

What (Should) Teachers Assess in Elementary Geometry?

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Abstract. Instructional practices based on teachers' personal, physical and instructional/institutional resources deeply influence the processes of assessment. How do these resources shape a situation of assessment on an open task and how does assessment of pupils' answers indicate particular teachers' resources? This study examines these questions on the topic of reflection and symmetric figures in elementary geometry in grade three in a primary school in Berlin by connecting the theory of didactic situations (TDS, Brousseau, 1997) with components of teachers' knowledge by Levenberg & Patkin (2014). The empirical insights offered in the paper may, further on, inform a selection of content for the purpose of a research-based design of a PD program.

Keywords: geometry, assessment, theory of didactic situations, teachers' personal resources, professional development.

INTRODUCTION

The complexity of mathematics teaching practices has already been addressed from several aspects as for example, teachers' knowledge and beliefs, or the role of personal, physical and instructional resources and their relationships (e.g. Topics of the TWG 19 and 20 at CERME 9). “Teachers' actions and meaning-making as they relate to instruction, including task selection and design, classroom communication, assessment, etc.” are of a particular interest in the current debates about teacher education (Call of the ERME TC3). This paper focuses on assessment as an action which is deeply influenced by instructional practices and based on *teachers' personal, physical and instructional resources*. Although the Call distinguishes between teachers' “personal resources, on the one hand, and physical and instructional/institutional ones, on the other hand” (TWG 19), I would rather refer to them as three different kinds of resources. By personal resources I mean primarily teachers' knowledge in mathematics besides other (according to Levenberg & Patkin, 2014), whereas by physical resources I refer to physical objects regardless if they are natural or man's creations. Instructional resources may be of diverse nature, for example, the content of the curricula and textbooks, or mathematical visualizations and geometric representations of mathematical concepts in narrow sense. This certainly does not state that the physical resources cannot be used for instructional purposes, on the contrary, physical objects or pictures and drawings of them are often used in classroom instruction. A textbook is also a physical artifact by itself though its existence is meaningless if it is not used for instruction, and therefore, I consider it as an instructional resource. I argue for my insistence on such triple distinction of the resources by an epistemological and didactic analysis of three dimensions of mathematics, about which I talk in the next section. Further on, I use the

Theory of Didactic Situations (TDS) to show the presence of these connections in a mathematics classroom related to assessment about reflection in grade three. Then, I explain how such assessment could indicate the quality of teachers' resources. The paper finalizes with a suggestion for the 'what' question by van den Heuvel-Panhuizen (2005) and Prediger et al. (2015) about the content which “teachers should learn and multipliers need to know” (p. 233).

THEORETICAL FRAMEWORK

Teachers' Personal, Physical and Institutional/Instructional Resources Regarding Reflection and Symmetric Figures

Are there (at least) three different kinds of mathematics (contemporary, school and every day) and if so, how do they differ from each other, is a question tackled by Civil (2002; see also Sfard, 1998). Although such a strong differentiation may seem artificial, it may be valuable for investigating the interplay between different types of resources for teaching. Here is an attempt to exemplify such investigation by concepts in geometry as reflection and symmetric figures.

In contemporary mathematics, the basic Euclidean isometries, reflections (both, mirror or line reflection and point reflection-mirror for 2π radians), rotations, translations, and combinations of these, are distance-preserving geometric transformations in two- or three dimensional Euclidean space. Any congruence transformation can be represented as a composition of maximum three reflections, and therefore the reflection is considered to be a fundamental concept. Further on, a figure is called symmetric if there exists a reflection which maps it to itself. This exemplifies the importance of symmetry, which has different meanings throughout different mathematical contents (e.g. a property, a relation) and everyday contexts. Symmetry is “not only a key idea in geometry [...] but also a key organizing principle in mathematics” (Jones, 2002, p. 131). While a use of formal concept definitions is a necessity for the introduction of concepts at the university level mathematics, an everyday application may be sufficient for an initial introduction of the same concept in school.

Symmetry is a model topic for study in school. It is embedded in reality, it is conceptually simple for younger pupils, and concrete examples abound. Its study yields many useful results, applicable in the real world. Equally important, it is a rich subject whose study is an excellent practice ground for mathematicians and scientists (Ellis-Davies 1986, p. 30).

In everyday life, through the senses for vision and touch children enter the world of mathematics (without any formalities, e.g. definitions). For example, a painting is beautiful because of the 'hidden' symmetric properties in it (regardless of the person's mathematical knowledge about reflection). Meanwhile, in primary school mathematics, e.g. in Berlin, reflection and symmetry as a property of figures are introduced even in grade one, through everyday contexts and with the aid of physical and instructional

resources. A common example is the use of life creatures in the nature, e.g. butterflies (physical resources) or objects made of paper, e.g. stars, hearts, etc. The teaching and learning is also supported by the use of different tools such as mirrors or special rulers (instructional resources). Axes of symmetry are mentioned in grade three and pupils are expected to identify and draw axes of symmetric figures and also reflect figures. In comparison, according to the K-12 Standards for Mathematical Practice in geometry for grade 4, pupils have to “recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts [...], identify line-symmetric figures and draw lines of symmetry” (NCTM, 2000). The differences in the curricula and standards open a new question for a possible consideration of the institutional as a fourth type of resource (separate from the instructional, in contrast to the statement in the Call, TWG 19).

The Role of the Teachers in Considering Different Resources in Assessment

In reality, absolute symmetry as it is defined in mathematics does not exist. To what extent are primary school teachers aware of the validity of this statement and the co-existence of the different facets of mathematics (as a science or as knowledge to be applied and studied in and out of school)? What is the role of the teacher in bringing together all of them in the classroom? How far can the teacher push everyday mathematics in the classroom instruction and assessment? We may debate that the pupils get motivated when they engage in everyday situations but how much mathematics do they really learn on the way? Do the teachers sometimes forget all the mathematics that they not only have to teach but also assess? For example, how much are the primary school teachers familiar with the specific content knowledge about geometric transformations, in particular reflection?

I start the discussion mainly pointing out *teachers' knowledge* as a part of *teachers' personal resources* (which also include beliefs and identity but are not discussed in this study). Namely, Levenberg & Patkin (2014, p. 94) identify *six components of teachers' knowledge*: *knowledge of the subject matter*, knowledge of the learner, background knowledge of the school environment, curricular knowledge, didactic knowledge and self-knowledge. In this paper, I prefer to talk about *teachers' content knowledge* in mathematics. To refer to this particular component, i.e. teacher's individual knowledge of the subject matter, I use the term “knowing” (which belongs to a person) and distinguish it from (generally available) “knowledge” (borrowed terminology by Brousseau, 1997 and González-Martín et al., 2014) meaning mathematical knowledge about reflections as described in the previous subsection. I also use the rest of the components as indicators in the analysis of teacher's personal resources (in order to answer the research question in this study).

Now, a backward look at the above questions allows a presentation of some answers already existing in literature. Geometry is perceived as a subject matter in which many teachers demonstrate a knowledge gap and is therefore difficult for them to create and

evaluate rich mathematical tasks (Ribeiro, 2011). In particular, “the meaning of symmetry is not precisely defined” (Leikin, Berman & Zaslavsky, 1998 and see also Leikin, 2003) which may be a source for teachers' difficulties. Despite such insufficiency, there are teachers who do not comprehend this issue in its fullness (Levenberg & Patkin, 2014, p. 97). Further on, in relation to the first question in this subsection,

“During geometry lessons, the use of all types of visual displays, pictures, presentations and movies, which show geometry in the pupils' environment (both natural and hand-made), constitutes a bridge between the concrete and the abstract” (Patkin & Levenberg, 2012, p. 14).

Regarding to the rest of the questions, it seems that a *Didactic Situation* (TDS) is a suitable model to analyze the teachers' roles in bringing all three resources at one place, and moreover, not only in instruction but also in assessment. Namely, an *open task* (I come to this term in the next subsection) asking for naming symmetric figures ensures several adjusted conditions of a *Situation* (see González-Martín et al., 2014, p. 118). For example, it targets reflection and symmetric figures as mathematical knowledge to be gained and it does not make any reference of the targeted knowledge (e.g. by stating a particular figure). Further on, pupils may name inadequate solutions which at a particular time of cognitive development or accessible mathematics may be accepted as adequate. Then, the solutions named by one pupil may be discussed and verified by the others. An exemplary task fulfilling these conditions of a DS is given and elaborated in the section Findings and Discussion (see Figure 1).

Next questions that arise are how do teachers evaluate pupils' achievements and understanding about symmetric figures in grade three, what kind of data provide pupils' answers and finally, how could the assessment influence further learning for both parties?

Assessment through Open Tasks

Due to the limitations of this paper, I would not go into details about how are open or open-ended tasks defined and classified or which are the advantages of their usage (for some specifications see Kwon, Park & Park, 2006; Yee, 2002). Rather, I emphasize that, in this study, assessment through open tasks is not seen as a process of collecting data about pupils' achievements, instead as a process of learning and in particular beneficial for both the pupils and the teachers. In addition, future steps in a larger study may aim to design rich tasks for multi-age classes in the first three years of schooling. A question which arises from this initial study is the following. How should a teacher evaluate answers on an open task, when it appears as one, on a test which has been created by authorities and not by her/himself? A particular test can be considered as an instructional/institutional and not as a personal resource because it has been suggested to the teacher as accompanying material to the textbook in use. If the pupils' answers

coincide with the examples in the textbook or with those discussed by the teacher during the lecture, they will certainly be evaluated as correct. In such case, it seems that pupils are granted for memorizing and repeating of what has already been stated during the instruction, which is certainly, not one of the most important goals of the teaching of mathematics. If they do not coincide, they may be evaluated as incorrect, which is another threat.

RESEARCH QUESTIONS AND METHODOLOGY

Research question (RQ). How could a teacher's assessment of pupils' responses on an open task in grade three indicate teacher's personal resources, in particular, teacher's knowing of the mathematical knowledge about reflection and symmetric figures?

The RQ goes beyond analyzing how does a teacher assess pupils' knowledge about reflection and line-symmetric figures. Namely, the investigations on the RQ do not only look at how does a teacher decide about true or false answers or distribute points. They also try to examine how does this distribution indicate the teacher's awareness of the co-emergence of the three aspects of mathematics in connection to the three kinds of resources in an assessment situation and in particular what is the teacher's knowing of the mathematical knowledge about mirror symmetry. This indication relates to the components of the teachers' resources.

For the analysis regarding the RQ in this theoretical paper I refer to the *core elements of TDS* (Brousseau, 1997 and González-Martín et al., 2014), where the DS is an assessment situation on an open task by analyzing the “relationships between students, a teacher and a *milieu*” (González-Martín et al., 2014, p.119). The *milieu* is defined as “the set of material objects, knowledge available, and interactions with others, if any, that the learner has in the course of said activity” (González-Martín et al., 2014, p.119). In the DS in this study, “the set of material objects” is consisted of the personal, physical and instructional resources (used during instruction) on which the pupil can reflect on when solving the open task in the DS (during assessment). “Knowledge available” refers to the coherence between the three types of mathematics that the teacher has (or has to a certain amount) brought in the classroom. This directly relates the teachers’ personal resources, specifically their knowing of the subject matter. Since the DS is an assessment situation, there are no direct “interactions with others” when solving or evaluating the open task. Yet, “the learner” in this Situation is not perceived as “a learner”, rather learners, i.e. both the pupil and the teacher. I continue the discussion based on the core elements of the TDS in the next section.

FINDINGS AND DISCUSSION

The open task in this DS is one out of five tasks on a written assignment about recognizing and drawing line-symmetric figures in the third grade primary school in Berlin. The analysis does not only focus on the way a teacher has assessed pupil's

answers and compares the assessment with a plausible one. Moreover, it tries to identify some indication which may lead to answering the RQ. The task is the following [1].

Task: Name 3 figures or representations from your surrounding, that are symmetric (Figure 1) [2]. Pupil's answers are: a circle, a heart and a triangle (Figure 1).

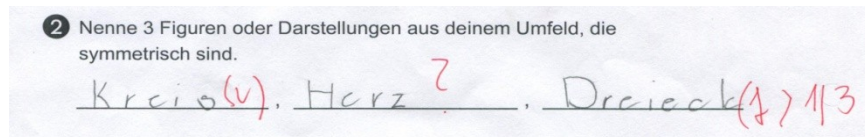


Figure 1: Assessment on an Open Task about Symmetric Figures

The task is an open-ended task for the reason that it does not have one fixed answer and it requires divergent thinking, although the formulation of the task may necessitate additional information. For example, are “figures” physical or mathematical objects, and are “representations” any kind of drawings (e.g. a drawing of a butterfly) or mathematical geometrical concepts (e.g. a square) or pictures and visual displays of physically existing objects (e.g. a window)? Further on, what does a “surrounding” (environment) for a pupil mean? Is it the classroom, or the school yard with all natural and man-made physical objects or the mathematical world the pupil lives in, or something else? These questions are relevant for an eventual design of rich tasks (steps 4 and 5, according to Prediger et al., 2015 which is discussed in the section Conclusions).

The teacher evaluated the solution with 1 out of 3 points (Figure 1), accepting only the “**circle**” as a correct answer. A short analysis of the textbooks for grade one to three “Einstern” 1, 2, and 3, which were in use, shows that the circle does not appear as an example of a symmetric figure in any of them. I see the infinite number of axes of symmetry of the circle (and the complexity of teaching it) as a reason for this absence of the circle as an example of a symmetric figure from these textbooks. The question is whether it has been discussed by the teacher. If not, this answer shows a possible higher pupil's knowledge than what is expected at this level of education. This refers to a core TDS tool named as a *didactic contract* which is “the implicit set of expectations that teacher and students have from each other regarding mathematical knowledge...” (González-Martín et al., 2014, p.119). Namely, if the teacher is aware of the absence of the circle as a symmetric figure from the corresponding curriculum for grade three, and the reasons therefore, (*curricular knowledge* – one of the six components of teachers' knowledge according to Levenberg & Patkin, 2014, p. 94), he/ she may acknowledge this pupil's answer. The *adidactic level* of the DS (the other level is called a *didactic level*, according to Artigue, 2000) concerns pupil's possible engagements involving interactions with the *milieu* and, as this answer “a circle” shows, involves maybe a posteriori enrichment which does not necessarily involve relationships between the pupil and the teacher but between the pupil and the *milieu* alone. The rest of the tasks in the exam show that figures with finite number of axes of symmetry have been discussed during instruction but with no more than two axes. Figures as n -sided regular convex polygons having reflection symmetry in n axes, for n greater or equal to three do not

appear among the tasks in the exam and it is not clear whether they have been discussed during instruction (regardless using physical and instructional resources or not). This assumption brings into focus *teacher's personal resources* and maybe the knowing of the knowledge about the symmetry group of an n -sided regular polygon being a dihedral group of order $2n$ (n reflections and n rotations). Finally, figures with infinite number of axes of symmetry as the circle seem to have remained out of the instructional scope and therefore the above-mentioned desirable acknowledgment seems to be grounded.

The third answer “**a triangle**” was rejected to be correct by the teacher. This indicates teacher's content knowledge. Yet, what is a triangle into the pupil's mind? The topic about reflections is on the beginning, while the one about the existence of three different kinds of triangles is at the end of grade three. Short analysis of the textbooks for grades one to three, which were in use in the classroom, shows that there are only a few irregular triangles (dominance of prototypes). This fact is enough reason to think that the pupil perceives the geometric shape of a triangle as being either equilateral or isosceles, and as a consequence a symmetric figure. Therefore, the answer may be considered as an adequate one at this particular moment, although such answer cannot be accepted as correct in the fourth and any other later grade. This indicates possible insufficient teacher's *curricula* and *didactic knowledge* but moreover *knowledge of the learner* (components by Levenberg & Patkin, 2014). In the vocabulary of the TDS, the feedback provided by the *milieu* (pupil's *validation* - “a triangle” being a symmetric figure) shows that the *milieu* which was in the current use of the DS appears to have been “insufficient to ensure didacticity in terms of adding new pieces of knowledge” (González-Martín et al., 2014, p. 119). This, further on, means that the *institutionalisation* as “the ultimate phase of a Situation in which the teacher brings the students back to the *didactic level* and makes the necessary links with the aimed knowledge and provides the semiotic tools to present this knowledge, especially if these were not produced” (González-Martín et al., 2014, p. 120) does not seem to have taken place.

The second pupil's answer is “**a heart**” which is also evaluated as an incorrect one by the teacher, probably because it is a non-visible object. The reasons for stating such probable interpretation are the following. Looking at other pupils' answers as “a window”, “a board”, “a door”, etc. which have been considered as correct by the teacher, it seems that physical resources in the classroom have been discussed a lot during instruction. It may be the case that the most of the pupils have been granted for reproducing such examples. However, the symmetry of these physical objects is really discussable and it is a question if this has been spotted by the teacher and pointed out to the pupils. Namely, a window may as well be used as an example of a non-symmetrical object, because the handle “ruins” the symmetry. Even if we consider the window without its handle, its ‘parallel’ sides are not exactly equal in length in reality. It is the ideal (imaginary) rectangular shape of the physical object “a window” which is symmetric, and not the realistic object itself. So, have counter examples been discussed?

I now discuss further examples which may appear as pupils' answers on the given task. How can a teacher evaluate an answer as “a house”? (S)he cannot know if the pupil has an image of a symmetric or a non-symmetric house (the problem of reality vs. representations). Such answers can not immediately be assessed and, as a consequence, seek deep teacher's involvement in asking additional questions or requesting drawings from the pupils. In such situations the role of the teacher in bringing all 'three aspects' of mathematics comes into focus. Depending on the imagination and creativity of teachers pupils develop interest and improve their understanding in geometry (Patkin & Levenberg, 2012, p. 14).

Dilemmas related to the task, that come on my mind now, are can a teacher expect an answer as “there are no such objects in my surrounding” or “symmetric objects do not really exist in our surrounding”. How would the teacher evaluate such answers? Could such answers be viewed as signs for pupils' giftedness in mathematics, and likewise the answer “a circle”? These are questions which require further analyses.

Further on, does the utilization of physical resources for instructional purposes make the school mathematics real to an extent that it is not possible to 'avoid' them (see Boaler, 1993 and a more extreme view by Lockhart, 2009)? Does not it seem that their usage may sometimes even prevent eventual early insights into mathematics? What is it with those pupils (like the one in this study) who are already able to think of and manipulate with abstract mathematical objects (e.g. circles, triangles) but are further 'forced' to use concrete objects (physical resources, e.g. windows)?

CONCLUSIONS

Although “many concepts of symmetry are not firmly established before twelve years of age” (Genkins, 1975), the answers of the participating seven-year old pupil in this study show her/his understanding of reflection and symmetric figures. They show a development of “mathematical learning as the result of the students' work and ideas – and not as a result of imitating the teachers' actions” (González-Martín et al., 2014, p. 118).

The paper opens a question for the need of a precise definition of three (or more) types of resources for teaching and assessment which have by now been perceived as two distant groups. The theoretical part of this study shows why and how is the content about reflections and symmetry (as a property of figures) relevant and suitable for implementing different types of resources related to the diversity of aspects of mathematics in the classroom. This conclusion may be considered as the first step in the five step approach for content specification according to Prediger et al., (2015, p. 239). The empirical findings based on a one to one (a teacher - a pupil) case study offer insights in the “concrete professional demands” required for assessing pupils' knowledge about reflection (step 2 in the same approach, p. 239). They specify teacher's difficulties with tasks in geometry (Ribeiro, 2011) related to the exact content of reflection and symmetric figures. This shows an answer of the 'what' question (van den Heuvel-

Panhuizen, 2005) - content knowledge about reflections. The empirical explorations in this study may be widened by a greater range of cases taking into account teachers' perspectives (step 3 in a development of a design of a PD program according to the same group of authors). Further on, it confirms the existing difference between “a theoretical Situation, as an ideal-type model and its actual implementation in the classroom which allows assessment of the students' actual work about mathematics” (González-Martín et al., 2014, p. 120). Moreover, this “assessment of the students' actual work” indicates teacher's insufficient *personal resources* about reflection in relation to the components of teacher's knowledge (Levenberg & Patkin, 2014, p. 94) which directly meets the main RQ in this study. “What teachers assess” is not only the pupils' knowledge but also, although implicitly, their personal resources and specifically their own knowing of the knowledge in geometry. “What teachers should assess” is the overall understanding of a concept (in geometry, e. g. reflection) on the basis of pupils' individual (imaginary and abstract) resources and/ or physical and instructional resources, and regardless if it has been achieved by interactions with the *milieu* and with or without the teacher.

NOTES

1. The total scored points on the written assignment, according to the evaluation of the particular teacher, is 19 out of 22 points.
2. The task and the pupil's answers are translated from German to English by the author of this paper.

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