge.

Building on the foundational work of Shulman (1986) and Ball et al. (2008), we retain the dichotomy between subject matter knowledge (here mathematical knowledge) and pedagogical content knowledge, PCK, andadd a third domain encompassing beliefs. Each of these three principal domains is further subdivided, as described below.

Mathematical knowledge consists of three subdomains which reflect the nature of mathematical content. Firstly, there is necessarily the knowledge associated with specific mathematical topics. This includes such things as definitions, phenomenology, properties, meanings and the operations that can be carried out. Together, these elements constitute the subdomain Knowledge of Topics (KoT).Second, there is the knowledge of the complex web of interrelations which bind these topics and associated concepts together. It is a kind of topographical knowledge, or to take another metaphor, it concerns the understanding of the edification of mathematical knowledge, whence the name Knowledge of the Structure of Mathematics (KSM). Finally, the third subdomain, Knowledge of the Practice of Mathematics (KPM), comprises knowledge of the syntax of mathematics (Schwab, 1978) and heuristic techniques related to problem-solving.

With respect to the domain of Pedagogical Content Knowledge, we follow tradition in focusing on teaching, students and the curriculum, albeit with variousspecific interpretations deriving from our particular position on certain issues (see below). The first of the three subdomains into which this area is divided, Knowledge of Mathematics Teaching (KMT), concerns, on the one hand, the knowledge the teacher can draw on regarding resources, methods, strategies and modes of presenting content to students, and on the other, the theories, whether personal or institutional, which underpin their approach to teaching mathematics. The second subdomain is Knowledge of the Features of Learning Mathematics (KFLM). This is the knowledge relating to how mathematics is learned, and includes the different kinds of difficulties, obstacles and strengths that students encounter, the types of interaction they tend to have with the content, and theories of learning mathematics which provide teachers with the tools for understanding learning. Finally, Knowledge of Mathematics Learning Standards (KMLS) comprises all reference material within the teacher's scope that enables them to decide what should be taught at any particular point along the educational spectrum, and how it should be taught. This includes, for example, knowledge of the curriculum, different professional standards, and even the opinion of senior teachers.

Finally, there is the domain of beliefs, which include beliefs about mathematics itself, as well as those about how it is learned and how it should be taught (Ernest, 1989).

PRE-DESIGN CONSIDERATIONS

This section summarises the considerations which went intothe design of the MTSK model. First,we discusswhat prompted the decision to create a new model in the first place. This is followed by a focus on the notion of specialisation, which is central to our conceptualisation, and how this affects our view of the role of mathematics in the model. Finally, we offer a brief account of our beliefs about mathematics teaching and learning and how these have influenced features of the model.

Why a new model?

Currently, the MKT model (Ball et al., 2008) is widely used in the research literature as an effective tool for capturing the knowledge which teachers deploy as they go about the business of their profession. Nevertheless, the model has been observed to have certain limitations deriving from the overlap between subdomains, and from problems in the notion of specialized content knowledge (Silverman, Thompson, 2008). The model also takes an evaluative perspective towards teachers' knowledge, with the ultimate aim that of assessing the quality of the knowledge under consideration. Another shortcoming is that it is designed to be used primarily in the classroom and is less effective as a tool for analysing teachers' knowledge outside this context.

Such issues as these, which affect the theoretical principles of the model, could be addressed by recasting it with modifications to the definitions of the subdomains, but in deciding to adopt a whole new model and to eliminate completely the subdomain of specialized content knowledge, we aim to draw attention to the intrinsically professional nature of all the teacher's knowledge. Furthermore, as shall be seen, the MTSK model is predicated on an interpretative perspective, giving it a significantly different orientation to its predecessor, and is designed to offer a multi-methodological access to data, in the sense that it allows the analysis of teachers' knowledge to be effected beyond the classroom, providing the opportunity for a fuller understanding to be achieved.

The role of specialisation

As suggested above, our conceptual basis takes a different view to that of MKT with respect to the way in which teachers' knowledge can be regarded as specialised, maintaining that certain aspects of their knowledge is exclusive to the teaching profession.

Specialised is understood, within the ambit of our model, to refer to any knowledge relating to the discipline in question deployed in a professional context, whether that be the actual activity of classroom teaching, or by extension lesson preparation, some type of procedure for reflecting on one's teaching, or teaching-based conversations with peers. In this sense specialisation is holistic, a feature intrinsic to all elements of teachers' knowledge included in the model, including the network of connections between subdomains, to which we also give prominence. In this respect, we extend the early ideas of Shulman (1986) regarding PCK to encompass the entire domain of mathematical knowledge.

The role of Mathematics

One drawback levelled at some models is that their conceptual framework is not specific to mathematics. In the MTSK model, as detailed in the sections below, the domain of SMK is reconfigured and renamed MK. This is more than just a change of label. The new domain is conceived of and structured in terms of categories which reflect the different features of mathematical knowledge. Likewise, the domain of PCK only addresses elements of knowledge pertinent to mathematics or to specific questions of teaching and learning the subject, leaving aside any unrelated item. Thus, for example, the subdomain Knowledge of the Features of Learning Mathematics excludes elements that can be considered general pedagogical knowledge, such as being aware that the types of activities appropriate to the latter part of the school day should take into account the pupils' tiredness.

Our beliefs about mathematics, and its teaching and learning

In line with Schoenfeld (2011), we recognize that our own beliefs about mathematics and how the subject is taught and learned (Ernest, 1989)permeate the very conceptual basis of the MTSK model. This is true of any researcher or research group. The position of any group or individual can always be perceived behind any theoretical proposal, not merely when the model is put into practice, but in the actualconceptual base itself where decisions about what categories to include and what form they should take are necessarily determined by the researchers' beliefs.In our case, a dynamic vision of mathematics (Ernest, 1989) leads us, for example, to accord a significant role to heuristic strategies, which we locate in the subdomain Knowledge of the Practice of Mathematics; the same belief prompted us to include a subdomain comprising knowledge of the different types of connections and associations in mathematics (Knowledge of the Structure of Mathematics). By the same token, a belief in the value of active inquiry in mathematics education (Carrillo and Contreras, 1994) resulted in our emphasizing, among other things, the interaction of the pupil with the subject, which in turn demands that the teacher be aware of how to contrive circumstances favourable to such interaction, as included I the subdomain Knowledge of the Features of Learning Mathematics. In general, a relativist and social view of knowledge accounts for the inclusion of varying curricular proposals and the experience of fellow teachers as sources of Knowledge of Mathematics Learning Standards.

DESIGN CHOICES

In this section we outline the considerations which were taken into account at the design stage of the model. The organization of the section follows the structure of the model itself, starting with the research perspective that was adopted, then moving on to the specific features of the three domains which make up the model – Mathematical Knowledge, Pedagogical Content Knowledge, and Beliefs – and finally discussing the methodological questions which were considered relevant to using the MTSK model for research. To this we add a brief observation about the educational levels we envisage MTSK being used with.

Research perspective/paradigm

Models of professional knowledge tend to exhibit a certain orientation according to their use. For example, studies using the MKT model (Ball et. al, 2008) are most frequently directed towards an evaluation of teachers' mathematical knowledge within the ambit of primary education in the USA. In the course of developing MTSK, our objective was to construct a model that would allow us to study the nature of mathematics teachers' knowledge without making any judgements about perceived gaps or the closeness of fit to some ideal profile (e.g. Godino, 2009). Rather we have aimed to work consistently within an interpretative paradigm (Bryman, 2001), albeit aware that once such an analysis has been undertaken with an individual or group of teachers, there always remains the option of setting about identifying areas where knowledge is lacking with a view to supplying the deficiencies.

Organisation of the domain of mathematical knowledge

In the model proposed by Ball et al (2008), the distinguishing feature by which knowledge is considered common or specialised is whether it is shared with other professions or not. In other words, the domain of mathematical knowledge is characterised in terms of the person using the knowledge, and is thus extrinsic to mathematics itself. Consequently, when we set about designing MTSK we made a commitment to characterising the domain of mathematical knowledge not in terms of the user but in terms of the ways that the subject is understood. To this end, KoT is defined as knowledge of specific mathematical topics, at both basic and advanced levels. By the same token, KSM aims to capture an overarching vision of mathematical knowledge, in which the teacher makes connections between different concepts and topics. Finally, KPM concerns what has been termed 'syntactic knowledge' (Schwab, 1978). The scheme (Figure 1) we have adopted in dividing up the domain of mathematical knowledge has proved its worth in the field, allowing us to characterise the knowledge displayed by different mathematics teachers in the realms of fractions (Rojas, Flores, Carrillo, 2015), infinity (Montes, Carrillo, 2015), algebra (Vasco, Climent, Escudero-Ávila, Montes, Ribeiro, 2016) and geometry by means of analyzing the teachers' reflections on their teaching for indications of the nature of this knowledge, and by making inferences about the particular way they understand a topic from how they manage the learning taking place in their classroom.

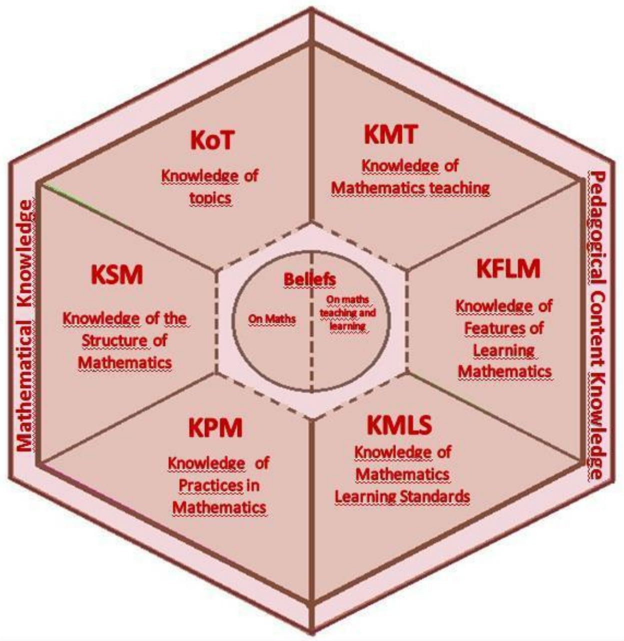

Figure 1: MTSK scheme

The organisation of the PCK domain

In the domain of Pedagogical Content Knowledge, we see no cause for rethinking the tripartite configuration of teacher-pupil-curriculum inherited from Shulman (1986), although the MTSK model does seek to characterise the specific contribution of mathematical content to this area of knowledge. Thus, in our efforts to develop descriptors for each subdomain(Sosa, Flores-Medrano, Carrillo, 2015) all pedagogical considerations unrelated to mathematics were excluded. In the same spirit, when considering the conceptual basis of the domain, and especially the subdomains, the influence of research on teachers' knowledge was acknowledged, recognising that, in addition to the habitual sources, teachers can develop their knowledge through an interest in research into mathematics education.

Theinclusion of beliefs

A significant aspect of the way in which the MTSK model was conceived, and shared with the COACTIV model (Baumbert, Kunter, 2013), is the inclusion of teachers' beliefs as an element permeating their knowledge. This inclusion derives from our research into beliefs (Carrillo, Contreras, 1994), the experience of which has confirmed the interaction of beliefs and knowledge in multiple ways (Aguilar, 2016).

Methodological aspects

Unlike previous models, MTSK does not assume that the chief source of data for research into teacher knowledge is what goes on in the classroom. Rather, following Baxter and Lederman (2001), our premise is that any research aiming to truly explore professional knowledge should take a multi-methodological approach, as a significant part of this knowledge is not necessarily manifested through the actual teaching that takes place. For this reason, our research using MTSK typically involves a combination of data sources which includes interviews, questionnaires, discussions of events during the lesson and hypothetical scenarios, analysis of contributions to online forums, as well as, of course, analysis of actual classroom practice, although this latter may not always be practicable. Likewise, the methodological approach which was adopted can be divided into two phases.In the first of these, based on Grounded Theory (Strauss and Corbin, 1998), various doctoral theses contributed to the development of the theoretical construct, specifying the content of subdomains, refining descriptors and working through actual examples, the detailed consideration of which helped to clarify the conceptual base. In the second phase, content analysis using MTSK is being carried out along with educational studies seeking to establish the model's potential for identifying areas for attention in initial and in-service training.

Applying MTSK to different educational stages

As mentioned above, the development of MTSK was carried out at the same time as various doctoral studies focused on diverse contexts from primary, through secondary to university education. In each instance, with appropriate adaptations to the specific mathematical context involved, the model proved itself to be useful. The model is also currently being applied to teachers working in nursery education, and there are projects in the pipeline which aim to study the adaptability of the model in analysing the knowledge deployed by teacher trainers. In our view, this kind of versatility is an additional bonus, although we are aware that, especially in the domain of mathematical knowledge, determining what is KSM and what is KPM could result in certain difficulties given the nature of the mathematics to be covered at each educational stage.

FINAL REFLECTIONS

This paper has outlined the theoretical position and the decisions taken by the research team regarding the conceptual base of the MTSK model. The aim has been to go beyond a simple description of the structural organisation of the model to provide a full and clear account of the conceptual groundwork on which the model is based for any researcher wishing to know more about it.

In our view, detailed accounts such as this, consistent with that proposed by Lerman (2013), can lead to a more appropriate use of models predicated on different conceptual foundations, and attuned to the aims and standpoint of the researchers in question. Where a new use is proposed for a model, the need for a process of reflection becomes evident, which can lead to a shift in the conceptual base, and which in itself would represent a contribution to the discipline.

Finally, we believe that amongst the different theoretical positions adopted by the various models that have been developed, there is a certain amount of common ground to be found, and consequently the search for common elements among them would not be fruitless and could potentially lead to Networking processes (Prediger, Bikner-Ashbahs, Arzarello, 2008).

ACKNOWLEDGEMENTS

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Reasoning and proving in mathematics teacher education

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There is evidence for recommendations to link mathematics teacher education (MTE) closely to school mathematics and to emphasise proving why *rather than* proving that *when teaching reasoning and proof (R&P) in schools. In spite of that we suggest not to take the implication that MTE focuses on* proving why *to extremes. We outline the background, framework, and results of a pilot to an intervention study that seeks to address the problems of R&P in MTE. The results suggest that teachers face more problems with R&P than expected and have difficulties just selecting situations from school in need of a mathematical justification, let alone developing justifications and supporting their students' learning of R&P. This supports our suggestion that a dual emphasis on* proving that *and* proving why *is needed in MTE.*

Keywords: Mathematics teacher education, reasoning and proof, Patterns of Participation (PoP), *proving that – proving why*.

Recommendations for teacher education increasingly emphasise issues that are specific to the profession. This is so in suggestions that academic mathematics does not suffice as teachers' content preparation (e.g. Ball, Thames, & Phelps, 2008; e.g. Rowland, Turner, Thwaites, & Huckstep, 2009). It is apparent in the explicit emphasis on the tasks of teaching in the college based parts of programmes, which involves a shift "from a focus on what teachers know and believe to a greater focus on what teachers do" (Ball & Forzani, 2009, p. 503). And the professional emphasis shows in suggestions that teaching-learning processes in mathematics teacher education (MTE) should model those envisaged for school mathematics if reform recommendations are to materialize in school (Krainer, 1998; Lunenberg, Korthagen, & Swennen, 2007).

We present the pilot to an intervention study for teachers in primary and lower secondary school in Denmark. The study is on reasoning and proving (R&P), a notoriously difficult topic for all students, including prospective teachers. In line with the general trend outlined above, it has been suggested to use approaches recommended for schools in mathematics teacher education and shift the emphasis from the disciplinary practice of *proving that* to *proving why* (Rowland, 2002). Our study, *R*easoning *a*nd *P*roving *i*n *T*eacher *E*ducation (RaPiTE), is in line with this recommendation, but building on studies of practising teachers we argue that it should not be taken too far. In a sense to be explained later, we suggest that prospective teachers need to become engaged in R&P processes that are "sufficiently close" to classroom practice as well as to the discipline of mathematics.

As the rationale of RaPiTE is based on research with practising teachers, the pilot addresses the question of whether the dual emphasis on school and academic mathematics is suitable for MTE. One result of the pilot is that research participants face even greater difficulties with R&P than expected and have problems identifying classroom situations in need of a mathematical justification, let alone devising justifications and engaging their students in working with them. This supports that MTE needs to be close to both instruction in schools and to the disciplinary practice of *proving that*, if teachers are to facilitate their students' proficiency with R&P.

Below we outline recent scholarship on R&P in school mathematics and MTE. We then present our framework, Patterns of Participation (PoP), and elaborate on the approach of RaPiTE, before describing the organisation, methods, and results of the pilot. We finish with a discussion of how the results relate to the main study.

R&P IN SCHOOL AND IN MATHEMATICS TEACHER EDUCATION

It is generally agreed that R&P are treated with less care than they deserve in schools and that mathematical argumentation at times degenerates into authoritative proof schemes (Harel, 2007). Proofs are often dealt with in secondary geometry only and for the purpose of verification of results that are presented ready-made. A.J. Stylianides (2007a) suggests that the late introduction of proofs may cause a disconnect in students' mathematical experiences that contributes to later problems with R&P. Yackel & Hanna (2003) argue that current uses of proof do not allow students to develop understandings of and proficiency with the multiple purposes of proving and do not facilitate their understanding of the contents in question.

Addressing these problems, Yackel and Hanna (2003) focus on the exploratory and communicative functions of proof. NCTM (2008) locates proof in a reasoning-and-proof cycle of exploration, conjecture, and justification. Emphasising proving as a specifically mathematical mode of justification, A.J. Stylianides (2007b) suggests that proving in school (and elsewhere) be understood as making mathematically valid inferences on the basis of what is or may become taken-as-shared in terms of content and modes of argumentation in the community in question. These recommendations seek to engage students at all school levels in a range of R&P-processes, including developing specifically mathematical justifications. They also intend to develop students' understanding of the meaning of mathematical reasoning as well as of the topic under investigation. The latter of these intentions implies a shift of emphasis from *proving that* to *proving why* in school mathematics.

One suggestion for how to focus on *proving why* is to rely on generic arguments. They can be based on a *single-case key idea inductive argument* (Morris, 2007). As an example, consider the case of Larry, a grade 5 teacher whose class is working on perfect squares (Skott, in press). The students have previously made geometrical representations of square numbers with centicubes (cubes that may be assembled and used e.g. for teaching place value). The class has now made a table of the natural numbers from 1 to 14 and their squares on the board. This leads to the observation that $5^2–4^2=9=5+4$. A single-case key idea inductive argument (not developed in the class) may build on the geometric representations used before. Placing two squares with side lengths 4 and 5 on top of each other with two pairs of sides aligned provides an explanation of the result and may be used to develop a generic argument that the difference between two consecutive perfect squares is the sum of their bases.

While it may alleviate some problems with R&P to engage students in investigating and conjecturing and focus on *proving why*, it may not sufficiently address others. In particular, students may become involved in the first two phases of the reasoning-and-proof cycle, but still rely on empirical or other justifications that do not qualify as mathematical. In Bieda's (2010) multiple case study experienced middle school teachers use a textbook that emphasizes R&P. In class, students produce conjectures in response to textbook tasks, but only in about half the cases do they provide some form of justification. Further, students' example-based justifications are accepted as much as their more general ones, and they had little opportunity to develop understandings of the specifics of mathematical reasoning and proof. A possible explanation, Bieda says, is that the teachers become involved in a reform agenda that prioritises "student-centred teaching", which requires them to play a relatively unobtrusive role in relation to the students' learning.

Similarly, Larry (cf. above) never capitalised on the students' conjecture that $(n+1)^2–n^2 = (n+1)+n$, and sought to develop a mathematically valid justification, generic or otherwise. Also, the first author's longitudinal study of a teacher, Anna, suggests that her intention of supporting the development of students' proficiency with mathematical communication and R&P is often submerged by other concerns, e.g. not to jeopardise her relationship with the students (Skott, 2013). Consequently she accepts arguments and justifications that do not qualify as mathematical.

The studies mentioned above suggest that teachers find it difficult to capitalise on the R&P potential of situations that "arise naturally from students' work as they explore mathematical phenomena, examine particular cases, discuss alternative hypotheses, and generate conjectures" (A. J. Stylianides & Ball, 2008, p. 312). As a result of the difficulties, mathematical R&P may lose its content specificity. We suggest that PoP may be able to explain why, and we let these explanations inform our development initiatives on R&P in MTE.

THE PATTERNS-OF-PARTICIPATION FRAMEWORK AND RAPITE

The PoP-framework adopts a participatory approach to human functioning, drawing on social practice theory (e.g. Holland, Skinner, Lachicotte Jr, & Cain, 1998; Lave, 1997; Wenger, 1998) and on Sfard's theory of commognition (Sfard, 2008). These frameworks focus, respectively, on emerging social processes (e.g. romance at a US university campus, cf. Holland & Eisenhart, 1990) and on well-structured cultural practices (e.g. mathematics, cf. Sfard, 2008). Rather than focusing on the practices per se, however, PoP re-centres the individual and asks how a teacher's involvement in unfolding school and classroom events relates to and is transformed by her re-engagement in other past and present practices and discourses. We have found the I-me distinction in symbolic interactionism (Blumer, 1969; Mead, 1934) helpful for this purpose. It allows us to focus on how the teacher takes the attitude to herself of different individual and generalized others as classroom processes unfold.

If, for instance, a teacher seeks to develop a good mathematical argument with a group of students, who appear to be weak and vulnerable in the situation, she may simultaneously take the attitude to herself of colleagues who focus on creating trusting relationships with the students; of the school leadership or of parents, who emphasise students' performance on standardized tests; or of her teacher education programme that focuses on the use of manipulatives to facilitate student learning with understanding (Skott, 2013, 2015; Skott, Larsen, & Østergaard, 2011). The teacher's engagement with each of these social constellations – or others – may transform or subsume her involvement in the practice of mathematical R&P and for instance have her accept justifications that do not qualify as mathematical. PoP provides a perspective on if and how this is the case.

PoP has so far framed studies conducted "in the perspective of teacher education" (Krainer & Goffree, 1998). These studies are not on MTE, but develop understandings of teaching-learning practices in schools and may raise questions about MTE and inform decisions on how to address them. As indicated above, the results suggest that even when teachers engage students in elements of the R&P cycle, modes of justification may lose their subject specificity. To avoid this it seems that MTE needs to fulfil two requirements. First, it must be close to teaching-learning processes in schools, as R&P practices are otherwise too distant from classroom interaction for teachers to draw on them when teaching. This is in line with the suggestion to emphasise *proving why* using generic arguments (Rowland, 2002). Second, and in spite of that, MTE must be close to the disciplinary practice of R&P and include significant elements of *proving that* so as to make mathematical R&P a practice for teachers to draw on as they interact with their students and to limit the risk of classroom processes losing their subject specificity. The assumption of RaPiTE, then, is that MTE needs to avoid the two extremes of focusing either on academic mathematics or school mathematics, not by reducing the emphasis on either but by transforming both (Skott, in press).

To be "sufficiently close" to both school mathematics and academic mathematics we use tasks and conjectures that may be used in or developed from tasks used in school and take them beyond the school level. Examples include:

(1) *Does 8 always divide* n^2*-1, if n is an odd integer?* (This is from an interaction in Larry's grade 5, cf. the previous example on perfect squares (Skott, in press));

(2) *Assume that you have a set of rods similar to Cuisenaire rods representing the positive integers from 1 to* n. *For what values of n can you make two "trains" of rods of equal length? Three trains?* m *trains?*

THE PILOT STUDY

The pilot study takes place at prestigious college in Denmark. The student teachers (from now on: teachers) have all performed fairly well in secondary school, and according to curricular documents they have worked with mathematical reasoning both in primary and secondary school. At the college, they need to specialize in Danish or mathematics. The research participants are a class of 31 prospective teachers for grades 4-9, who are among the 35 %, who specialize in mathematics.

Organisation and methods

The pilot consists of two parts, a questionnaire and a short teaching-learning sequence on R&P in connection with the teachers' first practicum. We do not expect the questionnaire and the observations of the teaching-learning sequence to shed light on relatively stable and context-independent mental constructs. Also, we do not assume any causal relation between responses to the questionnaire and teachers' contributions to classroom practice. At best, the questionnaire allows us to understand how the teachers react discursively to R&P in a setting in which they are not challenged by other concerns that may emerge in classroom interaction. From a PoP perspective it is an empirical question, whether teachers orient themselves towards such a discourse as they engage with their students in the classroom. However, if teachers face significant problems with R&P in the questionnaire, we consider it unlikely that they engage proficiently with these processes when teaching.

At the beginning of the academic year the teachers fill in the questionnaire, which consists of open items on why they decided to go into teaching, why they chose to specialize in mathematics, and what their general experiences are with school mathematics. They are also asked about specific experiences with R&P (e.g. "Describe how you felt about reasoning and proofs in mathematics") and to consider situations from school mathematics with an element of mathematical reasoning.

The second part of the pilot, the focus of the present paper, is the teaching-learning sequence on R&P, which is connected to the teachers' first practicum. As part of their first course on mathematics at the college, they are re-introduced to R&P in a 12-lesson sequence, organised as two sessions of six 45-minute lessons. This sequence was not taught by the authors of the present paper, but the second and third authors planned it and developed the teaching-learning materials. The intentions and the contents were discussed in detail with the colleague, who taught the sequence.

In the sequence the teachers are introduced to different types of arguments (cf. G. J. Stylianides & Stylianides, 2009), which leads to discussions of why R&P is taught in school, of what to expect in terms of student learning, and of the relationship and possible transition from empirical arguments to proofs. Also, the teachers watch and discuss a video of school students making and justifying conjectures about a number pattern in a sequence of geometric figures. This leads to discussions about the quality of the students' arguments and how students may be supported in developing them further. Subsequently, the teachers become involved in all three parts of the R&P cycle, for instance as they work on a version of the second task mentioned previously on making "trains" of equal length. As part of this they are to make geometrical or number theoretical justifications for their claims. They also discuss comments from school students, who have previously worked on the same task. One of these reads:

> "If I am to make two trains of equal length the sum must be even. If the sum is odd, I would have one left over. If the sum is even there could be other problems [...]. We do not know if it is sufficient that the sum is even."

After these sessions on mathematical R&P, the teachers form eight groups of three or four, each group going on a two-week practicum in a middle or lower secondary school. Before and during their practicum the students are to (1) plan for their students' involvement in R&P; (2) video record each other's teaching; and (3) select one video clip from the practicum in which the students are particularly involved in R&P. After the practicum, the teachers discuss the video clips and the inherent potentials for and problems with R&P in a whole-day session. Below we focus on the teachers' response to the last requirement and on the subsequent discussion.

The sequence on R&P before and after the practicum was video recorded and transcribed. Like the responses to the questionnaire, the transcripts were analysed with no pre-developed set of codes, using coding procedures inspired by grounded theory (Charmaz, 2006). The initial coding and categorization of the data material was first done by authors 2 and 3 independently. The coding included word-by-word, line-by-line, incident-to-incident and in Vivo coding (Charmaz, 2006). Memo-writing was used increase the level of abstraction. Subsequently codes and categories were compared and discussed among all authors and inconsistencies were resolved.

The analysis resulted in categories on (1) teachers' reasons for selecting the specific video clips; (2) the character of R&P in explicit discussions of these processes; (3) student learning and its possible relation to R&P; (4) the significance of R&P in school mathematics; (5) "blackboard-talk", that is, whole-class teaching as it relates to student learning in general and to R&P in particular.

Results

The results of the questionnaire support previous findings that many teachers have difficulties with deciding what a valid mathematical argument is. This is the case also for a large proportion of the teachers, who in the context of the questionnaire claim to be good at mathematics and to like engaging in mathematical R&P.

In the observations from the college classroom, the teachers face considerable problems arguing how or why the video clips they selected from their practicum is related to R&P. Four of the groups do not provide a coherent explanation for why they selected the episode, and three of the other groups select the clips for reasons that are unrelated to mathematical reasoning. The last group claims that their clip is on reasoning, but it shows students making number stories for tasks on fractions.

Looking at the clips themselves, rather than at the teachers' reasons for selecting them, three have no connection to mathematical reasoning (e.g. the teacher presents the solution to a procedural task on the board). The other episodes have some potential for student involvement in R&P, but the teachers do not emphasise aspects of R&P in the discussion in the classroom.

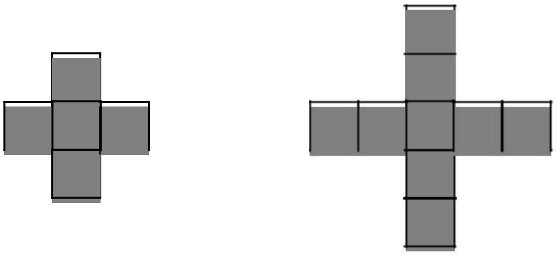

Figure 1

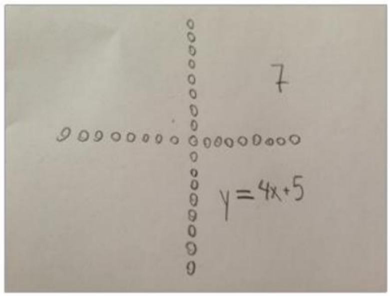

Figure 2

One example with some potential for involving the students in the phases of the R&P cycle concerns the pattern in the number of squares in a sequence of figures (fig. 1). The students, who are in grade 6, use cookies to represent the squares. The video clip shows two students, who have 33 cookies. They have written "7" and "y=4x+5" and made the drawing in fig. 2. However, they have trouble linking the equation to the geometric representation, and as the teacher joins them, the emphasis of the discussion is on the meaning of x and the length of the arms of the cross. This becomes a major concern also in the discussion at the college, which also revolves around the work of other students, who have written the same equation, but begun to solve it for different values of y, and around more general pedagogical issues such as how much support students should have in a situation like this. At no point does it become an issue if and how the episode could become the starting point for formulating a conjecture in the form of statement that could be verified. In this sense, mathematical justifications in the form of proving never become an issue.

Another example with R&P-potential concerns finding the point equidistant from the vertices of a triangle. In the video clip, a school mentor, the teacher normally teaching the class, unintentionally shows the students an incorrect procedure for constructing perpendicular bisectors. The students use the incorrect procedure but having measured the distances on their drawing they realise that something is wrong. They then shift their attention to the question of how to draw a perpendicular bisector and pay no attention to why it may help them solve the initial problem. In the discussion at the college the teachers discuss the episode, focusing on what they describe as lack of conceptual understanding on the part of the students and on the use of whole-class instruction in general. It does not become an issue if and how the episode may become a starting point for an exploration of the problem, for formulating a conjecture on properties of perpendicular bisectors, let alone developing justifications for such conjectures.

DISCUSSION AND CONCLUSIONS

The pilot confirms that teachers often face problems with R&P. The questionnaire establishes a setting remote from the classroom and the results do not in and by themselves indicate how the teachers react to similar questions when teaching. However, a PoP perspective suggests that the risk of not engaging sufficiently with R&P is greater in classrooms with many other pressing concerns beyond the quality of a mathematical argument. Further, our observations indicate that the teachers face problems identifying classroom situations with R&P, and in episodes with some potential for mathematical justification, modes of argumentation lose their subject specificity and conjectures are not subjected to mathematical verification.

In the cookie-episode the students engage in the important task of finding and generalising a pattern. Their difficulties may have been alleviated by moving the arms of the cross into a rectangular array with four columns and the centre cookie left over. This may be used as a single-case key idea inductive argument that builds only on rectangular representations of multiplication, and which may be turned into a generic argument that shows why the equation is right for all n. For this to happen, the teacher needs sufficient experiences with proving why in school contexts for it to become a mathematical practice (s)he can draw on in the interaction. It is of obvious importance that programmes for teacher education provide such experiences.

In other situations students' suggestions do not lend themselves as easily to generic arguments that *prove why*. This is the case for instance with the conjecture from Larry's classroom that if n is odd, 8 divides n^2-1. However, straightforward algebraic arguments and proof by induction may be used to prove the conjecture. However, if teachers are not sufficiently familiar with such arguments they have no alternative but to rely on the empirical ones used by the students.

From a PoP perspective these examples indicate that teachers need significant experiences with both *proving that* and *proving why*, if they are to support R&P activities in the classroom. The emphasis on *proving that* does not advocate a return in MTE to standard university courses with no relation to classroom practice; the mathematical practices involved are in that case too remote from school mathematics for teachers to draw on them in classroom interaction. However, using examples from school mathematics to develop means of *proving that,* including the much criticised proof by induction (Rowland, 2002), is necessary if teachers are to develop sufficient proficiency with dealing with all aspects of the reasoning and proof cycle in the classroom and capitalise on the potentials of their students' conjectures.

The pilot study examined the feasibility of proposals for MTE that are based on research with practising teachers. Our conjecture is that MTE needs to be close to both school and academic mathematics for teachers to link mathematical proficiency to instruction. In the case of R&P this means drawing on genuinely mathematical modes of justification in the classroom. The pilot supports the conjecture.

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Using Concept Cartoons to investigate future teachers' knowledge – new findings and results

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In the study presented here we address the issue of how to use Concept Cartoons for investigating future primary school teachers' mathematics content knowledge. We focus on various types of composition of Concept Cartoons, and on advantages and disadvantages associated with using these types of Concept Cartoons for diagnostic purposes.

Keywords: Concept Cartoons, future teachers' knowledge, mathematics content.

INTRODUCTION

The findings presented here continuously extend our contribution from last CERME conference (Samková & Hošpesová, 2015). The referred study belongs to a three-year project focusing on opportunities to influence professional competences of future primary school teachers through experienced *inquiry based mathematics education*. Within this project we aim to implement inquiry based methods into a set of university mathematics courses for future primary school teachers, and observe what impact the implementation has on students' knowledge and beliefs. As one of the diagnostics tools in this project we use an educational tool called *Concept Cartoons*.

In the previous contribution (Samková & Hošpesová,2015) we introduced Concept Cartoons in their diagnostics role. We showed that suitably chosen Concept Cartoons allow us to distinguish between subject matter knowledge and pedagogical content knowledge in the sense of Shulman (1986), and also between procedural and conceptual knowledge in the sense of Baroody, Feil and Johnson (2007).

In this particular study we concentrate more deeply on relation between the composition of a given Concept Cartoon and its suitability for investigating teachers' knowledge. Our research question is: What attributes of Concept Cartoons allow us to use them for investigating future teachers' mathematics subject knowledge?

THEORETICAL BACKGROUND OF THE

RESEARCH Teachers and their knowledge

Teachers and their knowledge needed for proper conduct of teaching are the focus of many educational researches. In this contribution we shall pay attention to areas related to Shulman's *knowledge base for teaching* (1986, 1987), and Rowland's *knowledge quartet* (Rowland, Turner, Thwaites, & Huckstep, 2009).

From Shulman's concept we focus on categories called *subject matter content knowledge* (SMK) and *pedagogical content knowledge* (PCK). The category of SMK can be understood as "knowledge for oneself", i.e. knowledge that the teacher can use during his/her own learning of a given topic. The category of PCK can be understood as "knowledge for helping others", i.e. knowledge t hat the teacher can use during teaching a given topic to someone else. In general, SMK and PCK are unequal but not disjunctive.

According to Grossman (1990), PCK consists of four components: *conceptions of purposes for teaching subject matter*, *curricular knowledge*, *knowledge of pupils' understanding*, and *knowledge of instructional strategies*. Our study relates to the last two. Knowledge of pupils' understanding refers to knowledge of pupils' conceptions and misconceptions related to a given topic, and to extend and limits of pupils' understanding of this topic. Knowledge of instructional strategies refers to strategies and representations needed for teaching a given topic.

Rowland's concept of *knowledge quartet* relates specifically to knowledge needed for teaching mathematics at primary school level. It consists of 20 categories grouped to four dimensions: *foundation*, *transformation*, *connection*, and *contingency*. The *foundation* dimension refers to teacher's theoretical background and beliefs, *transformation* to knowledge-in-action with central focus on representations (analogies, examples, explanations, etc.), *connection* to ways the teacher achieves coherence within and between lessons, and *contingency* to teacher's responses to unpredictable events in the classroom. Our study relates to all of these dimensions.

Rowland et al. (2009) also specify how knowledge quartet relates to SMK and PCK: foundation knowledge includes most of SMK, transformation knowledge belongs mostly to PCK, connection and contingency knowledges both combine SMK and PCK.

Concept Cartoons

In the research referred here we study a primary-school educational tool called *Concept Cartoons* (Naylor & Keogh, 2013). Each Concept Cartoon is a picture showing a situation well known to pupils from school or everyday reality, and a group of children in a bubble-dialog. The texts in bubbles present alternative viewpoints on the situation or alternative solutions to a problem arising from the situation (see Fig. 1).

Originally, Concept Cartoons were created as a classroom tool oriented on pupils, their goal was to support teaching and learning in science classroom by generating discussion, stimulating investigation, and promoting learners' involvement and motivation. Later the tool also expanded to mathematics (Dabell, Keogh, & Naylor, 2008). When working with Concept Cartoons, the pupils have to choose all children in the picture that are right, and justify their choice.

According to research conducted by authors of Concept Cartoons (Naylor, Keogh, & Downing, 2007), the lack of agreement amongst the children pictured in the Concept Cartoon encourages pupils to join the discourse with their own opinions, and such discourse can take a form of sustainable and purposeful argumentation.

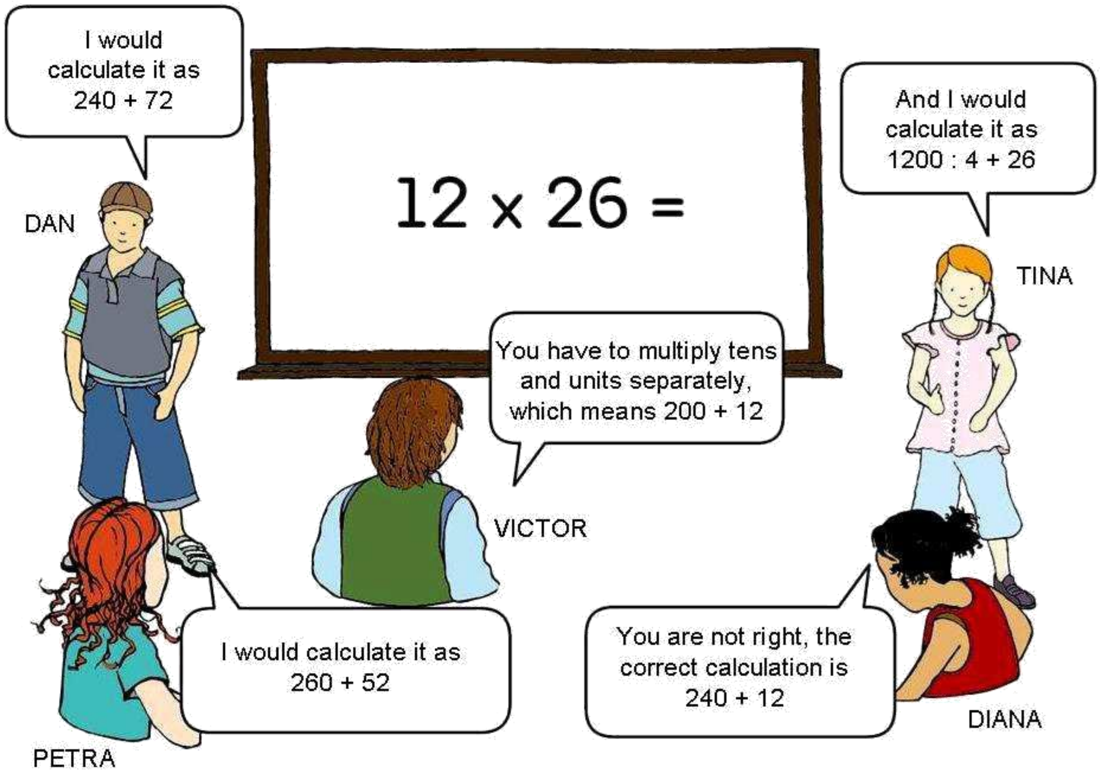

Figure 1: Concept Cartoon on multiplication; template with empty bubbles and empty board taken from (Dabell, Keogh, & Naylor, 2008), names of children added

In our project, we aspire to use Concept Cartoons as a diagnostic instrument for investigating various aspects of future primary school teachers' mathematics content knowledge. We understand each Concept Cartoon as a model of a classroom situation, and observe how the future teachers respond to such a situation. The situation modelled by the Concept Cartoon is predictable for an experienced teacher but may be unpredictable for an inexperienced future teacher. That means that from future teachers' training perspective, Concept Cartoons may be considered as models of contingent situations in the sense of Rowland's knowledge quartet, i.e. as models of situations where both SMK and PCK come into play. This attribute of Concept Cartoons shall ensure that data collected with Concept Cartoons might refer to both SMK and PCK.

We also need to be sure that Concept Cartoons allow us to collect enough data. For this purpose we shall supplement Concept Cartoons by a set of investigative questions, and we hope that the above mentioned way how Concept Cartoons encouraged pupils to present their own opinions during science lessons will also work in the case of future teachers during mathematics lessons. Hopefully the lack of agreement amongst the pictured children shall lead future teachers to responsiveness towards Concept Cartoons on mathematics topics, and towards willingness to contribute to the discussion pictured in them.

But still we are aware that Concept Cartoons were originally created for a different purpose, so that it is important to verify whether and what Concept Cartoons are suitable for the knowledge diagnostics.

DESIGN OF THE STUDY

Participants

Participants of the research were 129 university students, future primary school teachers, in full-time or distance form of study. In our country, primary school teachers are not math specialists, they teach all primary school subjects.

Course of the research

The research was conducted in two separate stages.

The first stage

For the first stage of the research we selected four Concept Cartoons from the original set created by Dabell, Keogh, and Naylor (2008). We picked out Concept Cartoons that differed in several factors:

- type of the pictured situation
    - classroom event;
    - everyday event;
- type of the text in bubbles
    - proposal of a result;
    - proposal of a procedure and a result;
    - advice to a pupil who made a mistake;
- number of bubbles with correct alternatives.

These Concept Cartoons were assigned to students on a worksheet with four common questions:

1) Which child do you strongly agree with?
2) Which child do you strongly disagree with?
3) Decide which ideas are right and which are wrong. Give reasons for your decision.
4) Try to discover the cause of the mistakes, and advise the children how to correct them.

Students worked on worksheets individually, approximately 80 minutes.

Data from worksheets were processed qualitatively, using open coding (Miles & Huberman, 1994). We focused on displays of SMK and PCK related to provision and recognition of right and wrong answers, to recognition of procedures used by pictured children, and to identification of the causes of mistakes. Detailed description of analysis of data connected with two Concept Cartoons that proved to be suitable for investigating SMK and PCK, and particular results belonging to data from 64 future primary school teachers in full-time form of study were reported in (Samková & Hošpesová, 2015).

The second stage

Based on findings from the first stage, we created 22 own Concept Cartoons, and used them in the second stage. 11 of them were modifications of original Concept Cartoons (just some texts in bubbles were adjusted to suit better our purpose), 11 were brand new (new pictured situation, new perspectives in bubbles). For some of these Concept Cartoons we also established new types of the text in bubbles:

- proposal of a procedure;
- proposal of a statement (e.g. a rule);
- opinion on the validity of a statement;
- opinion on the number of solutions;
- reference to an absent schema.

When creating the new Concept Cartoons, we put into bubbles various more or less usual pupils' conceptions or misconceptions, descriptions of various ways of solving (correct, incorrect), or intentionally prepared authentically looking misconceptions (for a sample of such a misconception see Samková & Tichá, 2015). We searched for inspiration in our own teaching experience and in teaching experience of our colleagues (e.g. Tichá & Hošpesová, 2010), in results of educational research (e.g. Ryan & Williams, 2011; Bana, Farrell, & McIntosh, 1995), in books and textbooks (Ashlock, 2002, 2010). The process of creation of one of the Concept Cartoons is described in detail in (Samková, Tichá, & Hošpesová, 2015).

These Concept Cartoons were assigned to various groups of participants.

Data from both stages of study were again processed qualitatively. In this time, we focused on displays of SMK and PCK described above but this time in relation to composition of given Concept Cartoons. We also monitored the amount of relevant data obtained from participants to various compositions of Concept Cartoons.

FINDINGS

For better comprehensibility of this section we shall illustrate the reported findings by means of a set of Concept Cartoons that are all based on common strategies for mental multiplication of integers. Such topic belongs to primary school curriculum, where we can find recommended strategies as rearranging numbers (e.g. counting 2 x 5 instead of 5 x 2), rearranging operations (e.g. counting 14 x (2 x 5) instead of (14 x 2) x 5), using repeated operations (e.g. counting 120 : 8 as 120 : 2 : 2 : 2), adjustment (e.g. counting 18 x 6 as (18 x 5) + (18 x 1), or counting 18 x 9 as (18 x 10) – (18 x 1)), using inverse relationships (e.g. calculating 12 x 25 as 12 x (100 : 4) = (12 x 100) : 4 or as 12 : 4 x 100), etc. For a detailed overview see e.g. (DfE, 2010).

Bubbles with just results

One of the studied types of Concept Cartoons contained bubbles proposing various results (solutions) of a pictured task. If the task was in a form of a calculation (e.g. as in Fig. 2, except Petra's bubble), then the respondents too often tended just to compare the correct result of the calculation with numbers in bubbles, and did not seek procedures hidden behind the incorrect results. Even though all four questions were assigned with the Concept Cartoon. These Concept Cartoons often emerged as not enough thought-provoking, providing little data, and thus not suitable for diagnosis.

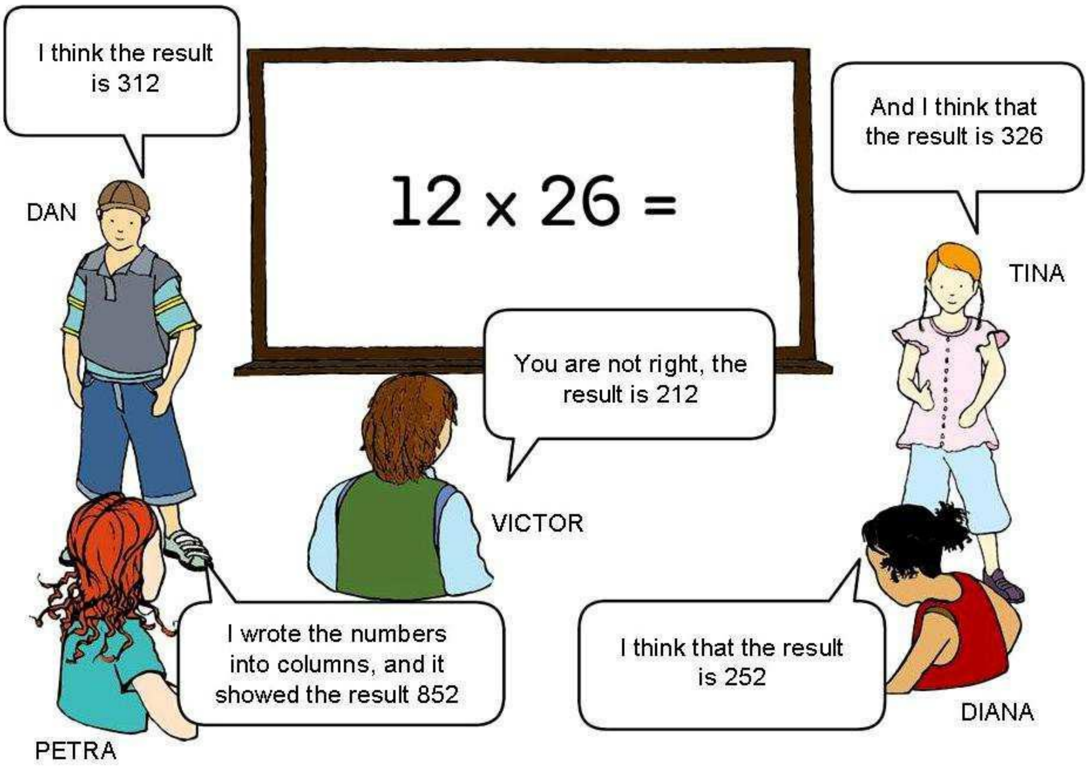

Figure 2: Concept Cartoon with results in bubbles; template with empty bubbles and empty board taken from (Dabell, Keogh, & Naylor, 2008), names of children added

Among Concept Cartoons in our research we found just one exception – a Concept Cartoon containing various results to a calculation 5904 + 5106, where all proposed results were composed only from digits 1 and 0. This unusual composition of numbers attracted respondents' attention, it showed as very thought-provoking, and we got a lot of relevant data (for more details see Samková & Hošpesová, 2015).

In Concept Cartoons containing bubbles with results we also used tasks in the form of a word problem. In case when the word problem is rather difficult to solve (e.g. unequal partition problem with compared quantities unknown), the Concept Cartoon is suitable for investigating respondents' knowledge of solving strategies as well as their grasping of a situation: the respondents who tend to avoid solving the difficult word problem, try to verify all offered alternatives instead, and this activity can reveal the level of their grasping of the situation (for more details see Samková & Tichá, 2015).

Bubbles with reference to an absent schema

As diagnostically valuable appeared bubbles introducing a result and referring to an absent schema leading to this result (e.g. Petra in Fig. 2). These bubbles were often though-provoking, respondents made attempts to find out what schema the child was talking about, they often proposed their own schemas that could lead to the result. In this case the Concept Cartoons played a similar diagnostic and developmental role as problem posing (namely as posing problems corresponding to a given calculation, see Tichá & Hošpesová, 2010).

Bubbles with procedures / with procedures and results

Another of the studied types of Concept Cartoons contained bubbles proposing various solution procedures of the pictured task, either with a result or without it (Fig. 1). With this type of bubble content, respondents can comment described results and procedures, look for errors in procedures leading to incorrect results (and also in procedures leading to correct results). This kind of Concept Cartoons proved to be thought-provoking for respondents. Unlike the previous case with just results, now a lot of concrete facts is offered to the respondents to judge and discuss, so that the respondents' responses provide a lot of relevant data on various dimensions of knowledge:

S11: Dan decomposed 26 as 20+6, and then added 12x20 to 12x6. His procedure is similar to "column" multiplication.

Petra: the same as Dan, she just multiplied 26 by 10+2.

S19: I agree with Dan and Petra. And also with Victor – but he should not use the formulation "have to", better would be "may".

Victor: his procedure is not the only one that is right. But he is right. 10x20=200, 2x6=12. Transparently and quickly solved! Correct.

For the diagnostic purposes, it showed profitable to include into one of the bubbles a procedure that is "clever" (unusual, tricky, advant ageously using a certain attribute or relation) – e.g. as in Eve's bubble in Fig. 3. Such bubble allows to reveal good knowledge when a respondent is able to decode the background of the procedure (S4 below), and also poor knowledge when the respondent offers inappropriate explanation and/or blames the child to count randomly (S15 below):

S4: 12x100=1200, 100:4=**25**, **1** is missing to **26**, 1x12=12

300+12=312

S15: Eve replaced 26 by 00, and got 1200. Then she divided 1200 by number of digits (4). In the end she added those 26 that had been previously replaced by zero.

The result is correct, but with different numbers the rule (the procedure) does not hold. A coincidence.

To investigate knowledge more intensively, we may include into one of the bubbles a procedure that is “clever” but contains a mistake i n the last step, e.g. as in David’s bubble in (Samková & Hošpesová, 2015), or in Tina’sbubble in Fig. 1. But we must be careful with including this kind of bubbles – wh en a “clever” procedure is hard to decode, then the same procedure with a mistake might be undecodable. It happened in our research with Tina’s bubble in Fig. 1: only 1 of 34 respondents was able to find the background of Tina’s procedure, the others left her bubble without any comments or with responses like “I do not know what she is doin g”. To clarify the situation (both to researchers and to respondents), we assigned the respondents a supplementary Concept Cartoon with Tina’s and Eve’s bubbles together (Fig . 3). The presence of the correct version helped several students to decode the Tina’s procedure as well.

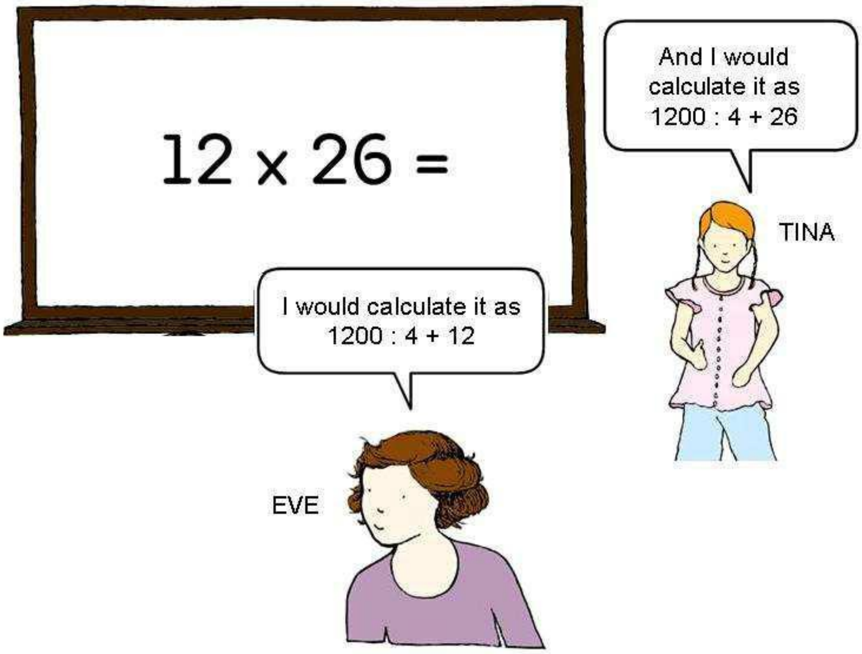

Figure 3: A supplementary Concept Cartoon to Fig. 1

“Clever” procedures (with or without mistakes) appe ar in classroom only rarely but they always announce a pupil that is thoughtful enough to produce an unusual idea, or courageous enough to try an unusual procedure that might be profitable. Even if being rare, we consider such moments as very important steps in the pupil’s learning process, and the teacher should be prepared for them.

CONCLUSIONS

In this study we focused more deeply on an educational tool called Concept Cartoons, and on its possible usage in diagnosing mathematics content knowledge of future primary school teachers. We conducted a large survey with more than 100 participants, and tested with them more than 20 different Concept Cartoons. During data analysis we observed what attributes of Concept Cartoons allow to collect enough data that are suitable for the diagnosis. Many of the original Concept Cartoons appeared to be unsuitable for these purposes, so that we prepared and tested also our own Concept Cartoons.

Findings show that the decisive attribute is the form of the texts in bubbles. The greatest amount of relevant data we got while using Concept Cartoons containing procedures in their bubbles. This type of bubble content was thought-provoking, and provided a lot of data on both SMK and PCK aspects related to provision and recognition of right and wrong answers, to recognition of procedures used by pictured children, and to identification of the causes of mistakes.

Concept Cartoons with just results in their bubbles appeared often as unsuitable, especially when the task was in the form of a calculation. In these cases the respondents tended to compare the results in bubbles with the correct result, and did not attempt to seek procedures hidden behind the results. Such Concept Cartoons provided little relevant data. Nonetheless, it was possible to take advantage of this type of Concept Cartoons in diagnosing – we used them for investiga ting just SMK, or took them as a basis for a new Concept Cartoon with a special bubble(s). The special bubbles contained unusually looking numbers that attracted respondents' attention, or referred to an absent schema to provoke respondents to pose their own schemas.

We also introduced special bubbles with a "clever" procedure, and with a "clever" procedure containing a mistake. In this case the difficulty of the bubbles must be determined carefully to optimize cases when the respondents are not able to decode the procedure and thus the bubble provides little relevant data.

Our research confirmed that Concept Cartoons are able to encourage future teachers to present their opinions on mathematical topics and display their mathematics content knowledge through this activity, and established a typology of Concept Cartoons suitable for such purposes.

NOTES

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Teacher's knowledge and the use of connections in the classroom

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This paper discusses the relationship between teacher's knowledge and the use of learning opportunities emerging from classroom interactions. In the context of non-standard measurements in primary school, we identify and characterize several connections related to the mathematical foundations of measurement. Our results confirm that different types of knowledge can help teachers to take advantage of the learning opportunities stemming from connections made in the classroom context. The link between extra- and intra-mathematical connections is also discussed.

Keywords: *Connections, Teacher Knowledge, Measurement, Learning opportunities*

INTRODUCTION

Recent body of research on mathematics education has been fruitful in understanding teachers' knowledge conceptualizations (e.g., Mathematical Knowledge for Teaching–MKT; Knowledge Quartet–QK; Mathematics Teachers Specialized Knowledge–MTSK). When analyzing the content of such models, connections are perceived in diverse ways (and are given different roles), either implicitly or explicitly. For example, Rowland, Turner, Thwaites, and Huckstep (2009) consider connections as a domain. In the case of Mathematical Knowledge for Teaching (Ball, Thames & Phelps, 2008), connections allow utilizing content of some of the sub-domains of such framework.

When focusing on the way connections are established, several categorizations have been developed (e.g., De Gamboa & Figueiras, 2014; Montes, Ribeiro, & Carrillo, 2016) as a means of clarifying the kind of linkages that can be generated by teachers and students. In order to deepen our knowledge of connections and their role in practice, as well as conceptualize ways to improve the effectiveness and utility of such connections, it is necessary to study the concretization of the aforementioned categorizations, as well as expand them in relation to several mathematical topics.

In practice, concretization of connections typically results in two distinct cases—anticipated situations, prepared by the teacher in advance; and those that could not anticipated, as they are triggered by students' comments (e.g., linked with contingency moments, as noted by Rowland et al., 2009). These concretizations are perceived, from a research perspective, as an opportunity for learning and developing a deeper and broader understanding of the content of teachers' knowledge, taking into consideration the specificities of such knowledge.

In this work, we focus on a prospective primary teacher's actions, and revealed knowledge. We analyze an episode she has prepared in advance, aiming to explore non-standard length measurement units in the classroom. In that sense, we discuss the use of mathematical connections as a way to create and explore learning opportunities, and analyze how teachers' knowledge can foster the use of such opportunities in the classroom.

THEORETICAL BACKGROUND

Attaining understanding of the measurement process is not a straightforward process, as in many mathematical topics (e.g., adding or dividing fractions) comprehension of the steps involved may be more complex than the process itself. Piaget (1972) noted that acquiring the notion of magnitude, for example, requires going through different stages, from the use of words to (correctly) express the magnitude, through grasping the necessary concepts, until the knowledge about the measurement of such magnitude is finally attained. The path through such stages is complex and involves a broader and deep understanding of the concept image and definitions involved (Tall, 1988).

Stephan and Clements (2003) posited that six key concepts have to be mastered to develop a full understanding of and skills required for measurement: (i) *Equal partitioning*—mental process of division of an object into equal parts, requiring the acknowledgement of the divisibility of the object; (ii) *Unit iteration*—skill to exhaustively repeat the unit successively to cover the object; (iii) *Transitivity*—recognition of the mathematical property of measure; (iv) *Conservation* of the measure through some transformations; (v) *Addition and accumulation of distance*—recognition that the measurement process outcome is the measure of the object (how many units have to be repeated to equal the measurement of the object); and (vi) *Relationship between number and measure*, implying accepting that a variation of the unit of measure would generate a change in the measurement outcome (total amount of units). These six key concepts are considered when discussing the nature of the connections employed by teachers and those emerging from the answers and/or comments students make in response to different problems.

Teachers' knowledge of measurement is essential for ensuring that the students develop requisite knowledge and awareness of the topic. In that sense, a broader understanding of the content of such knowledge is a crucial aspect for conceptualizing ways for improving education. Considering the MKT conceptualization, in the scope of this study, four sub-domains are of crucial importance, namely Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teachers (KCT).

The CCK is associated with knowing what instruments to use (in the sense of standard instruments/measurement) for measuring different entities. It also involves being able to use the instruments to correctly perform a certain measurement, with the understanding that no empty space must be left, as well as that no overlaps should exist (instrumental knowledge). Developing teaching tasks requires complementary content knowledge, in particular SCK related to the knowledge of the six key concepts involved in measurement (considered by Stephan and Clements, 2003). Possession of such knowledge facilitates understanding the *whys* and *hows* of their importance for the children to learn (while linking it with future learning goals). Thus, SCK corresponds to the core aspect of the knowledge that allows teachers to give meaning and interpret students' solutions and comments (as a part of the interpretative knowledge, recommended by Ribeiro, Mellone and Jakobsen, 2013).

Complementarily to CCK and SCK, when teachers conceptualize and implement mathematical tasks, they are required (in the view of many scholars and practitioners) to anticipate students' difficulties, as well as to pay attention and appropriately respond to students' comments. Such aspects of teachers' knowledge are an integral part of the KCS, which includes the ability to anticipate student difficulties in differentiating the measurement instrument (non-standard unit, such as the hand) and the measuring unit (e.g., finger, span), as well as in using non-standard measuring units (e.g., using the finger length or width). Moreover, teachers must be able to use, prepare, and implement all classroom tasks with mathematically significant aims, as well as perceive (and use) students' comments as an opportunity to make mathematical remarks. This approach is directly related to teachers' KCT, including the knowledge required to take advantage of students' questions, comments, and errors about the measuring units and the measurement procedure as a means of broadening the instruction and increasing potential for student understanding. Teachers are advised to incorporate students' comments and consider errors they make when solving problems as a starting point to clarify some key (and possibly problematic) ideas on measurement.

Similarly to teachers' knowledge, different categorizations have been developed when attempting to elucidate connections. For example, De Gamboa and Figueiras (2014) considered two principal connection types—*extra-mathematical connections* and *intra-mathematical connections*. According to the authors, *extra-mathematical connections* are formed between a mathematical content and a non-mathematical situation, such as a real life problem or content from another school subject. *Intra-mathematical connections*, on the other hand, can emphasize *transverse processes*, such as generalization, communication, or heuristics for problem solving, or can be *conceptual connections* that link different features of concepts, such as representations, procedures, or properties.

In their recent work, Montes et al. (2016) focused on the relationship between such connections and a concrete task, highlighting the differences between the intra- and extra-mathematical connections, which they label as *outside-mathematics connections*. In the present study, two types of intra-mathematical connections are considered, following the framework proposed by Montes et al. (2016). More specifically, we analyze *connections internal to the task*, which occur when the teacher uses various aspects related to the concept included in the task, as various representation systems, or the relationship between a measure and a unit of measurement. We also focus on *transverse connections*, which are linked to the nature of some mathematical concepts that emerge in different mathematical contexts. This is necessary, as the notion of measure is first introduced in primary school, but is also discussed and expanded upon in subsequent educational stages (e.g., while using absolute values or in elucidating the meaning of definite integrals).

The relationship between quantities of length and numbers generates several mathematical questions (Stephan & Clements, 2003). Such questions are related to some other mathematical connections, such as the need for using the same unit and the procedure when performing a measurement; the inverse relationship between the length of the unit and the final result (number of units); the possibility of obtaining slightly different results using the "same" unit (e.g., hands, finger, feet) and the same procedure; the approximate nature of any measure and the ways we can express it (using whole numbers, fractions, irrational numbers, etc.). The answers to those mathematical questions are the pillars on which the *extra-mathematical connection* can be built. When shifting the focus from the real world problem to its mathematical model, several *intra-mathematical connections* can emerge, and their use can foster the development of students' deep understanding of length in both mathematical and real context. Whenever such *intra-mathematical connections* occur, they are considered sub-connections (as they are imbedded in the *extra-mathematical connection* – solving the "real life" problem).

CONTEXT AND METHODS

Data has been collected as part of the 4th year of the teachers' training program – at University Autònoma of Barcelona. Prospective Teachers (PT) are required to record some of their classes during the field practice (a total of 10 hours out of 240 hours). They have then to select one of such classes and select what they consider rich episodes in terms of the mathematical content approach and the students understanding. For selecting such episodes, they should use Sherin, Linsenmeier and van Es' (2009) criterias: window, clarity and depth.

In order to select and analyze the episodes using Sherin, Linsenmeier and van Es' (2009) criteria, PTs are expected to activate specific professional knowledge, namely Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT). In our analysis of PTs practices and analysis, we have also used Sherin, Linsenmeier and van Es' (2009) criteria for characterizing the knowledge PT's activate, in terms of the subdomains of MKT (Ball et al, 2008).

Amongst the several case studies that have been developed as part of a broader research project, in this paper, we focus on a PT, Carla, and her exploration of three tasks pertaining to length measurement using non-standard units, where Sherin, Linsenmeier and van Es' (2009) criteria were used at a high level.

Thus, in the discussions that follow, we focus on an episode involving three tasks exploring the use of non-standard units for measuring the length in a "real life" (classroom) context, as it helps reveal Carla's knowledge of connections. It is important to note that, although the analysis tends to focus on identifying missed instruction opportunities, such situations are perceived as learning opportunities, and as a powerful tool for use in education. Thus, their inclusion enhanced the discussion presented in the subsequent sections.

When analyzing Carla's work, focusing on connections, we adopt categorizations proposed by De Gamboa and Figueiras (2014) and Montes et al. (2016) to provide analytical sensitivity for detecting emerging connections, in terms of their nature and their relationship with the task performed. These connections are characterized with respect to the learning opportunities they entail. Finally, we analyze how teachers' knowledge can help students to take advantage of the learning opportunities that emerge within the classroom.

ANALYSIS AND DISCUSSION

The analysis that follows pertains to an episode in which the teacher generates a discussion about how to measure some lengths using non-standard measures. Her goal in implementing the three tasks explored here is to prompt the children to conclude that standard units of length have to be used in all measurements. In these tasks, students had to measure height of a book using their hands, width of their classroom using their feet, and the height of a pot of glue using their fingers.

As the task involved measuring lengths in a "real life" context, an *extra-mathematical connection* emerges, linked to a problem that needs to be answered using a mathematical perspective (how can we compare and represent lengths?). This connection relates the length with the numerical value representing such length, and it is grounded on the comparison of the particular length to be measured with the length of an object that can be used as a reference (hands, feet, etc.), referred to as unit of measurement. In the particular case we focus on, the *extra-mathematical connection* underpinning the episode concerns the relationship between the understanding of length as an attribute of real world objects, and the mathematical foundation of length's measurement.

From the analysis of the episode, four sub-connections pertinent to the task (in particular, *intra-mathematical conceptual sub-connections*) and one transverse sub-connection can be identified.

One example of *conceptual sub-connection* emerged when Hugo pointed out the relationship between the length of the unit of measurement and the numerical result obtained (connecting the notion of quantity/size and its numerical representation):

> Teacher: Can anyone tell me why we have obtained different results?” […]
>
> Hugo: Each person has a bigger hand and the bigger it is, the less we get. And the smaller they are (the hands), the more they fit.

This connection emerged from the task, presenting the teacher with an opportunity for deepening students’ understanding of the inverse relationship between the size of the unit of measurement and the number of units needed, which should result in the same absolute value if the same procedure is used (Stephan & Clements, 2003). It corresponds to an anticipated connection, revealing Carla’s knowledge of the mathematical foundations of length’s measurement related to the relationship between number and measure (Stephan & Clements, 2003), a particular aspect of the content of CCK.

Another opportunity for this type of sub-connection to emerge arose when Isaac asked what would happen if they changed the way they open their hands (change in the measurement procedure –how to use the resource–, while using the same resource):

> Isaac: Is it possible to do it like this as well? (Opening the hand completely)
>
> Teacher: Of course, another thing is how we put our hands. If some of you put them like this (partially opened) and some of you put them as Isaac did (completely opened), Isaac will get less . . . . But it doesn’t mean that it is wrong; it simply indicates that we have different hands and we have measured differently.

Such sub-connection is made possible owing to the teacher’s knowledge of the relationship (similarities/differences) between two different uses of the same resource for measuring the same distance. Isaac’s question, and the connection elaborated by the teacher, creates the opportunity for the students to understand that if the procedure is not the same, the results will not (necessarily) be comparable. In that sense, although Carla eludes at the connection, her explanation remains only at a superficial level, thus not allowing students to deepen their knowledge (and conception) of length. In fact, by saying “*it doesn’t mean that it is wrong,*” Carla is likely confusing the students, who consequently may not appreciate the importance of following the same procedure when measuring length. In that sense, although she refers to the specific sub-connection, the opportunity for sustaining/developing students’ learning and understanding of the notion of length is lost (lost connection opportunity). The elaboration of such sub-connection is sustained in Stephan and Clements (2003) categorization that mandates knowing

(i) *equal partitioning* and (vi) *relationship between number and measure*, as both are necessary to appreciate that using different procedures and/or resources many produce different results, even if the same unit is used (CCK). On the other hand, being able to make such connection requires teachers to possess knowledge related to the difficulties that children experience when learning how to measure length (KCS). Greater KCS could have also helped Carla to notice the relationship between understanding the procedure and the correct understanding of the notion of length.

While discussing the results of the task concerning measuring the width of the classroom using students' feet, another intra-mathematical connection emerged. When sharing the results, Miguel commented that, if one separates the feet while performing the measurement, the result would not be the same:

> Miguel: If we separate the feet we are not doing (measuring) anything.
> Teacher: Of course, we have to put them next to each other.

This sub-connection corresponds to the relationship between the measurement result and the accuracy of the procedure. The teacher fails to take this opportunity to continue exploring such sub-connection by emphasizing that the procedure must be performed exhaustively (intuitive introduction of the notion of algorithm), in order to obtain the same answer when using the same measurement unit and representational system. Clearly, in-depth knowledge of the mathematical foundations of length's measurement (CCK) is required in order to observe (become aware of) the importance of using an algorithmic approach that would yield exhaustiveness, which in Stephan and Clements' (2003) key concepts would correspond to *unit iteration*. Complementarily, knowing that many students experience difficulties (KCS) in the iteration procedure, as well as in understanding the reasons behind the possible differences, would be useful to identify and explore such an important feature of the learning of measurement. Although these last two sub-connections may be perceived, at first glance, as elements of the same subset, the focus of the former was on how to place the resource (the unit), while in latter the focus is on how to join the object along the length. This sub-connection, and the associated knowledge and notions, are crucial to correctly perform length measurements, using either a non-standard unit or a standard one.

Another connection related to the task emerged during the discussion of the way students measured the height of a pot of glue using their fingers (Ribeiro, Badillo, Sánchez-Matamoros, & Artès, 2016). Miguel's answer was different from that offered by the rest of the class. While majority of students obtained numbers close to 10, Miguel's answer was one. When Carla asked him to explain how he obtained that result, Miguel elucidated his reasoning by placing his finger perpendicularly to the base of the pot, while the rest of the class was placing their fingers parallel to the base:

> Teacher: No? How many fingers did you get?
> Miguel (putting the finger vertically along the glue package): One!
> Teacher: One? Like this? (The teacher repeats the measurement process using the indicator finger horizontally)
> Miguel: No, two . . .

Teacher: Two? With two, you can cover the entire distance?
Miguel: No . . . ah . . . four . . .
Teacher: I don't know what you are measuring . . .
Miguel: Ah, four, four . . .
Teacher: No! It can't be . . . you should get eight; you are doing it wrong.

This situation provides an excellent opportunity to discuss the importance of establishing a common procedure when conducting measurements (an important sub-connection pertinent to the task). In this case, Miguel's answers show the use of non-standard units in a non-standard way (Ribeiro et al., 2016). However, Carla's arguments and exemplification indicate her sole focus on the standard measurement process. In that sense, she seems to be taking for granted the underlying procedures, thus disregarding an answer that differs from her own (Jakobsen et al., 2014). Finally, she also fails to grasp (or at least discuss with her students) the relationship between the number obtained and the measurement method used (in line with the key concepts on measurement proposed by Stephan and Clements, 2003).

Ability to make such a sub-connection is grounded in different aspects of teachers' knowledge. Awareness of such particularities could have helped Carla, as she could have expanded on Miguel's ideas to attract the attention of other students to the importance of using both the same units and the same measurement procedures in order to obtain comparable results. Once more, such knowledge is related to the knowledge of the mathematical foundations of measurement (CCK), but also to the need to keep in mind the importance of the unit and procedure used, and the difficulties students usually experience in grasping these concepts (KCS). Moreover, ability to recognize situations (when and why) that would benefit from the use of children's ideas for emphasizing an important concept (KCT) is also required for helping students to overcome the identified (or anticipated) difficulties.

The transverse connection referred to at the start of the analysis arose during the class discussion, and corresponds to an implicit connection between the measurement and the numerical value that represents such measurement. It highlights that, even if all the students use (exactly) the same unit and procedure, it is possible to obtain different results, which need to be linked both with the approximate nature of any measurement and the constraints of the number set used for the representation (natural, rational or real). This particular connection emphasizes the importance of taking into account the students' academic level, as it will influence the precision with which they will be able to represent the result of the measurement process. For example, eight feet or eight and a half feet can correspond to the same measurement obtained using one's foot as the unit of measurement, if we are using only natural numbers. However, considering the differences between those two results can create the opportunity to reflect upon the necessity to follow the measurement procedures accurately, and emphasizes the importance of being able to recognize two different numerical results as a correct measure of the same length. Moreover, having the knowledge that would facilitate formation of such connection can support children's understanding of the necessity of introducing new "unknown" numbers (different modes of representation), pointing to the need to represent non-exact quantities in terms of natural numbers. It is in that sense that this connection can be considered as transverse.

To use this transverse connection to broaden students' understanding of measurement notions, Carla required SCK related to the approximate nature of measurement as well as to how such nature relates to the different number sets. Obviously, in order to make such relationship explicit in the classroom, the teacher is required to be in possession of a KCS pertaining to the difficulties students encounter (as was the case in this example, as mentioned earlier). Finally, it requires sufficient KCT related to being able to recognize opportunities for including students' ideas in the ongoing practice.

FINAL REMARKS

The final set of sub-connections, when using real world contexts to develop mathematical knowledge and insights is perceived as one more contribution for deepening our understanding of teachers' knowledge content. The analysis of the knowledge linked to the learning opportunities triggered by connections shows that the use of extra-mathematical connections may rely on underlying intra-mathematical connections that require diverse kinds of knowledge to be used.

As a result, the usefulness of the extra-mathematical connection for the construction of mathematical knowledge depends, at least partially, on the use of the intra-mathematical connections for the understanding of the mathematical foundations of length. In this case, four connections pertinent to the task examined in this work are related to the foundations of length's measurement that are at the core of the activity. In addition, one transverse connection emerged, related to the understanding of the approximate nature of measurements.

However, if those intra-mathematical connections are not enhanced or misused, students' learning opportunities will be underutilized. In the case presented in this paper, starting from three tasks requiring students to perform non-standard measurements, five intra-mathematical connections emerged, but they were lost opportunities. In that sense, the extra-mathematical connection was misused in relation to those learning opportunities, as students' attention was primarily drawn to units, while omitting to address some core aspects of measurement, such as equal partition and unit iteration.

Even though there are several reasons for the teacher to misuse those learning opportunities, e.g. classroom management, knowledge analysis reveals that there are some types of knowledge that would help the teacher to take greater advantage of the learning opportunities that arose from the intra-mathematical connections. This knowledge is related to the mathematical foundations of measurement (CCK), the possible difficulties students can face (KCS), the relationship between measurements of continuous magnitudes and different types of numbers (SCK), and knowledge of when and how to use students' ideas to make some important remarks in the classroom. It is important to emphasize that all aforementioned types of knowledge are intertwined, as CCK and SCK are the pillars on which KCS and KCT are based. Thus, coordination of different kinds of knowledge allows teachers' knowledge to acquire its specialized dimension.

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WG 3 - Working Group:

Teacher Practice and Classroom Interaction

Frameworks supporting the coding and development of mathematics teachers' instructional talk in South Africa

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In this paper, we present two frameworks – for secondary and primary levels - aimed at coding and supporting the development of mathematics teachers' instructional talk in South Africa. Both frameworks draw from sociocultural bases focused on mediating categories within teacher talk, and quality indicators within them, and a press towards mathematics viewed as a network of scientific concepts. Both frameworks provide greater disaggregation, at the lower extreme in particular. This is important in turns of the imperative to support development of teacher talk in a context where disconnection, ambiguity and gaps in teachers' mathematical knowledge are described as relatively common.

INTRODUCTION

Several frameworks are available in the international literature that characterise the quality of instruction in mathematics classrooms. For example, Hill et al's (2008) Mathematical Quality of Instruction framework features aspects like lesson format and links to learning alongside teachers' mathematical talk. For a range of reasons outlined below, our attention, in the context of linked research and development projects aimed at improving the quality of mathematics teaching across ten primary and ten secondary schools in South Africa, is more specifically on the quality of the mathematics that is made available to learn within instruction in mathematics classrooms. The format of instruction and the nature of learner participation therefore fall outside our central scope of attention.

A number of issues - some overlapping and some differing - mark the research and development context when looking across secondary and primary mathematics. A key focus in the overlapping area is on what we call 'mathematical discourse in instruction' (MDI) – which, in parallel with the issues, is articulated in different ways across the two phases. Important differences relate to the much greater use of physical artefacts in primary mathematics, in comparison with secondary mathematics. Working developmentally in secondary and primary mathematics teacher education in this context is premised on our being able to characterise the pedagogic range of MDI at secondary and primary levels on the ground and build from this ground upwards. A focus on teachers' mathematical talk has been central to this focus and a function of a range in the South African context that is broader than is commonly described in the international literature.

Key issues have been identified as concerns relating to teachers' mathematical talk in South Africa. Some of these issues are linked with, and characterised, in frameworks in the international literature base. This is particularly true at the upper end of our concerns across primary and secondary levels where we have episodes of teaching that focus broadly on 'rules without reasons' (e.g. Skemp, 1987), This kind of teaching is widely critiqued in the international literature base as procedural and limiting of access to mathematical discourse. At the lower end though, the international literature contains much more limited disaggregation. Across our work, we have described episodes of teaching in both phases where concerns relate more fundamentally to mathematical coherence. In this teaching, we see episodes that sometimes confirm answers as though they are already known in the classroom space rather than deriving them, teacher talk about knowns as if they are unknowns, and talk that is infused with ambiguity, error and high levels of disconnection (Adler & Ronda, in print; Venkat & Naidoo, 2012).

While typically, the instructional triad views teaching as mediating between students and the mathematical object in focus, the range of problems identified above, coupled with evidence of significant content and pedagogic content knowledge gaps amongst South African teachers, leads to our attention to the teacher – mathematical object relation as the key initial link to both describe and strengthen in order to support teaching development on the ground.

The range overviewed above meant that we needed frameworks that allowed for adequate description and categorization of the ground. This involved the identification of key categories within instructional talk, and characterising quality markers that could also serve as developmental pathways within these categories. In this paper, we present and discuss the categories of instructional talk that we have focused on within MDI at secondary and primary levels, and the quality markers within them. Looking across the two framings of mediating talk, we comment on the ways in which they are linked by a concern with incoherence and error at the lower extreme, and with mathematics viewed as a network of scientific concepts at the upper extreme, with focus on structural relations and generality as key indicators of mathematics worked with in these ways within instruction. We go onto present episodes of teaching drawn from the lower and upper level of concerns and outline our ways of coding them using our respective coding frameworks.

MDI FRAMEWORKS

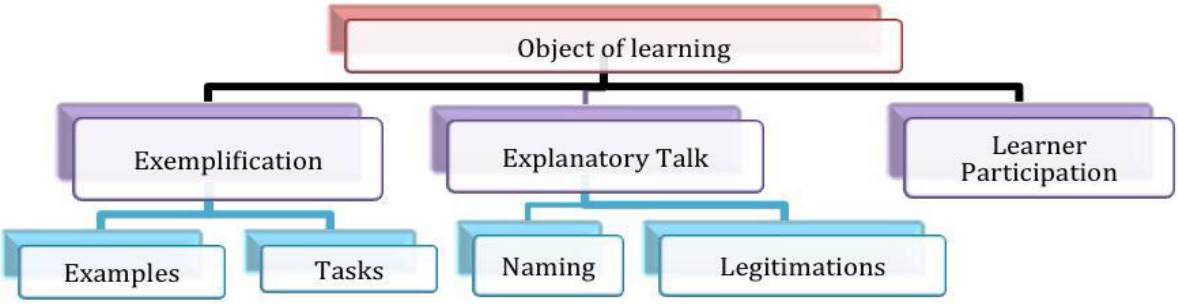
Across both phases, the concerns outlined earlier led to an emphasis on the view that that learning is always about something. Bringing into focus what this is, in terms of what learners are expected to know and be able to do, is central to the work of teaching. Marton and Tsui (2004) refer to this 'something' as the object of learning: 'The object of learning … is defined in terms of the content itself … and in terms of the learner's way of handling the content' (p. 228). Foregrounding the connection between 'object' and 'learning' is central, and contrasts with lesson 'goal' formulations. An object of learning in a mathematics lesson could be a concept, procedure or algorithm, or meta-mathematical practice. It goes without saying that the object of learning needs to be in focus for the teacher.

Juxtaposing primary and secondary level frameworks allows us to highlight ways in which the two frameworks differ in the aspects they focus on within their overall commonalities of focus on the mediation of mathematics predicated on the need for structure and generality.

MDI-Secondary (MDI-S) and mediating talk

In the MDI-S framework captured in Figure 1, the key generative mechanisms for the work of teaching are exemplification, explanatory talk and learner participation (for detail see Adler & Ronda, 2015). What stands between (i.e. mediates) the object (and here of learning) and the subject (the learner) are a range of cultural tools: examples and tasks, word use and the social interactions within which these are embedded. In this paper, our focus is on teachers' explanatory talk and how we think about quality within its two key features: naming and legitimating criteria.

Figure 1: Constitutive elements of MDI-S and their interrelations



Explanatory talk

Our emphasis on explanatory talk draws on Bernstein's (2000) notion of evaluation[1]. For Bernstein, any pedagogic discourse, and hence the discourse in mathematics lessons, transmits criteria as to what counts as mathematics. The transmission of criteria occurs continuously, be it implicitly or explicitly, through messages that are communicated as to what is valued with respect to the object of learning i.e. what is to be known or done, and how. We call this *explanatory talk*[2], the function of which is to name and legitimate what is focused on and talked about i.e. related examples and tasks. Analyzing how objects[3] focused on are named, and what is legitimated in an episode is key to being able to describe the mathematics made available to learn

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[1] Bernstein's notion of 'evaluation' is not to be conflated with assessment.

[2] The name here draws attention to the mathematical quality of the explication or elaboration offered – we could

[2] The name here draws attention to the mathematical quality of the explication or elaboration offered – we could equally have named this explicatory or mediating talk.

[3] Our use of 'object' here is in the most general sense and includes all that is in focus e.g. words, symbols, images, pictures, material objects, etc.

through explanatory talk, as well as reach a summative judgment on naming and legitimating as these accumulate over time in a lesson.

- ***Naming***

Learners' encounters with mathematical objects also occur through how these are named. We define naming to mean the use of words to refer to other words, symbols, images, procedures, or relationships, in the course of instruction. The tension in managing both formal and informal ways of talking mathematically, and thus naming what is focused on in class is now widely recognized. In WMC-S, we noticed some teachers' reluctance to use formal mathematical language as it is "abstract and the learners are put off", and others' over reliance on formal talk with neglect of connecting mathematical ideas to colloquial meanings.

We categorise naming within episodes as either *colloquial / non-mathematical* (and here we include *everyday language* e.g. 'over' in division, and/or *ambiguous pronouns* such as this, that, thing, to refer typically to what is being pointed to on the chalkboard) or *mathematical*. In this latter category we distinguish *mathematical words* used as *labels* or *name on*ly e.g. to read a string of symbols from *formal mathematical language* used. For example, in the first lesson extract below, *transpose* is categorised as *non-mathematical,* despite its common use in our mathematics classrooms. This is not because the word *transpose* should not be used when solving equations and inequalities. Our point is simply that if this is used exclusively to describe an algebraic transformation, with no accompanying mathematical justification (e.g. we subtract 6 from both sides of the equation) then underlying principles or properties like maintaining equivalence are never made explicit. Our purpose is to see the extent of *both* colloquial and formal mathematical talk and *the movement between these*.

- ***Legitimating criteria***

We distinguish criteria of what counts (or not) as mathematical that are particular or localized, or call on memory (L) (e.g. a specific or single case, an established shortcut, or a convention) from those that have some generality (e.g. equivalent representation, definition, previously established generalization; principles, structures, properties), distinguishing partial (PG) e.g. variables described as "letters which represent numbers which we do not know"; from full generality (FG) e.g. variables described as "letters representing any number". We are also interested in non-mathematical criteria (NM), everyday knowledge or experience (E), visual cues
(V) as to how a step, answer or process 'looks' (e.g. a 'smile' as indicating a parabola graph with a minimum, or memory devices that aid recall (e.g. FOIL)); or when what counts is simply stated, thus assigning authority to the position (P) of the speaker, typically the teacher. We further indicate errors in legitimating talk, which fell largely within NM by a negative sign e.g. V- .

The significance of these varying criteria is the opportunities they open and close for learning. Most obvious are the extremes of legitimations based on the one hand on principles of mathematics, thus with varying degrees of generality, and possibilities for learners to reproduce or reformulate what they have learned in similar and different settings. On the other hand, appeals to the authority of the teacher) and/or visual cues produce a dependency on the teacher, on memory (this is what you must do); or on how things 'look', requiring imitation that is local or situational (Sfard, 2008). While imitation might be necessary in aspects of mathematics learning, these cannot be the endpoint of learning. The criteria for what counts as mathematics that emerge over time in a lesson are thus key to what is made available to learn in terms of movement towards scientific concepts.

Table 1 summarises the categories and coding for explanatory talk. The categories themselves do not form a hierarchy – they distinguish different kinds of talk that emerge over a lesson in varying ways. In the second row are the levels we assign when we look at the accumulating categories across a lesson. The levels are hierarchical and reflect our privileging of mathematical names and principled criteria. We emphasise here that the assignment of a level in our analysis is an interpretive judgment, reflecting our privileging of generality through exemplification, mathematical names and principled criteria, and as these unfold over a lesson.

Figure 2: Explanatory talk – MDI-S

Explanatory talk	
Naming	**Legitimating criteria**
Within and across episodes word use is: *Colloquial* **(NM***)* *e.g.* everyday language and/or ambiguous pronouns such as this, that, thing, to refer to objects in focus *Math words used as name only***(Ms)** e.g. to read string of symbols *Mathematical language used appropriately **(Ma)*** to refer to other words, symbols, images, procedures	Legitimating criteria: *Non mathematical **(NM)*** *Visual* **(V)** – e.g. cues are iconic or mnemonic*;* *Positional **(P)*** – e.g. *a* statement or assertion, typically by the teacher, as if 'fact'. *Everyday **(E)*** *Mathematical criteria:* *Local* **(L)** e.g. a specific or single case (real---life or math), established shortcut, or convention *General* **(G***)* equivalent representation, definition, previously established generalization; principles, structures, properties; and these can be partial **(GP)** or 'full' **(GF)**
Use of colloquial and mathematical words Level 1: NM – there is no focused math talk – all colloquial/ everyday Level 2: movement predominantly between NM and Ms, some Ma Level 3: movement between colloquial NM, Ms & formal math talk Ma	Criteria for what counts as mathematics that emerge over time in a lesson and provide opportunity for learning geared towards scientific concepts. Level 0: all Criteria are NM i.e. V, P, E Level 1: criteria include L – e.g. single case, short cut. Level 2: criteria extend beyond NM and L to include Generality, but this is partial GP Level 3: GF math legitimation of a concept or procedure is principled and/or derived/proved

Episodes and their analysis[4]

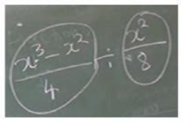
Secondary – Episode 1, lower end

Solving quadratic inequalities is included in the Grade 11 curriculum. For the inequality $x^2 > 4$ the teacher instructed the class that to solve for x, we "do exactly the same" as the steps followed in solving the equation $x^2 = 4$. After "transposing 4" to obtain $x^2 - 4 > 0$, "you then factorise to obtain $(x - 2)(x + 2) > 0$." He then wrote $x - 2$, looked to learners some of whom called out "greater than", and completed the inequality $x - 2 > 0$. He then asked the class whether he should write 'and' or 'or' and while some learners called 'and' and others 'or', he said "I will take 'and'". He continued with $x - 2$ and with some learners offering 'less than', and others 'greater than', he wrote x <---2 and said "x is less than negative two". The answer produced on the board was $x > 2 \text{ and } x < -2$. Having produced this answer, he then asked learners: "Now how come is it that the sign changed?" (pointing to > in the second part of the answer) and worked with learners testing various numbers to confirm the two inequalities.

Naming: With the exception of "this 'and' or 'or' thing" (and so demonstrative pronouns) both the teache and learners used mathematical words as *labels or to name the symbol strings* they were talking about, hence coded as follows: **(Ms).**
While in MDI---S we do not level an episode, for our purposes here, mathematical words are used, but only for labeling or reading symbol strings. If this persisted through the lesson, naming would then be**level 2.**
Legitimating criteria: The legitimating talk accompanying the steps taken to write down the answer $x > 2 \text{ and } x < -2$ for the inequality $x^2 > 4$ were at the level of assertions with no rationale for obtaining the inequality relations, nor the erroneous connector 'and'. (**P**---) The interpretive judgment, if restricted to this episode, would be that the legitimation was by assertion, and erroneous, and so NM and **level 0**. While the teacher proceeded to test various numbers, these were used to confirm an asserted solution and not to derive it.

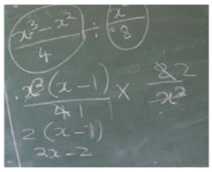
Secondary – Episode 2, more familiar

In a Grade 9 lesson introducing the division of algebraic fractions, the teacher used $\frac{2}{7} \div \frac{2}{3}$ as a first example to recall the rule "change the sign and swapover". The same rule was applied to $\frac{2x}{6x} \div \frac{2x}{3x}$ and then she put up the third example $\frac{x^3 - x^2}{4} \div \frac{x^2}{8}$ and said : "It's one and the same thing. They give you something like this (writes symbols on board), ok? … Over here (points to $\frac{2}{7} \div \frac{2}{3}$) you just have two numbers, a fraction divided by a fraction, ok? (Learners chorus 'yes'). Over here (pointing back to example 3) is the same thing. I've got, here's one fraction divided by one fraction (circles each fraction).



4 Both episodes have been described previously in papers differently focused, and where they form part of a full lesson analysis.

She then asked learners what they needed to do to complete the division, and continued "… before you divide you factorise, because over here it concerns the common factor. Why? Because we want to have one, one term at the top and one term below, ok?"



After completing the steps illustrated on the right, she concluded: "you just apply the same principle, it's just that when it looks complicated just pause and say what must I do here?" Together with contributions from the learners, she says we "take out the common factor x squared and we get x squared bracket x minus 1 close bracket" and she writes: $x^2(x-1)$. The class continues to call out with her the next steps i.e. "change the sign and swap", and then "cancel common factors"

Naming: In this episode, non---mathematical talk **NM** through use of ambiguous pronouns (e.g. this), was accompanied by mathematical words used mainly to read strings of symbols (x squared bracket xminus 1 close bracket) **Ms**. There was also some appropriate formal naming of objects (e.g. a fraction divided by a fraction, one term, common factor) **Ma.** This episode, again with the limitation that we do not assign levels to episodes, would be **Level 2**.

Legitimating criteria: The overarching **legitimating criteria** in this episode were to previous examples as the 'same thing' and their general structure – one algebraic fraction divided by another **(GF).** The "top" and "below" **(V)** of the fractions were pointed to as each needing to be "one term", and so expressed as factors which were defined in Episode 2 as "dividing without remainder" **(GF).** The division follows a short cut **(L)** (remembered from previous work … change signs and 'swap') with rules and procedures (factorise first, take out common factor, I cannot just go and say …) that were stated, notderived **(P).**

In overview, the criteria for recognizing the *form of the expression* were general, but the criteria for the procedure for division were dominantly localised, as there was reliance on rules, shortcuts, and in some cases assertions by the teacher. Hence, again with the limitation that it is a single episode, as there is some generality at least at the level of form, we would assign this as **Level 2**.

MDI-Primary (MDI-P) and Mediating Talk

Mediating for mathematical learning in relation to focal objects, and with a drive towards mathematics viewed in terms of a connected network of scientific concepts, was central to our work as well, but the key analytical foci, for better fit with the early primary years where much of our dataset was located, differed. In the primary years, a broad swathe of evidence points to the importance of using situations, diagrams, and physical artefacts to provide strong visualizable and imaginable underpinnings for the more abstract symbolic mathematical language that is to come. Mediating for ***connection*** is central to this work, with *physical artefacts*, *inscriptions*, and *talk* then being the key empirical phenomena in the context of tasks and example spaces for examining the nature and extent of connections seen in teachers' MDI. We look, across these phenomena, for features related to the extent to which mathematical structure and generality are made available for appropriation in instruction.

As with MDI-S, we focus specifically on the ways in which mediating talk is categorised, and the markers of quality developed within each of the MDI-P talk categories. The categories we have focused on relate to: generating solutions; building mathematical connections; building learning connections through explanation and evaluation. These categories and the quality markers within them are detailed in Figure 3.

The ‘generating solutions’ category is focused on teachers’ problem-solving methods and strategies within the task and example space in that episode. The hierarchy in this category marks, at the lower end, some of the problems outlined earlier with incoherence and disruptions to mathematical problem-solving processes. At the upper end, quality is viewed in relation to the offer of methods of solving that have generality beyond the example space being worked with, and without restriction to the particular artefact or inscription being worked with in that episode.

Figure 3: Explanatory talk – MDI-P

Method for generating/ validating solutions	Building mathematical connections	Building learning connections: explanations and evaluations --- of errors/ for efficiency/ with rationales for choices
No method or problematic generation/validation 0 Mixing of knowns and unknowns	Disconnected and/or incoherent 0 Disconnected /incoherent treatment of examples OR Oral recitation with no additional teacher talk	Pull---back 0 Pull back to naïve methods OR No evaluation of incorrect offers
Singular method/validation 1 Provides a method that generates the immediate answer; enables lr to produce the answer in the immediate example space	Every example treated from scratch 1	Accepts/evaluates offers 1 Accepts lr strategies or offers a strategy OR Notes or questions incorrect offer
Localized method/validation 2 Provides a method that can generate answers beyond the particular example space	Connect between examples or artefacts/ inscriptions or episodes 2	Advances or verifies offers 2 Builds on, acknowledges or offers a more sophisticated strategy OR Addresses errors/ misconceptions through some elaboration, e.g. ‘Can it be ------------?’ ‘Would – this be correct, or this?’ Non---example offers
Generalized method/validation 3 Provides a strategy/method that can be generalized to both other example spaces AND without restriction to a particular artefact/inscription	Vertical and horizontal (or multiple) connections made between examples/ artefacts/ inscriptions / episodes 3	Advances and explains offers 3 Explains strategic choices for efficiency moves OR Provides rationales in response to learner offers related to common misconceptions OR Provides rationale in anticipation of a common misconception

The ‘building mathematical connections’ category is focused on the ways in which examples, in that episode’s example space, are connected within instruction. At the lower end, disconnected/incoherent treatment of examples within episodes, or episodes involving oral recitation pupil responses (relatively common in a context where chorused chanting of answers is relatively common) with no teacher talk, are represented. At the upper end, multi-directional connections within the example space – which is treated as a linked set in the ways described in Watson & Mason's (2006) work, and focused on structure and generality, are aimed at.

In the 'building learning connections: explanations and evaluations' strand, our attention is on instruction focused on progression and explanation – teaching that presents mathematical discourse as having both progressions and rationales. Much of the coding in this strand is seen in the empirical space of teacher responses to learner offers. At the lower end, teaching that 'pulls back' towards more naïve strategies, or fails to offer any evaluation of learner inputs, is described – with both of these phenomena described in South African writing (see Ensor et al, 2009, for the former, and Hoadley, 2006, for the latter). At the upper end, instructional talk works to advance mathematical offers, and provide rationales for choices of steps.

Episodes and their analysis

Primary – Episode 1, lower end

Halving' is the topic being dealt with in a Grade 2 class. Initially, learners are given boxes/bottle tops and asked to make half of 12, 10, 8 and 4. In the following exercise, with bottle tops still available, learners are asked to work out half of the following numbers: 2, 4, 8, 16, 22, 24, 26, 32. In fieldnotes, the observers note that in the early examples, some children appear to 'know' the answer, but have trouble with halving two--digit numbers. The teacher steps in to explain how to work out 'Half of 26'.Each student pair in the class is asked to make 26 balls from clay – which they do taking extended time and, predictably, making balls of different sizes. The teacher draws 26 circles on the board in a line. Her explanation for how to work out half of 26 proceeds as follows: 'I want us to count to 13, and move those balls aside (marks divide on the board). How many balls are on the other side?13 as well. So 13 is half of 26.'
Method for generating/validating solutions: 0 (teacher's explanation introduces the solution, 13, at the outset of the problem---solving process, and then verifies its correctness, rather than working with given quantities to deduce the unknown) **Mathematical connections: 0** (through much of the episode, there is no additional teacher talk relating to the example space; where talk comes in, the example is dealt with in incoherent ways described above) **Learning connections: 0** (no evaluation of learner working in this episode)

Primary – Episode 2, more familiar

Within a lesson focused on working on place value based 'breaking down' and 'building up' of numbers the first episode with this focus (following some work on counting and number bonds) involves a task asking the class to 'break down the numbers:13, 19, 27, 45, 67, 93, into their place value by quantity, and following this being written up for all examples, then represent the tens and units quantities with ten strips and unit squares on the board'. Learners' offers of the symbolic breaking down are written in by the teacher on the board: (e.g. 13 = 10 + 3). The teacher's associated commentary included emphasizing the horizontal equivalences in each example, and working with the example space as a set to note that: 'we have two digits this side (gesturing down the 'tens' break down values), and 'now the remainder is one' (gesturing down the 'units' break down values). Multiple learner offers across this episode all involve correct answers, but teacher incorporates checks of these offers in two instances through making a counter---offer and asking learners to explain their choices e.g. when a learner states that 'one ten' strip is needed for 19, the teacher picks up one unitsquare, asks if this is okay, and then probes why not.
Method for generating/validating solutions: 2 (while the methods offered for generating solutions are

coherent and fit the example space, this talk would not generalise beyond two---digit numbers, and would also not deal well with either single---digit examples or multiples of ten where the breakdown need not necessarily have a 'ones' component)

Mathematical connections: 3 (horizontal and vertical connections made consistently)

Learning connections: 3 (teaching proceeds smoothly in alignment and with elaborations of learner offers; a common misconception is anticipated in her offer of a unit square instead of the ten strip suggested by the learner, with probing of why the teacher's choice is incorrect)

DISCUSSION

Our focus in developing and using our frameworks is on the quality of mathematics made available in the classroom MDI. This focus contrasts with the broader scope in frameworks such as Hill et al's (2008) Mathematical Quality of Instruction where features like lesson format and links to learning are incorporated alongside teachers' mathematical talk. Our narrower focus includes more disaggregation between the two levels of concern (incoherence and error at the lower extreme, and structural relations and generality at the upper extreme). Thus, while across both frameworks, categories are theoretically informed, the levels within them are empirically derived with a view to allowing description across the pedagogic range. We have needed key indicators of mathematics worked with across this wider range in instruction than is typical in available frameworks in the international literature. This disaggregation assists with our goals for being responsible in our coding of what is present in instruction, and then being able to be developmentally responsive in our work with teachers.

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Faults in preservice teachers problem posing: What do they tell us?

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This study investigates preservice teachers' faults in problem posing. Based on earlier theoretical and empirical contributions in the field of problem posing, the study focuses on preservice teachers' ability to pose problems based on a given representational settings within area of multiplication and division. Qualitative analysis of problems produced by 95 preservice teachers unveils typical mistakes. Classification of faults in problem posing primarily points to limitations in mathematical content knowledge but also in pedagogical content knowledge of prospective teachers. I discuss implications of the findings: what these mistakes tell us and how these findings can help us in prospective teachers' training.

Keywords: preservice teachers, problem posing, representations, multiplication and division.

INTRODUCTION

Could problem posing activity be used as a diagnostic tool for revealing students' knowledge of mathematics and/or knowledge of math pedagogy? Expertise of prospective mathematics teachers includes complex set of different elements of psychology, pedagogy, mathematics, philosophy, and other sciences. Ma (1999) discusses teachers' "profound understanding of fundamental mathematics" as expertise in mathematics and how to communicate with students. Ball and colleagues (2008) categorized mathematics knowledge for teaching into subject matter knowledge and pedagogical content knowledge (Ball et al., 2008). The later includes knowledge of content and students, content and teaching, and content and curriculum. One of newly recognized elements in math pedagogy is problem posing (Brown & Walter, 1990; Margolinas, 2013; Singer et al., 2015). Researchers recognize that knowledge about students, mathematical content and pedagogical content knowledge in teaching are intertwined (e.g. Ball et al., 2008; Liu, 2005). I presume that "problem posing proficiency" should not be looked upon as general element of teaching pedagogy separated from the math domain which is a subject of interest in the problem posing activity. Therefore, I focused on pre service faults in teacher's problem posing in particular area of whole number multiplication and division. Note that evaluating teachers' competences by studying faults is already recognized as a valuable approach in mathematics education research. For example, Ma (1999) examined and compared mathematics teachers in the United States and China by drawing attention to the type of mistakes they make.

In the following paragraph I explain how we can analyse design of math problems. Then, I will discuss in more details about what I mean by "proficiency in problem posing" and why I believe that it is important to attend to problem posing as a key teachers competence.

Any problem can be characterized by context (real or formal mathematical), given information (quantities, relations), requirements and mathematical environment (Malaspina et al., 2016). Math problems are described within a problem space in terms of its context, of givens and unknown elements and of the relations between the elements. Psychologically, problem space is defined as a mental representation of a problem that contains knowledge of the initial state and the goal state of the problem. The context may be abstract as well as realistic. Stoyanova and Ellerton (1996) discuss problem posing situations in terms of the source of ideas (e.g. classroom activities or textbook). A collection of problems with the same problem space may be posed by setting or varying 1) what is given, 2) what is searched for (unknown), or 3) the context (Milinković, 2015). In my study students were challenged to pose multiplication and division problems initialized by information set within various representations (pictorial, tabular and in words given numerical values to be used in problems).

Proficiency in problem posing might be considered by someone as a part of pedagogical content knowledge (i.e. math pedagogy) while for others it is part of subject matter knowledge. Kilpatrick (1987) thinks that problem posing should be seen not only as a means of instruction but as a goal of instruction. In problem posing activities, teaching competences such as fluency and flexibility of subject matter knowledge as well as inventiveness become visible. In the literature on relations among subject matter knowledge, pedagogical (didactical) knowledge and curricular knowledge we find that problem posing was an underestimated issue. For Shulman and Grossman subject matter knowledge consists of understanding of concepts, facts and principles as well as rules of evidence and proof (Shulman & Grossman, 1988). For them pedagogical content knowledge includes understanding of how to represent subject matter in ways suitable to the needs and abilities of learners. Malaspina et al. (2016) show that problem posing has beneficial impact on the development of teachers' didactic and mathematical competencies.

Representations in problem posing

Our training route in problem posing is based on a representational approach. The idea that representations are "tools in thinking" is well documented in the literature (Cuoaco & Curcio, 2001). Representations may be informally explained as different ways to represent a problem. Different representations are more often seen as tools in problem solving than as means to problem posing. Some physical representations such as counters or beads, or pictorial representations, such as number line or place value table provide good contexts for posing various problems. Tichá & Hošpesová (2016) explored graphical representations called branched chains to solve and pose word problems. They found that for both pre-service and in-service teachers it was more difficult to pose a problem to match a chin model than to create a picture to illustrate the problem in a process of solving it.

Friedlander and Tabach (2001) maintain that the teacher's presentation of a problem situation in different representations encourages flexibility in students' choice of representations. Representations might be seen as ways of presenting a problem on different levels of abstraction (Milinković, 2015). It means thinking about particular math idea in different paradigms. I inquired the results of problem posing efforts based on a range of representations which set a stage for the activity.

METHODOLOGY

The sample was drawn from students preparing to become elementary school teachers in a large university in Serbia. There were 95 students, all enrolled in the course Methodology of Teaching Mathematics which I taught during the fall semester of the third year of studying. In the course, participants learned about elementary school mathematics curriculum and teaching methods. They had 2 lessons per week over 15weeks long semester involving activities of problem posing and problem solving. A teaching assistant attended all my lectures and was the second person (beside myself) who assessed student's productions with scoring guidelines set in advance.

The course was designed to focus their attention to studying structure of math problems. As a part of regular class activities they were encouraged to analyse problems in different math domains (arithmetic, algebra, geometry, fractions) and explore possibilities to create variations of them. Each week they were asked to pose problems. In some occasions they designed variations of problems found in math textbooks. In others, they were asked to pose own problem from scratch, fulfilling some requirement (problems of different levels of difficulty, problems set in specific real context, etc.). As they learned about representations in mathematics they were asked to create problems using different representations or to solve problem by using different representations. Their productions were part of portfolios assessed at the end of semester. For example, they were asked to pose a problem involving numbers 32 and 4. Students came up with simple problems of division and multiplication (Figure 1).

Figure 1 From student portfolios, problems with numbers 32 and 4.

Without doubt the most common representation of a problem situation to be modelled is through word problems. Students need to develop competencies in constructing appropriate mathematical formulations, often in a form of math expression on the basis of syntactical surface tools. Another way of representing problems is in pictures (diagrams, graphs, or tables). In this case, problems are related to analysing pictures and understanding what the information given in pictures is telling them and how it can be used to provide answer or find a solution.

I set our investigation within the area of multiplication and division. Earlier, investigations of teachers' knowledge of representations of multiplication pointed to limitations in their knowledge (Barmby & Milinkovic, 2011). It was found that teachers tend to use simple contexts when posing problem (distributing flowers in cases or quantitatively describing a problem given in picture).

The problem posing test was administered at the end of the semester. The problem posing items were set in different representational frame. Two items referred to pictorially set contexts with jars and cookies (Figure 2). In the first one they were asked to pose 'multiplication' problem, in the second they were asked to pose 'division' problem.

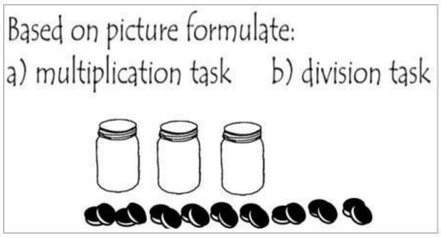
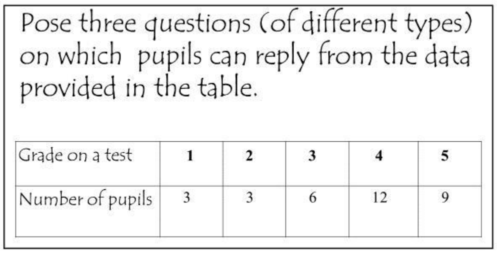


Figure 2 Pictorial contextual frame for designing task.

In the next, students had to design "equation with unknown divider". The final item assessed students' ability to pose different problems based on information provided in a tabular form (Figure 3).


Pose three questions (of different types) on which pupils can reply from the data provided in the table.

Grade on a test	1	2	3	4	5
Number of pupils	3	3	6	12	9

Figure 3 Tabular contextual task frame.

Two evaluators (myself and the teaching assistant) evaluated test independently. We compared and discussed our judgments about students' productions. The data were sorted and analysed qualitatively. During evaluation emerged categories of faults in posing problems. Consequently, faulty posed problems were analysed along three identified characteristics: context, math formulation (meaningfulness of question) and language. Note, “math formulation” is what defines relations between given and unknown elements within problem space. Also, we earlier identified context as one of key elements in problem space. In the analysis offered in this paper I focus on those two elements: context and math formulation, as they appear in problems designed by students.

RESULTS AND DISCUSSION

Before turning to the qualitative analysis let me mention that only 15 out of 95 (about 16%) students designed all problems without faults. Equal number of students correctly posed multiplication and division tasks based on the picture (Figure 2). Yet, only 52 out of 95 posed both problems correctly. Exactly the same number of students successfully posed three different problems based on the data given in the table. Slightly less, 49 students managed to pose a problem with which could be solved by equation with unknown divisor. (Remark that this type of requirement is commonly found in elementary school textbooks.) I turn now to the analysis of examples of faulty problems designed by students.

Analysis of tasks based on pictorial representations

First, I attend to results of posing multiplication tasks based on a picture. Here are some problems with faults made by students. I discuss how students use pictorially set context.

‘Multiplication’ problems

M04: There are 20 cookies, and 5 jars. You need to split them so that you have equal number of cookies in each jar. How many cookies will you put in each jar?

M19: Bogdan’s aunt has bought 5 jars with equal number of cookies. After turning over all cookies, she divided them into 2 equal parts. Bogdan got 10 cookies. How many cookies was in jars all together?

M51: Ana had18 cookies. She divided them into 3 jars. How many cookies did she put in each jar?

M84: How many cookies will be in a jar if there are equal number of cookies in each jar?

As we can see, M04 is designed correctly. But, to find the answer we need to use division instead of multiplication. This is an example of faulty defined relation between given and unknown elements within problem space. Problem M19 is a complex problem involving multiplication and division. But, from the text one cannot discern how many cookies were involved in the story at the beginning. If one should rely on picture when solving this problem, then the first part of the problem is not needed. If not, than it is not clear were those 10 cookies all cookies involved in the story or half of the total amount. Here, the student made mistake in creating context. The idea of “equal grouping” is missing in problem M51. In Problem M84, quantitative information is missing (or reference to the picture).

In the cases of designing 'division' problems based on pictorial representation students made similar mistakes. First, they could not discern for which word problem solution may be solved by multiplication and which one by division; second, they often forget to mention that objects need to be split equally (e.g. problem D09). In some, students omitted some elements of the context or added some which were not present on the picture.

'Division' problems

D09: Sara bought 12 cookies. She needs to split them into 3 jars. How man jars will be in each jar?

D95: There were 3 jars on a shelf. In the first there was 2 times less cookies than in the second. How many cookies was in the third jar if there was total of 9 cookies.

Analysis of tasks based on numerical representation

Next, I analyse problems posed based on defined relation between numerals whereas students were free to define context. To start up, look at E14. The model equation for E14 is an equation with unknown addend (not divider). Similarly nonmatching example is E67. The model equation for E67 is not the one which the student wrote, which unveils students' failure to model realistic problem situation into mathematical formula. Deficiency in subject matter knowledge prevented her from being successful in attempt to pose problem.

I remarked cases in which a student did not set problem space properly as he failed to give sufficient information. For example, in E71 and E89 quantitative information were missing. In addition, the student who wrote E89 actually did not know what the model equation for the designed problem should be (again having flows in knowledge of mathematics).

'Equation with division' problems

E14: Marko have had few stickers when his mother brought him 10 more. Now he has 15 stickers. How many stickers he had before he got stickers from his mum?

E67: How many grandchildren does granny have if she gave to each of them 5 apples and she had 2 leftovers? (Student wrote equation x : 5=2!)

E64: Jovan had few marbles out of which he gave half to his younger brother. How many marbles he had at the end?

E71: Marko have had few stickers when his mother brought him 10 more. Now he has 15 stickers. How many stickers he had before he got stickers from his mum?

E89: Novak had few candies. He gave to each of his friend 5 candies. He had 4 left over. How many candies did mother gave to Novak? (Student wrote: x: 5=4!)

Analysis of tasks based on 'Tabular' representation

Finally, I examine problems designed whereas the information was given in a table. Most of the constructed problems were in the domain of statistics. These problems appear to be good at first glance. Yet, they were out of reach of pupils aged 7 to 11 given the fact that statistics was not part of the elementary school curriculum in Serbia (e.g. E48). This indicates lack of knowledge of PCK (content and curriculum). In addition, there were students who were making logical mistakes (e.g. wording of problem T63; it is not possible to get two different grades at the same time). In the problem T77, as I looked at the student's solution, I found that he did not recognize that there were multiple solutions for the problem.

T48: What is the mean value on this test?

T63: How many pupils got grade 3 and 5?

T77: In one class out of 30 students 12 of pupils received a grade lower than 3 on a test in mathematics; number of pupils whose grade was 4 more than double the number of pupils who got grade 5. How many pupils got grade 4?

Classification of mistakes when posing problems

The problems discussed above exemplified different types of mistakes student teachers made such as providing insufficient information, incorrect direction for solving problem, creating impossible contexts or problems which were not part of primary school curriculum. As I went through all faulty problems I found that certain mistakes repeat. I recognized and classified them as follows.

Here is a categorization of faults in problem posing with reference to how we define problem space:

1. Not using (all) elements of the defined context
2. Using ill-chosen relations for given elements
3. Using unfitting context
4. Math semantics - using inappropriate mathematical formulation
5. Posing problem within math domain out of reach of children (unfamiliar math relations)
6. Syntax faults - using inappropriate sentence structure (not discussed in this paper)

To summarize, along lines of Shulman and Grossman (1988) and of Ball and her colleagues (2008), problem faults predominantly pointed to weaknesses in subject matter knowledge. Particularly, mistakes of type 1, 2, 4 and 6 indicate limitations in subject matter knowledge. On the other hand, mistakes of type 3 and 5 belong to pedagogical content knowledge.

CONCLUSIONS

The problem posing activities based on variation of representational problem setting exposed in this paper could be a good methodological tool in teacher training. Our study shows how the activity of problem posing may also be a diagnostic tool in evaluating student teachers knowledge. This activity can help us recognize hidden students' limitations either in subject matter knowledge or in pedagogical content knowledge. The identification of faults could give us clue how we could help students to overcome weaknesses them by analysing problems in details.

Our classification of faults in problem posing points primarily to preservice teachers' partial comprehension of multiplication and division. Thus, the results cannot be generalized. But, future research could explore students' competences by studying mistakes they make when posing problems in other math domains. From a research point of view, it would be valuable to examine further whether specific form of representation used to set a stage for the activity of problem posing may influence teachers' ability to design appropriate task. As I look about the next generation of students, I am thinking about ways to discuss potential mistakes in problem posing before they happen.

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Planning, teaching and reflecting on how to explain inverse rational numbers: The case of Ana

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To promote understanding of rational numbers is challenging to teachers and prospective teachers, and this led us to study the planning, teaching, and reflection processes of prospective teachers (grades 5-6) on that topic. The aim of the paper is to analyse how Ana, a prospective teacher, prepared, developed, and reflected about communication, with special focus on instructional explanations and on mathematics ideas about the unit on multiplication of fractions. Ana prepared and developed different explanations to reach pupils' ideas. We discuss the nature of her didactic knowledge as she seeks to promote pupils' understanding of the multiplication of inverse rational numbers.

Keywords: Prospective teachers, Teaching practice, Communication, Rational numbers, Multiplication.

INTRODUCTION

Rational numbers are a fundamental topic in the mathematics curriculum. Teaching it presents a strong challenge to teachers' knowledge and practice and they must use different representations and meanings in order to promote pupils' understanding of this concept. However, we know little about how teachers use different representations and meanings, for what purpose and facing what difficulties (Mitchell, Charalambous & Hill, 2013). We especially want to understand grade 5-6 prospective teachers' knowledge and practice during supervised teaching practice, as this enables a close look at the nature of their knowledge. In analysing teaching practice we focus on tasks, classroom communication and prospective teachers' actions, striving to understand the nature of their didactics knowledge. In this paper *the aim is to analyse how a prospective teacher, prepared, developed, and reflected about communication, with special focus on instructional explanations and on mathematics ideas about the unit on multiplication of fractions.*

PROSPECTIVE TEACHERS' KNOWLEDGE AND COMMUNICATION

Prospective teachers' knowledge may be considered from different perspectives. Both mathematics and didactics knowledge are critically important and deeply interconnected within teaching practice. Besides mapping both kinds of knowledge, it is important to understand their nature and how they relate to teaching practice. Didactics knowledge has two essential dimensions: knowledge about both tasks and pupils (Ponte & Chapman, 2015). Teachers must be able to select, design, and sequence tasks and to explore pupils' strategies, establishing learning sequences and recognizing learning opportunities. They also must anticipate pupils' common mistakes and misconceptions, listen and interpret their ideas, anticipate their solutions, and know what they will consider challenging, interesting, or confusing (Son & Crespo, 2009). A fundamental idea about rational numbers is that they "have multiple interpretations, and making sense of them depends on identifying the unit" (Barnett-Clarke et al., 2010, p. 17). Regarding representations, teachers should know how pupils deal with pictorial and symbolic representations (verbal, fractions, decimal and percentages) and how to relate them, making sense of the numerical set as a whole (Barnett-Clarke et al., 2010). So, when preparing tasks, teachers should recognize the pros and cons of using certain representations and know how to take advantage of pupils' strategies and representations to promote mathematics ideas (Ball et al., 2008; Stylianou, 2010). However, the use of particular representations may raise challenges to teachers since they may induce pupils into mistakes or incomplete conversions and may be far from pupils' initial knowledge.

Tasks and communications are essential aspects of teaching practice (Ponte, Quaresma & Branco, 2012) and they need special attention from teachers when preparing, teaching and reflecting about teaching practice. Communication is a fundamental element of teaching practice and it is inherent to the process of building knowledge (Menezes et al., 2014). Communication involves sharing something and, to do so, we make use of gestures, images and symbolic representations, explanations and questions. Communication may be oral or written, and it includes both linguistic and mathematics representations (Ponte & Serrazina, 2000). One important aspect of communication is questioning using confirmation, focus, and inquiry questions (Ponte & Serrazina, 2000). Communication also includes those representations that are used to aid in solving a task, such as building or illustrating objects, concepts, and mathematics situations. These representations may arise from pupils or not (Mitchell et al., 2013). Instructional explanations are another important aspect of communication. Far from being mere "transmissions of content," instructional explanations support the establishment of relationships between mathematics concepts. Active, pictorial (iconic and drawings), and symbolic representations (Bishop & Goffree, 1986; Bruner, 1999), together with verbal communication (Ponte & Serrazina, 2000) may be used to convey concepts and procedures to pupils. Instructional explanations may have different purposes and characteristics, focusing on procedures and/or concepts, and may be carried out at different times during a lesson. According to Charalambous, Hill, and Ball (2011), explanations can be used to introduce new content, answer pupil questions, or support pupils with difficulties. A good explanation may eliminate erroneous ideas, meanings and processes. Charalambous et al. (2011), in a study of prospective teacher education looked at the issue of the quality of explanations and concluded that an incoherent, incomplete, or unclear explanation may affect pupils' learning. On the other hand, a "good explanation" is meaningful and easy to understand. Thus, prospective teachers must: (i) keep the audience in mind, using language suitable for pupils; (ii) define appropriately the key terms and concepts; (iii) highlight the main mathematics ideas while explaining the process step-by-step; (iv) use appropriate examples and representations while also modelling procedures and concepts; and (v) clarify the issue in question, showing how it should be answered.

RESEARCH METHODOLOGY

This is a qualitative and interpretative case study. Ana is a 24-year old prospective teacher of a School of Education, doing supervised teaching practice on rational numbers (grade 5) in her last year of studies. She studied mathematics 12 years before coming to teacher education and is regarded as a good student. She is visibly insecure about what she intends to carry out in her practice and feels torn between direct and exploratory teaching. Ana was interviewed at the beginning (IE) and end (FE) of her supervised teacher practice. Ana's classes were observed and video taped for later analysis and video-stimulated recall interviews before and after each lesson (BCiE, ACiE) (Nguyen, McFadden, Tangen & Beutel, 2013). Her lessons plans and personal notes were also analysed. Data analysis is descriptive and interpretative searching to understand the processes of planning, teaching, and reflecting about the product of two inverse fractions. During planning, we analysed the strategies for solving tasks and representations that she prepared to support her explanations. During teaching, we focused on how Ana provided explanations and highlighted several aspects. During reflection, we emphasized her view of the explanations that she gave and the mathematics ideas that emerged. The categories used for analysis were taken from framework presented formerly in this paper.

PLANNING, TEACHING AND REFLECTING ON

EXPLANATIONS Class preparation

Ana taught several lessons about rational numbers, four of which introduced new concepts. In the first, the key idea was to explore the inverse of a rational number. As the pupils already knew how to multiply rational numbers, the aim of the task was that they pictorially represent expressions to visualize that "the product of a number by its inverse is 1." For Ana, it was essential that the pupils understand the rule:

> So that there might be a logical sequence to the classes. Because when they were doing multiplication, they identified the rule themselves and when they were adding as well… That way they will really realize what they are doing instead of memorizing…. I wanted to try to use the process of understanding rather than memorizing. As this is a new attempt, for me, let's say... I'll just experiment to see what'll happen. To see what works better, let's say. (BC1E)

Ana prepared a plan which briefly described what she intended to accomplish. She defined objectives and general ideas about the activity that was supposed to happen in different moments of the class. The plan was sent to her supervisors and did not anticipate possible pupils' solutions to the tasks. Therefore she did not discuss with her supervisors her potential reaction to pupils' answers. However, Ana solved the task in a personal notebook. She did not include the answers in the lesson plan as she was reluctant to expose possible weaknesses to her "supervisors/evaluators." Our analysis of the different records found that she solved the problems in different ways (symbolically and pictorially), as shown in Figure 1.

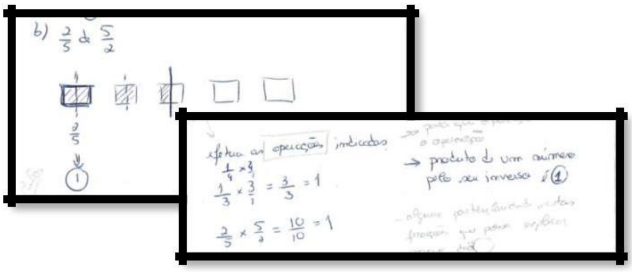


Figure 1. Solution of the proposed task (1st version).

Doing symbolic computations, she answered the problems using the algorithm $a/b \times c/d = ac/cd$ and when she obtained integers she turned them into fractions. When solving the problem “2/5 of 5/2”, using a pictorial representation, she used a unit of five that she divided into two equal parts by colouring two rectangles and a half. She divided the new unit into five parts and coloured two of them in the same manner as the rectangle. Thus, Ana began to interpret the fraction 5/2 as a quotient when she divided 5 rectangles in half. She then thought of “2/5” as an operator when she took one fifth to be a new whole and then she took two parts. Thus the product of 2/5 by 5/2 were two parts of five, which were equivalent to a rectangle.

Class 1: Solving 2/5 of 5/2

In class, Ana asked the pupils to represent the expressions 1/4 of 4 and 1/3 of 3. In these expressions, the fractions have the operator meaning, which posed no problems to the pupils or to her. She then wrote “2/5 of 5/2” on the board and again asked the pupils to represent the expression “with a drawing or a diagram.” The pupils began to solve the task and Ana moved around the room, emphasizing the need to illustrate the expression pictorially. At a certain point, she realized that there were recurrent questions from pupils, and she decided to discuss the task with the whole class. She began to focus pupils’ attention on the fraction 5/2, focusing their attention on 5, as the starting unit, and dividing the unit into two parts. A pupil proposed to divide each of the 5 rectangles into halves and Ana drew Figure 2 and explained:

Ana: Gabi wanted to divide each of the 5 units in half. But is that what they want us to do?... We have five units… And we’re going to divide them into two parts... [Gabi’s idea] will help us find out where the half of our five parts is... What will it be? [draws a line through the middle] Why? We have here two and a half units and another 2 units and a half... We will now have five halves. And now these 2/5? Now we have to represent 2/5 of 5/2. That is, when we represent the 5/2 we find our unity for 2/5. In other words, when we will represent 2/5. Why? Because now our universe will now turn into just one part. That's where we'll represent 2/5… What shall we colour? 2 parts of 5...

Gabi: I thought about putting just 5/2 (five halves).

Ana: You're right. These 2/5 will only belong to this part. We don't need to represent 2/5 of our entire universe, of the 5 units. It's just this part here. And now what will these 2/5 be? They will be these two parts, these 5. What will that give us?

Gabi: One.

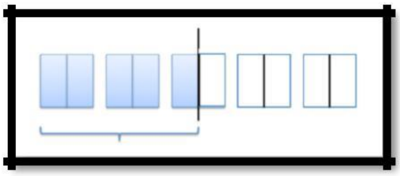


Figure 2. Ana's representation in the 1st class.

Ana had planned to discuss the pupils' answers but during class she rushed ahead and explained the problem herself. She explained that the improper fraction 5/2 is a quotient where the unit is 5 and is divided into two parts. She defined the terms of the expression emphasizing the importance of indicating the reference unit and stressing the word *of* to know that a multiplication was involved. She then told pupils that they had gotten to a new unit and considered two fifths of this new unit. Thus, using step-by-step modelling, she explained the main mathematics idea that she intended to illustrate. However, the pupils seemed to interpret the fractions as a part-whole relationship. Ana did not reject this view, nor did she explore these two perspectives, missing the opportunity to clarify how her explanation fitted the pupils' previous ideas.

Reconsidering the explanation

After class, at the request of the school supervisor, Ana tried to recast her explanation. In her notebook, we found another attempt at solution where we realize that she had not decided on the way she wanted to represent 5/2 (Figure 3):

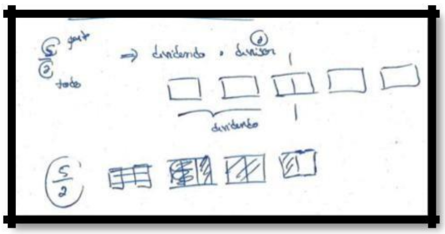


Figure 3. Second solution of 2/5 to 5/2.

When we analysed her notes, we realized that Ana thought of 5/2 as a fraction meaning a part-whole as well as a quotient where the numerator is the dividend and the denominator is the divisor. Thus 5 is the dividend and 2 is the divisor and that is why she divided the five rectangles in half. In the second illustration (Figure 4) we see that she scratched out two rectangles as being extra and used only two rectangles and a half or five half rectangles. She seemed to be satisfied with the solution about the fraction 5/2. In this second solution she thought of the rectangle as a unit and interpreted the fraction as part-whole. So, in trying to decide on how to illustrate the expression 2/5 of 5/2 she hesitated between the part-whole and quotient meanings depending on the unit identified and the representation used. From a didactics point of view, this question may have an impact on pupils' understanding of the concept.

To review the explanations for pupils, Ana wrote down in her notebook some ideas to point out. We note that she had anticipated potential ways to illustrate the concept and to clarify the issue. Figure 4 appeared in notes that she produced before making a PowerPoint that the showed in the second class about inverse rational numbers.

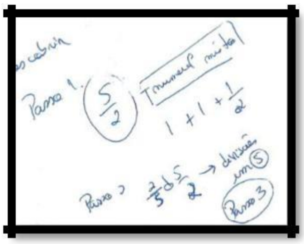
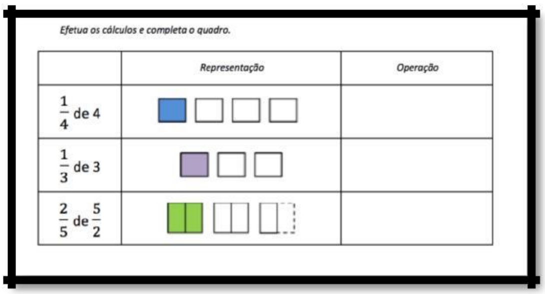


Figure 4. Solution of the proposed task for class 1 and 2 (2nd version).

When we analysed her log, we realized that Ana had kept the final idea and considered the fractions to mean part-whole, drawing three rectangles. It is worth noting that the right side of the figure predicted the explanation she later gave. She had anticipated identifying the unit 5/2 (improper fraction) and intended to decompose it into 1 + 1 + 1/2. In the second step she planned to split the five halves into five parts and then taking two parts of this whole.

Class 2: Continued discussion of the solutions focusing on procedures

In the next day, Ana began by handing out a form to systematize the ideas explored in the previous lesson and reviewed the work done. She projected the hand out on the whiteboard (Figure 5) and explored the expressions 1/4 of 4 and 1/3 of 3. In this explanation the fractions took on the meaning as operators.


Efetua os cálculos e completa o quadro.

	Representação	Operação
$\frac{1}{4}$ de 4		
$\frac{1}{3}$ de 3		
$\frac{2}{5}$ de $\frac{5}{2}$		

Figure 5. Summary table projected on the whiteboard.

Ana asked a pupil how to multiply using calculation procedures. Thus, although the proposed expressions are the same we have a new task with a different nature focused in procedural skills. Note that pupils were asked to solve the expressions using the multiplication calculation procedures.

At the end of the solving process, Ana repeated the operation and the respective product, leading the pupil to verbalize that 4/4 is equal to 1. Next a pupil went to the board to solve "1/3 of 3" and another pupil "2/5 of 5/2" using the same multiplication procedures. Finally, and to systematise the operation of 2/5 of 5/2, Ana used a PowerPoint and explained again the solution (Figure 6).

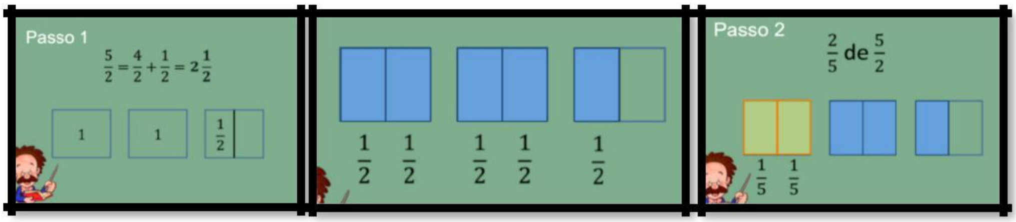


Figure 6. PowerPoint situation 2/5 of 5/2.

Ana: We figured it out by the calculation. But yesterday how did we do it? By a diagram. So let's clarify what we were doing yesterday. What do we see in 5/2? If we use the mixed number numeral, what are we going to get? We'll have 2/2 + 2/2 + 1/2. What does this tell us?

Pupil: 2 1/2.

Ana: 2 1/2, which is what we have here (pointing to the first slide). In other words, we have two units, which we have here, 1 plus 1 plus a half, which is what we have represented here, right? And what we want to know is 2/5 of 5/2. So how do we do? We have represented our whole. We have 2 units and then we have the 5 in total, don't we? Each unit is divided into two and what do we want? 2/5 The two parts of five. So what do we get? One, two... will correspond to how much? How much will 2/5 of 5/2 be?

Pupil: One.

Ana began her explanation by reviewing the work done previously in order to focus pupils' attention again on the expression 2/5 of 5/2. Then she said that 5/2 can be represented by a mixed numeral but she did not explain why. In order to explain the step-by-step process, she decomposed the mixed numeral so that pupils could see why the unit are two and a half rectangles. Finally, she represented pictorially each step, showing how the multiplication can be seen as two pieces of five (5/2) which corresponds to two rectangle halves, namely a unit. She then confirmed with the pupils that the issue in question was clarified.

Reflecting on how to explain 2/5 of 5/2

At the end of the process, in her reflection, Ana explained what had happened and what it meant, saying:

> [The explanation of the first class] goes beyond the procedure and may lead to a conflict of ideas. Ideas...Conflicting ideas is good for discussion. However, confusing pupils is something else. So, I think that's more what happened, the kids were confused. Why? . . . In this case, since I was asking something that went a bit beyond what we were working

> on, talking about, it demanded a little more and they ended up just feeling a little "what's just happened here ?!"… [In fact] it was halves of 5. This was the problem... At the time I clarified their confusion... (FE)

In this reflection Ana felt that her explanation did not consider the children's knowledge. The pupils began by interpreting the fraction 5/2 as a part-whole relation and she stuck to her plan and interpreted the fraction as a quotient. Maybe that is why she felt she confused the pupils and the meaning of the quotient might have become confusing. She described a dialog from the first class:

Ana:	She said we had five units, which were five square [rectangles] and we had to split them in half. And the question was "how to divide them in half?" Because I did not know if she was going to divide each unit in half or if she wanted to divide the set of units in half. She went to the board to divide each unit in half.
Researcher:	So she believed there were five units?
Ana:	Rather than five, only one unit. And that's what I think I failed to take advantage of. Because I wanted them to realize that the five was a unit that could be divided in half. And after this, they were going to be thinking of a unit for two-fifths, let's say... (AC1E)

In this reflection we realize that Ana had difficulty in understanding what unit the pupil was thinking of. In this interview, just after the first class, she appeared to be insecure about her explanation since the pupils had not fully understood her unit of reference. After this interview, she reflected with the school supervisor, who helped her to understand the pupils' perspective. As a result of this conversation and as described, Ana rethought her explanation. While reflecting on the second class and planning future attempts, she said:

> For me this process was so logical it didn't occur to me that they would be thinking of the five parts...I thought it would be easy for them to get here. Why? Because at the time I did not divide them into five, only at the second step did I divide them. I missed out on one of the strategies, so to speak… My way made more sense… Then, when the [new] proposal made sense… I think the kids realized where we wanted to go, but maybe the strategy should have been explored another way… [We could have] compared the two strategies, both proposals... That would have even been ideal. Perfect! (AC2E)

In this excerpt, Ana still did not feel completely confident about the explanation given in the second class. For her, it made more sense to focus on half of 5 as a reference of five halves. But she found that focusing on the meaning of the part of the whole, and thus on the reference unit "rectangle" made more sense for the pupils. So her final solution was to combine the two perspectives without explaining how.

Ana's reflections show that she did not question the purpose of these classes. She was confident in the tasks she had designed. However, this confidence did not extend to her explanations. She did not foresee alternative solutions to the first task and the difficulties the pupils might feel. So, when she faced an unexpected solution, she found it hard to understand the pupils' thinking.

Conclusion

This paper presents the case of Ana, who, as a part of her supervised teaching practice, was supposed to explore the concept of inverse rational numbers. To this end, she planned and carried out a lesson. So that the pupils might visualize the product obtained in a task and not be "given" a rule to memorize, she asked them to draw an illustration of the expressions. During planning, she solved the expressions both symbolically and pictorially but did not anticipate possible pupils' solutions. Relating what she planned with what she accomplished in the classroom we conclude that Ana shows weaknesses in her didactical knowledge about pupils because she did not anticipate that her pupils could identify one rectangle as the unit and interpret 2/5 as a part-whole relationship. Note that Ana did not talk with her school supervisor and did not realize that the pupils could interpret the fractions as a relation part-whole as a result of their mathematics experience. When Ana designed the task she proposed an expression that could have multiple interpretations. This issue raises questions about her didactical knowledge about tasks.

When Ana was faced with an unexpected interpretation from her pupils she chose to stick to her plan. So, she did not consider the pupils' perspective nor did she compare the two views. In this situation, she was not able to apprehend the pupils' understanding and adapt her approach. She planned to discuss the pupils' solutions but rushed ahead and took "control" of communication and built an instructional explanation supported in her ideas of the reference unit and her interpretation of 2/5 of 5/2. To convey concepts and procedures to pupils she used pictorial and symbolic representations (Bishop & Goffree, 1986) connected with verbal communication (Ponte & Serrazina, 2000). According to Charalambous, Hill, and Ball (2011) Ana gave a "good explanation" but, as these authors highlight, she did not offer a meaningful and easy to understand explanation because she did not take in account her pupils' previous experience and knowledge. Ana later planned a second, 45-minute class. After talking to the cooperating teacher and reflecting, she rethought her explanation. In this second stage, her explanation was more confident and she was able to deal with unforeseen conceptual issues and was more attentive to pupils' ideas. Throughout the process, Ana reflected on these issues, became aware of the complexity of teaching rational numbers, and developed her didactic knowledge about pupils.

It is not our aim to analyse issues related to the supervision process, but some reflections may be made. Ana's case illustrates the complexity of teachers' knowledge for teaching rational numbers (Barnett-Clarke et al., 2010). As a teaching practice that focuses on understanding concepts is complex and requires careful planning (Serrazina, 2012), both prospective teachers and their educators must be aware of issues related to planning and carrying out such teaching practice. During planning, prospective teachers need to discuss alternative solutions with their supervisors to be able to deal with unforeseen situations. Such issues are related to didactics knowledge regarding pupils and tasks and about classroom communication. Just as Mitchell et al. (2013) indicate, prospective teachers sometimes think that pictorial representations illustrate concepts by themselves. However, this does not always happen and it is important to reflect on the most appropriate representations for which purposes and how they might support pupils' learning.

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Promoting the understanding of representations by grade 3 pupils: Two teachers' practices

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In this paper we analyse the practice of two grade 3 teachers in order to understand how they promote their pupils' understanding of representations. Two lessons were video recorded and we focused on teacher-pupils' interactions. Data collected from three different moments (introduction of the task, pupils' autonomous work, and whole class discussion) were analysed through content analysis. The results show that the way the teachers organize the different classroom moments are related to their perception of pupils' needs and difficulties and that to promote their pupils' understanding of representations both adapt their actions and questioning to this perception.

Keywords: teachers' practices, teachers' actions, representations.

INTRODUCTION

The way teachers use representations in their practice has a great influence in pupils' understanding of representations (Stylianou, 2010). Faced with a challenging situa-tion it may be very difficult to pupils to choose a suitable representation to handle that situation. In addition, the fact that mathematical representations are related to each other enhances pupils' difficulties in understanding and learning about repre-sentations (Goldin, 2008). Tripathi (2008) suggests that teachers must use several types of representations to promote pupils' understanding of a given concept. How-ever, Acevedo Nistal et al. (2009) refer that the use of too many representations may be in the origin of pupils' difficulties in making a suitable choice. In this study we look at the practice of two grade 3 teachers aiming to understand how they promote their pupils' understanding of representations.

TEACHERS' PRACTICES AND REPRESENTATIONS

"Representation" includes the process of representing as well as the resulting product (NCTM, 2000). Bruner (1999) categorizes representations as active, iconic or symbolic. Thomas, Mulligan and Goldin (2002) refer three types of representations: pictorial, iconic, and notational. Webb et al. (2008) categorize representations in informal, preformal and formal and Ponte and Serrazina (2000) also refer to oral lan-guage.

The importance of the role of the teacher in supporting pupils' learning of representations has received attention from several authors. For example, McClain (2000) analyses a grade 1 teacher's practice, showing how she translates what pupils say in-to symbolic representations and how it influences the development of pupils' nota-tions and symbolizations. According to the author, the teacher tries to get her pupils to use more formal representations, introducing the notation of addition and subtrac-tion based on their answers. She concludes that the representations proposed by the teacher were gradually adopted and adapted by her pupils, contributing to the en-richment of whole group discussions. In a similar perspective, Stylianou (2010) re-fers to teachers' introduction of representations as a way to feature new concepts, il-lustrations and processes in solving problems. She states that creating links between these concepts is a crucial element to support pupils' learning. She suggests that teachers should use more than one representation related to the same concept, select-ing those that they find more adequate.

For Swan (2007), the success of a task varies according to teachers' actions, how teachers lead pupils in doing it, the role that they assume, how they introduce the task, and the questions that they make during the whole class discussion. Teacher's actions can be analyzed regarding how they promote pupils' understanding of repre-sentations while they are involved in different kinds of activity, namely choosing or designing a representation, using and transforming a representation, or reflecting about representations (Table 1).

Students' activity regarding representations	Teachers' actions
Designing/Choosing	Promoting free choice
	Hinting through questioning
	Providing explicit suggestions or examples
Using/Transforming	Challenge students through open questioning
	Asking to explain in a structured way
	Guiding to establish connections
	Guiding to make conversions or treatments
Reflecting	Promoting the evaluation of the work done
	Promoting systematizations
	Informing

Table 1 – Teachers' actions in different moments of the pupils' activity.

Thus, to support their pupils' design or selection of a representation, teachers may (i) promote free choice, by letting them to decide about which the most appropriate representations are; (ii) hint pupils about the representation they should use; or (iii) give suggestions or examples. To promote pupils' use or transformation of a given representation, the teacher may (i) pose them open questions to make them think about hypothetical transformations (conversions or treatments) of representations; (ii) ask pupils in a more structured way to explain what they did; (iii) guide pupils to establish connections between representations; or (iv) guide pupils to make conversions and treatments of representations. Finally, teachers may enhance their pupils' reflection about representations by (i) promoting the evaluation of the work that has been done; (ii) systematizing information; or (iii) informing about new representa-tions and their connections with used ones.

METHODOLOGY

This study is part of a qualitative research on the practices of primary school teachers concerning their work with mathematical representations and was undertaken in a school near Lisbon with the first author as a non-participant observer. The partici-pants are Sofia and Sara, two young grade 3 teachers. They were on a team of grade 3 teachers that worked together very often. In this paper we present some episodes of their classes, showing how they strive to promote pupils' understanding of representation as they work on the following task: "In a theatre play performed by grade 3 pupils, João, Pedro and Ulisses wanted to be the King. On the other hand, Ana, Inês and Estrela wanted to play the Queen. How many pairs of King/Queen may be formed?"

The teachers chose this task taking into account that they felt that their pupils were struggling with problem solving with whole numbers. Data was collected by video recording and by collecting pupils' written work. We analysed data through content analysis in the moments of introduction of the task, pupils' autonomous work and whole class discussion (Ponte, 2005). Pupils' representations were categorized according to Bruner (1999), Thomas at al. (2002), Webb et al. (2008) and Ponte and Serrazina (2000). We categorized as informal representations all pictorial representations (drawings), as preformal we consider iconic representations (non mathematical symbols and schemes) and verbal representations (words) and as formal representation the symbolic representations (mathematical symbols). The teachers' actions were categorized according to the framework indicated in Figure 1.

SOFIA'S CLASS

Introduction of the task

Sofia reads the statement of the task, stressing the information that she finds im-portant (number of boys and girls, awareness that a problem may have more than one answer), thus providing hints to the pupils. Noticing that some pupils struggle to un-derstand the meaning of the verbal representation (the word "pair"), she challenges the pupils through open questioning ("Can I have two pairs and a half?", "What is a pair?"), but as the pupils remain silent, she decides to question them in a structured way ("How many persons do I have in a pair?"), getting an interpretation from one pupil ("A group of two!").

Pupil's autonomous work

As Sofia notices that pupils are still struggling to find a strategy to solve the task, she decides to explore the task statement once more, addressing its key points. She gives hints and makes suggestions leading the pupils to review some of the task conditions like who can be Queen or King ("Who can be the King and the Queen?", "Only one of the boys can be the King?"). She reinforces what she considers to be a complete solution ("So I want you tell me all the possibilities… All the ways of making a pair!"). To help the pupils to interpret the task statement she also suggests an active representation ("Imagine that . . . I am going to pick the King and Queen!… These three girls raise their arms . . . And these three boys want to be the King… And now… Which are the possibilities?").

Sofia walks through the class, observing and questioning in detail almost all pupils. She challenges Angelo through open questioning to explain his mixed representation ("Can you explain me what is this? …"). After noticing that the pupil has an incomplete answer (he says that there are three possible pairs – figure 1a) Sofia challenges him through open questioning ("Why João does not like Inês or Estrela? Is he angry with them?"), and then she informs him ("How many are the possibilities! It does not say: 'Tell me three [possibilities]…'"). When the pupil understands that his answer is incomplete, she lets him to continue to work autonomously. Later, Sofia comes back and challenges Angelo again, through open questioning, to explain his new mixed representation (figure 1b) ("What are you doing?", "And what are you repeating here?"). Faced with his teacher's challenge, Angelo explains to her why he now considers nine pairs and he easily describes his representation.

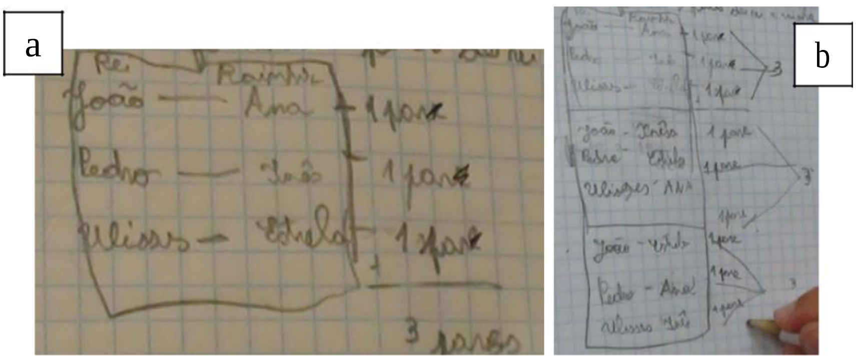

Figure 1a e 1b – Angelo's mixed representations before and after Sofia actions (verbal and iconic).

Another pupil, Joaquim assumes that he has to use a pictorial representation (he is drawing every Queen and King) and starts to complain. Noticing that more pupils are also using pictorial representations, Sofia questions the class hinting them ("Did anyone told you: Spend a lot of time on drawings!? Or to draw all the Kings and Queens?). As another pupil answers her questioning ("No! Why [should we draw]?! They have names!") she reinforces that the pupils may choose freely their representation. Later, she returns to see how he is doing and she notices that Joaquim followed the advice of his colleague and he drew an mixed representation (figure 2).

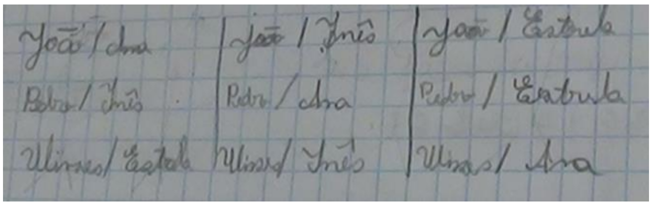

Figure 2 – Joaquim's mixed representation (verbal and iconic).

This time, she challenges Joaquim (figure 2) through open questioning to explain his representation ("What are you doing?") and he does it easily. During the pupil's autonomous work, most solve the task by using different types of informal and preformal representations. At that time, Sofia decides to begin the whole class discussion.

Whole class discussion

Sofia begins by inviting Luís to present his solution (he had an incomplete answer, as he indicated that there were six different pairs) and write it on the board. She asks him in a structured way to explain his representation ("Why did you not considered João and Estrela?", "Can João be paired with someone else?"). During the discus-sion, through an iconic representation that Sofia made on the board, Luís and other pupils acknowledge that they forgot some pairs, and identify them easily ("Ah! He can [also be paired] with Ana!").

Then, Sofia decides to pose to the whole class a follow up question ("If one of the girls drop out, how many pairs would be possible?"). This is a question that was solved during the autonomous work, only by the fastest pupils. One of those pupils, Laura, has no difficulty in presenting her answer and explaining to the class how she thought. Sofia then decides to transform Laura's oral representation into a mixed and then a symbolic representation (figure 3c). At the same time, she tries to guide the pupils to establish connections between the representations that were written on the board (Figure 3a and 3c).

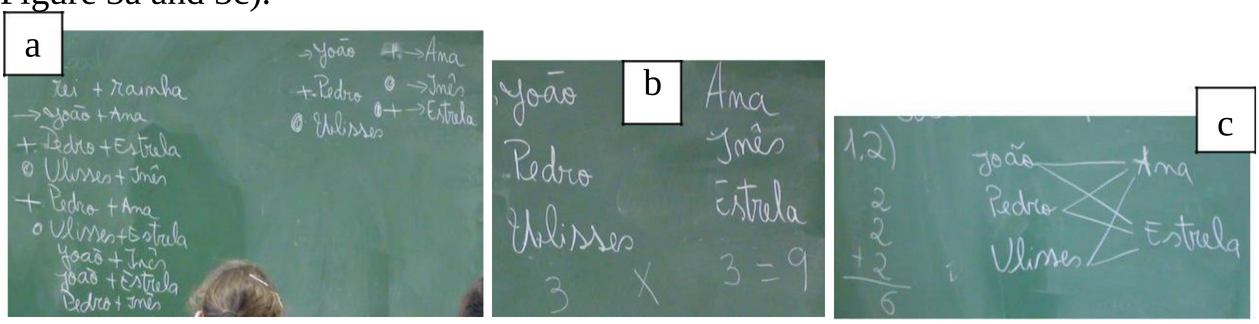

Figure 3a,3b and 3c– Sofia's iconic, symbolic and mixed representations.

At the end of the discussion, Sofia introduces the multiplication sign ("If we have… Three boys [she writes "3" below the boys' names] and three girls ([she writes "3" below the girls' names]… I have (she puts the × sign writing 3×3)… Nine! Nine possibilities!") (Figure 3b).

SARA'S CLASS

Introduction of the task

Sara challenges a pupil, André, to explain to her the statement of the task ("What did you get from the exercise?"). Faced with André's difficulty in answering to her challenge, she hints him ("How many pairs… What is a pair?"). At a certain point she notices that the pupils are having difficulty in understanding the meaning of the verbal representation "pair" and she informs the pupils ("We need to have a King and a Queen!"). Afterwards she guides the pupils to focus into the information that she finds important (each pair must have a King and a Queen, who are the eligible boys and girls, there are several possible pairs). When the introduction of the task is al-most finished, some pupils try to answer it orally without writing the answer ("I did it! It is…"!) and Sara reinforces the importance of writing and justifying all the an-swers in their notebook ("So do it!... In your exercise book!", "I want you to explain me which are the pairs! And why!").

Pupil's autonomous work

As some pupils try to answer Sara orally, she reinforces the importance of writing down their answer. Other pupils present their incomplete answers and she hints them, by saying "there are more pairs to be found". Most pupils get the right answer by us-ing a verbal representation similar to the answer of Carlos (Figure 4).

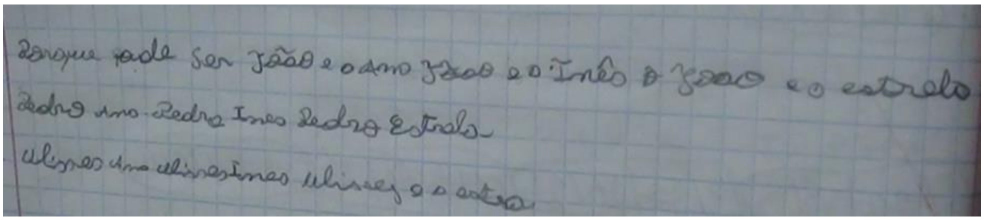

Figure 4 - Verbal representation used by Carlos.

Sara challenges Carlos to explain his representation ("And why? How did you saw it?") and he does it easily. She continues to walk through the class and observes her pupils' work. When she finds answers with different representations, she questions them with more detail.

At some point Sara notices Mauro's mixed representation (Figure 5). She challenges him through open questioning to explain how he solved the task ("Explain it to me…") which he does with no difficulty. She praises his representation loudly ("Good work!") in order to induce other pupils to also find different representations.

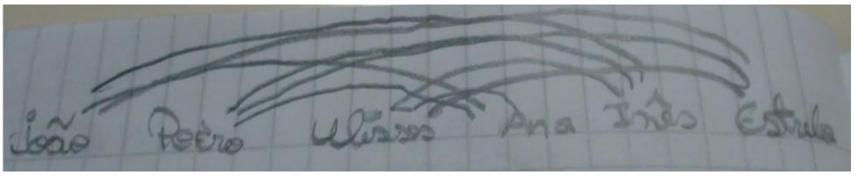

Figure 5 - Mixed representation (verbal and iconic) used by Mauro.

After, Sara questions Mariana, the only pupil that uses a symbolic representation (3+3+3 as a vertical computation) to solve the task. She challenges her to explain the representation ("I am not understanding [your representation]… Could you explain it to me?"). Most specifically, she wants to know if Mariana understands the meaning of each portion. As the pupil points to each portion and explains it (Ana with the three [boys] (points to the first line), Inês with three (points to the second line) and Estela with all three (points to the third line)… And it's nine!!"), Sara is pleased with her answer and continues walking through the class.

Then Sara questions Leonardo, a pupil that felt compelled to find a "different representation" (Figure 6):

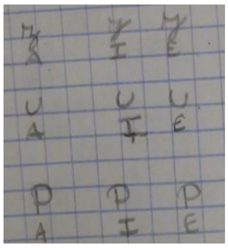

Figure 6 – Leonardo's mixed (verbal and iconic) representation

Sara challenges Leonardo to explain his mixed representation which he does easily ("J" from João… "I" from Inês!…So… (as he points to each capital letter) Ana, Inês and Estrela. U is Ulisses… and Ana, Inês e Estrela! (points to P) This is Pedro with Ana, Inês e Estrela… Nine pairs!"). Then, Sara praises him loudly, and, once again, she tries to motivate other pupils to find different representations.

Whole class discussion

Sara asks several pupils to present their answer to the class. The first is Jonas, a real-ly shy and insecure pupil with whom Sara had been talking during pupils' autono-mous work, noticing that he had a right answer (figure 7). In the beginning of the whole class discussion Sara challenges Jonas to explain his answer ("Explain to me…", "Why?"). However, faced with the difficulty of the pupil in answering, she decides to question him in a more structured way ("You did the pairs… Do you know why?"). She ends by guiding Jonas, giving him some information related to his first explanation ("You were trying to join a boy and a girl… Was it?").

Afterwards, Sara challenges Mauro to show his answer (an iconic representation where he connects, in a scheme, the different characters' names) ("How did you did that?") and, sometimes she questions him in a more structured way ("What is that [connection]?”). In the end of Mauro’s presentation, she guides the pupils in establishing connections between Jonas’ and Mauro’s representations.

The last pupil to present her answer is Mariana, who used a symbolic representation. This is also a very shy pupil and Sara begins by question her in a more structured way. Although Mariana explained perfectly her representation during pupils’ auton-omous work, now she feels the need of using an active representation (counting her fingers) to assure that her answer is right. This leads Sara to change her actions and inform the class about Mariana’s explanation. Next, Sara teases pupils to catch their attention (“I am going to teach you a trick!”. When she starts talking it seems like she is guiding pupils to interpret the statement of the task (“How many boys?”, “How many girls?”). However, a glimpse of information (“Each boy can be in three pairs…”) is actually a challenge that triggers pupils to convert the presented repre-sentations into a symbolic representation of multiplication (“Teacher! There are three pairs of three!”, “It is three times three!”).

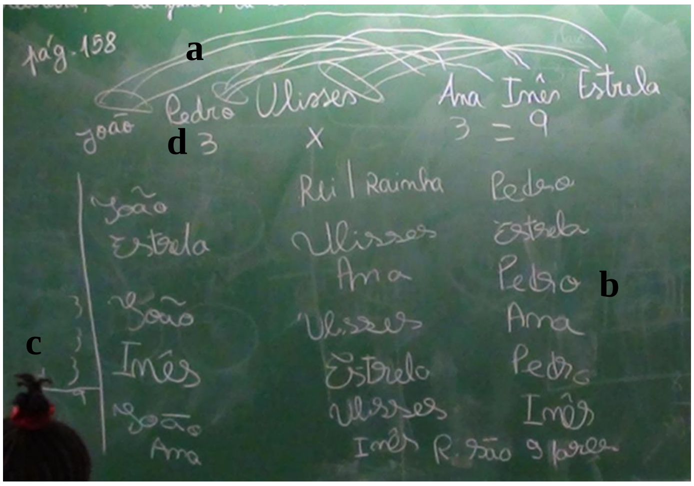

Figure 7 – Mauro’s iconic representation (a), Jonas mixed representation (b), Mariana symbolic representation (c), and the class symbolic representation (d).

Pleased with her pupils’ answers, Sara writes the symbolic representation (3×3=9) above Mauro’s representation (figure 7).

CONCLUSION

In the introduction of the task, both teachers lead pupils in interpreting the statement of the task, focusing some key elements (number of boys and girls, characters names, main condition to have a pair). In both classes pupils struggle to interpret the mean-ing of the verbal representation “pair”, and both teachers felt the need of negotiating the meaning of “pair”. The main differences between Sofia and Sara concern their actions, as Sofia mainly hints through questioning (Who? How? How many?) and Sara often challenges her pupils.

During pupils’ autonomous work, Sofia and Sara (i) ask their pupils to write down their answers, despite the efforts of some to answer only orally; (ii) promote their pupils' free choice of representations; and (iii) do not suggest alternatives nor guide their pupils to find conversions or treatments, even when they are struggling. Apparently, these actions would enable the emergence of a large variety of representations, but that does not happen in both classes. Thus, while Sofia's pupils use several types of representations (mainly informal and preformal), most pupils in Sara's class use an identical mixed representation and just a few use the symbolic representation of adding. The different results from their classes, seem to constrain the actions of Sofia and Sara. In Sofia's class, when a pupil shows her a wrong or incomplete answer she first challenges and questions the pupil, then she lets him to solve the task autono-mously, and later she comes back to question that pupil again. In Sara's class, when a pupil shows her a wrong or incomplete answer she briefly advises him or her to re-view their answer. It seems that she is searching for pupils that are using different types of representations (as she also tries to motivate pupils to do that). When she finds someone that, according to her, has an interesting representation, she questions the pupil lingeringly, in order to understand if he or she is understanding his/her rep-resentation and is able to explain it.

In whole class discussions, both teachers register on the board all representations presented and that facilitates the establishment of connections between representations. Sofia and Sara also guide the pupils to establish connections between the representations presented and the symbolic representation of multiplication that no pupil has used during the autonomous work (Stylianou, 2010). As during pupils' autonomous work, teachers' actions in whole class discussions are also constrained by pupils' results and difficulties. That way, Sofia decides to ask a pupil with an incom-plete answer to present his answer and then her actions are mainly informing, as she felt the need of guiding pupils to formal representations (her pupils used different types of representations but mainly informal and preformal ones) as in MacClain (2000). At the same time, Sara asked some key pupils to present their answers that included different representation types (her pupils used mainly the same iconic rep-resentation). At the end of whole class discussion, Sofia challenges pupils so they can find by themselves that 3×3 is also a representation that can be used to answer the task.

During the class, the success of the task was influenced by the teachers' actions that changed according to pupils' activity (Swan, 2007). Regarding representations, Sofia and Sara moved towards more formal or more informal representations according to their perceptions of their pupils' difficulties. Regarding teachers' questioning, both tend to change their questions in what we may consider as a low or high level of challenge according to pupils' difficulties. That way, they usually started by challenging their pupils (a higher level of questioning) but, sometimes they felt that they had to decrease their questioning level into questioning in a more structured way.

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What (Should) Teachers Assess in Elementary Geometry?

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Abstract. Instructional practices based on teachers' personal, physical and instructional/institutional resources deeply influence the processes of assessment. How do these resources shape a situation of assessment on an open task and how does assessment of pupils' answers indicate particular teachers' resources? This study examines these questions on the topic of reflection and symmetric figures in elementary geometry in grade three in a primary school in Berlin by connecting the theory of didactic situations (TDS, Brousseau, 1997) with components of teachers' knowledge by Levenberg & Patkin (2014). The empirical insights offered in the paper may, further on, inform a selection of content for the purpose of a research-based design of a PD program.

Keywords: geometry, assessment, theory of didactic situations, teachers' personal resources, professional development.

INTRODUCTION

The complexity of mathematics teaching practices has already been addressed from several aspects as for example, teachers' knowledge and beliefs, or the role of personal, physical and instructional resources and their relationships (e.g. Topics of the TWG 19 and 20 at CERME 9). "Teachers' actions and meaning-making as they relate to instruction, including task selection and design, classroom communication, assessment, etc." are of a particular interest in the current debates about teacher education (Call of the ERME TC3). This paper focuses on assessment as an action which is deeply influenced by instructional practices and based on *teachers' personal, physical and instructional resources.* Although the Call distinguishes between teachers' "personal resources, on the one hand, and physical and institutional/instructional ones, on the other hand" (TWG 19), I would rather refer to them as three different kinds of resources. By personal resources I mean primarily teachers' knowledge in mathematics besides other (according to Levenberg & Patkin, 2014), whereas by physical resources I refer to physical objects regardless if they are natural or man's creations. Instructional resources may be of diverse nature, for example, the content of the curricula and textbooks, or mathematical visualizations and geometric representations of mathematical concepts in narrow sense. This certainly does not state that the physical resources cannot be used for instructional purposes, on the contrary, physical objects or pictures and drawings of them are often used in classroom instruction. A textbook is also a physical artifact by itself though its existence is meaningless if it is not used for instruction, and therefore, I consider it as an instructional resource. I argue for my insistence on such triple distinction of the resources by an epistemological and didactic analysis of three dimensions of mathematics, about which I talk in the next section. Further on, I use the

Theory of Didactic Situations (TDS) to show the presence of these connections in a mathematics classroom related to assessment about reflection in grade three. Then, I explain how such assessment could indicate the quality of teachers' resources. The paper finalizes with a suggestion for the 'what' question by van den Heuvel-Panhuizen (2005) and Prediger et al. (2015) about the content which "teachers should learn and multipliers need to know" (p. 233).

THEORETICAL FRAMEWORK

Teachers' Personal, Physical and Institutional/Instructional Resources Regarding Reflection and Symmetric Figures

Are there (at least) three different kinds of mathematics (contemporary, school and every day) and if so, how do they differ from each other, is a question tackled by Civil (2002; see also Sfard, 1998). Although such a strong differentiation may seem artificial, it may be valuable for investigating the interplay between different types of resources for teaching. Here is an attempt to exemplify such investigation by concepts in geometry as reflection and symmetric figures.

In contemporary mathematics, the basic Euclidean isometries, reflections (both, mirror or line reflection and point reflection-mirror for 2pi radians), rotations, translations, and combinations of these, are distance-preserving geometric transformations in two- or three dimensional Euclidean space. Any congruence transformation can be represented as a composition of maximum three reflections, and therefore the reflection is considered to be a fundamental concept. Further on, a figure is called symmetric if there exists a reflection which maps it to itself. This exemplifies the importance of symmetry, which has different meanings throughout different mathematical contents (e.g. a property, a relation) and everyday contexts. Symmetry is "not only a key idea in geometry […] but also a key organizing principle in mathematics" (Jones, 2002, p. 131). While a use of formal concept definitions is a necessity for the introduction of concepts at the university level mathematics, an everyday application may be sufficient for an initial introduction of the same concept in school.

> Symmetry is a model topic for study in school. It is embedded in reality, it is conceptually simple for younger pupils, and concrete examples abound. Its study yields many useful results, applicable in the real world. Equally important, it is a rich subject whose study is an excellent practice ground for mathematicians and scientists (Ellis-Davies 1986, p. 30).

In everyday life, through the senses for vision and touch children enter the world of mathematics (without any formalities, e.g. definitions). For example, a painting is beautiful because of the 'hidden' symmetric properties in it (regardless of the person's mathematical knowledge about reflection) . Meanwhile, in primary school mathematics, e.g in Berlin, reflection and symmetry as a property of figures are introduced even in grade one, through everyday contexts and with the aid of physical and instructional resources. A common example is the use of life creatures in the nature, e.g. butterflies (physical resources) or objects made of paper, e.g. stars, hearts, etc. The teaching and learning is also supported by the use of different tools such as mirrors or special rulers (instructional resources). Axes of symmetry are mentioned in grade three and pupils are expected to identify and draw axes of symmetric figures and also reflect figures. In comparison, according the K-12 Standards for Mathematical Practice in geometry for grade 4, pupils have to “recognize a line of symmetry for a two- dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts [...], identify line-symmetric figures and draw lines of symmetry” (NCTM, 2000). The differences in the curricula and standards open a new question for a possible consideration of the institutional as a fourth type of resource (separate from the instructional, in contrast to the statement in the Call, TWG 19).

The Role of the Teachers in Considering Different Resources in Assessment

In reality, absolute symmetry as it is defined in mathematics does not exist. To what extent are primary school teachers aware of the validity of this statement and the co-existence of the different facets of mathematics (as a science or as knowledge to be applied and studied in and out of school)? What is the role of the teacher in bringing together all of them in the classroom? How far can the teacher push everyday mathematics in the classroom instruction and assessment? We may debate that the pupils get motivated when they engage in everyday situations but how much mathematics do they really learn on the way? Do the teachers sometimes forget all the mathematics that they not only have to teach but also assess? For example, how much are the primary school teachers familiar with the specific content knowledge about geometric transformations, in particular reflection?

I start the discussion mainly pointing out *teachers' knowledge* as a part of *teachers' personal resources* (which also include beliefs and identity but are not discussed in this study). Namely, Levenberg & Patkin (2014, p. 94) identify *six components of teachers' knowledge*: *knowledge of the subject matter*, knowledge of the learner, background knowledge of the school environment, curricular knowledge, didactic knowledge and self-knowledge. In this paper, I prefer to talk about *teachers' content knowledge* in mathematics. To refer to this particular component, i.e. teacher's individual knowledge of the subject matter, I use the term “knowing” (which belongs to a person) and distinguish it from (generally available) “knowledge” (borrowed terminology by Brousseau, 1997 and González-Martín et al., 2014) meaning mathematical knowledge about reflections as described in the previous subsection. I also use the rest of the components as indicators in the analysis of teacher's personal resources (in order to answer the research question in this study).

Now, a backward look at the above questions allows a presentation of some answers already existing in literature. Geometry is perceived as a subject matter in which many teachers demonstrate a knowledge gap and is therefore difficult for them to create and evaluate rich mathematical tasks (Ribeiro, 2011). In particular, “the meaning of symmetry is not precisely defined” (Leikin, Berman & Zaslavsky, 1998 and see also Leikin, 2003) which may be a source for teachers' difficulties. Despite such insufficiency, there are teachers who do not comprehend this issue in its fullness (Levenberg & Patkin, 2014, p. 97). Further on, in relation to the first question in this subsection,

> “During geometry lessons, the use of all types of visual displays, pictures, presentations and movies, which show geometry in the pupils' environment (both natural and hand-made), constitutes a bridge between the concrete and the abstract” (Patkin & Levenberg, 2012, p. 14).

Regarding to the rest of the questions, it seems that a *Didactic Situation* (TDS) is a suitable model to analyze the teachers' roles in bringing all three resources at one place, and moreover, not only in instruction but also in assessment. Namely, an *open task* (I come to this term in the next subsection) asking for naming symmetric figures ensures several adjusted conditions of a *Situation* (see González- Martín et al., 2014, p. 118). For example, it targets reflection and symmetric figures as mathematical knowledge to be gained and it does not make any reference of the targeted knowledge (e.g. by stating a particular figure) . Further on, pupils may name inadequate solutions which at a particular time of cognitive development or accessible mathematics may be accepted as adequate. Then, the solutions named by one pupil may be discussed and verified by the others. An exemplary task fulfilling these conditions of a DS is given and elaborated in the section Findings and Discussion (see Figure 1).

Next questions that arise are how do teachers evaluate pupils' achievements and understanding about symmetric figures in grade three, what kind of data provide pupils' answers and finally, how could the assessment influence further learning for both parties?

Assessment through Open Tasks

Due to the limitations of this paper, I would not go into details about how are open or open- ended tasks defined and classified or which are the advantages of their usage (for some specifications see Kwon, Park & Park, 2006; Yee, 2002) . Rather, I emphasize that, in this study, assessment through open tasks is not seen as a process of collecting data about pupils' achievements, instead as a process of learning and in particular beneficial for both the pupils and the teachers. In addition, future steps in a larger study may aim to design rich tasks for multi-age classes in the first three years of schooling. A question which arises from this initial study is the following. How should a teacher evaluate answers on an open task, when it appears as one, on a test which has been created by authorities and not by her/himself? A particular test can be considered as an instructional/institutional and not as a personal resource because it has been suggested to the teacher as accompanying material to the textbook in use. If the pupils’ answers coincide with the examples in the textbook or with those discussed by the teacher during the lecture, they will certainly be evaluated as correct. In such case, it seems that pupils are granted for memorizing and repeating of what has already been stated during the instruction, which is certainly, not one of the most important goals of the teaching of mathematics. If they do not coincide, they may be evaluated as incorrect, which is another threat.

RESEARCH QUESTIONS AND METHODOLOGY

Research question (RQ). How could a teacher's assessment of pupils' responses on an open task in grade three indicate teacher's personal resources, in particular, teacher's knowing of the mathematical knowledge about reflection and symmetric figures?

The RQ goes beyond analyzing how does a teacher assess pupils' knowledge about reflection and line-symmetric figures. Namely, the investigations on the RQ do not only look at how does a teacher decide about true or false answers or distribute points. They also try to examine how does this distribution indicate the teacher's awareness of the co-emergence of the three aspects of mathematics in connection to the three kinds of resources in an assessment situation and in particular what is the teacher's knowing of the mathematical knowledge about mirror symmetry. This indication relates to the components of the teachers' resources.

For the analysis regarding the RQ in this theoretical paper I refer to the *core elements of TDS* (Brousseau, 1997 and González-Martín et al., 2014), where the DS is an assessment situation on an open task by analyzing the "relationships between students, a teacher and a *milieu*" (González-Martín et al., 2014, p.119). The *milieu* is defined as "the set of material objects, knowledge available, and interactions with others, if any, that the learner has in the course of said activity" (González-Martín et al., 2014, p.119). In the DS in this study, "the set of material objects" is consisted of the personal, physical and instructional resources (used during instruction) on which the pupil can reflect on when solving the open task in the DS (during assessment). " Knowledge available" refers to the coherence between the three types of mathematics that the teacher has (or has to a certain amount) brought in the classroom. This directly relates the teachers' personal resources, specifically their knowing of the subject matter. Since the DS is an assessment situation, there are no direct "interactions with others" when solving or evaluating the open task. Yet, "the learner" in this Situation is not perceived as "a learner", rather learners, i.e. both the pupil and the teacher. I continue the discussion based on the core elements of the TDS in the next section.

FINDINGS AND DISCUSSION

The open task in this DS is one out of five tasks on a written assignment about recognizing and drawing line-symmetric figures in the third grade primary school in Berlin. The analysis does not only focus on the way a teacher has assessed pupil's answers and compares the assessment with a plausible one. Moreover, it tries to identify some indication which may lead to answering the RQ. The task is the following [1].

Task: Name 3 figures or representations from your surrounding, that are symmetric (Figure 1) [2]. Pupil's answers are: a circle, a heart and a triangle (Figure 1).

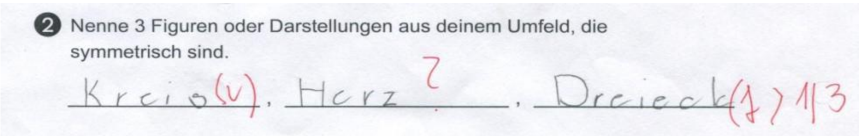

Figure 1: Assessment on an Open Task about Symmetric Figures

The task is an open-ended task for the reason that it does not have one fixed answer and it requires divergent thinking, although the formulation of the task may necessitate additional information. For example, are “figures” physical or mathematical objects, and are “representations” any kind of drawings (e.g. a drawing of a butterfly) or mathematical geometrical concepts (e.g. a square) or pictures and visual displays of physically existing objects (e.g. a window)? Further on, what does a “surrounding” (environment) for a pupil mean? Is it the classroom, or the school yard with all natural and man-made physical objects or the mathematical world the pupil lives in, or something else? These questions are relevant for an eventual design of rich tasks (steps 4 and 5, according to Prediger et al., 2015 which is discussed in the section Conclusions).

The teacher evaluated the solution with 1 out of 3 points (Figure 1), accepting only the “**circle**” as a correct answer. A short analysis of the textbooks for grade one to three “Einstern” 1, 2, and 3, which were in use, shows that the circle does not appear as an example of a symmetric figure in any of them. I see the infinite number of axes of symmetry of the circle (and the complexity of teaching it) as a reason for this absence of the circle as an example of a symmetric figure from these textbooks. The question is whether it has been discussed by the teacher. If not, this answer shows a possible higher pupil's knowledge than what is expected at this level of education. This refers to a core TDS tool named as a *didactic contract* which is “the implicit set of expectations that teacher and students have from each other regarding mathematical knowledge...” (González-Martín et al., 2014, p.119). Namely, if the teacher is aware of the absence of the circle as a symmetric figure from the corresponding curriculum for grade three, and the reasons therefore, (*curricular knowledge* – one of the six components of teachers' knowledge according to Levenberg & Patkin, 2014, p. 94), he/ she may acknowledge this pupil's answer. The *adidactic level* of the DS (the other level is called a *didactic level*, according to Artigue, 2000) concerns pupil's possible engagements involving interactions with the *milieu* and, as this answer “a circle” shows, involves maybe a posteriori enrichment which does not necessarily involve relationships between the pupil and the teacher but between the pupil and the *milieu* alone. The rest of the tasks in the exam show that figures with finite number of axes of symmetry have been discussed during instruction but with no more than two axes. Figures as *n*-sided regular convex polygons having reflection symmetry in *n* axes, for *n* greater or equal to three do not appear among the tasks in the exam and it is not clear whether they have been discussed during instruction (regardless using physical and instructional resources or not) . This assumption brings into focus *teacher's personal resources* and maybe the knowing of the knowledge about the symmetry group of an *n*-sided regular polygon being a dihedral group of order 2*n* (*n* reflections and *n* rotations). Finally, figures with infinite number of axes of symmetry as the circle seem to have remained out of the instructional scope and therefore the above-mentioned desirable acknowledgment seems to be grounded.

The third answer "**a triangle**" was rejected to be correct by the teacher. This indicates teacher's content knowledge. Yet, what is a triangle into the pupil's mind? The topic about reflections is on the beginning, while the one about the existence of tree different kinds of triangles is at the end of grade three. Short analysis of the textbooks for grades one to three, which were in use in the classroom, shows that there are only a few irregular triangles (dominance of prototypes). This fact is enough reason to think that the pupil perceives the geometric shape of a triangle as being either equilateral or isosceles, and as a consequence a symmetric figure. Therefore, the answer may be considered as an adequate one at this particular moment, although such answer cannot be accepted as correct in the fourth and any other later grade. This indicates possible insufficient teacher's *curricula* and *didactic knowledge* but moreover *knowledge of the learner* (components by Levenberg & Patkin, 2014). In the vocabulary of the TDS, the feedback provided by the *milieu* (pupil's *validation* - "a triangle" being a symmetric figure) shows that the *milieu* which was in the current use of the DS appears to have been "insufficient to ensure adidacticity in terms of adding new pieces of knowledge" (González-Martín et al., 2014, p. 119). This, further on, means that the *institutionalisation* as "the ultimate phase of a Situation in which the teacher brings the students back to the *didactic level* and makes the necessary links with the aimed knowledge and provides the semiotic tools to present this knowledge, especially if these were not produced" (González-Martín et al., 2014, p. 120) does not seem to have taken place.

The second pupil's answer is " **a heart**" which is also evaluated as an incorrect one by the teacher, probably because it is a non-visible object. The reasons for stating such probable interpretation are the following. Looking at other pupils' answers as "a window", "a board", "a door", etc. which have been considered as correct by the teacher, it seems that physical resources in the classroom have been discussed a lot during instruction. It may be the case that the most of the pupils have been granted for reproducing such examples. However, the symmetry of these physical objects is really discussable and it is a question if this has been spotted by the teacher and pointed out to the pupils. Namely, a window may as well be used as an example of a non-symmetrical object, because the handle "ruins" the symmetry. Even if we consider the window without its handle, its 'parallel' sides are not exactly equal in length in reality. It is the ideal (imaginary) rectangular shape of the physical object "a window" which is symmetric, and not the realistic object itself. So, have counter examples been discussed?

I now discuss further examples which may appear as pupils' answers on the given task. How can a teacher evaluate an answer as “a house”? (S)he cannot know if the pupil has an image of a symmetric or a non-symmetric house (the problem of reality vs. representations). Such answers can not immediately be assessed and, as a consequence, seek deep teacher's involvement in asking additional questions or requesting drawings from the pupils. In such situations the role of the teacher in bringing all 'three aspects' of mathematics comes into focus. Depending on the imagination and creativity of teachers pupils develop interest and improve their underlining in geometry (Patkin & Levenberg, 2012, p. 14).

Dilemmas related to the task, that come on my mind now, are can a teacher expect an answer as “there are no such objects in my surrounding” or “symmetric objects do no really exist in our surrounding”. How would the teacher evaluate such answers? Could such answers be viewed as signs for pupils' giftedness in mathematics, and likewise the answer “a circle”? These are questions which require further analyses.

Further on, does the utilization of physical resources for instructional purposes make the school mathematics real to an extent that it is not possible to 'avoid' them (see Boaler, 1993 and a more extreme view by Lockhart, 2009)? Does not it seem that their usage may sometimes even prevent eventual early insights into mathematics? What is it with those pupils (like the one in this study) who are already able to think of and manipulate with abstract mathematical objects (e.g. circles, triangles) but are further 'forced' to use concrete objects (physical resources, e.g windows)?

CONCLUSIONS

Although “many concepts of symmetry are not firmly established before twelve years of age” (Genkins, 1975), the answers of the participating seven-year old pupil in this study show her/his understanding of reflection and symmetric figures. They show a development of “mathematical learning as the result of the students' work and ideas – and not as a result of imitating the teachers' actions” (González-Martín et al., 2014, p. 118).

The paper opens a question for the need of a precise definition of three (or more) types of resources for teaching and assessment which have by now been perceived as two distant groups. The theoretical part of this study shows why and how is the content about reflections and symmetry (as a property of figures) relevant and suitable for implementing different types of resources related to the diversity of aspects of mathematics in the classroom. This conclusion may be considered as the first step in the five step approach for content specification according to Prediger et al., (2015, p. 239). The empirical findings based on a one to one (a teacher - a pupil) case study offer insights in the “concrete professional demands” required for assessing pupils' knowledge about reflection (step 2 in the same approach, p. 239). They specify teacher's difficulties with tasks in geometry (Ribeiro, 2011) related to the exact content of reflection and symmetric figures. This shows an answer of the 'what' question (van den Heuvel-

Panhuizen, 2005) - content knowledge about reflections. The empirical explorations in this study may be widened by a greater range of cases taking into account teachers' perspectives (step 3 in a development of a design of a PD program according to the same group of authors). Further on, it confirms the existing difference between “a theoretical Situation, as an ideal - type model and its actual implementation in the classroom which allows assessment of the students' actual work about mathematics” (González- Martín et al., 2014, p. 120). Moreover, this “assessment of the students' actual work” indicates teacher's insufficient *personal resources* about reflection in relation to the components of teacher's knowledge (Levenberg & Patkin, 2014, p. 94) which directly meets the main RQ in this study. “What teachers assess” is not only the pupils' knowledge but also, although implicitly, their personal resources and specifically their own knowing of the knowledge in geometry. “What teachers should assess” is the overall understanding of a concept (in geometry, e. g. reflection) on the basis of pupils' individual (imaginary and abstract) resources and/ or physical and instructional resources, and regardless if it has been achieved by interactions with the *milieu* and with or without the teacher.

NOTES

1. The total scored points on the written assignment, according to the evaluation of the particular teacher, is 19 out of 22 points.

2. The task and the pupil's answers are translated from German to English by the author of this paper.

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Mediating mathematics teaching for connections and generality.

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This paper is located within the South African context of large class numbers and limited resources. Taking a sociocultural theoretical perspective the paper examines the mediational means–artefact-based and human–employed by a teacher in two Grade 2 lessons separated by three years. Differences are noted in the extent to which in each lesson the teacher's mediation focuses on mathematical connections and generality–on mathematics as a 'scientific' discipline in the Vygotskian sense. The differences observed show that even within the culturally dominant practice of whole class, teacher-centred, pedagogies with limited resources the mathematical 'objects' brought into being in lessons can be connected both within the mathematics, and across examples.

Keywords: artefact and human mediation, sociocultural theory, primary.

INTRODUCTION

A breadth of research evidence highlights challenges in raising levels of attainment in primary mathematics in many South African schools. Besides international evidence for the low standing of mathematical achievement in South Africa in comparison to other nations, national research points to low levels of attainment in primary schools (Department of Basic Education, 2014) and also to possible underlying reasons, two of which are relevant here. First is evidence of teaching that treats mathematics as unconnected procedures or facts that learners need to remember rather than make sense of (Askew, Venkat & Mathews, 2012). Secondly, is teaching characterised by 'extreme localisation' (Venkat & Naidoo, 2012), whereby mathematical tasks neither build upon what learners have previously learned, nor connect together different aspects of mathematics: tasks are approached by teachers and pupils alike as having to be answered through naïve, practical methods, most commonly unit counting or tallying (Ensor et al., 2009).

In this context the Wits Maths Connect – Primary project (WMC–P) focuses on developing and investigating interventions aimed at improving the teaching and learning of mathematics in ten government primary schools. Baseline data collection in 2011 involved observing and videotaping single numeracy lessons across all the Grade 2 classes in the ten schools. Follow-up video data in 2014 of lessons again in Grade 2 saw many of the 2011 teachers recorded again. While the majority of the 2011 lessons appeared, on the surface, to unfold smoothly, closer analysis revealed a disconnected sequencing of actions and operations leading to ambiguity in and obscuring of the mathematical learning objects. Many of the teachers videoed in 2014 had worked with the project team on professional development activities and many of the 2014 lessons were more coherent and demonstrated a strong shift in the extent to which the object of learning was brought into being by teachers explicitly drawing attention to different connections between tasks, examples and representations. This paper presents two such lessons–both from the same teacher–to highlight and explore the nature of such differences.

THEORETICAL FRAMING

The theoretical background framing the analysis that follows is based in Vygotskian-based sociocultural theory and the assumption that learning comes about through mediated transactions (Wertsch, 1991). Kozulin (2003) describes mediation as occurring through two key forms–via artifact-based mediation and via human mediation: both of these forms are attended to in the following analysis. Sociocultural theory is helpful in the examination of whether teachers present mathematical classroom examples as set within networks of scientific concepts (Vygotsky, 1987). Kozulin (op. cit.) notes that viewing disciplines as networks of scientific concepts is revealed in teaching where there are explicit mediation moves towards generality. This counterpoints concerns about localization that we have observed in the project and noted above, that is teaching approaches that may enable learners to provide answers within the context of the support provided in the lesson but which are unlikely to be appropriated by learners as methods to be used beyond the lesson.

Teachers' choices of examples are important in mediating towards generality. Watson & Mason (2006) have studied and theorized about examples and example spaces, emphasizing the importance of connecting between sets of examples within mathematics lessons in order to draw attention to structure and generalization. They too note, however, that teaching often only focuses on generating solutions to immediate problems rather than abstracting generalities across examples, thus the issue of localization is not unique to the South African context, although it may be more extreme there. Adler & Venkat's (2014) analysis of secondary mathematics teaching also shows that a focus on structure and generality can be brought about through mediating actions that attend to connections.

The notion of connections has wide support as central to mathematics instruction. For example, Askew and colleagues (1997) in their study of effective teachers of numeracy (ETN) (numeracy defined, essentially, as number, operations and applications), examined factors that might contribute to gains in learner attainment (measured as class mean gains across a year in pre- and post-test assessments). From questionnaires given to 100 primary school teachers and case studies of 18 of these teachers three archetypes of teacher orientations towards teaching primary numeracy emerged. Two of these archetypes–transmission and discovery orientations–were identified as associated with narrower learner gains. In contrast, many of the teachers whose classes showed the highest learning gains over the year, displayed characteristics of the third orientation–connectionist–characterised by beliefs and practices that assumed amongst other things that: teaching not only needs to help learners connect different aspects of mathematics but also mediated mathematical content through a variety of connected words, symbols and diagrams. This paper looks at a teacher's mediating means–artefact-based and human–in two lessons and the extent to which these focused on making such connections.

METHODOLOGY

The WMC-P project has collected videos of lessons taught to the 2011 Grade 2 cohort tracked through into Grade 3 in 2012. Then from 2013, attention turned to lessons taught to the new Grade 1 cohort, who were tracked through Grades 2 and 3 across 2014-15. Thus, at the teacher level, there is a body of data of two lessons taught by the same teacher to the same grade, with one drawn from the early years of the project (2011/12), and one drawn from the later years (2013-15).

Our analysis of the videos starts by dividing lessons into episodes identified through shifts in the mathematical task focused on. Within episodes, we list the example spaces, along with any evidence of incorrect or inefficient offers from learners, as the absence of these raises the possibility that the lesson was revising previous learning, which could then reduce the need for teacher mediation. Each episode is then examined for evidence of teacher mediation, both artifact-based (number charts, counters and so forth as well as teacher inscriptions) and human, in particular talk and gesture.

While overall the videotaped lesson observation dataset consists of almost one hundred lessons, the data drawn upon here is of one teacher, Mrs. S and her lessons from 2011 and 2014, both with a Grade 2 class. Mrs S is an experienced Foundation Phase teacher, teaching in 2011 one of three grade 2 classes in an urban Johannesburg school, with approximately 100 learners in the grade cohort, and, in 2014, one of four grade 2 classes, with a cohort of approximately 170 learners: relatively large, and increasing class sizes thus adding to the pedagogic challenges. In both years the language of instruction was English. Mrs. S is chosen as a 'telling case' as her lessons are not only typical of what we observed at the two times, but they also display differences in the sorts of mediation enacted at each time. It is beyond the scope of this paper to address the means and reasons for the changes observed in Mrs. S teaching and hence the analysis addresses the following research questions:

1. How does teacher mediation operate in historically disadvantaged foundation phase classrooms?
2. Can differences in teacher mediation be observed over time?

FINDINGS

Lesson 1 (2011): The lesson comprised two main episodes: forward and backward skip counting, then what the teacher called repeated subtraction – finding the answers to calculations where a number was repeatedly subtracted, for example, 10 – 2 – 2 or 20 – 5 – 5 – 5. The initial whole class counting forward in 2's to 100 appeared to present no difficulties to learners and it was followed by class counting backwards in 2s from 100, again unproblematic. The final count was back from 50 in 5s. Some learners could be heard making errors, the teacher got the class to repeat the count and errors could still be heard. The teacher turned to a 1-100 chart on the board:

Mrs S: Uh uh, wait [T claps her hands] wait, I want to use my number chart. I want you to look at the board. I saw some people were missing one out. We are counting back in...?

Mrs S+Class: Fives

Mrs S: From fifty. Where's fifty?

Class: Here [some learners point to the board]

Mrs S: Here. Ok here. Let's go. [T points to 50 on the number chart on the board.]

Class: [Count in unison back to 5, while T points to the numbers, with some learners also saying 'zero'.]

Mrs S: Zero isn't it? [The last number said.]

Class: Yes.

Mrs S: All right. So when we count back in fives, how many numbers do we skip? [T points to the number chart. A learner shouts 'four'.] How many? [Learners shout 'five', or 'four'.] We skip four, isn't it, one, two, three, four, then we get to the next one. One, two, three, four, [T demonstrates on the chart.] the next one is the answer. One, two, three, four, then we get to the next one. One, two, three, four. The next one because we are counting in…?

Mrs S+Class: Fives

Mrs S: So the fifth one is the answer, until you get to five, to zero at the end there. All right, that's enough of that. [T removes the number chart from the board.]

Here the teacher provided a localized explanation for how to count back in fives: it meant skipping over four spaces on the 100 chart (counting the spaces but not the numbers in those spaces), the number in the fifth space providing the answer. The teacher's mediating talk made no moves towards the 'fading' of the resource, either through talk of expecting recall of the backward number word sequence, or of ways in which this action is particular to the 100 square and would need to be adapted for a different resource or worked with differently in the absence of the chart. Further to this the teacher made no reference to the patterns in the answers obtained – either numerically or spatially (as in their positioning in vertical columns on the 100 square). Nor was there any exploring with learners how this might be extended to other skip counts, say counting back in 6's. Thus, the teacher's mediation provided learners with a method to remember, but a method that was primarily contingent on the availability of a practical resource: a localised, non-scientific, method. With the chart available this method enables the production of the answers but in the way explained here it is likely to be difficult for learners to appropriate in that the spaces were counted rather than the numbers landed on articulated. So while there is artefact-based mediation, this was not used to draw attention to structure or generality.

The lesson then moved into the main part, repeated subtraction. The first example comprised answering 10 – 2, and the teacher advised the learners thus:

> Mrs S: Ten minus two [Writes 10 – 2 on board] Ten minus two. Use your number chart. You count back, isn't it? This is subtraction, you count back. Anyone needs counters? [Many learners raise their hands.] If there's a problem, you can use the number chart. Others you can, if you want to, use your counters…

This followed soon after the learners had, it would appear, successfully orally counted back in twos from 100, but neither here nor in any subsequent dialogue did the teacher refer back to this counting. And although she told the children they could use their 100 squares, she did not make any reference to the method for counting back in 5s that she just modelled, nor any advice on how best to use counters. Thus the teacher's mediation was again localized in that it did not draw learners' attention to the network of mathematical ideas in play within and across the example spaces.

As the learners were getting organised with paper, charts and counters the teacher engaged them in singing a song about ten pawpaws on a tree and the wind blowing them away one at a time. She subsequently used this to explain how to count back two–one pawpaw blown away followed by a second. Here we see a 'pulling back' to a more naïve method–the learners had just demonstrated that they could count back in twos, but the teacher encouraged a return to counting back in single units. Again, an opportunity was missed to make a connection with what the learners already knew. The next examples were 10 – 2 – 2, then 10 – 2 – 2 – 2, each worked by counting back in ones from the previous answer. The calculations and the answers 8, 6, 4 were listed under each other on the board, but no comment made on the connections between or patterns within the answers. The teacher immediately turned to 10 - 5.

> Mrs S: Now I'm changing. The pawpaws are no longer ten. I want my tree to have fifteen. My tree has fifteen now. Can you do this one for me? [T writes 15 – 5] My tree's having fifteen pawpaws now. [Learners work this problem out, many using counters] Hey? Yes [T walks around, some learners raise their hands.] Mm. Ok, P? What's the answer?
>
> Learner: Ten?
>
> Mrs S: Is it 10?
>
> Learners: No.
>
> Mrs S: Who says no? [Learners shout out answers like 'Yes', '12', '6', '10'.] Ok, the answer is ten, he is correct. [T completes '15 – 5' with '= 10'.]

Again the mediating approach is one of localisation: all learners had 100 charts available but no backward referencing was made to what the teacher had previously modelled on the board, rather learners not knowing the answer were encouraged to use counters. No attempt was made to work with any incorrect answers: again this is typical of many lessons observed where the teacher largely ignored incorrect answers, with more answers being sought until the correct answer was elicited. Once the correct answer was in the public space, then any incorrect answers were set aside. 15 – 5 was followed by 15 – 5 – 5 and in a similar fashion, the answer was established from first principles, taking 5 away from 15 and then from 10.

The lesson proceeded with the teacher modelling 10 – 2 – 2 – 2 by taking away two from ten (again counting back in ones) and writing a small 8 above the second 2 from the right, then taking two from eight and writing 6 above the third two from the right, and so forth. While the teacher made some reference to prior answers in this episode, she did count again in every case The learners were then given the following to work through individually: 10–2–2–2–2; 15–5–5; 15–5–5–5; 20–5–5; 20–5–5–5.

The lesson can thus be characterized as presenting the mathematics contained within it as a series of discrete tasks, each of which was to be answered in isolation of any other task, either within the example set (20–5–5–5 was not linked to 20–5–5 for instance) nor across the tasks (the subtractions not being linked to the oral counting back). If learners did make any connections then that would only be a result of them 'discovering' them, as the teacher did not draw their attention to the potential of there being connections. This is consistent with the sequential working of individual examples highlighted by Venkat & Naidoo (2012) where the dominant practice focused in temporally localised ways on each current example.

Lesson 2 (2014): This lesson comprised two main episodes: rapid recall of number bonds for twelve, followed by an extended sequence on place value. The second episode is focused on her as it contained a number of sub-episodes covering: partitioning two-digit numbers into tens and ones and linking this to base ten blocks, recording the partitioning in extended form (27 = 20 + 7) and with the T U notation, building up numbers using ten strips and single squares, identifying the value of a digit in a two-digit number, ordering numbers, and adding ten to a number. (An extended analysis of this lesson's connections is given in Askew (2015) – the analysis here focuses in particular on the teacher's mediational means.)

Prior to the lesson Mrs S had listed in a column on the board: 13, 19, 27, 45, 67, 93. After the class had read out the numbers the teacher said she wanted learners to break the numbers down. A girl asked to break down thirteen replied 'ten plus three'. Alongside the '13' Mrs S wrote '= 10 + 3'. Other learners were asked to break each number down similarly until '93 = 90 + 3' was written on the board.

Mrs S: Very interesting, eh?

Class: (chorus) Yes.

Mrs S: This number [Pointing to '13'.] is now ten plus three [Moves her hand, tracing under '= 10 + 3' written on the board.] And this? [Pointing under '19'.]?

Class: Ten plus nine. [T moves hand under '= 10 + 9' along with the chorus.]

Mrs S: Now here? [Sliding her hand down to under '27'.]

Class: Twenty plus seven [T continues to run her hand down to the next numeral, and along underneath the expanded notation in time with the class chorusing the expansion.]

Mrs S: Now there is something happening here. Look here [gestures down the column of tens]. Now we have two digits this side, now the remainder is one [gestures down the column of ones]. These are tens [points to column of tens] and here we have? [Points to the ones, questioning intonation]

Class: Units

Here the teacher explicitly drew attention to a set of connections that are both 'horizontal' and 'vertical' (Watson & Mason, 2006) through talk and gestures drawing attention to the horizontal expansion of the notation and to the vertical commonalities across the examples. The teacher then picked up a stick of ten interlocking cubes, joined to make a 'ten-stick' and attached one stick to the board, close to the left of the '10' in '13 = 10 + 3'

Mrs S: And here [pointing to the '3'] we need?

Class: Three ones.

Mrs S: Okay, three, am I okay? [Holding three ten sticks up next to the digit '3']

Class: Noooo.

Mrs S: So what can I use?

Class: [Some say 'three ones', some 'three units']

Mrs S: So where are the ones? [Child comes to teachers' desk and hands over three single cubes.] I thought these [holding up the three ten-sticks] were the ones because this [holding up a single ten stick] is one. Okay, the small ones. Why? Because ten of them will make one ten. I must put how many?

Class: Three

Mrs S: Three of them [Attaches three single cubes to the board, to the right of and close to the digit 3 in '13= 10 + 3']

Here the teacher explicitly addressed two foci. First, her actions and talk raised the issue of the possible confusion between referring to a ten-stick as 'one ten' and needing three 'ones': her playing at getting it wrong drew attention to the need to be clear about the different referents of 'three' in the talk. Second, the careful positioning of the artefacts near the symbols, the literal proximity of the concrete and symbolic, reinforced the connection between these two representations, and the underlying mathematical structure. The artefact-based mediation here thus goes beyond the use of materials in a localized fashion as was seen in the 2011 lesson.

The lesson continued similarly for the other numbers, with each number treated as an opportunity to check and extend understanding. For example in partitioning 19, when the nine single cubes had been established, the teacher asked what would happen if one more cube were added to the nine, the ensuing conversation focusing on it becoming ten and the change from 19 to 20. The teacher thus used the example to check the learners' understanding: in questioning what would happen in an imagined example she went beyond the 'immediate answer' for the 'immediate example'.

Once all the numbers had been partitioned the teacher continued:

> Mrs S: Now we are going to do a similar activity using the same numbers. I just want to see whether you have observed something. I will underline the number and then you will tell me the value, what does it stand for? Don't tell me that it's tens or units or ones, here I want the value, how many. [Makes a circular cupping motion with hands]. Okay?

Mrs S mediation here drew attention to the fact that what was coming up was not completely new but connected to prior learning, that there is something common across the examples. Learners were expected to have agency in appropriating what the teacher is working on – to note patterns they may have observed not simply remembered. Another distinction was marked through the emphasis on saying the value designated not simply which place a digit is in.

> Mrs S: What is the value of that one? [Underlining '1' in '13'.] The answer is there already. In breaking down we show the value in another way. Okay? Now I want you to tell me the value of that one [in '13'.]
>
> Learner 1: Ten.
>
> Mrs S: It's a ten, that's (unclear) isn't it? So the value of that number is ten. [Writes '10' to the right of the equation.] What is the value of nine in that number? [Underlines '9' in '19'.] A?
>
> Learner 2: Nineteen.
>
> Mrs S: She is saying nineteen. Is she correct?
>
> Class: No.
>
> Mrs S: Can somebody come here and explain?
>
> Learner 3: Nine.
>
> Mrs S: Nine. Why is it nine?
>
> Learner 3: 'cos it's in the unit.
>
> Mrs S: Just as a reminder, remember (learner 2), it is like this, tens, units. [Writes T U above each number.] So nine is under the units, under the ones [pointing to the position of '9' relative to the label 'T U'] so the value of this number [circling the '9' in '19'] is only nine [writes '9' to the right]. There it is (learner 2). [Underlines the '9' in '10 + 9'] Okay? There, okay? There it is. Nine, so the value of this number [Circling gesture around the '9' in '19'] is nine [writes over the '9' to the side again.]

Here we see human mediation connecting together the different representations, and extending the range of representations. The teacher directed attention to another connection - ‘the answer is there already’ – and by reframing ‘showing the value’ as associated with ‘breaking down’ not only were two potentially discrete ideas (breaking down a number into its place value partitions, and identifying the values associated with digits) connected, but also learners were encouraged to connect with what they already knew. Multiple links between place value features are again made explicit. As the lesson continued the connection with breaking numbers down was repeatedly reiterated with the teacher drawing attention to learners connecting what they already know and the connections between representations.

DISCUSSION

The teacher’s mediational moves employed in the 2014 lesson stand in marked contrast to the 2011 lesson. In the earlier lesson one idea was focused upon (repeated subtraction) but only one task type was engaged in–finding the answer to a calculation– and whilst the numbers used in each example were varied, each example was treated in a localized fashion. The potential for mediation that attended to the connections within and across the examples and to the already appropriated act of forward and backward counting was not realised, and hence the treatment of mathematics as ‘scientific’ (in the Vygotskian sense) limited: a sequential working with individual examples dominated the 2011 lesson.

In 2014, in contrast, a rich and connected experience of place value and how to work with it was woven through the range of tasks, many of which kept coming back to the same example space, enabling Mrs S to repeatedly and explicitly draw attention to a set of connections that were both ‘horizontal’ and ‘vertical’ (Watson & Mason, 2006) in that her mediating talk and gestures drew attention to the horizontal expansion of the notation and to the vertical commonalities across the examples. Mediation that draws on learner misconceptions as a way of engaging with learner understanding is rare in the classrooms we have studied but here we see the teacher effectively anticipating a misconception (her ‘error’ in distinguishing ‘three tens’ from ‘three units’) and working with that.

CONCLUSION

The differences noted across the two lessons of Mrs S are typical of differences observed in the broader data set. The evidence from Mrs S, and other teachers, shows that it is possible for rich, connected teaching to be enacted in classes with large numbers of pupils and limited resources. Furthermore mediation focused on connections can be established without disrupting the culturally dominant practice of whole class, teacher-centred, pedagogies. Given the evidence for differences the research team is developing a framework for describing and interpreting different levels of empirical phenomena related to mediation - the Mathematical Discourse of Instruction - Primary framework – that will enable analysis of the full data set, which, if revealing of changes across the years to mediation more focused on structure and generality, will enable the exploration of the data with respect to the professional development activities that the teachers engaged in in the intervening years.

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The didactician as a model within classroom activities: investigating her roles

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In this paper we address the issue of teacher professional development, with reference to how to support teachers in activating effective and aware approaches to be adopted during the lessons to foster the students' use of algebra as a thinking tool. We hypothesize that the didactician, intervening during class activities, could act as a role model for the teacher. We analyse a teaching episode, by means of a combined theoretical framework, to highlight, on one side, the way the didactician acts as a model and, on the other side, moments of achieved harmony between the didactician's and the teacher's interventions.

INTRODUCTION

Since the nineties, research studies have pointed out that algebraic language should be presented and treated in classroom as a tool for representing, exploring relationships, interpreting and developing reasoning (see, as paradigmatic example, Arcavi, 1994). In tune with these research studies, both the authors have investigated the design and implementation of activities of proof construction through algebraic language (Cusi & Malara, 2009; Morselli & Boero, 2011) aimed at promoting algebra as a tool for thinking (Arzarello, Bazzini & Chiappini, 2001).

Few studies have focused on the role played by teacher's actions and interventions in fostering an effective and aware development of reasoning by algebraic language and on the interrelations between these roles and the thinking processes developed by the students. In (Cusi & Morselli, 2016) we addressed this issue, combining two theoretical lenses - the construct of "Model of aware and effective attitudes and behaviours" (MAEAB) and Habermas' construct of rational behaviour - in the analysis of a class discussion from a teaching experiment. The analysis showed that the teacher is crucial in catching and deepening occasions of meta reflection, so that students may become aware of their rational behaviour and share it with their mates.

Aware of the complexity connected to the teacher's task of acting as a model in the effective use of algebra as a thinking tool and in promoting students' rational behaviour, we turned our reflection to possible ways to promote the teacher's development on this issue and we focused on situations where teacher and didactician (we use this term in the sense of Jaworski, 2012) collaborate in all phases of the teaching and learning process, from the planning to the implementation and analysis of teaching sequences. Our methodology of working with teachers involves an active role of the didacticians in fostering teachers' analysis of their practice and the use of specific theoretical constructs as tools to support this joint analysis and the communication between teachers and didacticians (Cusi & Malara 2016). In this paper we focus on another important moment in which the teachers and the didacticians interact, that is when the didactician participates to classroom activities. In particular, we are interested in studying the ways in which the didactician could behave, during classroom activities, to foster the teacher's aware activation of the roles that could be played to support students in the use of algebra as a thinking tool.

In the following, we organize our theoretical framework in two sections: at first we illustrate relevant references on the relationship between theory and practice and the possible collaboration between teachers and didacticians to frame our methodology of work with teachers; afterwards we present the theoretical tools we combine to study the actions and interventions of the didactician and the influence of the didactician's actions and interventions in terms of teacher's activation of the different roles that could be played to support students in the use of algebra as a thinking tool.

TEACHERS AND DIDACTICIANS WORKING IN COLLABORATION: THE INTERPLAY BETWEEN THEORY AND PRACTICE

In the last years there has been an increasing interest towards the crucial role played by collaborative ways of working with teachers within teacher education processes. The model of collaboration to which we refer is the one introduced by Jaworski (2003), who has stressed the value of, on one side, fostering teachers' critical reflection about their practice, and, on the other side, sharing these reflections between didacticians and teachers within a community of inquiry. She stresses that this kind of research programs foster the *co-learning* for all the participants: "in co-learning, the learning of one is dependent on the participation and learning of others: mathematics teachers and educators learn together with different roles, goals and learning outcomes, while engaged in common activity for mutual benefit" (Wagner, 1997, quoted in Jaworski, 2003, p. 250). We put ourselves in a perspective of co-learning, since, in this work, we, as didacticians, are reflecting on our roles of teacher educators within the teacher education program in which we are involved.

Jaworski (2012) suggests that, in order to reflect elements of learning and development for teachers and didacticians, the usual didactic triangle (teacher-student-mathematics) should be extended to a didactic tetrahedron (the didacticians representing the fourth vertex), the *expanded didactic triangle*. The expanded didactic triangle enables to focus both on: (a) the traditional didactic triangle, which characterises elements of the relationships involved within a community of teachers, their students and mathematics; (b) a meta-level triangle, which highlights the developmental processes that involve teachers and didacticians. In this paper we will adopt the model of the didactic tetrahedron to describe the focus of our research.

As stated above, we are interested in studying how the actions and interventions of the didactician during classroom activities may influence the teacher's activation of different roles to support the use of algebra as a thinking tool in their students. In particular, we claim that the didactician's interventions during teaching experiments could represent a fundamental way of supporting teachers in activating effective and aware approaches to be adopted during the lessons. This perspective is in tune with Mason's (2008) stress on the teacher educators' role in directing teachers' attention toward constructs, theories, and practices that can inform and guide their future choices, in order to lead them to become aware "not simply of the fact of different ways of intervening, but of the fact of subtle sensitivities that guide or determine choices between types and timings of interventions" (2008, p. 49). In tune with Mason's description of what happens to a student who internalizes the stimuli received by his/her teacher, we claim that, in the same way, the interventions of the didactician during the teaching experiments could foster shifts of attention for teachers and their internalization of the received stimuli, so that the activity of reflection moves from a process "in themselves" to a process "for themselves".

THEORETICAL TOOLS FOR THE ANALYSIS OF THE ROLE PLAYED BY THE TEACHER WITHIN CLASSROOM ACTIVITIES

The MAEAB construct is the result of a study aimed at highlighting the delicate role played by the teacher in effectively guiding his/her students to the construction of reasoning through algebraic language. A set of roles (summarised in the following table) have been identified (Cusi & Malara, 2009, 2016) to outline the approach of a teacher who consciously behave constantly aiming at "making thinking visible" (Collins et al., 1989), in order to make his/her students focus not only on syntactical or interpretative aspects, but also on the effective strategies adopted during the activity and on the meta-reflections on the actions that are performed.

<table>
<tr><td rowspan="4">A first group of roles are those performed when the teacher tries to carry out the class activities posing him/herself not as a "mere expert" who proposes effective approaches, but as a learner who faces problems with the main aim of making the hidden thinking visible, highlighting the objectives, the meaning of the strategies and the interpretation of results.</td><td>Investigating subject and constituent part of the class in the research work being activated: when the teacher asks students to give suggestions about how to go on with the activity, intervening with the aim of making them feel involved in the activity as a group;</td></tr>
<tr><td>Practical/ Strategic guide: when the teacher poses herself, in front of the problem, as an inquirer who aims at sharing the thinking processes and discussing the possible strategies to be activated;</td></tr>
<tr><td>"Activator" of interpretative processes: when the teacher makes the students activated proper conceptual frames (Arzarello, Bazzini & Chiappini, 2001) to interpret the different algebraic expressions constructed when solving a problem;</td></tr>
<tr><td>"Activator" of anticipating thoughts (Boero, 2001): when the teacher makes the objectives of the manipulation of algebraic expressions explicit and recall them during the discussion, in order to enable the students to share these objectives, monitor and control the activated strategies;</td></tr>
</table>

The second group of roles refers to the phases during which the teacher becomes also a point of reference for students, to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed.	*Guide in fostering a harmonized balance between the syntactical and the semantic level*: when the teacher makes the students focus on the importance of controlling both syntactical and interpretative aspects and she discusses possible problems arisen when the syntactical or the interpretative level is not controlled;
	Reflective guide: when, in front of a student who proposes an effective approach to the resolution of a problem, the teacher asks him/her to make his/her thinking processes explicit, or she repeats what has been said by the student stressing on the reasons subtended to his/her approach, or she asks to other students to interpret what he/she said;
	"Activator" of reflective attitudes: when the teacher poses meta-level questions aimed at making the students evaluate the effectiveness of a strategy and reflect on the effects of a choice that was made during the resolution process.

Table 1: Characterisation of the roles played by a teacher as a MAEAB

The second theoretical tool to which we will refer in our analysis is Habermas' construct of rationality. Drawing from this construct, Morselli & Boero (2011) propose that the discursive practice of proving encompasses: an epistemic aspect (conscious validation of statements according to shared premises and legitimate ways of reasoning); a teleological aspect (conscious choices to be made in order to obtain the aimed product); a communicative aspect (conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture). When proving through algebraic language, epistemic rationality consists of modeling requirements, inherent in the correctness of algebraic formalizations and interpretation of algebraic expressions, and systemic requirements, inherent in the correct application of syntactic rules of transformation; teleological rationality consists of the conscious choice and management of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity; communicative rationality consists of the adherence to the community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions. The student must combine the adherence to syntactical rules on one side, and the goal-oriented management of the processes of formalization, transformation and interpretation, on the other. Still related to teleological rationality, the student must be aware of the fact that proving by algebraic language means deriving from algebraic manipulation a new algebraic expression, whose interpretation gives new information concerning the truth of the statement.

RESEARCH QUESTIONS AND RESEARCH METHODOLOGY

The aforementioned theoretical tools were already used to analyse the teacher's complex role as a model for fostering students' rational behaviour when dealing with algebra as a thinking tool (Cusi & Morselli, 2016). We plan to analyse the didactician-teacher interaction and the teacher professional development throughout a 10 years process. The objective of this long-term study will be to analyse the teacher's development, highlighting the ways in which the didactician, collaborating with the teacher, may promote the teacher's awareness of her role in the classroom and, more in general, her professional development.

In this paper we start this analysis, focusing on the didactician-teacher-students (D-T-S) interaction during classroom activites. The model of the didactic tetrahedron (Jaworski, 2012) is helpful in describing the focus of our research. In particular, it enables to describe the complexity of the interactions that our methodology of work with teachers involves. In addition to the traditional didactic triangle (T-S-M) and the meta-level triangle (D-T-M), in fact, the other facets of the tetrahedron introduce new levels at which our analysis can be performed: the triangles D-S-M and D-T-S, in fact, highlight the levels of the interaction between the didactician, the teacher and the students during classroom activities. In this work, our aim is to investigate the ways in which the dynamics that can be analysed looking at the triangles D-S-M and D-T-S may influence, on one side, the interaction between the teacher and her students (triangle T-S-M) and, on the other side, the developmental processes highlighted through the meta-level triangle. For this reason, we will use our theoretical tools for a double aim: studying the way the didactician acts as model for the students, and studying the way teacher is influenced buy the model of the didactician. We use the construct of rational behaviour to discuss the rational behaviour in using algebra as a thinking tool during the activity, and the MAEAB to analyse the role of the didactician as a model for the students. More specifically we focus on moments during which the didactician, thanks to the activation of specific roles connected to the MAEAB construct, fosters the shift of attention in the teacher, who, consequently, tries to activate the same roles. It is a preliminary analysis, mainly aimed at investigating the use how our theoretical tools to highlight these dynamics. We, in particular, focus on an episode during which the didactician and the teacher orchestrate collaboratively a mathematical discussion. Data at disposal are video recordings of the classroom discussion, pictures of the whiteboard, students' written productions.

ANALYSIS OF AN EPISODE

The context we refer is that of the long-term project "Language and argumentation" (Morselli, 2013), aimed at designing and experimenting task sequences with a special focus on argumentation and proof. Within the project, the didactician and the team of teachers collaborate in task design and process analysis. The didactician takes part to all the class sessions, co-conducting the lesson with the teacher. After each lesson there is a brief meeting between the teacher and the didactician, so as to comment the session and plan possible variations for the subsequent session. Regular meetings with all the team of teachers are organized, so as to analyse the processes and compare the teaching experiments in the different classes.

The episode comes from a teaching experiment performed in grade 7. The teacher, who holds a university degree in Chemistry, had more than 10 years of experience in teaching mathematics at lower secondary school level. She was at her fifth year of collaboration with the didactician within the project. She had already taken part to the design and implementation of task sequences for the first approach to algebraic language as a proving tool, but she was at her first experience with the task at issue. The students already had performed some activities on argumentation and first approach to algebra as a proving tool.

Students worked in group on the following task: "*What can you tell about the sum of three consecutive numbers?*". In the subsequent class discussion, the groups shared their answers and explanations with all the class. Only one group (Edel, Sonia and Giulia) attempted an argumentation with letters, proposing two different algebraic representations (the second being an amendment of the first one): $n+n+n=n/3$; $n1+n2+n3=n/3$. Next to the two expressions, the group proposed a verbal explanation: "*Three consecutive numbers can be added and the result is multiple of 3. The sum of these numbers is divisible by 3 because the added numbers are 3. The middle number is given by the division of the sum of the three numbers*". The following excerpt refers to the discussion on their solution, with a specific focus on the algebraic representations. This solution was presented after another group expressed its conjecture (the sum is always divisible by 3) and proposed a pragmatic explanation, made up of numerical examples.

The discussion starts with Edel, one of the elements of the group that proposed the algebraic expressions, writing at the blackboard the expression $n+n+n=n/3$.

1 Edel: I do number plus number plus number, equal n divided by 3.

Bos raises his hand and starts criticizing, but the teacher stops him.

4 D (didactician): I ask you a question: this thing that you wrote … did you write it to express the property or to justify, to motivate it?

5 Edel: To try and explain what we did, to try and explain the way three consecutive numbers can be summed up and give a number that is divisible by 3. To try to explain what we did before, that is the three numbers, the numbers are three and then this is why they are divisible by 3.

6 D: Ok. After, we will reason on her representation. In the meanwhile, what can we find of really different from what we wrote before? (*R is referring to the pragmatic explanation proposed by the previous group*) …That she does not use…

D, referring to the activities performed during the previous school year, guides the students highlighting that the use of letters enables to reason in general terms.

13 Edel: At first we had written number plus number plus number, without 1,2,3, but after one had to add 1,2,3 in order to show that they are consecutive.

14 D: Ok, in order to show that they are different and you say, if we call them n1, n2, n3 I give the idea that they are three consecutive numbers. Ok.

Another student, Alb, proposes to use the two expressions $n \cdot 2+ n \cdot 2+1+ n \cdot 2$ and *$n \cdot 2+1+ n \cdot 2+ n \cdot 2+1$, which are written on the whiteboard.*

18 T (teacher): Ok, this is when we start with an even number, the other one when we start with an odd number.

19 D: What do you think about this proposal, in comparison with the former one?

D guides Alb in making the meaning of the two expressions explicit. Alb, helped also by T, stresses that the two expressions represent two different cases: when the first number is even and when it is odd.

24 D: What do you think of this representation? Do you find it convincing?

25 Vic: It doesn't specify that they are consecutive.

26 D: It does not specify that they are consecutive, that is to say if I get into the room right now and I see the sum written on the whiteboard, do I understand that it is the sum of three consecutive numbers?

27 Vic: It is the sum of an even number plus an odd number or an odd number plus an even number.

28 Alb: You can write first, second and third. As we said before.

29 T: In this way? (*T adds Roman numbers on the top*)… Does this help to understand that they are consecutive?

30 Voices: no.

The students, supported by D who suggests to substitute specific values to n, are able to highlight that the representations proposed by Alb are characterised by the fact that the first and the third numbers are the same, therefore they do not represent three consecutive numbers. Moreover, the expression n1+n2+n3, proposed by Edel's group, are too general because they only represent the sum of three numbers.

42 Pir: I can write n and after I change the letter. Different letters.

43 D: But it is the same objection I did for n1, n2, n3… how can I know they are consecutive numbers?

45 Vic: We can write… in the first case $n \cdot 2$, after $n \cdot 2+1$, after $n \cdot 2+3$.

46 T: Plus?

47 Vic: +2.

T writes on the whiteboard the expression $n \cdot 2+ n \cdot 2+1+ n \cdot 2+2$.

48 D: Did you understand what is it? Vic, could you explain it?

49 Vic: $n \cdot 2$ is an even number, $n \cdot 2+1$ is the consecutive…

50 D: Let's try and give some numeric values.

Vic proposes to substitute n=3 in the expression. Other students declare that Vic's expression is right.

55 D: Is this ok? This is a way of writing three generic consecutive numbers, isn't it?

Other students agree with Vic's proposal. D asks whether 5+6+7 can be written in that way and Bes proposes to change the representation into n+n+1+n+2.

64 T: Here, let's check whether we can write also an odd number.

65 Bes: n=5, you can do n=5, n+1=6 and n+2=7.

66 Vic: Or you can modify the above case, the second case says that…

D promotes a comparison between Vic's idea of representing two cases and Bes' idea of creating a more general representation and asks to the class whether it is necessary, to the aim of proving the divisibility by 3, to distinguish the two cases.

68 Voices: No.

69 D: Then, we can write only one, that will be for instance $n+n+1+n+2$. By now what did we do? We just represented the sum of three consecutive numbers… By now we just wrote the sum of three consecutive numbers. What do we do with that writing? Now we can go on and write $n/3$ or something similar, but… I let you think in which way, using this writing, we can go on with the justification.

70 T: Why does writing it in this way is useful for us?

71 Vic: Because modifying we would get number + number + number +1+2 and then… *T writes at the whiteboard*

72 Vic: Summing up we would get number +number + number +3.

73 D: And $n+n+n$, how can we write it?

74 Voices: $n\cdot 3$.

75 D: And at this point do I see that is a number divisible by 3?

76 Voices: yes.

If we focus on D's interventions, we can observe that, from the very beginning of the discussion (4), she often poses herself at a meta-level, acting as an *activator of reflective attitudes,* bringing to the fore the *teleological* dimension. Specifically, in line 4, D wants to elicit the aim of writing the algebraic expression (communicating or proving) because her objective is to intervene at two different levels: at *epistemic* level, enabling the students to realise that the representations are not correct; at *teleological* level, enabling them to highlight that the algebraic representation should not contain also the "resulting property" (divisibility by 3), that should be derived from the transformation of the algebraic expression "sum of three consecutive numbers". When she asks to the students to compare Edel's group's approach with the approach analysed previously (6), she also acts as a *reflective guide*, fostering the comparison between two different ways of facing the activity. This role is activated by D also when she asks the students to compare Alb's proposal to Edel's (19).

During the discussion, D often acts also as an *activator of interpretative processes*, trying to support the students in highlighting the meaning of the algebraic representations they propose (14, 24, 43, 55). At the same time, D acts as a *reflective guide* and as an *activator of reflective attitudes* because her aim is to make the students catch if the different representations are really correct or not. In this part of the discussion, therefore, D focuses on the *epistemic* aspects, disentangling them with the *communicative* and *teleologic* ones: the algebraic representation must be correct, not only easy to understand, and "transformable". The effectiveness of this approach is evident when Vic is able to highlight a problem connected to the expressions proposed by Alb (25), to make the meaning of these expressions explicit (27) and to propose a possible modification of these expressions to represent consecutive numbers (45). D acts as a *reflective guide* also in helping Vic express the meaning of his proposal to the classmates (48, 50).

The influence of D's approach on T's activation of roles that should be played is evident when, instead of commenting on Alb's proposal of distinguishing the three numbers simply writing "first, second and third" (28), T re-launches this suggestion to the whole class (29), acting as an *activator of* both *reflective attitudes* and *interpretative processes* with the aim of making the students identify the problem. Starting from this moment, it is possible to highlight what we call "*achieved harmony*" between T's and D's interventions, that is an evidence of T's intention of supporting D's approach through her interventions. When, for example, Bes, referring to Vic's observations, correctly suggests to write a more general expression that really represents the sum of three generic consecutive numbers, T supports Bes in checking the correctness of her algebraic expression and in explaining the effectiveness of her proposal (60-62-64).

After having acted again as a *reflective guide*, making the meaning of Vic's and Bes' suggestions more explicit (67), D shifts students' attention on the effectiveness of the last expression (n+n+1+n+2) in supporting the construction of a mathematical justification of the fact that this sum is always divisible by 3 (69). In particular, focusing on this objective, D is acting as an *activator of anticipating thoughts* because she wants the students to transform this expression with the aim of highlighting the observed property. Here again we can observe an *achieved harmony* between T's and D's interventions, because T acts to make the teleological level arise, re-launching D's question to the class (70). In this way she enables the students to highlight how to transform the expression n+n+1+n+2 to show that it always represents a number that is divisible by 3.

COMMENTS AND CONCLUSIONS

Our working hypothesis was that the didactician, by her interventions during class discussions, may help the teacher carry out efficient ways to promote the student's rational behaviour in the use of algebra as a thinking tool. To test this hypothesis, we analysed a teaching episode, showing that the teacher, while working with the didactician acting as a MAEAB for the students, gradually activated specific roles in tune with the MAEAB construct. In particular, we introduced the idea of *"growing harmony"* to indicate those moments when the teacher starts proposing interventions, attitudes and behaviours in tune with the didactician's approach. In our opinion this "*growing harmony"* could represent an indicator of a deeper teacher's awareness about the ways in which she should behave to foster students' aware and effective use of algebraic language as a thinking tool.

In order to test this hypothesis, we will compare this discussion with other subsequent discussions carried out by the didactician and the teacher and by other teachers involved in the project. Moreover, we will interview the teachers to collect their narratives about their professional development path and, in tune with the methodology proposed by Cusi and Malara (2016), we will make the teachers refer to specific theoretical lenses (in particular the M-AEAB construct and Habermas' levels of rationality) in their a-posteriori reflections on the written transcripts of class discussions. In this way, it will be possible to highlight the teacher's growing awareness about both the meaning of the researcher's interventions and the crucial roles that should be played.

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Studying secondary mathematics teachers' attempts to integrate workplace into their teaching

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This paper focuses on 3 cases of mathematics teachers' attempts to integrate the workplace into their teaching while participating in a professional development (PD) program. We draw on the work of 5 groups of mathematics and science teachers who collaborated for a school year to design and implement tasks related to workplace non-routine situations. Teachers' activities are analysed under an Activity Theory (AT) perspective. The results indicate different forms of interaction between the activity system of workplace and the one of mathematics teaching.

Keywords: Workplace, teachers' goals and actions, Activity Theory.

INTRODUCTION

There is a great deal of research supporting claims that workplace settings may offer pedagogical opportunities for teachers to introduce authentic situations into their school classrooms activities (e.g., Nicol, 2002; Wake, 2014). These pedagogical opportunities refer to making mathematics meaningful to students by preparing them to explore open and unstructured problems that connect mathematical knowledge as taught at school and as used out-of-school. Our understandings of the nature of mathematical activity in workplace are informed by a number of research studies (e.g., Hoyles & Noss, 2001). These studies indicate that mathematical notions underlying professionals' practices in workplace settings are mostly hidden and embedded in the particulars of the situations. This makes any attempt to connect workplace and mathematics teaching highly demanding. A challenge for a teacher, in this case, is to connect situations, symbol systems, technological and workplace tools, contextual constraints/rules and personal and professional knowledge to help students make sense of work processes.

How teachers can use the workplace as a context for designing and using lesson activities in the classroom remains an open question that has only received partial answers from small scale studies mostly on prospective teachers. For instance, Nicol (2002) found that a teacher education program including visits to workplace sites helped prospective teachers to keep the mathematics contextualized when designing activities for their students. Frykholm and Glasson (2005) suggested that teacher education courses involving collaboration between science and mathematics prospective teachers provide a fertile ground for them to develop interdisciplinary units connecting both topics. In this direction, Potari et al. (2016) argue that workplace seems to provide a context for collaborative work of mathematics and science practicing teachers that helps them to connect meaningfully the two subjects.

The study reported in this paper took place in the context of a European project, mascil (see: www.mascil-project.eu), aiming to integrate workplace in the teaching and learning of mathematics and science through implementation of inquiry-based tasks in classrooms. Thirteen partner countries participated in the project and developed a body of exemplary classroom and teacher education materials as a basis for the organisation of PD activities and classroom implementations. Our aim is to examine how practicing mathematics teachers integrate workplace tools and practices when designing and implementing problem-solving classroom activities and what factors facilitate or constrain this integration. We adopt an AT perspective to focus on how workplace situations – considered as activity systems - interact with the mathematics teaching activity in the context of mascil.

THEORETICAL FRAMEWORK

Thinking beyond dichotomies such as school versus work, Bakker (2014) argues on the importance of developing research-informed understanding of what happens at the boundaries of schools and workplaces settings. However, the task of building connections between the two is rather demanding from an epistemological and didactical point of view. From an epistemological point of view, a number of studies emphasize the extent and depth of mathematical concepts and sophisticated mathematical skills encountered in the workplace. However, the conventional epistemological view of mathematics fails to capture this richness (Hoyles & Noss, 2001; Triantafillou & Potari, 2010). At the level of teaching, viewing the workplace context as non-mathematical might eliminate teachers' opportunities to explore its pedagogical potential. Wake (2015) argues that modelling the structure of a contextual situation could provide teachers an opportunity to create a nexus of mathematics and reality.

We adopt Engeström's (2001) approach to investigate mathematics teachers' activity when they are challenged to integrate workplace into their teaching. We consider two activity systems: the system of workplace and the system of mathematics teaching in which the teachers have been engaged to study the interaction between the two. The "activity system" is a basic concept of AT that is collective, tool-mediated and needs a motive and an object. Individual and group actions are studied and interpreted against the background of entire activity systems. Activity systems are transformed over lengthy periods of time when the object and the motive of the activity are reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity. Central to the process of transformation are contradictions within and between activity systems emerging when a new element comes from the outside. Figure 1 shows a representation of two interacting activity systems under Engeström's (2001) perspective. The two triangles represent the two activity systems considered in the present study.

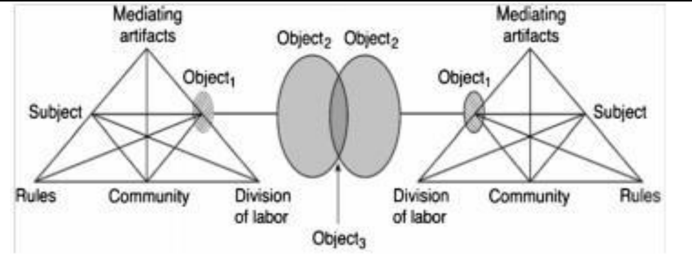

Fig. 1. Interacting activity systems (Engeström 2001, p. 136)

Each system involves the basic dimensions of AT with elements the subject and the object of the activity (object 1) that is constructed through the mediation of tools and it is framed by the community in which the subject participates, its rules and the division of labor. In the interaction of the two systems object 1 moves from an un-reflected and situationally given goal to a collectively meaningful object constructed by the activity system (object 2) and to a potentially shared or jointly constructed object (object 3).

In this study, we analyze teachers' goals and actions when acting as subjects into the activity systems of workplace and mathematics teaching. Our aim is to explore the role of tools (workplace artefacts, teaching resources) and the specificities of the workplace and classroom contexts (rules and division of labour) in the formulation of a new object incorporating elements of both activity systems.

METHODOLOGY

The context of the study

In mascil implementation in Greece, thirteen groups of practicing secondary teachers (about 10 in each group) from mathematics, science and technology have been established to work in the spirit of lesson study (Hart, Alston & Murata, 2011). In each group, teachers collaborated with the support of a teacher educator for a school year to design and implement inquiry-based tasks related to workplace non-routine situations and reflect on their teaching. Before and after each implementation of the designed lessons PD meetings took place. In the initial PD meetings, the teacher educator informed teachers about the rationale of the project and introduced them to the main principles of inquiry-based tasks and to the nature of workplace mathematics. In the subsequent meetings, teachers were asked to collaborate in transforming the exemplary mascil tasks or designing new ones in the same spirit, share their experiences from the implementations and discuss emerging issues. In the highly centralized Greek educational system, mathematics teaching in secondary school is rather traditional with a strong emphasis on mathematical content without connections to real life contexts. Moreover, PD activities are often limited to lectures and short term courses of a top down philosophy. Thus mascil was a rather innovative project both in terms of its teaching objectives and its PD approach.

Participants

In this paper, we focus on five groups of practicing teachers (22 mathematics teachers, 14 science and 9 technology). Teachers in these groups worked in upper or lower secondary schools and they had long teaching experience (more than ten years). We analyze the work of the mathematics teachers in these groups who collaborated together and/or with science and technology teachers. Participation in mascil was on a voluntary basis and most of the teachers had qualifications beyond those required by their profession (e.g., master or PhD degrees in mathematics, science or technology education).

Data collection and analysis

The data collected from the five teacher groups included: audio and/or video recordings of the PD meetings (7 two- hour meetings per group – 35 in total) and classroom implementations (71 in total, 2 teaching hours each); teachers' portfolios (tasks, worksheets, written accounts/journals, power -point presentations, digital materials, students' work, students' evaluation reports) and selected interviews with teachers and teacher educators.

In this paper, under a grounded theory approach (Charmaz, 2006) we analyse the discussions in the PD meetings and teachers' portfolios. Initially, we identified parts of the data concerning the activity of mathematics teaching and the activity of workplace. Then we analysed teachers' goals and actions looking for possible intersections between the objects of the two activity systems identifying emerging contradictions and convergences in relation to: (a) the origins of their ideas for tasks (e.g., personal experiences); (b) the tasks and resources by which they targeted students' familiarisation with workplace (role playing, workplace tools, representations used); (c) the links they made between workplace and mathematics; (d) the supportive factors and/or constraints in the process of integration; and (e) the teachers' reflections on the contribution of workplace in improving their teaching. The rationale of the goals and actions was analyzed by taking into account the bottom elements of the extended mediational triangles of the activity systems (community, rules, division of labor).

RESULTS

In this section, we present the case studies of three teachers from different groups indicating three emerging ways of interaction among the elements of the activity systems. In case 1, the workplace context is mostly used for motivating students to see the applications of mathematics while the teaching goals are not linked to the workplace activity. In case 2, the workplace context is smoothly integrated into the classroom teaching through a modelling process linking the workplace activity with problem solving in the classroom. In case 3, a simulation of the workplace activity in the classroom facilitated a strong integration of workplace into mathematics teaching.

Case 1

The teacher studied in this case, James, is a mathematics teacher with more than 20 years of teaching experience who participated in one PD group consisted of four mathematics, one technology and four science teachers.

The initial idea of his design was based on a contextual textbook task: “Two villages are situated on the opposite sides of a river and their distances from the sides are unequal. In which place do we have to construct a bridge perpendicular to the sides of the river so that the two villages to have the same distance from the bridge”. His proposal was negotiated in the group and he was challenged to make more explicit the workplace connection. The mathematics teachers invited a landscape engineer to inform them about the design of a bridge and the main issues involved in it. The engineer pointed out that at his workplace context the main goal was to reduce the cost of the bridge construction. The cost was related to the width of the river and that the distance from the villages did not matter. The science teachers started to propose non mathematical parameters from the realistic situation to take into account in the task design such as “rivers with varying width” or “rocky landscape”. After this exchange of ideas, James did not feel happy with this workplace complexity: “it would be better not to have all these factors interfering”.

James reformulated the problem of the design of the bridge by referring to a specific very old bridge that it had been awarded a prize for its original construction. The students were asked to find the parabolic curve given the length and the height of the bridge that a technician could use to build it. This task was an extension of a similar textbook problem. In the group discussion conflicts emerged as the other teachers and the teacher educator could not see any connection with the workplace. James presented to the group a technical method that he had found in the internet about the construction of this bridge.

In the classroom implementation (11^{th} grade students, 17-year -olds), James took the following teaching actions: (a) familiarized the students by asking them to read information about the history of the bridge; (b) engaged them is solving a textbook task (drawing the graph of $y=-x^2+6x$ and find its maximum value); (c) asked them to find the formula of a parabola when they knew that it passed through three points; and (d) explained on the board the technical process of joining together different parts of a bridge. He closed the lesson by asking the students “What would you recommend to the constructor of the bridge?”

James based his task design on a familiar to him tool, the school textbook. In the collective process of transforming this task the inputs from the science teachers in the PD group brought realistic factors that he could possibly include into his design. However, for him it was not easy to take these factors into the account in his implementation. Although, he tried to be familiarized with the specific workplace context and tools (talking with the professional, finding relevant information about different techniques of bridge construction) the gap between his teaching goals and the workplace goals still remained. This can also be explained by the fact that mathematics teaching practice in upper secondary education in Greece is characterized by norms and rules targeting students' conceptualization of abstract mathematical ideas while connections with contextual situations are rather limited.

Nevertheless, James made an attempt to introduce a contextual task into his teaching but did not succeed in overcoming norms and rules established in his professional community.

Case 2

The mathematics teacher, Elena, had 15 years of teaching experience. In mascil she participated in a group of five mathematics, three science and two technology teachers. She chose to use a task (the *Solar Cells*) that was included in the exemplary mascil materials (www. mascil-project.eu). The task concerned the installation of solar panels on a house rooftop. In this task, the students had to decide whether a specific installation of solar panels on a house rooftop was a profitable choice for a family in relation to the cost of electrical supply provided by the National Electricity Company. In this process, students had to explore how to place the panels on the roof in order to maximize their number by studying their projections. In terms of mathematics, the problem required students to visualize relations between the three-dimensional context of the task and its two-dimensional representation.

Elena collaborated with the science teachers during and between the PD meetings in order to be familiarized with the scientific context of the task. Also she discussed specificities of panel installation with a professional working in a solar panel company. In her design, she used resources provided in the initial version of the task (e.g., actual panel dimensions, panel inclinations, video from the workplace). Furthermore, she adapted the problem to be closer to reality on the basis of the information that the professional provided to her (e.g., the distance between horizontal rows of panels).

During classroom implementation (8^{th} grade students, 14- year-olds), Elena supported students' familiarization with the scientific aspects of the problem by asking them to interpret authentic representations. For example, she provided the representation (Fig. 2) of sun's positions during the spring and the winter equinox and asked students "what case we could consider as important in order to decide about the shadow effect on the panels' installation?" Furthermore, she challenged them to consider the advantages of using solar energy as power supply for houses: "why making your house energy sustainable is a profitable investment?"

The main part of students' activity concerned the modelling of the problem through the development of different strategies such as: defining the rooftop area dimensions to be covered; translating the problem in the three-dimensional space by utilizing the projections of the panels on the rooftop through the use of trigonometric ratios; and examining alternative ways to place the panels and comparing the expenses in each case.

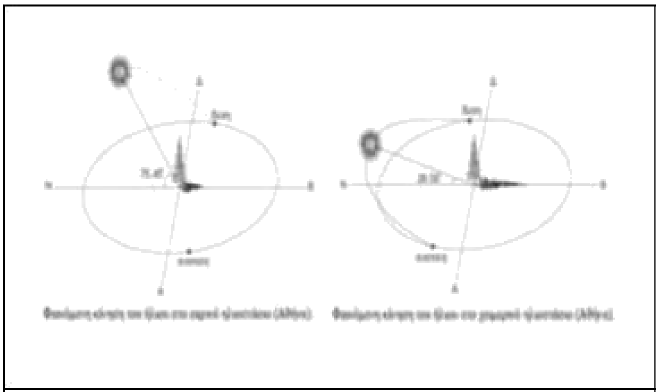

Fig. 2: Diagram of sun's route

In her reflection, Elena realized that the modelling process revealed unexpected students' weaknesses and strengths that she had not noticed in her day-to-day mathematics teaching.

Elena's willingness to integrate the workplace of solar cells in her teaching was followed by a number of actions such as her own familiarization with the workplace context (i.e. discussion with science teachers in the group and one professional) and students' familiarization with this context by emphasizing situational aspects of it (e.g., technicians' installation practices, how panels' energy capacity is related to sun's position). The emerging rich interaction between the two activity systems was unfolded as a multifaceted modelling process involving the use of workplace tools, scientific representations, mathematical concepts, strategies and inquiry processes.

Case 3

This case refers to a mathematics teacher, Katerina, who had about 10 years of teaching experience. She participated in a mascil group with thirteen members (eight mathematics, one technology and four science teachers). Katerina developed a task entitled *Seismologists for One Day* where the students had the role of a seismologist responsible to study main features of a specific earthquake (e.g., the epicentre).

The initial idea of the task was provided by a group member whose specialization was geology. The teacher educator had suggested collaboration between mathematics and science teachers as a way to help them integrate workplace context into their classroom teaching. The geology teacher designed and implemented a similar task in his classroom and shared his materials (e.g., description of the main features of earthquakes and how they are studied by specialists) with the PD group. Katerina was teaching mathematics and geography in the 7^{th} grade (13 -year-old students) in her school, so she found as a challenge to develop a task for integrating the context of seismologists into her teaching by combining mathematics and geography. Her familiarization with the context of earthquakes in the PD meetings allowed her to use it as a context for designing a task for her students.

In classroom implementation, Katerina presented and discussed scientific aspects of the earthquakes based on her knowledge from physics and geography and provided students with authentic data from the National Institute of Geodynamics. The data included: (a) the velocity of p (V_P) and s (V_S) waves and the exact time these waves were recorded in specific seismic stations; (b) the mathematical formula $D = t \cdot (V_P \cdot V_S) / (V_P - V_S)$ (1) where D is the distance (in Km) of the epicentre from the seismic station and t the difference of the time arrivals of the waves; (c) a geographical map indicating all the seismic stations in the country with the corresponding codes (e.g., LKD2 for the seismic station in Lefkada island); and (d) the specific measures recorded in the seismographs of six stations in western Greece (see Fig. 3).

	
Fig. 3: Data sheet from seismic stations	**Fig. 4: Students' actions as seismologists**

In terms of mathematics, the students had to identify that the epicentre of the earthquake was the common point of three intersecting circles whose centers were situated on three seismic stations (Fig. 4). In particular, they had to: substitute given quantities into the formula (1) to calculate the distance of the epicentre from the different stations; model the situation through the use of map scales; conceptualize the calculated distances as radii of different circles; and design them with the use of ruler and compass.

Katerina's attempt to integrate the workplace of seismology into her teaching was followed by a number of actions such as: her own familiarization with the workplace context through discussions with the geology teacher in the PD group and her involvement in teaching mathematics and geography in the same classes; her decision to connect the topic of earthquakes included in geography curriculum with aspects of the mathematics curriculum (e.g., scales, properties of geometrical figures); the use of authentic workplace worksheets and tools; and the assignment of the role of seismologist to the students simulating the actual workplace practice.

In the case of Katerina, we see that the two activity systems are strongly connected and a new object started to be formulated in the intersection of the two systems. In this case, sharing of artefacts, goals and actions between mathematics teaching and workplace emerged through the simulation of the workplace activity in the classroom.

DISCUSSION

Our study builds upon existing research indicating that integrating workplace situations in mathematics teaching is pedagogically sound in two ways: mathematics can be helpful to broaden students' understanding of a situation and conversely, the out-of-school situations provide students the opportunity to deepen their mathematical knowledge. Through the above case studies we explore how the three teachers attempted to integrate workplace in their teaching and what factors facilitated or constrained this integration.

Our results indicate different forms of interaction between the activity system of workplace and the one of mathematics teaching. We address here these forms of interaction by focusing on two dimensions: the process by which the teachers attempted to integrate workplace into their teaching and the factors that supported or hindered this integration.

As regards the process of integration, teachers' goals and actions included their familiarization with the workplace context, students' engagement in modelling activities, students' familiarization professional contexts and their attribution of a professional role. Modelling was a process that triggered all teachers' interest and, as Wake (2015) argues, it operated as a means of building connections between mathematics teaching and workplace. This process was adopted by Elena and Katerina who engaged students in mathematizing workplace situations such as panels' installation and identifying geographical maps' scaling. James, on the other hand, considered the modelling of the bridge construction as an application of mathematics by engaging his students in working in a ready- made model. Modelling in Elena's case was primarily based on a problem solving activity. In Katerina's case it was embedded in a process of simulating authentic workplace practice in the classroom while in James' case modelling remained bounded in the context of school mathematics. Teachers' attempts to familiarize themselves with workplace practices were carried out through either their personal communication with professionals or through their cooperation with science teachers in their PD groups. Familiarization of students with workplace was carried out in the following ways: engaging them in working with authentic contextual or scientific representations such as photos of bridges or diagrams of sun's route or geographical maps; and including in the task aspects of the broader scientific context (e.g. the solar energy). Finally, only one teacher (Elena) assigned her students a professional role (i.e. seismologist) and a task (i.e. to find the epicentre of an earthquake) strongly related to the workplace practice.

The analysis brings to the fore the following categories of supportive factors and constraints that facilitated and/or hindered the interaction between workplace and mathematics teaching: the collaboration between teachers from different disciplines in the PD groups in co -designing a task; the use of exemplary resources and materials provided by mascil; teachers' experiences in teaching subjects related to the workplace broader scientific context; and the rules underlying mathematics teaching. The collaboration between mathematics and science teachers supported the integration in the cases of Elena and Katerina since they co- designed the task with science teachers from their PD groups. This supports recent research findings that acknowledge workplace as a fertile ground for science and mathematics teachers' collaboration (Potari et al., 2016). In case 1, however, James and physics teachers in PD group did not find a ground for co-designing a task. The use of exemplary mascil materials favoured Elena's attempt to integrate workplace in her teaching. Katerina's teaching experiences of geography supported the smooth integration of the workplace practice of a seismologist in her classroom teaching of mathematics. Finally, the established rules of mathematics teaching in upper secondary level, as in the case of James, provided barriers to the integration of workplace in his teaching.

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Investigating the work of teaching geometric proof: The case of a Malawian secondary mathematics teacher

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This paper presents findings from a qualitative case study of a secondary mathematics teacher's work of teaching geometric proof. Data analysed in the paper was collected by interviewing the teacher about the lesson plan and observing the lessons. The results of our analysis of lesson observations as well as interview data indicate that the work of teaching proof construction includes the tasks of 1) initiating, supporting and emphasising common steps in geometric proof construction, 2) selecting, using and involving students in discussions of appropriate examples, 3) exposing student misconceptions in order to facilitate learning, and 4) initiating discussions in order to let students identify errors and mistakes in proofs.

Keywords: work of teaching mathematics, geometric proof, secondary school.

INTRODUCTION

Despite a continually rising interest in research on the mathematical knowledge that is specific to teaching mathematics, Hoover, Mosvold, Ball and Lai (2016) argue – based on their review of literature in the field – that "mathematical knowledge for teaching needs to be elaborated – for specific mathematical topics and tasks of teaching, across educational levels" (p. 18). Scheiner (2015) also argues that investigation of teachers' knowledge at the level of specific concepts is an important issue that needs more attention in the recent literature on teachers' knowledge. The present study aims at contributing to this field of research by investigating the instructional tasks involved in the work of teaching geometric proof.

Euclidean geometry is justified as a part of the high school curriculum in many countries on the basis that it provides many opportunities for proving (Herbst, 2002). A standard geometry proof question is of the form 'Given X, show that Y' with a figure, so to prove means to construct a sequence of argumentation from X to Y with supportive reasons (Cheng and Lin, 2009). According to Cheng and Lin (2009), the crucial point is to decide a theorem or geometric property that links X and Y in logical order and to justify every claim.

Previous research has investigated various aspects concerning mathematical knowledge for teaching proof. For instance, Stylianides and Ball (2008), in their investigation of mathematics teaching in an elementary school in the USA, argue for the importance of knowledge of different kinds of mathematical tasks that involve proving as well as knowledge of the connection between such tasks and proving activity. As another example, Herbst and Kosko (2012) draw upon investigations of classroom instruction and analyses thereof, in their attempt to develop instruments to measure mathematical knowledge for teaching high school geometry. Their measures involve, but do not have an exclusive focus on, knowledge related to proofs and proving. Herbst (2002) describes a scenario in which a teacher and her students together undertook a proof of a claim about angles but later on the teacher took over to provide a formal proof. Herbst (2002) argues that the teacher decided to take over because of her conception of proof as a two-column proving. The conception implies division of labour where the teacher's responsibility is to provide the question and a diagram and to make sure that students produce proofs by the end of the lesson. In another study, Herbst and Brach (2006) argued that the teacher's personal choices of tasks and diagrams depend on their conception that they are responsible for students' learning of proof. These authors focus on proofs and proving in USA high school geometry, and they ask what is going on for the students. In this study, we investigate proving in a Malawian secondary school and ask what is going on for the teacher. In Malawi, secondary school students struggle with constructing geometric proofs. Lack of teacher knowledge is considered to be the main contributing factor to students' failure to construct geometric proofs in Malawi (Malawi National Examinations Board [MANEB], 2013). The present study thus has a local relevance, but we also aim at contributing to the international literature in the field by approaching the following research question: What instructional tasks are involved in the work of teaching geometric proof at secondary school level? In our attempt to respond to this research question, we consider data from a case study of a particularly good secondary school mathematics teacher in Malawi. Before elaborating more on the methodological considerations in our study, however, we provide some information about the theoretical framework for the study.

THEORETICAL FRAMEWORK

Numerous frameworks have been developed in order to describe important aspects of the professional knowledge needed for teaching mathematics (e.g., Hoover et al., 2016), and most of these frameworks refer to the work of Lee Shulman. Shulman (1986) proposed that teacher effectiveness can be viewed as a combination of content knowledge and pedagogical knowledge. He developed a general framework for understanding teacher knowledge that categorised the combination into subject matter content knowledge, pedagogical knowledge, pedagogical content knowledge and curricular knowledge.

Ball, Thames and Phelps (2008) draw upon the work of Shulman in their practice-based theory of mathematical knowledge for teaching (MKT), and we adhere to this framework in the present study. Ball and colleagues focus on two core concepts: 1) the work of teaching mathematics, and 2) the tasks of teaching involved in this work. In fact, their definition states that MKT is "the knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395). When investigating the work of teaching mathematics, they focus in particular on the mathematical tasks of teaching – referring to core instructional tasks that the teacher is recurrently faced with in the work of teaching. These mathematical tasks of teaching constitute the work of teaching mathematics. Examples of such tasks of teaching include giving or evaluating mathematical explanations, and asking productive mathematical questions. In their efforts to design measures of mathematical knowledge for teaching high school geometry, Herbst and Kosko (2012) used the list of generic tasks of teaching from Ball et al. (2008) and developed a list of mathematical tasks of teaching, specifically related to the work of teaching high school geometry. For instance, Herbst and Kosko (2012) discussed the task of selecting appropriate examples for the geometric concepts taught, as well as the task of being attentive to students' misconceptions concerning mathematical practices specific to geometry. Our analysis of the work of teaching geometric proof draws upon these ideas and include attempts to investigate tasks of teaching geometric proof and the knowledge required to carry out such tasks.

METHODOLOGY

This study is part of a larger study on knowledge for teaching geometric proof. In this paper, we report from a qualitative case study. Ritchie, Spencer and O'Connor (2004) explain that the primary defining features of a case study are that it is rooted in a specific context, and it draws from multiple perspectives through either single or multiple data collection methods.

This paper focuses on two lessons presented by a Malawian secondary school teacher called Kim (a pseudonym). Kim is regarded as one of the best teachers in his school, because of his long teaching experience and because his students perform well at national examinations. The two lessons were considered for analysis because they illustrate the work of teaching mathematical proof. The lessons were observed and video recorded on two consecutive days. The duration of the first lesson was 80 minutes while the second lesson was 40 minutes. The teacher shared his overview of the first lesson after teaching the lesson because he was engaged in other duties before the lesson time. On the second lesson, the teacher was able to share his lesson plan before going to the class. These included reflections about the theorem to be proved, how he was going to prove the theorem with students and how he was going to assess students' understanding. The duration of these brief interviews were from five to ten minutes, and each briefing was recorded and transcribed by the first author. Both types of data were analysed separately using thematic analysis as the aim of the study was to capture and interpret sense and substantive meanings in the data (Ritchie et al., 2004). During interview analysis, thematic analysis involved reading transcribed data several times to make sense of it and identifying main points. These points were used as a structure or predetermined framework for deductive thematic analysis of the transcribed video data (Ritchie et al., 2004). The data material was also read several times to determine categories in which every portion could be coded. Although data analysed for the study is only from two lessons and lesson plans, we consider it sufficient for illustrative purposes and for proposing certain analytical generalisations. The data for the study was generated from real-life context, hence considered as rich data (Yin, 2009). As Yin (2009) argues, the goal of case studies is to expand understanding of social issues in their context and generate or generalise theories rather than recording frequencies.

RESULTS AND DISCUSSION

Lesson 1

The aim of the first lesson was to prove a theorem which states that an angle subtended by an arc at the centre is twice an angle subtended by the same arc at the circumference. Kim gave the students a diagram and a statement (see table 1), then asked them to go into their groups to draw similar diagrams and discuss how to come up with the proof.

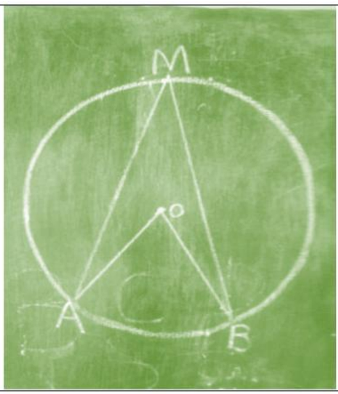
<table>
<tr>
<td>
</td>
<td>Given: a circle with centre O, with arc AB subtending angle AOB at the centre and angle AMB at the circumference. Prove that the angle at the centre is twice the angle at the circumference.</td>
</tr>
</table>

Table 1: Diagram and task given to students for a proving activity

After about ten minutes of group discussions, Kim moved around to check the students' progress. He found that some students were considering the reflex angle instead of the obtuse angle at the centre of the circle for their proofs. As a result, they were using wrong theorems, some groups proved for congruency while others proved for similarity theorem which made them get stuck. When Kim realised that the students were stuck because they were considering a wrong angle at the centre of the circle, he suspended the proving activity and asked the students to measure the two angles at the centre and the angle AMB at the circumference of the circle. When Kim asked the students to report their findings, it was noticed that the students in all groups had agreed that the angle at the centre was the obtuse angle. Students said that they reached this agreement upon finding that it was the value of the obtuse angle which was twice the value of the angle at the circumference. Then Kim asked the students to resume their discussion about proof construction for 15 minutes. During this time, students were able to construct the proof for the theorem using correct constructions and theorem. All groups joined MO and produced it to some point and then used a property of the theorem, which states that exterior angle of triangle equals sum of opposite interior angles to construct the proof. During lesson consolidation, Kim emphasised that there are three crucial points in proof construction: 1) analyse the given angles and see how they can be connected to the conclusion, 2) know the type of construction that will make the required proof to be possible, 3) justify every claim and link the claims to the required conclusion. The emphasis made by Kim mainly focused on geometric proof construction in general (Cheng & Lin, 2009). Later on, Kim and his students discussed how to solve two examples using the theorem that they had proved. The first example involved finding the angle at the centre when given the angle at the circumference, while the second involved finding the angle at the circumference when given the angle at the centre. The circle diagrams in both examples involved a minor arc and an obtuse angle at the centre. Towards the end of the lesson, Kim gave the students the following homework question as part of assessing their understanding of the theorem and its application: calculate the sizes of the marked angles (see figure 1).

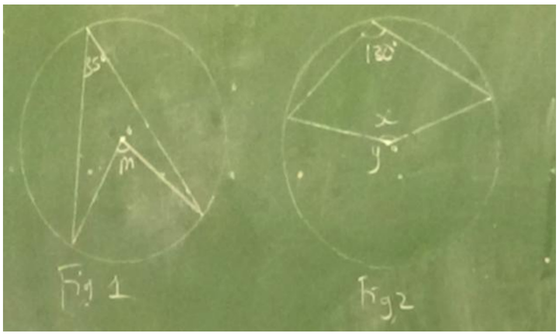


Figure 1. Calculating sizes of marked angles.

After the lesson, Kim reflected about the lesson and the students. He said that the lesson was successful, and he was confident that the students had understood the theorem and the proof. The following extract presents some of Kim's reflections about the students and how they learn geometric proofs.

Kim: If you just start proving without engaging students in an activity like measuring or discussions on how to prove, they just memorise the proof. So to avoid memorisation, I involved the students in discussions. When I found that they were referring to a wrong angle at the centre I did not tell them the angle, I wanted them to find out on their own by measuring the angles. Activities like measuring make the theorem to be established in their brain because they provide tangible evidence that the theorem is true. Apart from measuring I also ask them to prove on their own with my assistance of course, this helps to develop independent thinking, because if I do it for them and they memorise, then the moment the same question comes in a different situation they get confused.

The extract indicates that Kim was conscious about some instructional tasks involved in the work of teaching geometric proofs. We highlight some elements from the teacher's reflections that relate to tasks involved in the work of teaching. First, students can easily get confused if a teacher only tells them how to prove a theorem without involving them in a proving activity or group discussions. The teacher is then faced with a task of asking some productive questions that redirect the students' attention in the right direction. Second, when students construct their own proofs, they understand the theorem and can be able to apply it to any situation. In relation to this, Kim gave the students a statement and diagram and asked them to discuss and construct their own proofs. This involves a task of initiating discussion and reflection in order to help the students become independent thinkers. So, when students were able to come up with proofs, Kim assumed that the students have understood the theorem. When asked why he did not only give the students a statement and let them explore the diagram to be used before they construct the proof, Kim said that the focus of the lesson was on proof construction and not diagram construction. Giving students only a statement would make them spend more time on discussing how to draw the diagram and limit their discussion of how to construct the proof. This indicates that Kim gave the diagram because he wanted to fulfil his obligation of making sure that students construct a proof by the end of the lesson (Herbst, 2002).

Lesson 2

The following day before the lesson, Kim said that he was going to make some revisions because, when he was marking the students' homework, he discovered that most of the students were unable to find values of marked angles in the second figure (see figure 1). When asked why the students were able to calculate value of angle in the first figure but unable to do so in the second figure, Kim responded:

> It was because they thought that angle at the centre can only come as an obtuse angle but now when it came as a reflex angle they got confused. The problem was that the students got used to the figures where the minor arc was subtending angles, now when I turned the figure upside down so that it should be the major arc subtending the angles students got confused. So in class I will let the students expose their errors by asking them to answer same question before I give them their exercise books. I believe that when an error has been made by a student and another student recognises and corrects it, then it means that other students who made similar errors will learn from that. After the presentation from the student, I will ask the whole class if they have any problem or have identified a mistake with the solution. This will make the students to analyse their friend's work and find if there is something to correct.

In the extract, Kim explains what he thought was the cause of students' failure to answer the second question and how he was going to help students clarify their misconceptions. We make three observations from the extract. First, Kim thought that the students' misconception is that the angle at the centre is always an obtuse angle and that is why they were confused with the second question. This refers to a task of identifying students' misconceptions when failing to complete a proof. Second, Kim thought that it is necessary to let students expose their misconceptions. This relates to a task of letting a student expose his errors and having other students correct them. This, according to Kim, becomes a learning opportunity for students who made similar errors. The third observation relates to a task of initiating student discussion in order to let them identify errors and mistakes in a proof.

During the lesson, Kim began by writing the homework on the chalkboard. Then he asked for a volunteer to find the value of M for the first diagram. Many students raised their hands and the one who was nominated said that, “angle M = 70° because angle at the centre is twice angle at the circumference”. The rest of the students agreed that the answer was correct. Then Kim asked for volunteers to find angle X and Y in the second figure. Only two students raised their hands, and Kim asked both of them to write their solutions on the chalkboard. The table below presents what the students wrote on the chalkboard.

Student 3	**Student 4**
130° = X (opposite angles of a kite) *X = 130°* *X + Y = 360°* (angles at a point) *Y = 360° – 130°* *Y = 230°*	*Y = 2 (130°)* (angle at the centre is twice the angle at the circumference) *Y = 260°* *X + Y = 360°* (angles at a point) *X = 360° – 260°* *X = 100°*

Table 2: proofs constructed by students 3 and 4.

The following segment of a lesson episode presents a conversation between Kim and the students discussing the answers that were presented on the board.

Kim: Let us start with the first one, is this correct?

Students: Yes (some of them), no (others)

Kim: Is there anywhere given that this (pointing at the quadrilateral in figure 1) is a kite?

Students: No (chorus answer)

Kim: So why are you indicating the property of a kite?

Student 3: It’s because of the way it is drawn.

Kim: Are you telling me that you just look at a diagram and judge that it is a kite?

Students: No (chorus answer)

Kim: You need to either use the information that you are given, or prove first and convince us that it is a kite. But here we are not told that this is a kite and you haven’t proved to convince us that this is a kite. Why did you not use yesterday’s theorem to find x and y?

Student 3: There is no angle at the centre which is connected to the angle at the circumference.

Kim: Class is it true that there is no angle at the centre in diagram 2?

Student 2: There is angle x and y at the centre

Then Kim asked the students in the class to analyse solution 2 and say if it is correct. Only a few students said that it was correct; most students remained silent. The segment below presents Kim's explanation in trying to help the students to understand the theorem and its application to the question.

Kim: These two angles here (pointing at angle x and y) are at the centre. You needed not to be confused because this diagram (pointing at the second diagram) is the same as this diagram (pointing at first diagram). The only difference is that the radii in this diagram (pointing at second diagram) are facing upwards and that makes the angle at the centre to be bigger or to be a reflex angle while the radii in this diagram (pointing at first diagram) are facing downwards making the angle at the centre to be smaller or to be an obtuse angle. So you find the value of *x* the same way you found value of M and then proceed to finding *x* using property of angles at a point. Any questions?

Student 5: Yes, what if somebody takes y as x? What I am trying to say is that if we compare the radii, the M angle in question 1 looks like the x angle in question. So since we take the one which is smaller, I am thinking of x as equal to 2 times 130°.

Kim: No we take the one which is facing the direction the angle at the circumference is facing. It can either be the obtuse angle or the reflex angle. Had it been that there was an angle down here (pointing at the bottom of the circle) facing x then we would say that x is 2 times that angle.

In the segment of the lesson episode, student 3 thought that there is no angle at the centre that is connected to angle at the circumference. So the student decided to use the property of a kite. From the conversation between Kim and student 5, it is also noticed that the student appeared to have a misconception that when given two angles at the centre, they need to pick the one which is the smallest to be the angle at the centre. This agrees with the first point observed from interviews: the misconceptions displayed by the students seem to develop from the fact that they were only exposed to diagrams that contained an obtuse angle at the centre. The second point observed from interviews is that students expose their misconceptions when they explain their solutions. Kim asked student 3 to explain why he decided to use the properties of a kite. The student said that it is because of its appearance. This misconception might have developed because the student did not notice that they could apply the theorem learnt in the previous lesson. So Kim is pinpointing the main cause of the student's mistake with an aim of making the student reveal his misconception. Kim appears to think that this might be a learning opportunity for other students. The third observation made from interview analysis is that students seem capable of analysing a solution and correcting mistakes. In the lesson segments, it has been noticed that students were able to analyse the solution to question 1 and comment that it was correct. However, it is noticed that there were divisions among students concerning the solution given by student 3. When some students analysed the solution they said it was correct, while others argued that it was not correct. The students who responded that the solution was correct might have the same misconception as student 3. Kim clarified the misconception held by most of the students after student 4 explained his solution. Persistence of misconceptions was noticed even after Kim explained that an angle at the centre can be of any type and size. Student 5 revealed his misconception through a question which also centred on comparison of sizes of angles at the centre in the two figures. The misconception exposed by student 5 made Kim realise that the meaning of the theorem was missing in his explanations. He then clarified the misconception by emphasising the direction of the angles. What was missing in Kim's emphasis is that the angles can be subtended by either the major arc or the minor arc. When the angles are subtended by a minor arc, the angle at the centre is small, but when the angles are subtended by a major arc, the angle at the centre is big. Although the teacher believed that students mainly reveal their misconceptions through their explanations, it was also noticed that there were some misconceptions that were revealed through the questions that they asked.

CONCLUSION

In our analysis of the work of teaching geometric proof in a Malawian secondary school classroom, we have identified several potential mathematical tasks of teaching. These are tasks that we have observed in the lessons, and the teacher himself reflects upon these tasks before or after the lesson. First, we observe that Kim's work of teaching geometric proof involves the task of initiating, supporting and emphasising common steps in geometric proof construction – even by making these steps explicit to the students in his instruction. Second, we observe that the task of selecting appropriate examples that illustrate geometric concepts taught, identified by Herbst and Kosko (2012), also appears to include using the examples and involving students in discussing these examples. Third, Herbst and Kosko (2012) discussed the task of being attentive to student misconceptions. From our observations of Kim's work of teaching, we suggest that this task can be extended to exposing student misconceptions in class in order to facilitate learning among other students with similar misconceptions. Fourth, we suggest that the work of teaching geometric proof involves the task of initiating discussions in order to let students identify errors and mistakes in a proof. Unlike Herbst and Kosko (2012), our analyses of the tasks involved in the work of teaching geometric proof have not been done in order to support development of measures of this particular aspect of MKT. Instead, we suggest that such analyses might productively inform a further development of an elaboration of the knowledge needed for teaching specific mathematical topics at different levels – as called for by Hoover et al. (2016).

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Posters

Concept Cartoons created by prospective primary school teachers

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The poster will present creation of Concept Cartoons as possible way to development and refinement of prospective primary teachers' PCK and issues related to proper argumentation.

Keywords: problem posing, Concept Cartoons, argumentation.

BACKGROUND

The poster is a follow up to a presentation on CERME 9 (Tichá & Hošpesová 2015), which discussed posing problems of a given structure as a way of enhancement the quality of prospective primary teachers´ content knowledge. We develop the idea of problem posing further and we work with the tool called "Concept Cartoons" (CCs). CCs are a special type of tasks in which alternative statements on situation and/or of a problem are given. Solver has to decide which statement is right/wrong and justify this decision. In primary school teacher education, we use this type of tasks to promote "teachers´" view on the problems and necessity to use valid arguments in its analysis. An example of CC created by the prospective teacher is in Fig. 1. The poster aims to answer the question: How prospective teachers´ PCK and their awareness of the need of proper argumentation occurs in their creation of CCs?

METHODOLOGY OF THE STUDY

In this study, 35 prospective teachers created the CCs and made their didactical analysis (what is the objective of solution of the task, estimation of the correct and incorrect (but plausible) solution of the problem). The CCs were presented and reflected in a group of students, that means: the prospective teacher presented his/her CCs, reacted on the comments, especially explained the uncertainties and formulated the task more preciously. Then we coded the created CCs in terms of use of argumentation, and characteristics of the task.

SELECTED FINDINGS

Initially, the prospective teachers created mostly the tasks with simple arithmetic content. During a joint reflection of created CCs they started to realize how important is to get in touch with various methods of argumentation and gradually changed the character of the created tasks: (a) from easily solvable tasks, "textbooks´" type, often incorrectly formulated; (b) to tasks that are challenging for pupils, not ordinary, of colorful settings (graphs, tables, ...), enable different solution methods, approaches, require explanation and further consideration (open problems).

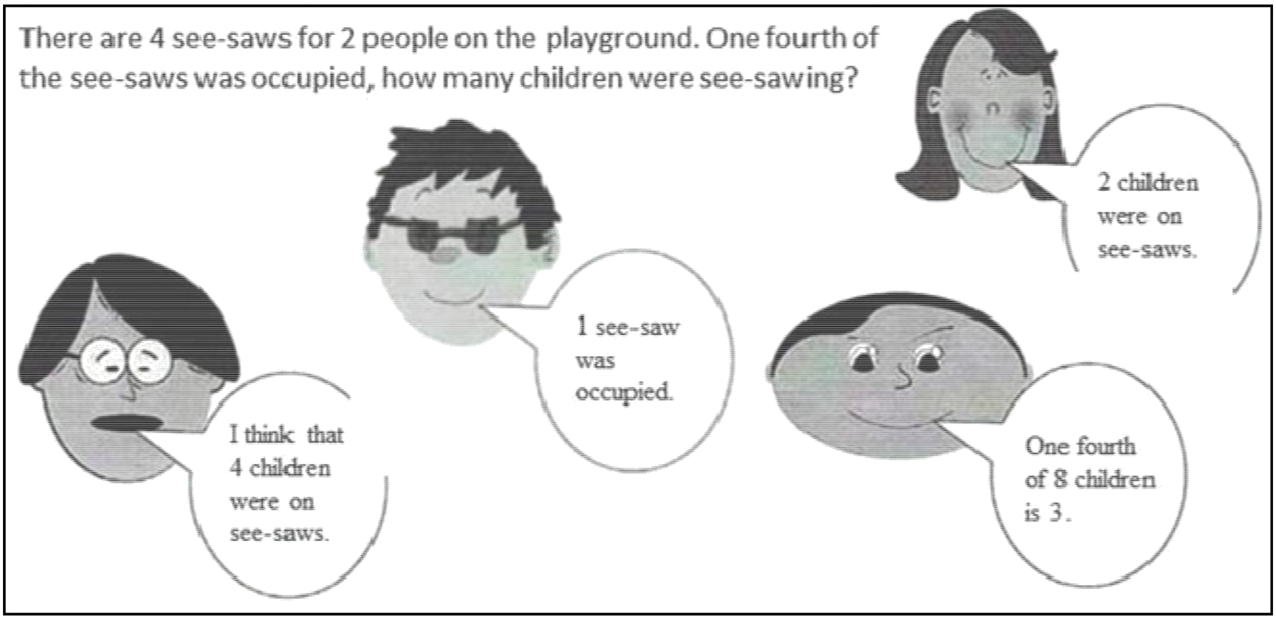

Fig. 1: CC created by a prospective teacher

In analyzing the PCK of participating prospective teachers we concentrated especially on (a) proper formulation of statements in bubbles, their correctness (for example in Fig. 1: does the statement "1 see-saw was occupied", that there are 2 children on it?); (b) comments to statements in following joint reflection and awareness of their potential ambiguity. Our experience showed that prospective teachers are not aware of the weaknesses of CCs created by them and they realized only in the course of joint reflection, that the statements can be grasped differently. The student, who created CC in Fig. 1, for example wrote in her commentary: *"If I were assigning the same problem in the second grade, I would simplify it for example in this way: There were four see-saws in the playground. One quarter of them was used. How many children were there on the see-saws?"* She did not see, that she revealed further misconceptions. That confirmed the diagnostic potential of CCs.

Creation of CCs and especially joint reflection motivates prospective teachers to deeper thinking about mathematical content of primary mathematics.

NOTES

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Design of Hypothetical Teacher Tasks (HTT) to Access Pre-service Elementary Teachers' Knowledge on Rational Numbers

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In this poster, we present the idea of the PhD-project of the first author: to use hypothetical teacher tasks (HTT), designed and analysed with the anthropological theory of the didactics (ATD) to study pre-service elementary teachers' mathematical and didactical knowledge on rational numbers, with a comparative focus on Indonesia and Denmark. An example of a HTT is also included.

Keywords: hypothetical teacher tasks (HTT), teachers' knowledge, rational numbers

Studies about pre-service and in-service teachers' knowledge on rational numbers have been done by many researchers, with various approaches (Ma, 1999; Hourigan & O'Donoghue, 2013). Most of the studies focus on teachers' mathematical subject matter knowledge (MSMK), content knowledge (CK) and pedagogical content knowledge (PCK) on rational numbers, based on cognitive paradigm - that is, focusing on individual knowledge.

Meanwhile, our study takes a different approach, based on the *anthropological theory of the didactic* (ATD) introduced by Chevellard (1992). In this framework, knowledge is considered as institutionally situated, and is studied through *praxeological reference models*. We develop *hypothetical teacher tasks* (HTT) about rational numbers based on the ATD framework (Durand-Guerrier, Winsløw & Yoshida. 2010). The aim is to develop a framework to study pre-service elementary teachers' mathematical and didactical knowledge on rational numbers. The framework will be applied to comparative pre-service elementary teachers from Indonesia and Denmark. We choose both countries because Danish students performed significantly above the OECD average compare to Indonesian students who performed significantly below the average (see the result of PISA 2012, OECD, 2014). We assume that the result has a link to teachers' mathematical and didactical knowledge. The subjects for this study are pre-service elementary teachers from the University of Riau, Indonesia and from the Metropolitan University College, Denmark. The results of this study are expected to contribute to develop our knowledge about teaching rational number models to pre-service elementary teachers.

HYPOTHETICAL TEACHER TASKS (HTT)

Five *hypothetical teacher tasks* (HTT) are designed to access pre-service elementary teachers' mathematical and didactical knowledge about rational numbers. The HTT are designed based on a *praxeological reference model* for the practical and theory blocks of both mathematical and didactical praxeologies. A practical block is formed by a type of tasks (T) and corresponding techniques (τ), and the theory block consists of a technology (θ) and a theory (Θ). So, each of HTT can be described based on two kinds of four tuples (T, τ, θ, Θ). As an example, we outline one HTT about multiplication and division of decimals. The task is given to the pre-service elementary teachers as follows:

As a teacher, you ask students to compute the following as homework: a) $0.25 \cdot 8 = \cdots$, b) $8 \div 0.25 = \cdots$. At the next meeting in the class, a student notices that when he enters $0.25 \cdot 8$ into a calculator, the answer is smaller than 8, and when he enters $8 \div 0.25$, the answer is bigger than 8. He is confused with this answer and thinks that the calculator must be broken. What can you do to help such students understand this result? (*discuss in pairs in 8 minutes, use the space below if necessary, and write your ideas to support the discussion*)

From the task we can derive *praxeological reference model* for a mathematical task (T) and for a didactical task (T^*) as follows:

T = given a decimal number a and an integer b, calculate $a \cdot b$ and $b \div a$.
T^* = given a type of task T (where $0<a<1$, $b>0$) explain determine what to do as a teacher to make students understand why $a \cdot b <$ and $b \div a > b$.

and then a-priory analysis can also be described for both tasks contains techniques (τ), technologies (θ) and theories (Θ).

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Developing the design of a role-play in a professional development

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The poster shows how a role-play conducted by teachers can be developed through the methodology of educational design research. The role-play represents a classroom situation and is an instructional activity as part of a two-year long teacher professional development program. The implementation focuses on some of the features identified as important in an effective professional development. The aim of the role-play is to practice and deepen mathematics teachers' knowledge and skills of stimulating mathematically gifted pupils higher order thinking. After the development of the role-play, the study can move forward and include; teachers' change in instruction and pupil learning in order to study the effectiveness of the role-play as a part of a professional development program.

Keywords: Professional development, educational design research, mathematically gifted pupils.

THEORY

Regarding the effectiveness of teacher professional development (TPD) Garet, Porter, Desimone, Birman, and Yoon (2001) identified; duration, collective participation, content focusing, active learning, and coherence as important features. To study the effectiveness of a TPD-program, Desimone (2009) included the identified features in a conceptual framework. The framework includes a link between four elements; the TPD-program, teacher knowledge, teacher instruction, and pupil learning. The elements are embedded in a context, here TPD on education of mathematically gifted pupils, as an important mediator and moderator. This poster presents how a role-play, an instructional activity belonging to the first element, the TPD-program, can be developed and thereby most likely be effective. Educational Design Research (EDR) (Gravemeijer & Cobb, 2013) is used as methodology in the development process of the role-play. The development of the role-play focuses on some features identified as important for TPD (Garet et al., 2001).

BACKGROUND AND AIM

It is shown that teachers have little knowledge on how to support mathematically gifted pupils (e.g. Leikin & Stanger, 2011). Several researches recommend TPD on gifted pupils: on those pupils' educational needs and on methods of how to develop their learning (e.g. Persson, 2015). Sheffield (2003) suggests special questions to ask mathematically gifted pupils to develop their higher order thinking. In the role-play some participants in the TPD-program present a mathematical task acting as teachers in a regular classroom, and the other participants act as pupils trying to solve the task. The aim of the role-play is to develop all participating teachers' (n=17) knowledge and skills on how to develop higher order thinking of mathematically gifted pupils. The research question guiding this poster is: How can a role-play, in a TPD-program, be developed to improve teachers' knowledge and skills of using special questions to improve higher order thinking in mathematically gifted pupils?

EXPECTED OUTCOME

The use of EDR (Gravemeijer & Cobb, 2013) to develop the role-play means that the role-play is repeated in an iterative process, after each iteration retrospective analysis is performed. In this study the analysis focuses on the five features identified as important for an effective professional development (Garet et al., 2001). The result of the analysis is expected to guide the development of the role-play.

The poster presents two iterations of the role-play and how the analysis has led to changes aimed to improve some of the features identified as important for professional development. Furthermore, the poster show how EDR might be used to develop, and thereby most likely improve, the effectiveness of the role-play as part of a TPD-program. Data in this study are video recordings from the two iterations.

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Glimpses of practice: pre-service teachers' evaluation of students' answers

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In initial teacher training, the contact with teaching practice is scarce, fragmented and often disjointed. Therefore, the challenge is to build tasks that constitute approaches to practice and place pre-service teachers in situations similar to the ones they will have to deal within the future. One of the tasks teachers have to perform in their practice is the evaluation of answers given by students. In this context, we developed a study in which we analyse how pre-service teachers (PST) evaluate answers given by students in solving certain tasks.

Keywords: mathematical teacher training, elementary mathematics, mathematical knowledge for teaching.

BACKGROUND

There are plenty of studies focusing on teacher's knowledge, trying to identify its components, analyse their characteristics and/or understand its complexity. One of the most influential works concerning teachers' knowledge was developed by Shulman (1986).

Much work has been done ever since (e.g., Ball, Hill & Bass, 2005; Gomes, 2003; Ma, 1999). In one of those works, developed by Hill, Ball & Schilling (2008), the notion of "mathematical knowledge for teaching," appeared. This conceptualization highlights the mathematical knowledge that teachers need to carry out their work as teachers and considers the specificity of specialized content knowledge.

Despite evidence linking teacher knowledge with the mathematical performance of their students (e.g,, Baumert et al, 2010), there is still no consensus regarding the content, nature and type of such knowledge. Therefore, further research into how can teachers be helped in order to increase/develop their knowledge is still needed.

METHODOLOGY

This study aimed to analyse how pre-service teachers (PST) evaluate answers given by students in solving certain tasks. To this end we considered the following research questions: (1) How do PST assess students' answers?; (2) What remediation strategies are suggested in case of wrong/inadequate answers?

Given the nature of the study, a qualitative approach was adopted. The study was developed within a course of Didactics of Mathematics, taught by the researcher.

This course is part of a Masters Degree designed to prepare future elementary school teachers (children aged 6 to 12). 21 PST participated in the study. These PST were faced with tasks consisting of inadequate or wrong answers to certain questions. They were asked to comment on the answers and evaluate them. They were also asked to propose remediation strategies.

Data was obtained through PST's work and interviews with three PST after the end of the course. Data was coded with a specific coding schema.

In this presentation we will focus on one task, related to the properties of rectangles and squares.

SOME RESULTS

Even though the data analysis is not yet complete, we can advance some results. These type of tasks were unfamiliar to PST. Generally, PST reveal major difficulties in assessing the answers given by students, being unable in many cases to identify the errors or argue about their possible causes. The type of arguments used to make the assessment was often common and had no mathematical nature.

This study shows that it is essential to challenge PST with tasks related to their future practice. In particular, facing errors students make urges PST to develop their specialized knowledge for teaching (Hill, Ball & Schilling, 2008).

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A Malawian preservice secondary school teacher's mathematical knowledge for teaching equations

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This qualitative study investigated a preservice secondary school teacher's mathematical knowledge for teaching equations. Data were generated using video recorded interviews and analysed using thematic analysis. The preservice secondary school teacher – who was among the best in his class – showed evidence of mathematical knowledge for teaching, but his knowledge seemed limited to knowledge of facts and procedures. These findings may inform preservice teacher educators about the content of mathematics teacher preparation.

Keywords: Mathematical knowledge for teaching, preservice teacher, secondary school, equations.

A body of research indicates that mathematical knowledge for teaching influences the quality of teaching and student learning, but less is known about how this knowledge develops (Hoover, Mosvold, Ball, & Lai, 2016). Although there appears to be general consensus that mathematics teachers need to know the content in ways that surpass the knowledge of educated people outside the teaching profession (Ball, Thames, & Phelps, 2008), more research is needed in order to investigate the different types of knowledge needed for teaching particular mathematical topics at particular levels (Hoover et al., 2016). From her review of literature on teaching and learning of algebra, Kieran (2007) suggests that researchers have barely begun to investigate the knowledge needed for teaching algebra. In light of this, the present study investigates a Malawian preservice secondary school teacher's mathematical knowledge for teaching algebra – and equation solving in particular – in a Malawian teacher education context. We approach the following research question: What mathematical knowledge for teaching is displayed by a Malawian preservice secondary school mathematics teacher?

THE STUDY

The purpose of this study was to explore a Malawian preservice teacher's mathematical knowledge for teaching equations. Ball's et. al. (2008) mathematical knowledge for teaching model and Kriegler's (2007) algebraic thinking model informed the study. Data were generated from one preservice secondary school teacher, Dinga Pseudonym), using semi-structured task based interviews. Dinga was a Diploma in education student at a college of education. He was in his final year of study the time the data were being generated. He was a particularly bright student. The interview lasted for one and half hours. We analysed the data using thematic analysis (Powell, Francisco & Maher, 2003). Themes were developed *a priori* and *a* *posteriori*. During the initial coding, some themes that were not in the theoretical framework were emerging from the data. Some of these themes were incorporated into the theoretical framework, while others were regarded as separate categories of the characteristics of Dinga's mathematical knowledge for teaching equations.

RESULTS AND DISCUSSION

Dinga solved an equation using two approaches – by factor method and the quadratic formula – thus indicating common content knowledge. He also indicated some specialised content knowledge, but his knowledge seems to be mainly knowledge of facts and procedures. Knowledge of several solution methods is important, and this procedural knowledge is an important prerequisite for a mathematics teacher, but we suggest that Malawian teacher education could benefit from focusing more on developing deeper common content knowledge, stronger specialised content knowledge as well as problem solving skills among preservice secondary mathematics teachers.

As far as pedagogical content knowledge is concerned, Dinga displayed some knowledge of analysing students' errors and anticipating their possible misconceptions, but he appeared unprepared to apply such methods in his teaching of algebra. It appears to us that Malawian teacher education might benefit from focusing more on developing pedagogical content knowledge among preservice secondary teachers.

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Preservice teachers' ability to assess student thinking and learning

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Successful teaching requires the ability to deal with various kinds of student heterogeneity. Therefore, assessing and promoting students' learning should be a core aspect in teacher education. Video has been suggested as a means to promote (preservice) teachers' abilities to assess student thinking and learning. In our project, video-analysis of student learning activities (e.g., while solving tasks) was employed into a mathematics and physics teacher education course. Various data were gathered from preservice teachers (PSTs): Written analyses of transcripts, questionnaires on self-concepts and experiences, videos of the PSTs working on assessment tasks and interview data. Data were used to investigate how the PSTs develop abilities to as-sess student thinking and learning as well as to analyse the impact of the courses.

Keywords: assessment, professional vision, classroom videos, student heterogeneity, mathematics and physics teacher education.

THEORETICAL FRAMEWORK

Teachers' ability to elicit and interpret student thinking is assumed to be a necessary prerequisite for adaptive instruction (e.g., Kang & Anderson, 2015) and has to be established during teacher training. Video can serve as a tool to promote especially preservice teachers' (PSTs') ability to assess student thinking and learning (e.g., Santagata & Guarino, 2011), even though research results vary due to the different conceptualisations and methodological approaches (e.g., Stahnke et al., 2016). In order to investigate how PSTs analyse different videos and how instruction on criteria to analyse student thinking is used, we utilised video ourselves as an assessment tool (similar approach conducted, for instance, by van Es & Sherin, 2008). With this design, we are able to pay specific attention to PSTs' processes of noticing and reasoning and how these change during instruction.

SETTING AND RESEARCH QUESTIONS

The study, supported by the German Telekomstiftung, has been conducted in two educational courses for preservice mathematics and physics teachers in the mid-dle/end of their initial teacher training. These courses are aiming to promote PSTs to value a student perspective, to establish criteria with which student thinking and learning can be assessed and to theorise how assessment and adaptive instruction are linked. During the courses, videos of student learning activities (e.g., working on physics experiments or solving mathematical tasks) were analysed and discussed by PSTs in small groups and with the entire group of PSTs. For PSTs who study the combination of physics and mathematics, the two courses follow each other, starting with physics education. While both courses deal with video analysis, they differ in two aspects. 1 – Order: In mathematics, the PSTs start with the development of instruction and then focus on assessing student thinking, whereas in physics assess-ment is targeted first and then instruction is designed. 2 - Involvement in videos: In mathematics, the PSTs analyse instruction that they had developed and taught to small student groups whereas in physics the analysis neither involve the PSTs as teachers nor does it focus on teacher activity in general.

We are aware that our research cannot be considered an intervention study. However, we expect that the similar general approach in the two courses (video as a tool to train PSTs) and the differences (order, involvement) help to explore the following research questions: (i) How do the PSTs approach videos under these different con-ditions? (ii) How do their approaches change while they learn? (iii) How do the PSTs employ criteria established for analysis within and across the courses? (iv) Which of the particular components of each course are experienced positively by PSTs, which are, to them, not contributing to their professional development?

DATA, ANALYSIS AND FIRST RESULTS

N=69 PSTs were enrolled in the first main study (45 in physics; 24 in mathematics, including 9 attending to both courses). Data gathered at different points of the courses comprise PSTs' written analysis of transcripts, questionnaires (e.g., self-concept, experiences), videos of the PSTs working in small groups on assessment-related tasks in about 30% of the courses and interview data. Qualitative data has been cod-ed with a specific coding schema, questionnaire data has been analysed with Rasch.

Preliminary results from the PSTs' pre-post-analysis indicate a shift from a focus on content and its accuracy and a more general pedagogical focus to using criteria closely related to domain specific pedagogical knowledge (questions i/ii). Criteria introduced within the courses were employed more frequently and also consistently across both courses but seem to narrow down the focus (question iii). The PSTs re-port that they have experienced the courses as relevant and feel more competent to assess student thinking and learning with a theoretical foundation (question iv).

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Researching school development programs through classroom culture

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Two rather similar ongoing large-scale projects are in this poster combined aiming to research possible tensions between them.Methodology for research are suggested.

Keywords: Social norms, Socio-mathematical norms, Assessment for learning.

BACKGROUND

Assessment for learning (Sv: Bedömning för lärande)

Since 2013 there has been a large-scale project initiated in Lunds kommun[1] encouraged Dylan Wiliams approach assessment for learning (Wiliam, 2011). All teachers are through regular group discussions supposed to (1) gain professional development, (2) develop daily teaching practice and (3) through this contribute to higher student achievements. The process involves systematic reflection in groups, based on a clear structure, supported by discussion leaders.

The Mathematics Boost (Sv: Matematiklyftet)

Between 2013-2016 the Swedish National Agency for Education is launching a 649M Skr curriculum-based professional development project. The Mathematics Boost aims to improve mathematical classroom teaching at scale in all of Sweden. This project is also based on a clear structure supported by discussion leaders and a digital platform. In cycles teachers have (1) collective planning with colleagues, (2) individual classroom teaching and (3) collective reflects with colleagues upon the classroom instruction. In Lunds kommun almost 180 of the teachers in mathematics have participated.

POSTER

Both projects are rather similar in approach even if they originated from different policymakers. They need to be explicitly connected even though they both aims to develop teaching practice at schools. In this poster I want to show one way how they might be connected using Cobb and Yackels (1996) classrooms-norms. I am aiming for investigation of norms from a student perspective. What kind of tensions will there be between these two projects, from a student perspective, expressed through Cobb and Yackels classrooms-norms? When teachers are implementing new teaching inventions, there will be potential tensions between students´ view of norms and teacher´s intention of new supporting norms (Wester, 2015). To be aware of these potential tension will

[1] Lunds kommun is a municipality in southern Sweden with a population of 120 000.

makes it possible to explicit encourage developments of new suitable social and socio-mathematical norms.

THEORETICAL FRAMEWORK

To be successful in school mathematics, it is not enough to know the mathematics. It is also necessary that students master the rules and values governing practice inside the mathematical classroom. If students do not share understanding of these, they likely not to be able to participate in the activities as intended. This will affects students' opportunity for learning. Cobb and Yackel (1996) express these rules and values through classroom norms, divided into social- and socio-mathematical norms. Existing classroom norms are possible to be investigated through observations and interviews.

I will mainly connect social norms to assessments for learning and socio-mathematical norms to the Mathematics boost. From there, they certainly have an influence on each other and also support each other in developing teaching practice. But there will probably also be tensions between them which will become hinders to successful development.

METHODOLOGY

Data are collected through videotaping 3 different cycles in Mathematics boost according to one participating teacher. Each cycle including recording teachers collective planning with colleagues, planned activity in classroom environment, and collective reflection of the outcome. Before and after each teaching activity, teacher will be shortly interviewed. Close to the teaching activity, some of the students are also interviewed in a focus group.

This research will take its starting point from student interviews. Analyzing interviews through Cobb and Yackels framework, what kind of classroom norms do students express? How will these norms relate to their teacher´s intentions of supporting norms, expressed in interviews, observations, and teachers collective planning and reflections? From there, what kind of tensions will there be between these two kinds of development projects?

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Tracing diagnostic strategies of teachers and pre-service teachers: an explorative study on interactive video-simulations

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The main focus of the research presented with this poster is on specific diagnostic situations and on diagnostic processes of teachers and pre-service teachers. Following a process-oriented approach, we are about to develop an interactive video-simulation tool to identify diagnostic strategies and to track pre-service teachers' development of their individual diagnostic strategies (e.g. in terms of attention and noticing, diagnostic sensitivity, and the ability to interrelate their general knowledge of content and students, KCS, to specific cases).

THEORETICAL FRAMEWORK

Following the competence model of Blömeke et al. (2015), situation-based diagnostic competence can be regarded as a continuum, including a performance dimension of teachers. The authors emphasize that competencies may be understood in the sense of observable behaviour. Furthermore, the dimension of performance is influenced by affect-motivational skills and cognitive ability as well as by situational-specific skills, such as perception, interpretation and decision making.

Set in the project *diagnose:pro* and based on the works of Reinhold (2015), we chose a process-oriented approach towards pre-service teachers' diagnostic competence instead of focussing on accuracy of teachers' judgements. Previous research on micro-processes and strategies in diagnostic processes used re-interviews with pre-service teachers about one-on-one diagnostic interviews they had conducted shortly before. Findings of these studies provided insight into individual approaches to diagnostic situations as they led to a model of strategic elements in pre-service teachers' diagnostic proceeding and suggested types of diagnostic strategies (e.g. the strategy of a "*concluding collector*", "*descriptive collector*" or a "*branched interpretation*") (cf. Reinhold, 2015).

The medium of video-vignettes seems appropriate for research concerning diagnostic processes in simulated (classroom-like) situations and provides interesting findings on diagnostic types (cf. Hoth et al., 2016). However, this methodological approach leaves the observer outside of the diagnostic situation. As a consequence, the aspect of interacting with and reacting on the child's activities and his or her comments on verbal stimuli or hands-on- manipulatives is completely neglected. To do justice to this fact, an interactive component needs to be implemented to involve facets of situational steering of the situation – i.e. by inquiry or provision of visual aids. Therefore, we develop interactive video-simulations which enable to influence the course of the video.

RESEARCH QUESTIONS

- To what extend can interactive video-simulation be an appropriate tool to trace individual micro-processes in diagnostic situations? Which constraints do we have to take into account?
- What kind of varieties concerning diagnostic strategies occur when pre-service teachers are engaged in the interactive-video simulation?
- How do the plots of several consecutive video-simulations differ when the pre-service teachers take part in specific courses which aim at the development of their diagnostic competence?

METHOD

In the explorative study, paths in interactive video-simulations are protocolled in the sense of plots and pop-up textboxes ask for arguments for the choices made by the diagnosing (pre-service) teacher. Furthermore a (final) statement, based on noticed incidents of various sources for interpretation is requested. Resulting data is analysed via qualitative analyses based on Grounded Theory methodology.

For the purpose of the explorative study, the mask for the tool is filled with a fictive case example of a child that shows symptoms of problems in numeracy learning (e. g. missing insight into the place-value system). University students who solely participate in the *Advanced Module: Primary Mathematics Education* at the University of Leipzig and those who additionally participate in a specific program
[1] are asked to engage in the interactive video-simulation. Excerpts from the results and the first findings will be presented during the conference.

NOTES

1. This project links theory and practice concerning early identification of children with specific difficulties in numeracy in an extended phase of grades 1 and 2.

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Visualising connections

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The poster presents an idea of how Bernstein's theory about horizontal and vertical discourses can be used with variation theory. In the long run, the goal is to develop mathematics education in Malmo taken teachers and students into account as well as to create a research based teacher development program. The starting point is that pedagogy need to be visible, meaning that the connection between what happens in the classroom and how mathematical knowledge is evaluated and graded is visible to the students. The purpose of the poster is to get feedback on the ideas and get ideas of how this issue can be dealt with.

Keywords: vocational education, variation theory, visible pedagogy, action research.

BACKGROUND

In order to close the achievement gap when it comes to mathematics there is a need to make the pedagogy visible (Dahl, 2014). Visible pedagogy, drawing on Bernstein (2000), means that the connection between what happens in the classroom and what is evaluated and graded must be clear to the students.

Mathematics education in upper secondary school in Sweden follows a national curriculum, launched in 2011 (Skolverket, 2011). Mathematics in this curriculum is divided into "core content" (what each course should cover) and "knowledge requirements" (what should be graded). Core content is for instance geometry and algebra while knowledge requirements are expressed as different competencies, for instance problem solving and conceptual understanding.

Furthermore, upper secondary school in Sweden is divided in different programs, some preparing for further education and some preparing for a specific vocation, such as building and construction or nursing and caring. It is well known in Sweden that vocational programs attract students who come from lower socio-economic backgrounds and who is at risk of become low achievers in mathematics (Broady & Börjesson, 2005). For these different kinds of programs there are different mathematics courses with some similarities but also some crucial differences. For the vocational programs (but not for the other programs) it is stated under "core content" that most of the mathematics should cover relevant mathematics for the "subjects typical of a programme" (Skolverket, 2012 p. 4). But, there is nothing about this in the "knowledge requirements". Because of the structure, there is a risk that the pedagogy becomes invisible to the students attending a vocational program.

A way to deal with this risk, is to use a material for supporting evaluation and grading for the vocational programmes that is constructed by Skolverket (Swedish national agency for education). These are under construction and the idea is that it should be optional for teachers to use them. Therefor we do not know to what extent they will be used or how this will affect the visibility of the pedagogy.

Another way to deal with this issue is to connect the mathematics education for the vocational programmes more tightly to the subjects typical of a program. In an on-going project in Malmö, the third largest city in Sweden, this is just the case. In this project, teachers in mathematics and teachers in building and construction work together and teach mathematics and construction simultaneously in the workshop.

These two ways of dealing with the risk of invisibility, deal with two different aspects of the issue: the first one is about the connection between the "core content" and the "knowledge requirements", while the second way deals with connection that students need to be able to make between the different aspects of mathematics, the "pure" mathematics in knowledge requirements and the "applied" mathematics in the core content for vocational programs.

In previous research (Dahl, 2014) Bernstein's division into horizontal and vertical discourse (Bernstein, 2000) were used in order to analyse the mathematics curriculum and the national tests in Sweden. In order to bring this research into the classroom, textbooks and teacher-made-tests will be analysed by as well as planning lessons with the teachers and classroom observations. In these analyses we will use both Bernstein's division into horizontal and vertical discourse, as well as variation theory (Marton, 2014). The aim is not to combine the two, rather to use different tools to unfold the students' opportunities to learn. It has been argued that teachers need an explanatory framework that can shed light on how their actions in the classroom affect student learning and help them to discover the features that make a difference to student learning in the classroom (Wernberg, 2009) It is crucial that it is the students' opportunities to experience the connection that needs to be in focus.

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Resistance seen as possibilities during professional development

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Previous studies have been approaching the relationship between a successful professional development and changes in the teaching of mathematics. This change is regarding the understanding, both in terms of the mathematics itself, but also the teaching of the subject. However, the potential resistance towards change hasn't been discussed to the same extent. The purpose of this study was therefore to investigate in what way collegial discussions could contribute to a change. This study will illustrate how 61 preschool teachers' understanding of mathematical activities constitutes in the preschool and preschool class during a professional development.

Few studies have been done focusing on professional development courses for preschool teachers who are working with younger children within the research field of mathematical education (Tsamir, Tirosh & Levenson, 2011). Previous research has suggested that preschool teachers perceive mathematics only to be about counting and measuring (Clements & Sarama, 2007). More broadly there is an argument made in favour of encouraging preschool children to think and make many mental relationships rather than to teach them specific subject content. Counting, measuring, patterns and geometry are mathematical contents and do not necessarily include the expectation of mathematical thinking, such as Playing (Bishop, 1988). This is in contrast to the Swedish preschool curriculum, which emphasises mathematical thinking. In a background document to the curriculum, mathematics in preschool is discussed on the basis of Bishop's (1988) six mathematical activities. Playing is the mathematical activity, which deals with aspects of mathematical thinking. Bishop (1988) considered Playing as characterized by three components, *thinking hypothetically* (imagining a potential action to take in the game and is the beginning to think abstractly), *modelling* (abstracting something for reality) and *abstracting* (identifying the relevant features to focus on within a situation), guessing, estimating, assuming or adopting. The role of playing in education is a major concern of early childhood educators and so even in Sweden. Playing has a long history in the preschool curriculum in Sweden, which could be the means of that the preschool teachers are unlikely to naturally connect it with mathematical thinking. There are traditions that suggest that when children play, they learn. As a result, it is possible that Swedish preschool teachers have difficulties understanding

Playing as a mathematical activity. However, teachers have different perceptions about their current teaching methods in different situations and therefore respond differently to a professional development.

METHOD

This study has, with inspiration from Engeströms (2014) Expanding theory, investigated to what extent resistance towards changes appeared, as well as how they, during collegial conversations, contributed to changing the views of preschool teachers on how mathematical activities are being constituted in the preschool. By using the tertiary contradiction of resistance, the empirical material consisted of the teachers completing written documentation were discussed. The question they wanted answered was what kind of resistance was evident during the changed understanding in that playing can be seen as a mathematical activity.

RESULT

Cultural and historical dimensions were, in combination with the perception that mathematical situations will occur as soon as children play, shown to be the main reason behind the resistance towards Playing. This result revealed itself in a way in which several documentations were done consisting of playing initiated by the children where the preschool teachers did not want to interfere. In other documentations it consisted of some playing initiated by children but including some material that was perceived as mathematical. The mathematical item did in turn encourage the children to count.

DISCUSSION

Despite the resistance in the beginning of the course, the teachers developed changes to perceive children play even as a mathematical learning situation.

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Assessment’ practices to regulate teaching: A theoretical approach

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This poster aims to present a theoretical framework that allows us to study how mathematics teachers adjust their teaching from the use of assessment for learning in the classroom.

Keywords: Assessment for learning; Teaching learning; Assessment´ practices; Mathematics teachers; Reflection.

Formative assessment encompasses all activities performed by teachers and/or their students, which provide information to be used as feedback to regulate the activity of teaching and to support learning (Black & Wiliam, 1998). Formative assessment has been a growing industry in recent years (Black, 2005) when the intention is to promote learning, but the teaching regulation has not received equal attention. It is therefore necessary to understand the way mathematics teachers seek to integrate the assessment´s practices to regulate teaching, an essential component of teaching learning (Ball & Even, 2009)

According to Black and Wiliam (2009), the effectiveness of formative assessment depends on the learner´s ability (student and teacher) to answer the following questions: “Where the learner is going?”; “Where the learner is right now?”; and “How to get there?”. A formative assessment practice depends on the teachers’ involvement in an inquiry and construction of knowledge cycle in which they identify student learning, the knowledge and competencies of teachers as professionals, the needs of students and they improve their learning (Timperley, 2014). A central aspect is the teachers’ capacity to collect relevant information from the students and to be able to interpret it properly. Without this step, the success of the cycle is immediately threatened. It is also necessary to engage students in new learning experiences, assess the impact of these experiences and return again to the cycle.

Reflection is considered as imperative for teachers’ learning (Ponte & Chapman, 2016). In this study we will consider the following components: the *reflexive posture*, the *reflect capacity* and the *nature of reflection* (Jorro, 2006). The reflective posture can be *containment, testimony and questioning*. The capacity to reflect is distinguished at three levels: *reproduction; interpretation; and critical*. The nature of the reflection, according to Jorro (2006), should be retrospective on the developed activity and integrate also an assessment dimension.

We adopted as theoretical framework a cycle that connects the questioning and the construction of teacher´s knowledge to engage students in formative assessment experiments, collect and analyze their productions, and reflect with colleagues and expert to regulate teaching.

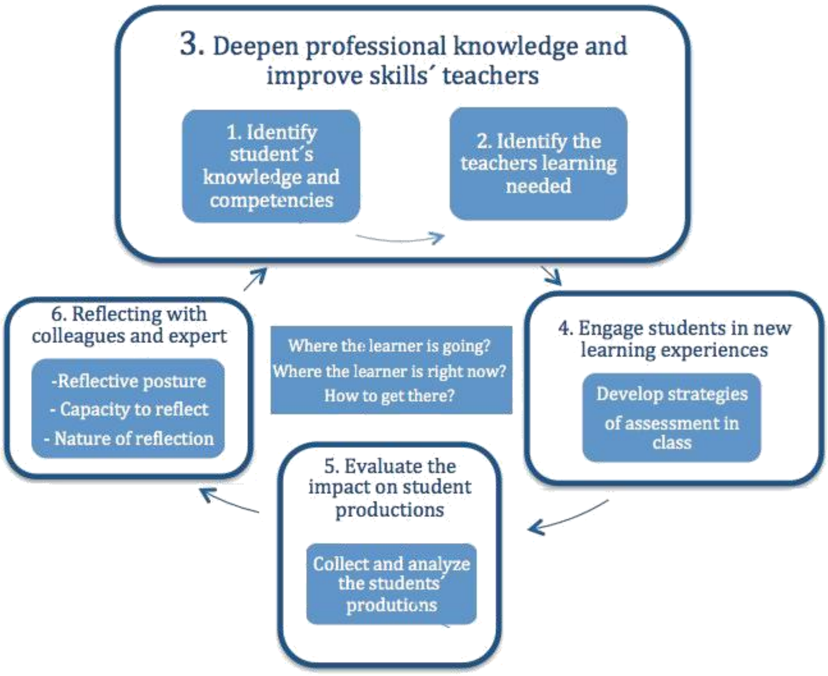

Table 1: The regulation teaching cycle

The next step of our study is to apply this theoretical framework to empirical data and to improve it from the results obtained

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Linking face-to-face and distance phases in professional teacher training – an explorative research

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A combination of face-to-face and distance phases in longer lasting teacher trainings is said to foster professional development. Yet, little is known about ways to link these two phases effectively. The poster gives insights into the work of a professional learning community (PLG) of the German Center for Mathematics Teacher Education (DZLM) that investigates practical examples in this context. The aim is to identify and describe quality criteria for linking tasks.

Keywords: continuing professional development, teacher training, design principles, sandwich-model, DZLM.

The German Center for Mathematics Teacher Education (DZLM) is an organization providing teacher training in mathematics. Among others, special Continuing Professional Development (CPD) courses for multipliers and in-service teachers are offered. These are based on six design principles identified as relevant for effective CPD (cf. Barzel & Selter 2015). The favored structural framework for these courses is the sandwich-model which combines alternating input with practical try-outs and reflection phases throughout the whole course (cf. Rösken-Winter et al. 2015). This type of training combines at least a sequence of two face-to-face sessions and an intermediate distance phase (see Fig. 1.).

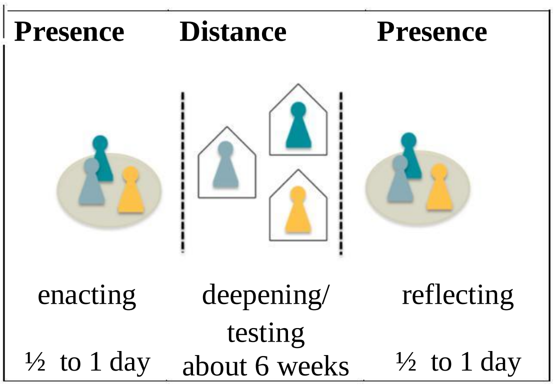

Figure 1: Sandwich-model of two face- to-face sessions and an intermediate distance phase (cf. DZLM 2015b, p. 4, translated by the authors)

Compared to one-day courses with only one face-to-face session, it can be expected that teacher training courses lasting several days could in general have positive effects on the level of acceptance as well as on the level of knowledge and beliefs (cf. Lipowski 2011). Thus, the altering between presence and distance phases can be identified as an essential aspect for the quality of teacher training (cf. DZLM 2015).

With regard to the conception and implementation of teacher trainings it can be noted that an effective link between face-to-face input and practical try-outs in the distance phases can be particularly challenging. Due to a lack of research in this field, a professional learning community (PLG) has been established to work on this topic in the context of the DZLM. The aim is on the one hand to investigate exemplary linking tasks connecting face-to-face and distance phases, on the other hand quality criteria of these linking-tasks shall be derived. With these aims in mind the following questions will guide an exploratory research:

- Which characteristics should linking tasks have?
- How should linking tasks be introduced in order to motivate participants?
- How could the participants' experiences with linking tasks be reflected and how could these experiences be integrated in the whole training course?

For this purpose, different methodological approaches were pursued: Linking tasks were analyzed, teacher trainings were observed and a questionnaire was developed and as well used.

The poster will present guiding questions, the research concept and the developed methods. Based on that, first key findings are presented and design ideas for further research will be discussed.

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Which mathematical contents make a good a primary teacher? Identification of specific contents and levels of school- related content knowledge relevant for the professional development of pre-service teachers

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The poster will be showing first ideas regarding the professional development of mathematics pre-service primary teachers. In order to adapt university courses in a way that they support a thorough development of school-related content knowledge relevant for future primary school teachers, specific mathematical contents and their levels need to be defined. On the basis of an interdisciplinary literature review and in line with various interviews among experts related to teacher training (professors, school teachers and mentors) and corresponding curriculum analyses, beneficial structures and contents of a lecture "Introduction to Arithmetic" are being determined.

Keywords: pre- service primary teacher education, professional development, school-related content knowledge, Arithmetic, PSI.

OVERALL OUTLINE OF THE PROJECT

Research in teacher professional development has shown that content knowledge cannot always be linked to its didactics or general pedagogical knowledge, which could eventually result in inadequate teaching practices (Wahl, 2006). Especially in Germany, this lack of coherence is particularly promoted by a strict division of university courses into exclusively content related lectures and rather detached didactical courses (Blömeke et al., 2004). For mathematics, the relevance of combining both kinds of knowledge required for teaching has been shown empirically (Blömeke et al., 2008), with COACTIV emphasizing the special role of content knowledge for the teaching of mathematics (Kunter et al., 2011).

Studies focusing on the dimensions of professional knowledge in school and university settings (e.g., COACTIV, KiL, TEDS-M, LMT) have shown that teachers' professional knowledge can be divided into different types of content knowledge (knowledge of the curriculum in school, content knowledge at university level and school-related content knowledge). However, they fail to address what specific mathematical contents are required for teaching mathematics successfully.

The quality initiative project at the university of Potsdam (PSI) tries to close this gap in research, focusing on desired contents and levels of mathematical content knowledge in the course "Introduction to Arithmetic" for primary teachers from two perspectives: A curriculum analysis of school curricula as well as university curricula serves as a descriptive instrument. In a complementary normative approach, experts in teacher education are being interviewed. From that, the overall concept of which adaptations are necessary in teacher training courses at university is derived.

In the semester to come, the current lecture “Introduction to Arithmetic” will be adapted to implement the findings of the previous interviews and analyses. Afterwards, a thorough evaluation will show which further adaptions are necessary. The lecture will then again be redesigned.

The poster will show the underlying concepts of a suitable school-related content knowledge concept and will introduce first ideas of which mathematical contents and actions can lead to a better development of such. It is also meant to serve as a prompt for discussions and will ideally help to acquire even more experts in the field of primary teacher education in mathematics.

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Learning to observe mathematical learning in lesson studies

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This poster deals with lesson study in pre -service teacher education. In particular how to prepare, carry out, and reflect upon observations of pupil learning. Observation is of crucial importance to the lesson study process, and here we present a study of observation features which enable or hinder fruitful lesson study.

Keywords: Lesson study, observation techniques, pre-service education.

BACKGROUND

A key element of lesson study is the joint observation of a lesson, and the entire lesson study process will not work as a means to improve teaching if the participants are unable to extract relevant information from the research lesson. Although the issue of performing structured observation to some extent is dealt with in many guidebooks for lesson study, e.g. Lewis and Hurd (2011); Stepanek, Appel, Leong, Mangan, and Mitchell (2006), it continues to be a pitfall of lesson study (Chokshi & Fernandez, 2004), something which is particularly evident in relation to pre-service teacher education (Bjuland & Mosvold, 2015), which will be our focus here. The basis for obtaining useful evidence from a research lesson begins with the lesson plan, and it is by no means a trivial matter to prepare a plan which enables pupil learning to be observed. Indeed to do so, the pre-service teacher requires knowledge about observational techniques (Artzt, Armour-Thomas, Curcio, & Gurl, 2015; Star & Strickland, 2007) as well as *awareness* and *noticing* (Mason, 2002; Mason & Davis, 2013; Scherer & Steinbring, 2006). While substantial research has been carried out in the general field of observing pupils' learning processes and teachers' pedagogical practice, little is known about this in the particular setting of lesson study

RESEARCH QUESTIONS

How do pre-service teachers observe didactic and pedagogical practice during research lessons and how do they look for specific qualities in this practice? Are certain observational methods recommendable for lesson study in mathematics with pre-service teachers?

CONTEXT

The research questions were investigated through the design, implementation and evaluation of a course for 20 pre-service elementary and lower secondary teachers in Copenhagen labelled: "Developing the didactics of mathematics using observational tools and techniques". The course consisted of a mix of lectures and lesson studies enacted in teams of the participants, whose findings were communicated in a final written report.

METHODS AND THEORETICAL FRAME

Data was obtained in the form of audio-recordings from the lectures and lesson study processes, which together with the written reports forms the basis for an analysis of elements which either furthers or hinders good research lesson observation practice. Criteria are developed both from existing theoretical and experimental literature (cf. Background section above) as well as more inductively from the data itself.

FINDINGS

We present salient observational techniques which have special characteristics when utilized in research lesson observation, and which are suitable to act as shared focal points of the corresponding reflection session. The poster will exemplify the findings by juxtaposing two of the pre-service teacher lesson study teams, whereby highlighting the particular conditions for an observational technique to be successful in the study of mathematical learning.

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